NNAMDI AZIKIWE UNIVERSITY, AWKA

FACULTY OF ENGINEERING

COURSE: FEG 303, ENGINEERING MATHEMATICS (III)

MATRICES AND APPLICATIONS

EQUATIONS AND MATRIX SOLUTIONS

Definition

A matrix is a set of real or complex numbers or (elements) arranged in rows and columns to form a rectangular array.

A matrix having m-rows and n-columns is called an m x n that is ('m by n') matrix and is referred to as having order m x n. However, a matrix is indicated by writing the array within brackets.

E.g. $\begin{pmatrix} 5 & 4 & 1 \\ 8 & 4 & 2 \end{pmatrix}$

This is a 2×3 matrix, that is a '2 by 3' matrix, where 5, 4, 1, 8, 4, 2 are the elements of the matrix.

It is important to state here that in describing the matrix, the number of rows is stated first and the number of columns second.

Thus $\begin{pmatrix} 6 & 4 & 2 \\ 4 & -3 & 2 \\ 9 & 8 & 9 \\ 5 & 6 & 7 \end{pmatrix}$ is a matrix of the order 4×3 which is 4 rows and 3 columns.

Also the matrix $\begin{pmatrix} 8 & 2 \\ 0 & 4 \\ 3 & 5 \end{pmatrix}$ is of order 3×2 which is 3 rows and 2 columns.

While the matrix $\begin{pmatrix} 3 & 5 & 7 & 8 \\ 4 & 6 & 8 & 9 \end{pmatrix}$ is of order 2×4 which is 2 rows and 4 columns.

More so, a matrix is simply an array of numbers, there is no arithmetical connection between the elements and it therefore differs from a determinant in that the elements cannot be multiplied together in any way to find a numerical value of the matrix. A matrix also has no numerical value. Again, rows and columns cannot be interchanged as was the case with determinants.

ROW MATRIX

A row matrix consists of 1 row only e.g. $\begin{pmatrix} 3 & 2 & 1 & 0 \end{pmatrix}$ is a row matrix of order 1 × 4.

COLUMN MATRIX

A column matrix consists of 1 column only e.g. $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$ is a column matrix of order 3 × 1.

A column matrix could also be written on one line but with 'curly' brackets e.g. $\{4 \quad 3 \quad 2\}$ which is the same column matrix of order 3 × 1.

For more clarifications look at the following examples stated below;

1. $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is a column matrix of order 2 × 1
2. $\begin{pmatrix} 8 & 0 & 6 & 2 \end{pmatrix}$ is a row matrix of order 1 × 4
3. $\begin{pmatrix} 4 & 2 & 1 \end{pmatrix}$ is a row matrix of order 1 × 3
4. $\{2 \quad 6 \quad 9\}$ is a column matrix of order 3×1 which is the same as $\begin{pmatrix} 2 \\ 6 \\ 9 \end{pmatrix}$ as earlier stated.

However, it is vital to note that in stating the x– and y– axis, we use simple row matrix.

Example: If P is the point (8, 6) then the 8 is the x– coordinate and 6 the y– coordinate. It is also interesting to know that in matrices generally, no commas are used to separate the elements.

Finally, matrices applications to Engineering are numerous, for example; Eigen values and vectors, the characteristics equation, the Cauley Hamilton theorem, Kronecker product, iterative solutions of Eigen values and vectors, Quadratic and Hermitian forms. Triangle decomposition via lower and upper mean (i.e. Lu Decomposition) and its applications, matrix transformation, rotation of axis, diagonalisation etc, however, we shall limit our lectures to suit the course content.

SINGLE ELEMENT MATRIX

A single number may be regarded as a 1×1 matrix that is having 1 row and 1 column.

DOUBLE SUFFIX NOTATION

In matrix each element has its own particular ‘address’ or location which can be defined by a system of double suffixes, while the first is indicating the row, the second indicated the column thus; $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

For example a_{24} indicate the element in the second row and fourth column, and for clarity sake see below an example that best explains the suffix notation stated above;

$$\begin{pmatrix} 6 & -5 & 1 & -3 \\ 4 & -4 & 8 & 5 \\ -4 & 6 & 4 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

MATRIX NOTATION

A whole matrix can be denoted by a single general element enclosed in brackets, or by a single letter printed in bold type where there is no ambiguity per say.

Example: $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$

Which can be denoted by (a_{ij}) or (a) or by A

Again $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

Which can be denoted by (x_i) or (x) or simply by x, however conditions exist as listed below;

a) For an (m x n) matrix, we use a bold capital letter e.g. A
b) For a row or column matrix, we use a lower case bold letter e.g. X

However, note that in handwritten work, we can indicate bold face type by a wavy line placed under the letter; e.g. $\underset{\sim}{\mathbf{A}}$ or $\underset{\sim}{X}$

Example 1: Assuming B represents a 2 × 3 matrix, write out the elements b_{ij} in the matrix using the double suffix notation solution.

$$\mathrm{B} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

EQUAL MATRICES

Two matrices are said to be equal if corresponding elements throughout are equal. Hence, the two matrices must also be the same order.

