ENGINEERING MATHEMATICS IV

(FEG 404)

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TOPICS

<u>FURTHER NUMERICAL METHODS:</u>

- **FIRST ORDER DIFERENTIAL EQUATIONS**
  - ✓ **EULER'S EQUATION**
- **SECOND ORDER DIFFERENTIAL EQUATIONS**
  - ✓ **RUNGE- KUTTA METHOD**

EULER'S EQUATION

Euler's method is the easiest approach of the Numerical Methods for solving First Order Differential Equations.

Euler's Equation is given as: $\boldsymbol{f(a+h) = f(a) + hf'(a) \ldots .(1)}$

Equation (1) can also be written as: $\underline{\boldsymbol{y_1 = y_0 + h(y')_0 \ldots .(2)}}$

Example

Given that $\frac{dy}{dx} = 2(1 + x) - y$ with the initial condition that at x = 2, y=5. Find an appropriate value of y at x = 2.2

<u>Solution</u>

$$x_0 = 2, \qquad y_0 = 5$$

$$\frac{dy}{dx} = y' = 2(1 + x) - y$$

$$at\ (y')_0, \qquad x_0 = 2, \qquad y_0 = 5$$

$$substituting, (y')_0 = 2(1 + 2) - 5 = 1$$

$$\boldsymbol{therefore, \quad x_0 = 2, \quad y_0 = 5, \quad (y')_0 = 1,}$$

$$\boldsymbol{x_1 = 2.2\ (given)}$$

$$\boldsymbol{h = x_1 - x_0 = 2.2 - 2 = 0.2}$$

$$\boldsymbol{from\ h = x_1 - x_0, it\ implies\ that\ x_1 = h + x_{0,}\ so\ x_{n+1} = h + x_n}$$

$$\boldsymbol{from\ Euler's\ equation\ \ldots.(2){:}\ y_1 = y_0 + h(y')_0}$$

$$\boldsymbol{y_1 = 5 + 0.2(1) = 5.2}$$

$$\boldsymbol{(y')_1 = 2(1 + x_1) - y_1 = 2(1 + 2.2) - 5.2 = 1.2}$$

$$\boldsymbol{y_2 = y_1 + h(y')_1 = 5.2 + 0.2(1.2) = 5.44}$$

$$(\boldsymbol{y}')_2 = 2(1 + x_2) - y_2 = 2(1 + 2.4) - 5.44 = 1.36$$

$$y_3 = y_2 + h(y')_2 = 5.44 + 0.2(1.36) = 5.712$$

$$(y')_3 = 2(1 + x_3) - y_3 = 2(1 + 2.6) - 5.712 = 1.488$$

$$y_4 = y_3 + h(y')_3 = 5.712 + 0.2(1.488) = 6.0096$$

$$(y')_4 = 2(1 + x_4) - y_4 = 2(1 + 2.8) - 6.0096 = 1.5904$$

$$y_5 = y_4 + h(y')_4 = 6.0096 + 0.2(1.5904) = 6.32768$$

$$(y')_5 = 2(1 + x_5) - y_5 = 2(1 + 3.0) - 6.32768 = 1.67232$$

Present the results in a tabular form

N	x_n	y_n	$(y')_n$
0	2.0	5.0	1.0
1	2.2	5.2	1.2
2	2.4	5.44	1.36
3	2.6	5.712	1.488
4	2.8	6.0096	1.5904
5	3.0	6.32768	1.67232

Example 2

Obtain a numerical solution of the equation $\frac{dy}{dx} = 1 + x - y \quad with\ initial\ condition\ that\ y = 2, x = 1, for\ the\ range\ x = 1.0(0.2)2.0$

<u>Solution</u>

$$The\ range\ x = 1.0(0.2)2.0\ means\ from\ x = 1.0\ to = 2\ at\ constant\ intervals\ (h) = 0.2)$$

$$here\ x_0 = 1, y_0 = 2, and\ h = 0.2, substitute\ to\ get\ (y')_0\ ,$$

$$and\ later\ y_1\ up\ to\ y_5\ and\ (y')_5$$

Then, tabulate as shown below:

N	x_0	y_n	$(y')_n$
0	**1.0**	**2.0**	**0**
1	**1.2**	**2.0**	**0.2**
2	**1.4**	**2.04**	**0.36**
3	**1.6**	**2.112**	**0.488**
4	**1.8**	**2.2096**	**0.5904**
5	**2.0**	**2.32768**	**0.67232**

SECOND ORDER DIFFERENTIAL EQUATIONS

The two formulae for solving the second order differential equations are:

$$\boldsymbol{y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y')_0 \dots.(1)}$$

$$\boldsymbol{(y')_1 = (y')_0 + h(y'')_0 \dots.(2)}$$

Example

Solve the equation

$\boldsymbol{\frac{d^2y}{dx^2} = \frac{xdy}{dx} + y \quad for\ x = 0(0.2)1.0, given\ that\ at\ x = 0, y = 1, and\ \frac{dy}{dx} = 0}$

