FACULTY OF ENGINEERING MATHEMATICS IV

(FEG 404)

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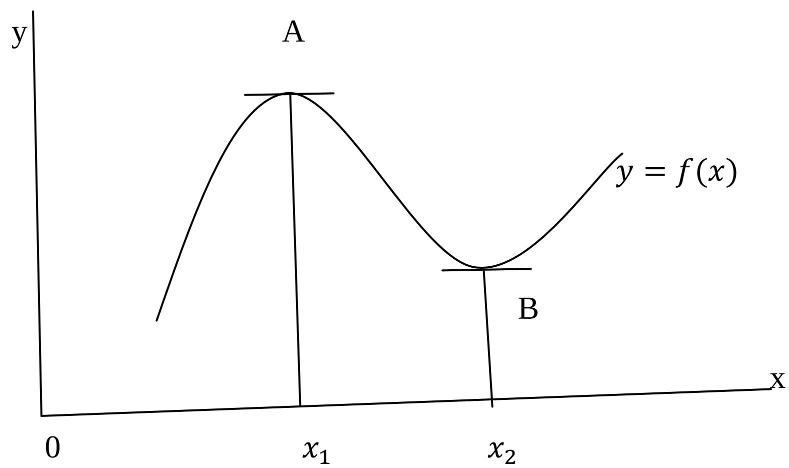
COURSE CONTENTS

Calculus of Variation and Further Numerical Methods

Lagranges Multipliers/Multiples

Euler Equation

Maximum and Minimum Values



At A and B $\frac{dy}{dx} = 0$

For maximum $\frac{d^2y}{dx^2}$ should be a negative value

For Minimum $\frac{d^2y}{dx^2}$ should be a positive value

Also, for a function $Z = f(x, y)$, the two conditions are as follows:

i. If $\frac{d^2F}{dx^2}$ and $\frac{d^2F}{dy^2}$ are both negative, it is a maximum;
ii. If $\frac{d^2F}{dx^2}$ and $\frac{d^2F}{dy^2}$ are both positive, it is a minimum.

Example

1. $If\ Z = x^2 + xy + y^2 + 5x - 5y + 3$
   a. Determine the stationary values of the function
   b. Establish if they are maximum or minimum;
   c. Find the actual maximum or minimum value of z.

Solution

a.

$$Z = x^2 + xy + y^2 + 5x - 5y + 3$$

$$\frac{\mathrm{dZ}}{\mathrm{dx}} = 2\mathrm{x} + \mathrm{y} + 5 = 0 \ldots . (1)$$

$$\frac{dZ}{dy} = x + 2y - 5 = 0 \ldots . (2)$$

Solve simultaneously:

From (2), $x = 5 - 2y \ldots . (3)$

Substitute in (1), $2(5 - 2y) + y + 5 = 0$

$$10 - 4y + y + 5 = 0$$

$$15 = 3y$$

$$y = 5$$

From (3), $x = 5 - 2(5) = -5$

The stationary values are (-5, 5)

b. from (1) $\frac{d^2Z}{dx^2} = 2$

also, from (2) $\frac{d^2Z}{dy^2} = 2$

since both $\frac{d^2Z}{dx^2}\ and\ \frac{d^2Z}{dy^2}$ are positive, it implies that the values are minimum.

c. To find the actual minimum value of Z, substitute $x = -5, and\ y = 5\ into\ Z.$

$$Z = 5^2 + (-5 * 5) + 5^2 + (5 * -5) - 5 * 5 + 3 = -22$$

Lagranges Multipliers

In Lagranges multiples you are expected to determine the stationary points of a function subject to a given constant. That is, two equations are given (U and ϕ).

Here, you should form two equations and then solve.

$$\frac{dU}{dx} + \lambda\frac{d\phi}{dx} = 0 \ldots. (1)$$

$$\frac{dU}{dy} + \lambda\frac{d\phi}{dy} = 0 \ldots. (2)$$

Example

Find the stationary points of the function

$$U = x^2 + y^2$$

$$\phi = x^2 + y^2 + 2x - 2y + 1$$

<u>Solution</u>

$$\frac{dU}{dx} = 2x, \qquad \frac{dU}{dy} = 2y$$

$$\frac{d\phi}{dx} = 2x + 2, \qquad \frac{d\phi}{dy} = 2y - 2$$

Next, the two given equations are formed.

$$\frac{dU}{dx} + \lambda\frac{d\phi}{dx} = 2x + \lambda(2x + 2) = 0,$$

$$x + \lambda(x + 1) = 0 \ldots. (1)$$

$$\frac{dU}{dy} + \lambda\frac{d\phi}{dy} = 2y + \lambda(2y - 2)$$

$$y + \lambda(y - 1) \ldots. (2)$$

$$from\ (1), x = -\lambda(x + 1) \ldots. (3)$$

$$from\ (2), y = -\lambda(y - 1) \ldots. (4)$$

$$from\ (3)\ and\ (4)\frac{x}{y} = \frac{-\lambda(x + 1)}{-\lambda(y - 1)}$$

$$cross\ multiplying\ gives - x = xy + y$$

$$y = -x$$

$$substituting\ in\ \phi\ gives\quad x^2 + ((-x)^2) + 2x - 2(-x) + 1 = 0$$

$$2x^2 + 4x + 1 = 0$$

$$applying\ d\ almighty\ formula, x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, where\ a = 2, \qquad b = 4, and\ c = 1$$

$$x = -1 \pm \frac{\sqrt{2}}{2}, recall\ that\ y = -x, therefore\ y = 1 \mp \frac{\sqrt{2}}{2}$$

Functions with three independent variables

Here, due to the three variables, you are expected to form three equations before solving.