Thus if $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{33} \end{pmatrix} = \begin{pmatrix} 8 & 6 & 2 \\ 6 & 4 & 3 \end{pmatrix}$

(Note: i is 1 @ 1st column and j changes from 1 while i is 2 @ 2^{nd} column and j changes from 1)

Then $\begin{matrix} a_{11} = 8 \\ a_{12} = 6 \\ a_{13} = 2 \\ a_{21} = 6 \\ a_{22} = 4 \\ a_{23} = 3 \end{matrix}$

Hence if $(a_{ij}) = (x_{ij})$ then

$a_{ij} = x_{ij}$ for all values of i and j

Thus if $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \begin{pmatrix} 5 & -7 & 3 \\ 1 & 2 & 6 \\ 0 & 4 & 8 \end{pmatrix}$

Therefore
$$\begin{aligned} a &= 5 \\ b &= -7 \\ c &= 3 \\ d &= 1 \\ e &= 2 \\ f &= 6 \\ g &= 0 \\ h &= 4 \\ k &= 8 \end{aligned}$$

Hence c – f = -3 and a – k = -3 etc.

ADDITION AND SUBTRACTION OF MATRICES

Note that two matrices must be of the same order for them to be added or subtracted. However, the sum or difference is then determined by adding or subtracting corresponding elements.

Example 1: $\begin{pmatrix} 6 & 4 & 3 \\ 7 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 6 \\ 8 & 6 & 10 \end{pmatrix}$

$$= \begin{pmatrix} 6+2 & 4+4 & 3+6 \\ 7+8 & 5+6 & 6+10 \end{pmatrix} = \begin{pmatrix} 8 & 8 & 9 \\ 15 & 11 & 16 \end{pmatrix}$$

Example 2: $\begin{pmatrix} 6 & 2 & 4 \\ 8 & 6 & 10 \end{pmatrix} - \begin{pmatrix} 5 & 7 & 9 \\ 6 & 8 & 10 \end{pmatrix}$

$$= \begin{pmatrix} 6-5 & 2-7 & 4-9 \\ 8-6 & 6-8 & 10-10 \end{pmatrix} = \begin{pmatrix} 1 & -5 & -5 \\ 2 & -2 & 0 \end{pmatrix}$$

Example 3: $\begin{pmatrix} 6 & 5 & 4 & 1 \\ 2 & 3 & -7 & 8 \end{pmatrix} + \begin{pmatrix} 1 & 4 & 2 & 3 \\ 6 & -1 & 0 & 5 \end{pmatrix} = ?$

Solution:

$$= \begin{pmatrix} 6+1 & 5+4 & 4+2 & 1+3 \\ 2+6 & 3+(-1) & -7+0 & 8+5 \end{pmatrix} = \begin{pmatrix} 7 & 9 & 6 & 4 \\ 8 & 2 & -7 & 13 \end{pmatrix}$$

Example 4: $\begin{pmatrix} 8 & 3 & 6 \\ 5 & 2 & 7 \\ 1 & 0 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = ?$

Solution:

$$= \begin{pmatrix} 8-1 & 3-2 & 6-3 \\ 5-4 & 2-5 & 7-6 \\ 1-7 & 0-8 & 4-9 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 3 \\ 1 & -3 & 1 \\ -6 & -8 & -5 \end{pmatrix}$$

MULTIPLICATION OF MATRICES

1. **Scalar multiplication**

Multiplying a matrix by a single number (i.e. scalar) each individual element of the matrix is multiplied by that factor.

For example: $4 \times \begin{pmatrix} 3 & 2 & 4 \\ 4 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 12 & 8 & 16 \\ 16 & 8 & 24 \end{pmatrix}$

Therefore in general, $K\left(a_{ij}\right) = \left(Ka_{ij}\right)$; which means that in reverse, we can take a common factor out of every element not just one row or one column as in determinants.

Thus; $\begin{pmatrix} 10 & 25 & 45 \\ 35 & 15 & 50 \end{pmatrix}$ can be written as; $5 \times \begin{pmatrix} 2 & 5 & 9 \\ 7 & 3 & 10 \end{pmatrix}$

2. **Multiplication of two matrices**

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

Example 1: If A = $\left(\mathrm{a_{ij}}\right) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

and b = $(\mathrm{b_i}) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

then $A \cdot b = \overrightarrow{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \downarrow$ row by column

$= \begin{matrix} a_{11}b_1 & + & a_{12}b_2 & + & a_{13}b_3 \\ a_{21}b_1 & + & a_{22}b_2 & + & a_{23}b_3 \end{matrix}$

That means each element in the top row of A is multiplied by the corresponding element in the first column of b and the products added. Again, the second row of the product is found by multiplying each element in the second row of A by the corresponding element in the first column of b.