Solution

From the given equation $y'' = xy' + y,\ x_o = 0, y_0 = 1, {y'}_0 = 0, h = 0.2$

Therefore $(y'')_0 = x_0{y'}_0 + y_0 = 0(0) + 1 = 1$

From (1), $y_1 = y_0 + h(y')_0 + \frac{h^2}{2!}(y'')_0 = 1 + 0.2(0) + \frac{(0.2)^2}{2*1}(1) = 1 + 0.02 = 1.02$

From (2), $(y')_1 = (y')_0 + h(y'')_0 = 0 + 0.2(1) = 0.2$

$$x_1 = 0.2$$

$$(y'')_1 = x_1{y'}_1 + y_1 = 0.2 * 0.2 + 1.02 = 1.06$$

$$y_2 = y_1 + h(y')_1 + \frac{h^2}{2!}(y'')_1 = 1.02 + 0.2(0.2) + \frac{(0.2)^2}{2*1} * 1.06$$
$$= 1.0812$$

$$(y')_2 = (y')_1 + h(y'')_1 = 0.2 + 0.2(1.06) = 0.412$$

$$x_2 = 0.4$$

$$(y'')_2 = x_2 y'_2 + y_2 = 0.4 * 0.412 + 1.0812 = 1.246$$

$$y_3 = y_2 + h(y')_2 + \frac{h^2}{2!}(y'')_2$$
$$= 1.0812 + 0.2(0.412) + \frac{(0.2)^2}{2*1}(1.246) = 1.18852$$

$$(y')_3 = (y')_2 + h(y'')_2 = 0.412 + 0.2(1.246) = 0.6612$$

$$x_3 = 0.6$$

$$(y'')_3 = x_3 y'_3 + y_3 = 0.6 * 0.6612 + 1.18852 = 1.58524$$

$$y_4 = y_3 + h(y')_3 + \frac{h^2}{2!}(y'')_3$$

$$= 1.18852 + 0.2 * 0.6612 + \frac{(0.2)^2}{2 * 1}(1.58524)$$

$$= 1.3524648$$

$$(y')_4 = (y')_3 + h(y'')_3 = 0.6612 + 0.2(1.58524) = 0.978248$$

$$\boldsymbol{x_4 = 0.8}$$

$$\boldsymbol{(y'')_4 = x_4 y'_4 + y_4 = 0.8 * 0.978248 + 1.3524648 = 2.135063}$$

$$\boldsymbol{y_5 = y_4 + h(y')_4 + \frac{h^2}{2!}(y'')_4}$$

$$\boldsymbol{= 1.3524648 + 0.2(0.978248) + \frac{(0.2)^2}{2 * 1}(2.135063)}$$

$$\boldsymbol{= 1.590816}$$

$$\boldsymbol{(y')_5 = (y')_4 + h(y'')_4 = 0.978248 + 0.2 * 2.135063}$$

$$\boldsymbol{= 1.405261}$$

$$\boldsymbol{x_5 = 1.0}$$

$$\boldsymbol{(y'')_5 = x_5 y'_5 + y_5 = 1.0 * 1.405261 + 1.590816 = 2.996077}$$

Tabulate the Answers as shown below:

n	$\boldsymbol{x_n}$	$\boldsymbol{y_n}$	$\boldsymbol{y'_n}$	$\boldsymbol{y''_n}$
0	**0**	**1.0**	**0**	**1.0**
1	**0.2**	**1.02**	**0.2**	**1.06**
2	**0.4**	**1.0812**	**0.412**	**1.246**
3	**0.6**	**1.18852**	**0.6612**	**1.58524**
4	**0.8**	**1.352465**	**0.978248**	**2.135063**
5	**1.0**	**1.590816**	**1.405261**	**2.996077**

Runge-Kutta Method for Second Order Differential Equations

Starting with the given equation $\boldsymbol{y'' = f(x, y, y')}$ **and initial conditions that at**
$\boldsymbol{x = x_0,\ y = y_0,\ y' = (y')_0, we\ obtain\ the\ value\ of y_1\ at\ x = x_0 + h\ as\ follows}$:

(a) We evaluate

$$\boldsymbol{k_1 = \frac{1}{2}h^2(y'')_0}$$

$$\boldsymbol{k_2 = \frac{1}{2}h^2\{\left(x_0 + \frac{1}{2}h\right)\left(y'_0 + \frac{k_1}{h}\right) + \left(y_0 + \frac{1}{2}hy'_0 + \frac{1}{4}k_1\right)\}}$$

$$\boldsymbol{k_3 = \frac{1}{2}h^2\{\left(x_0 + \frac{1}{2}h\right)\left(y'_0 + \frac{k_2}{h}\right) + \left(y_0 + \frac{1}{2}hy'_0 + \frac{1}{4}k_1\right)\}}$$

$$k_4 = \frac{1}{2}h^2\{(x_0 + h)\left(y'_0 + \frac{2k_3}{h}\right) + \left(y_0 + hy'_0 + k_3\right)\}$$