The equations are:

$$\frac{dU}{dx} + \lambda\frac{d\phi}{dx} = 0 \ldots.(1)$$

$$\frac{dU}{dy} + \lambda\frac{d\phi}{dy} = 0 \ldots.(2)$$

$$\frac{dU}{dz} + \lambda\frac{d\phi}{dz} = 0 \ldots.(3)$$

The three equations alongside the constraints will yield the values of x, y, and z.

Example

Determine the stationary points of the function:

$$U = x^2 + 2y^2 + z\ subject\ to\ the\ constraint$$

$$\phi(x, z) = x^2 - z^2 - 2 = 0$$

<u>Solution</u>

$$U = x^2 + 2y^2 + z$$

$$\phi(x, z) = x^2 - z^2 - 2$$

$$\frac{dU}{dx} = 2x, \qquad \frac{d\phi}{dx} = 2x$$

$$\frac{dU}{dy} = 4y, \qquad \frac{d\phi}{dy} = 0$$

$$\frac{dU}{dz} = 1, \qquad \frac{d\phi}{dz} = -2z$$

Next is the formation of the three equations

$$\frac{dU}{dx} + \lambda\frac{d\phi}{dx} = 2x + \ \lambda 2x = 0, therefore\ \lambda = -1$$

$$\frac{dU}{dy} + \lambda\frac{d\phi}{dy} = 4y + \lambda(0) = 0, therefore\ y = 0$$

$$\frac{dU}{dz} + \lambda\frac{d\phi}{dz} = 1 + \lambda(-2z) = 0, therefore\ \lambda 2z = 1, but\ \lambda = -1,$$

$$-2z = 1, z = -\frac{1}{2}$$

$$substituting\ the\ value\ of\ z\ in\ \phi, we\ have\ \phi = x^2 - (-\frac{1}{2})^2 - 2 = 0$$

$$x^2 = 2 + \frac{1}{4} = \frac{9}{4}, therefore\ x = \pm\sqrt{\frac{9}{4}}, = \pm\frac{3}{2}$$

$$the\ stationary\ points\ are\ \left(\frac{3}{2}, 0, -\frac{1}{2}\right) and\ \left(-\frac{3}{2}, 0, -\frac{1}{2}\right)$$

Let’s consider a practical example

A hot spherical storage tank is a vertical cylinder surmounted by a hemispherical top of the same diameter. The tank is designed to hold 400 M^3 of liquid. Determine the total height and the diameter of the tank if the surface heat loss is to be a minimum.

Solution

$$Area\ of\ a\ hemisphere = 3\pi r^2$$

$$Area\ of\ a\ cylinder = 2\pi rh$$

$$Total\ surface\ Area\ (A) = 3\pi r^2 + 2\pi rh \ldots .(1)$$

Equation (1) is the function that has to be minimum, the constraint is the volume = 400 M^3

$$Volume\ of\ a\ hemisphere = \frac{2}{3}\pi r^3$$

$$Volume\ of\ a\ cylinder = \pi r^2 \mathrm{h}$$

$$Total\ Volume\ (V) = \pi r^2 \mathrm{h} + \frac{2}{3}\pi r^3 = 400 \ldots .(2)$$

$$\frac{dA}{dr} = 6\pi r + 2\pi h, \qquad \frac{dA}{dh} = 2\pi r$$

$$\frac{dV}{dr} = 2\pi rh + \pi r^2, \qquad \frac{dV}{dh} = \pi r^2$$

$$\text{so,} \frac{dA}{dr} + \lambda \frac{dV}{dr} = 6\pi r + 2\pi h + \ \lambda(2\pi rh + 2\pi r^2) = 0,$$

$$= 3\pi r + \pi h + \lambda(\pi rh + \pi r^2) = 0 \ldots .(3)$$

$$\text{also,} \frac{dA}{dh} + \lambda \frac{dV}{dh} = 2\pi r + \lambda \pi r^2 = 0 \ldots .(4)$$

$$from\ (4), 2\pi r = -\lambda \pi r^2\ therefore\ \lambda = -\frac{2}{r}$$

$$substituting\ in\ (3)\ gives, 3\pi r + \pi h - \frac{2}{r}(\pi rh + \pi r^2) = 0$$

$$solving\ gives\ r = h$$

$$substituting\ in\ (2)\ yields: \pi r^3 + \frac{2}{3}\pi r^3 = 400$$

$$3\pi r^3 + 2\pi r^3 = 1200$$

$$5\pi r^3 = 1200, \qquad \pi r^3 = 240, \qquad r^3 = \frac{240}{\frac{22}{7}} = \frac{240 * 7}{22} = 76.4$$

$$r = \sqrt[3]{76.4} = 4.243$$

$$recall\ that\ h = r = 4.243$$

$$Total\ height = h + r = 4.243 + 4.243 = 8.49M$$

$$Also, d = r * 2 = 8.49M$$