Example :

$$\begin{pmatrix}4 & 2 & 1\\ 3 & 2 & 0\end{pmatrix}\cdot\begin{pmatrix}4\\ 2\\ 0\end{pmatrix} = \begin{pmatrix}4\times 4\ + & 2\times 2\ + & 1\times 0\\ 3\times 4\ + & 2\times 2\ + & 0\times 0\end{pmatrix}$$

$$= \begin{pmatrix}16\ + & 4\ + & 0\\ 12\ + & 4\ + & 0\end{pmatrix} = \begin{pmatrix}20\\ 16\end{pmatrix}$$

Example: $\begin{pmatrix}2 & 3 & 5 & 1\\ 4 & 6 & 0 & 7\end{pmatrix}\cdot\begin{pmatrix}3\\ 4\\ 2\\ 9\end{pmatrix} = \begin{pmatrix}2\times 3\ + & 3\times 4\ + & 5\times 2\ + & 1\times 9\\ 4\times 3\ + & 6\times 4 & 0\times 2 & 7\times 9\end{pmatrix}$

$$= \begin{pmatrix}6\ + & 12\ + & 10\ + & 9\\ 12\ + & 24\ + & 0\ + & 63\end{pmatrix} = \begin{pmatrix}37\\ 99\end{pmatrix}$$

Also if A = $\begin{pmatrix}3 & 6 & 8\\ 1 & 0 & 2\end{pmatrix}$ and b = $\begin{pmatrix}7\\ 4\\ 5\end{pmatrix}$

Then A$\cdot b = \begin{pmatrix}3\times 7\ + & 6\times 4\ + & 8\times 5\\ 1\times 7\ + & 0\times 4\ + & 2\times 5\end{pmatrix} = \begin{pmatrix}21\ + & 24\ + & 40\\ 7\ + & 0\ + & 10\end{pmatrix} = \begin{pmatrix}85\\ 17\end{pmatrix}$

Example 2: If A = $(a_{ij}) = \begin{pmatrix}1 & 5\\ 2 & 7\\ 3 & 4\end{pmatrix}$

and B = $(b_{ij}) = \begin{pmatrix}8 & 4 & 3 & 1\\ 2 & 5 & 8 & 6\end{pmatrix}$

then A$\cdot B = \begin{pmatrix}1 & 5\\ 2 & 7\\ 3 & 4\end{pmatrix}\cdot\begin{pmatrix}8 & 4 & 3 & 1\\ 2 & 5 & 8 & 6\end{pmatrix}$

$$= \begin{pmatrix}1\times 8+5\times 2 & 1\times 4+5\times 5 & 1\times 3+5\times 8 & 1\times 1+5\times 6\\ 2\times 8+7\times 2 & 2\times 4+7\times 5 & 2\times 3+7\times 8 & 2\times 1+7\times 6\\ 3\times 8+4\times 2 & 3\times 4+4\times 5 & 3\times 3+4\times 8 & 3\times 1+4\times 6\end{pmatrix}$$

$$\begin{pmatrix}8+10 & 4+25 & 3+40 & 1+30\\ 16+14 & 8+35 & 6+56 & 2+42\\ 24+8 & 12+20 & 9+32 & 3+24\end{pmatrix} = \begin{pmatrix}18 & 29 & 43 & 31\\ 30 & 43 & 62 & 44\\ 32 & 32 & 41 & 27\end{pmatrix}$$

Again note that multiplying a (3 × 2) matrix and a (2 × 4) matrix gives a product order (3 × 4) i.e. (3 × 2) × (2 × 4) = order (3 × 4)

Also the product of an (l x m) matrix and (m x n) matrix has order (l x n) since n – n is the same as was observed above.

Thus if A = $\begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$ & B = $\begin{pmatrix} 3 & 1 \\ -4 & 5 \\ 2 & 6 \end{pmatrix}$. Find A$\cdot B$

$$A\cdot B = \begin{pmatrix} 4\times 3+2\times -4+0\times 2 & 4\times 1+2\times 5+0\times 6 \\ 3\times 3+2\times -4+1\times 2 & 3\ \times 1+2\times 5+1\times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 12-8+0 & 4+10+0 \\ 9-8+2 & 3+10+6 \end{pmatrix} = \begin{pmatrix} 4 & 14 \\ 3 & 19 \end{pmatrix}.$$

Example 3: A matrix can be squared only if it is itself a square matrix which means that the number of rows equals the number of columns

E.g. If A = $\begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}\cdot\begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 4\times 4+7\times 5 & 4\times 7+7\times 2 \\ 5\times 4+2\times 5 & 5\times 7+2\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 16+35 & 28+14 \\ 20+10 & 35+4 \end{pmatrix} = \begin{pmatrix} 51 & 42 \\ 30 & 39 \end{pmatrix}$$

Note that multiplication of matrices is defined only when;

- The number of columns in the first equals the number of rows in the second.

However, $\begin{pmatrix} 1 & 5 & 6 \\ 4 & 9 & 7 \end{pmatrix}\cdot\begin{pmatrix} 2 & 3 & 5 \\ 8 & 7 & 1 \end{pmatrix}$ has no meaning, since the conditions stated above does not apply. Again if A is an (m x n) matrix and B is an (n x m) matrix. Then products A·B and B·A are possible. E.g. if A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ and B = $\begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$

Thus A$\cdot$ B = $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\cdot\begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix}$

$$= \begin{pmatrix} 1\times 7+2\times 8+3\times 9 & 1\times 10+2\times 11+3\times 12 \\ 4\times 7+5\times 8+6\times 9 & 4\times 10+5\times 11+6\times 12 \end{pmatrix}$$

$$= \begin{pmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & 40+55+72 \end{pmatrix} = \begin{pmatrix} 50 & 68 \\ 122 & 167 \end{pmatrix}$$

AND B·A $= \begin{pmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 7\times 1 + 10 \times 4 & 7 \times 2 + 10 \times 5 & 7 \times 3 + 10 \times 6 \\ 8 \times 1 + 11 \times 4 & 8 \times 2 + 11 \times 5 & 8 \times 3 + 11 \times 6 \\ 9 \times 1 + 12 \times 4 & 9 \times 2 + 12 \times 5 & 9 \times 3 + 12 \times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 7 + 40 & 14 + 50 & 21 + 60 \\ 8 + 44 & 16 + 55 & 24 + 66 \\ 9 + 48 & 18 + 60 & 27 + 72 \end{pmatrix} = \begin{pmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{pmatrix}$$

Again in matrix multiplication, A·B ≠ B·A which means that multiplication is not commutative. The order of factors is important, hence in the product A·B, B is pre-multiplied by A and A is post multiplied by B.