(b) $P = \frac{1}{3}(k_1 + k_2 + k_3)$

$$Q = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$$

(c)

$$x_1 = x_0 + h$$

$$y_1 = y_0 + hy'_0 + P$$

$$(y')_1 = (y')_0 + \frac{Q}{h}$$

Example

If $\boldsymbol{y'' = xy' + y \quad for\ x = 0(0.2)1.0, given\ that\ when\ x = 0, y = 1, and\ \frac{dy}{dx} = 0.\ \ Solve\ the\ equation\ using\ Runge - Kutta\ Method.}$

<u>Solution</u>

$$\boldsymbol{y'' = xy' + y}$$

From the initial conditions $\boldsymbol{{y''}_0 = x_0{y'}_0 + y_0}$

But $\boldsymbol{x_0 = 0,\ \ y_0 = 1,\ \ {y'}_0 = 1,\ \ and\ h = x_1 - x_0 = 0.2}$

Therefore $\boldsymbol{(y'')_0 = 0 * 0 + 1 = 1}$

Step 1

Calculate k_1 to k_4

$$k_1 = \frac{1}{2}h^2(y'')_0 = \frac{1}{2}(0.2)^2(1) = 0.02$$

$$k_2 = \frac{1}{2}h^2\left\{\left(x_0 + \frac{1}{2}h\right)\left({y'}_0 + \frac{k_1}{h}\right) + \left(y_0 + \frac{1}{2}h{y'}_0 + \frac{1}{4}k_1\right)\right\} = \frac{1}{2}(0.2)^2\left\{\left(0 + \frac{1}{2} * 0.2\right)\left(0 + \frac{0.02}{0.2}\right) + \left(1 + \frac{1}{2} * 0.2 * 0 + \frac{1}{4} * 0.02\right)\right\} = 0.0203$$

$$k_3 = \frac{1}{2}h^2\left\{\left(x_0 + \frac{1}{2}h\right)\left(y'_0 + \frac{k_2}{h}\right) + \left(y_0 + \frac{1}{2}hy'_0 + \frac{1}{4}k_1\right)\right\}$$

$$= \frac{1}{2}(0.2)^2\left\{\left(0 + \frac{1}{2} * 0.2\right)\left(0 + \frac{0.0203}{0.2}\right)\right.$$

$$\left. + \left(1 + \frac{1}{2} * 0.2 * 0 + \frac{1}{4} * 0.02\right)\right\} = 0.020303$$

$$k_4 = \frac{1}{2}h^2\left\{(x_0 + h)\left(y'_0 + \frac{2k_3}{h}\right) + \left(y_0 + hy'_0 + k_3\right)\right\}$$

$$= \frac{1}{2}(0.2)^2\left\{(0 + 0.2)\left(0 + \frac{2 * 0.020303}{0.2}\right)\right.$$

$$\left. + (1 + 0 + 0.020303)\right\} = 0.021218$$

$$P = \frac{1}{3}(k_1 + k_2 + k_3) = \frac{1}{3}(0.02 + 0.0203 + 0.020303)$$
$$= 0.020201$$

$$Q = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4) =$$
$$\frac{1}{3}(0.02 + 2 * 0.0203 + 2 * 0.020303) + 0.021218 =$$
$$0.040808$$

$$recall\ that \quad x_1 = x_0 + h =$$
$$0. + 0.2 = 0.2$$

$$y_1 = y_0 + hy'_0 + P = 1 + 0.2(0) + 0.020201 = 1.020201$$

$$(y')_1 = (y')_0 + \frac{Q}{h} = 0 + \frac{0.040808}{0.2} = 0.20404$$

Therefore from the given equation $\boldsymbol{y'' = xy' + y}$

$$\boldsymbol{(y'')_1 = x_1 y'_1 + y_1 = 0.2 * 0.20404 + 1.020201 = 1.061009}$$

<u>Second Step</u>

Here
$\boldsymbol{h = 0.2,\ x_0 = 0.2,\ y_0 = \ 1.020201,\ (y')_0 = 0.20404,\ (y'')_0 = 1.061009}$

Calculate $\boldsymbol{k_1\ \ to\ \ k_4}$

Next, evaluate P and Q.

Also, calculate for $\boldsymbol{x_1,\ \ y_1,\ \ and\ (y')_1}$

Then, find $\boldsymbol{(y'')_1}$ **from the equation** $\boldsymbol{(y'')_1 = x_1 y'_1 + y_1}$

Other Steps

Repeat the process for
$\boldsymbol{x = 0.4,\ \ 0.6,}$
$\boldsymbol{0.8.\, and\ 1.0\ and\ then\ tabulate\ your\ values\ as\ shown.}$

x	y	y'	y''
0	1.0	0	1.0
0.2	1.020201	0.204040	1.061009
0.4	1.083284	0.433316	1.256610
0.6	1.197208	0.718333	1.628208
0.8	1.377107	1.101706	2.258472
1.0	1.648677	1.648721	3.297398

Questions?

Thanks a lot for listening.