Hence if A $= \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and B$= \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$

Then A· B $= \begin{pmatrix} 5 \times 9 + 2 \times -2 & 5 \times 2 + 2 \times 3 & 5 \times 4 + 2 \times 6 \\ 7 \times 9 + 4 \times -2 & 7 \times 2 + 4 \times 3 & 7 \times 4 + 4 \times 6 \\ 3 \times 9 + 1 \times -2 & 3 \times 2 + 1 \times 3 & 3 \times 4 + 1 \times 6 \end{pmatrix}$

$$= \begin{pmatrix} 45 + -4 & 10 + 6 & 20 + 12 \\ 63 + -8 & 14 + 12 & 28 + 24 \\ 27 + -2 & 6 + 3 & 12 + 6 \end{pmatrix} = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}$$

While B·A $= \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 9 \times 5 + 2 \times 7 + 4 \times 3 & 9 \times 2 + 2 \times 4 + 4 \times 1 \\ -2 \times 5 + 3 \times 7 + 6 \times 3 & -2 \times 2 + 3 \times 4 + 6 \times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 45 + 14 + 12 & 18 + 8 + 4 \\ -10 + 21 + 18 & -4 + 12 + 6 \end{pmatrix} = \begin{pmatrix} 45 + 26 & 18 + 12 \\ 39 - 10 & 18 - 4 \end{pmatrix}$$

$$= \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$$

TRANSPOSE OF A MATRIX

This is when the rows and columns of a matrix are interchanged, hence the followings occur:

- The first row becomes the first column
- The second row becomes the second column
- The third row becomes the third column

However, the new matrix so formed is called the transpose of the original matrix. If A is the original matrix, its transpose is denoted by $\bar{A}\ or\ A^T$

Thus if $\bar{A} = \begin{pmatrix} 6 & 8 \\ 4 & 6 \\ 2 & 4 \end{pmatrix}$

Then $A^T = \begin{pmatrix} 6 & 4 & 2 \\ 8 & 6 & 4 \end{pmatrix}$

Also given that A = $\begin{pmatrix} 6 & 4 & 2 \\ 8 & 6 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 4 & 0 \\ 2 & 4 \\ 1 & 3 \end{pmatrix}$

Then A·B = $\begin{pmatrix} 6 \times 4 + 4 \times 2 + 2 \times 1 & 6 \times 0 + 4 \times 4 + 2 \times 3 \\ 8 \times 4 + 6 \times 2 + 4 \times 1 & 8 \times 0 + 6 \times 4 + 4 \times 3 \end{pmatrix}$

$= \begin{pmatrix} 24 + 8 + 2 & 0 + 16 + 6 \\ 32 + 12 + 4 & 0 + 24 + 12 \end{pmatrix} = \begin{pmatrix} 34 & 22 \\ 48 & 36 \end{pmatrix}$

A·B = $\begin{pmatrix} 34 & 22 \\ 48 & 36 \end{pmatrix}$

$(A \cdot B)^T = \begin{pmatrix} 34 & 48 \\ 22 & 36 \end{pmatrix}$

SPECIAL MATRICES

1. **Square matrix**

   Square matrix is a matrix of order m x m.

   E.g. $\begin{pmatrix} 1 & 1 & 2 \\ 2 & 2 & 1 \\ 3 & 1 & 3 \end{pmatrix}$ is a 3×3 matrix

A square matrix (a_{ij}) is symmetric if $a_{ij} = a_{ji}$

e.g. $\begin{pmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{pmatrix}$

It is symmetrical about the leading diagonal but note that A = A^T

More so, a square matrix (a_{ij}) is skew-symmetric of a_{ij} = -a_{ij}

e.g. $\begin{pmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{pmatrix}$ and in this case A = A^T

2. **Diagonal matrix**

This is a square matrix with all elements zero except those on the leading diagonal,

Thus: $\begin{pmatrix} \mathbf{5} & 0 & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{7} \end{pmatrix}$

3. **Unit matrix**

This is a diagonal matrix in which the elements on the leading diagonal are all unity (1 all through) i.e. $\begin{pmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & \mathbf{1} \end{pmatrix}$

The unit matrix is denoted by I

If A = $\begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then $\text{A}\cdot = \begin{pmatrix} 5\times1+2\times0+4\times0 & 5\times0+2\times1+4\times0 & 5\times0+2\times0+4\times1 \\ 1\times1+3\times0+8\times0 & 1\times0+3\times1+8\times0 & 1\times0+3\times0+8\times0 \\ 7\times1+9\times0+6\times0 & 7\times0+9\times1+6\times0 & 7\times0+9\times0+6\times1 \end{pmatrix}$

$\text{A}\cdot \text{I} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$ that is A· I = A

Again if we form the product I ·A we obtain:

I ·A = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 1\times5+0\times1+0\times7 & 1\times2+0\times3+0\times9 & 1\times4+0\times8+0\times6 \\ 0\times5+1\times1+0\times7 & 0\times2+1\times3+0\times9 & 0\times4+1\times8+0\times6 \\ 0\times5+0\times1+1\times7 & 0\times2+0\times3+1\times9 & 0\times4+0\times8+1\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 5+0+0 & 2+0+0 & 4+0+0 \\ 0+1+0 & 0+3+0 & 0+8+0 \\ 0+0+7 & 0+0+9 & 0+0+6 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 4 \\ 1 & 3 & 8 \\ 7 & 9 & 6 \end{pmatrix} = \mathrm{A}$$

Therefore A·I = I·A = A

Therefore, the unit matrix I behaves very much like the unit factor in ordinary algebra and arithmetic.

4. Null matrix

A null matrix is one whose elements are all zero.

e.g. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and is denoted by 0

Thus if A·B = 0, we cannot say that, therefore A = 0 or B = 0.

Hence, if A = $\begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$

Then A·B = $\begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 9 \\ 4 & -6 \\ 2 & 4 \end{pmatrix}$

$$\Big(= \begin{matrix} 2\times1+1\times4\pm3\times2 & 2\times9+1\times-6\pm3\times4 \\ 6\times1+3\times4\pm9\times2 & 6\times9+3\times-6\pm9\times4 \end{matrix}\Big)$$

$$= \begin{pmatrix} 2+4+(-6) & 18+(-6)+(-12) \\ 6+12+(-18) & 54+(-18)+(-36) \end{pmatrix} = \begin{pmatrix} 2+4-6 & 18-6-12 \\ 6+12-18 & 54-18-36 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore A·B = 0 but clearly A ≠ 0 and B ≠ 0.

ASSIGNMENT

a) If A = $\begin{pmatrix} 2 & 4 & 5 & 6 \\ 3 & 2 & 7 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 4 & 6 & 2 & -8 \\ 6 & 4 & -2 & 6 \end{pmatrix}$

Determine (i) A + B (ii) A – B

b) If A = $\begin{pmatrix} 4 & 3 \\ 2 & 7 \\ 8 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 5 & 7 & 4 \\ 3 & 0 & 8 \end{pmatrix}$

Determine (i) 5A, (ii) A·B and (iii) B·A

c) If A = $\begin{pmatrix} 2 & 8 \\ 4 & 6 \\ 6 & 4 \end{pmatrix}$ and B = $\begin{pmatrix} 1 & 4 \\ 3 & 6 \\ 2 & 8 \end{pmatrix}$

Determine A·B

d) Given that A = $\begin{pmatrix} 8 & 6 & 4 \\ 6 & 4 & 2 \end{pmatrix}$

Determine (i) A^T (ii) $A_I\ A^T$

NEXT LECTURE

- Determinant of a square matrix
- Cofactors
- Adjoint of a square matrix
- Inverse of a square matrix
- Product of a square matrix and its inverse
- Solution of a set of linear equations
- Lu- decomposition method for solving system of linear equations
- Gaussian elimination method for solving a set of linear equations
- Eigen values and Eigen vectors.

DETERMINANT OF A SQUARE MATRIX

The determinant of a square is the determinant having the same elements as those of the matrix.

E.g. the determinant of $\begin{pmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{pmatrix}$ is?

$$\begin{vmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{vmatrix}$$

First we look for the minors of the elements (2, 3, 4)

Element 2 Element 3 Element 4

$$5\begin{vmatrix} 6 & 3 \\ 4 & 7 \end{vmatrix} 5\,(42-12)-2\begin{vmatrix} 0 & 3 \\ 8 & 7 \end{vmatrix} 2\,(0-24)+\begin{vmatrix} 0 & 6 \\ 8 & 4 \end{vmatrix} 2(0-48)$$

Thus the values of this determinant are:

$5\,(42\text{-}12)\text{-}\,2(0\text{-}\,24)+1(0\text{-}\,48)$

$5\,(30)\text{-}\,2(\text{-}24 \quad +1(\text{-}48)$

$=150+48=150$

However, the transpose of the matrix = $\begin{pmatrix} 5 & 0 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 7 \end{pmatrix}$

And its determinants is $5\begin{vmatrix} 6 & 4 \\ 3 & 7 \end{vmatrix} 5(42-12)-0\begin{vmatrix} 2 & 4 \\ 1 & 7 \end{vmatrix} 0\,(14-4)+8\begin{vmatrix} 2 & 6 \\ 1 & 3 \end{vmatrix} 8(6-6)$

Its value is $5(42-12)-0(14-4)+8(6-6)5(30)=150$

Therefore, the determinant of a square matrix has the same value as that of the determinant of the transpose matrix.

However, note here that a matrix whose determinant is zero is called a singular matrix.

E.g. the determinant of the matrix $\begin{pmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{pmatrix}$

Has the value;

$$\begin{vmatrix} 3 & 2 & 5 \\ 4 & 7 & 9 \\ 1 & 8 & 6 \end{vmatrix} = 3\begin{vmatrix} 7 & 9 \\ 8 & 6 \end{vmatrix} 3(42-73) - 2\begin{vmatrix} 4 & 9 \\ 1 & 6 \end{vmatrix} 2(24-9) + 5\begin{vmatrix} 4 & 7 \\ 1 & 8 \end{vmatrix} 5(32-7)$$

$$= 3(-30) - 2(15) + 5(25)$$

$$= -90 - 30 + 125 = -120 + 125 = 5$$

Again, the determinant value of the diagonal matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ is the value

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2\begin{vmatrix} 5 & 0 \\ 0 & 4 \end{vmatrix} 2(20-0) + 0\begin{vmatrix} 0 & 0 \\ 0 & 4 \end{vmatrix} 0(0-0) + 0\begin{vmatrix} 0 & 5 \\ 0 & 4 \end{vmatrix} 0(0-0)$$

$$= 2\,(20) + \ 0 + 0 = 40$$

COFACTORS

If $A = \ (a_{ij})$ is a square matrix, we can form a determinant of its element

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & & & & a_{2n} \\ a_{31} & a_{32} & a_{33} & & & & a_{3n} \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ a_{n1} & a_{n2} & a_{n3} & & & & a_{nn} \end{vmatrix}$$

Each element gives rise to a cofactor which is simply the minor of the elements in the determinants together with its ‘place sign’

E.g. the determinant of the matrix $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

Solution $det\ A = |A| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$

$$2\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} 2(0-24) - 3\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} 3\,(0-6) + 5\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} 5\,(16-1)$$

$$+(-48) - (18) + 75$$

$$-30 + 75 = 45$$

The place sign is + as in Element 2

i.e. $\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = 0 - 24 = -24$

Therefore, the cofactor of the element is $+(-24)$ i.e -24

Again, the minor of the element 3 is $\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = 0 - 6 = -6$

The place sign is $-$. Therefore the cofactor of the Element 3 is $-(-6) = 6$

However, in each case the minor is found by striking out of the line and column containing the element in question and forming a determinant of the remaining elements the appropriate place signs are given by. $\begin{pmatrix} + & - & + & - & . & . & . \\ - & + & - & + & . & . & . \\ + & - & + & & & & \\ \vdots & & & & & & \end{pmatrix}$

Which shows alternate plus and minus from the top left hand corner which carries $a +$

ADJOINT OF A SQUARE MATRIX

Consider $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$ its determinant is, $\det A\ (A) = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$ from which we can form a new matrix of the cofactors.

$c = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ Where A_{11} is cofactor of a_{11} A_{ij} , is the cofactor of a_{ij} etc

$A_{11} = +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = +(0-24) = -24$

$A_{12} = -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -(0-6) = 6$

$A_{13} = +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16-6) = 15$

$$A_{21} = -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0-21) = 20$$

$$A_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0-5) = -5$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8-3) = -5$$

$$A_{31} = +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18-5) = 13$$

$$A_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12-20) = 8$$

$$A_{33} = +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2-12) = -10$$

Therefore, the matrix of cofactors becomes;

$$C = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

And the transpose of C;

$$C^T = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

This is called the adjoint of the original matrix A and B is written as adj A. Finally, it is important to note here that for us to find the following:

a) Form the matrix C of cofactors

b) Write the transpose of the formed matrix C of cofactors i.e. C^T

Exercise

Find the adjoint of the matrix $A = \begin{pmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{pmatrix}$

Solution

$\det A = |A| = \begin{vmatrix} 5 & 2 & 1 \\ 3 & 1 & 4 \\ 4 & 6 & 3 \end{vmatrix}$ From here we can form a new matrix of the cofactors

To form the new matrix we look for the new elements of the new matrix to be formed

$$A_{11} = +\begin{vmatrix} 1 & 4 \\ 6 & 3 \end{vmatrix} = +(3-24) = -21$$

$A_{12} = - \begin{vmatrix} 3 & 4 \\ 4 & 3 \end{vmatrix} = -(9-16) = 7$

$A_{13} = - \begin{vmatrix} 3 & 1 \\ 4 & 6 \end{vmatrix} = +(18-4) = 14$

$A_{21} = - \begin{vmatrix} 2 & 1 \\ 6 & 3 \end{vmatrix} = -(6-6) = 0$

$A_{22} = + \begin{vmatrix} 5 & 1 \\ 4 & 3 \end{vmatrix} = +(15-4) = 11$

$A_{23} = - \begin{vmatrix} 5 & 2 \\ 4 & 6 \end{vmatrix} = -(30-8) = -22$

$A_{31} = + \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = -(8-1) = 7$

$A_{32} = - \begin{vmatrix} 5 & 1 \\ 3 & 4 \end{vmatrix} = -(20-3) = -17$

$A_{33} = + \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = +(5-6) = -1$

$c = \begin{pmatrix} -21 & 7 & 14 \\ 0 & 11 & -22 \\ 7 & -17 & -1 \end{pmatrix}$

$Adj\ A = C_T = \begin{pmatrix} -21 & 0 & 7 \\ 7 & 11 & -17 \\ 14 & -22 & -1 \end{pmatrix}$

Home work

Find the adjoint of the matrix $A = \begin{pmatrix} 6 & 3 & 2 \\ 4 & 2 & 5 \\ 5 & 7 & 4 \end{pmatrix}$

INVERSE OF A SQUARE MATRIX

The adjoint of a square matrix is important, since it enables us to form the inverse of the matrix. However if each element of the adjoint of A is divide by the value of the determinant of A, i.e.$|A|$, provide $|A| \neq 0$, the resulting matrix is called the inverse of A and is denote by A^{-1}

Consider $A = \begin{pmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{pmatrix}$

$\det A = |A| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$

Therefore, $2 \begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} \ 2\,(0-24) - 3 \begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} 3\,(0-6) + 5 \begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} 5(16-1)$

$= 2\,(0-24) - 3\,(0-6) = +5\,(16-1) = 45$

Therefore the matrixes of cofactors C which we have done before now under adjoint of a square matrix are:

$$C = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

and its adjoint i.e. $C^T = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 15 & -5 & -10 \end{pmatrix}$

Hence the inverse of A is given by,

$$A^{-1} \begin{pmatrix} -24/45 & 20/45 & 13/45 \\ 6/45 & -5/45 & 8/45 \\ 15/45 & -5/45 & -10/45 \end{pmatrix} = 1/45 \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Therefore for us to form the inverse of a square matrix A we do the following,

1. Evaluate the determinant of A i.e. $|A|$
2. Form a matrix C of the cofactors of the elements of $|A|$
3. Unite the transpose of C, i.e. C^T to obtain the adjoint of A
4. Divide each element of C^T by $|A|$
5. The resulting matrix is the inverse A^{-1} of the original matrix A

E.g. find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

Solution

1. First we Evaluate the determinant of A i.e. $|A|$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{vmatrix} \Rightarrow 1 \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} 1\ (2-0) - 2 \begin{vmatrix} 4 & 5 \\ 6 & 2 \end{vmatrix} 2\ (8-30) + \ 3 \begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} 3\ (0-6)$$

$= 2 + 44 - 18 = 28$

i.e. $46 - 18 = 28$

2) We form a matrix c of the cofactors of the elements as we use to do in the past.

$$A_{11} = + \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = +(2-0) = 2$$

$$A_{12} = - \begin{vmatrix} 4 & 5 \\ 6 & 2 \end{vmatrix} = -(8-30) = 22$$

$$A_{13} = + \begin{vmatrix} 4 & 1 \\ 6 & 0 \end{vmatrix} = +(0-6) = -6$$

$A_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 2 \end{vmatrix} = -(4-0) = -4$

$A_{22} = +\begin{vmatrix} 1 & 3 \\ 6 & 2 \end{vmatrix} = +(2-18) = -16$

$A_{23} = -\begin{vmatrix} 1 & 2 \\ 6 & 0 \end{vmatrix} = -(0-12) = 12$

$A_{31} = +\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = +(10-3) = 7$

$A_{32} = -\begin{vmatrix} 1 & 3 \\ 4 & 5 \end{vmatrix} = -(5-12) = 7$

$A_{33} = +\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = +(1-8) = 7$

Thus our C becomes,

$$C = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & 7 \end{pmatrix}$$

(3) We write down the transpose of C, i.e. C^T, to obtain the adjoint of A.

$$adj\ A = C^T = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$$

(4) Finally, we divide the elements of adj A by the value of $|A|$ ie 28 to get A^{-1}, the value of A.

Thus $A^{-1} = \begin{pmatrix} 2/28 & -4/28 & 7/28 \\ 22/28 & -16/28 & 7/28 \\ -6/28 & 12/28 & -7/28 \end{pmatrix}$

Home work

Determinant the inverse of $A = \begin{pmatrix} 2 & 7 & 5 \\ 3 & 1 & 6 \\ 5 & 0 & 8 \end{pmatrix}$

PRODUCT OF A SQUARE MATRIX AND ITS INVERSE

However, we have seen from the previous example that when $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

$A^{-1} = 1/28 \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$ Which is the C^T of the above matrix and 28 is the determinant of the matrix

Then $A^{-1} \, . A = \frac{1}{28} \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} . \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$

$$\frac{1}{28} = \begin{pmatrix} 2 \times 1 & -4 \times 4 & 7 \times x6 & 2 \times 2 & -4 \times 1 & 7 \times 0 & 2 \times 3 & -4 \times 5 & 2 \times 7 \\ 2 & -16 & +42 & 4 & -4 & +0 & 6 & -20 & +14 \\ 22 \times 1 & -16 \times 4 & 7 \times 6 & 22 \times 2 & -16 \times 1 & 7 \times 0 & 22 \times 3 & -16 \times 5 & 7 \times 2 \\ 22 & -64 & +42 & 44 & -16 & +0 & 66 & -80 & +14 \\ -6 \times 1 & 12 \times 4 & -7 \times 6 & -6 \times 2 & 12 \times 1 & -7 \times 0 & -6 \times 3 & 12 \times 5 & -7x2 \\ -6 & 48 & -42 & -12 & 12 & +0 & -18 & 60 & -14 \end{pmatrix}$$

$$= \frac{1}{28} = \begin{pmatrix} 28 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathrm{I}$$

thus $\mathrm{A}^{-1}. \mathrm{A} = \mathrm{I}$

However $A. A^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} . 1/28 \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} = 1/$

$28 \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} . \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$

When we multiply out we have

$$A. A^{-1} = 1/28 \begin{pmatrix} 28 & 0 & 0 \\ 0 & 28 & 0 \\ 0 & 0 & 28 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1$$

Therefore $A_1 \, A^{-1} \, . A = 1$

Again let us note that the product of a square matrix and its inverse in whatever order the factors are written is the unit matrixes of the same matrix order.

SOLUTION OF A SET OF LINEAR EQUATIONS

Consider the set of linear equations

$$\begin{array}{cccccc} a_{11}x_1 & +a_{12}x_2 & +a_{13}x_3 & + \ldots & +a_{1n}x_n & = b_1 \\ a_{21}x_1 & +a_{22}x_2 & +a_{32}x_3 & + \ldots & +a_{2n}x_n & = b_2 \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ a_{n1}x_1 & +a_{n2}x_2 & +a_{n3}x_3 & + \ldots & +a_{nn}x_n & = b_n \end{array}$$

Then from our knowledge of matrix multiplication this can be written in matrix form as;

$$\begin{pmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & & & . \\ . & . & & & . \\ . & . & & & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

such that $\mathrm{A.X} = \mathrm{b}$

$$where\ A = \begin{pmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & & & & . \\ . & . & & & & . \\ . & . & & & & . \\ a_{n1} & a_{n2} & & & & a_{nn} \end{pmatrix}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{pmatrix} \ and \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{pmatrix}$$

If we multiply both sides of the matrix equation by the inverse of A, we have;

$A^{-1}.A.X = A^{-1}.b\ (note\ A.X =)b$

But $A^{-1}.A = 1\ as\ earlier\ stated$

Hence $1.X = A^{-1}.b\ ie\ X = A^{-1}.b(note\ A.A^{-1} = A^{-1}.A = 1)$

Therefore, if we form the inverse of the matrix of the matrix of coefficients and pre-multiply matrix b by it, we shall obtain the matrix of the solutions of x.

Example: Solve the set of equations by inverse method.

$x_1 + 2x_2 + x_3 = 4$

$3x_1 - 4x_2 - 2x_3 = 2$

$5x_1 + 3x_2 + 5x_3 = -1$

Solution

- First we write the set of equation in matrix form which will give us;

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \quad . \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$$

$ie\ A\ .x\ = b\ thus;\ x\ =\ A^{-1}\ .b$

- The next step is to find the inverse of A, where A is the matrix of the coefficients of x.

$$A^{-1} = \begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix}$$

$to\ find\ Det.A = |A| = 1\begin{vmatrix} -4 & -2 \\ 3 & 5 \end{vmatrix}\ 2\begin{vmatrix} 3 & -2 \\ 5 & 5 \end{vmatrix}\ 1\begin{vmatrix} 3 & -4 \\ 5 & 3 \end{vmatrix}$

$1(-20 + 6) = 14\ \ 2\ (15 + 10) = 50\ \ (1\ 9 + 20) = 29$

$|A| = -14 - 15 + 29 = 29 - 64 = -35$

$therefore\ |A| = -35$

Again for the cofactor we have;

$A_{11} = +(-20 + 6) = -14$

$A_{12} = -(15 + 10) = -24$

$A_{13} = +(9 + 20\ ) = \quad 29$

$A_{21} = -(10 - 3) = \quad -7$

$A_{22} = +(5 - 5) \ = \quad 0$

$A_{23} = -(3 - 10) \ = \quad 7$

$A_{31} = +(-4 + 4) \ = \ 0$

$A_{32} = (-2 - 3) \ = \quad 5$

$A_{33} = +(-4 - 6) \ = -10$

$therefore\ C = \begin{pmatrix} -14 & -25 & 29 \\ -7 & 0 & 7 \\ 0 & 5 & -10 \end{pmatrix}$

$thus\ adj\ A = C^T = \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$

$Now\ |A| = -35\ therefore\ A^{-1} = \frac{adj\ A}{|A|}$

$= -1/35 \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$

Again X = $A^{-1}.b = 1/35 \begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} . \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$

$= -4 \times 4 + -7 \times 2 + 0 \times -1$ ___(1)

$= -25 \times 4 + 0 \times 2 + 5 \times -1$ ____(2)

$= 29 \times 4 + 7 \times 2 + -10 - 1$ _____ (3)

Equation (1) becomes

$= -56 - 14 + 0 = -70$

$Hence - 70 \times -1/35 = 2$

$= (-70, 2)$ ____(1) Above

Equation (2) becomes

$= -100 + 0 + -5 = -105$

$Hence - 105 \times -\frac{1}{35} = 3$

$= (105, 3)$ ______(2)

Equation (3) becomes

Finally 166 + 14 +10 = 140

Hence 140 $\times -1/35 = -4$

$= (140, -4)$ ___(3) above

$therefore\ X = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

$Hence\ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$

$Thus\ x_1 = 2,\ x_2 = 3\ and\ x_3 = -4$

DETERMINANT METHOD OF SOLVING SYSTEM OF EQUATIONS

Determinant method seems to be faster than the inverse method because of its straight forwardness. However, care must be taken while observing the sign conventions to avoid making mistakes that can make the whole matrix equation wrong. The few steps to take in this method are listed below;

1. Put the set equations in matrix form
2. Find the determinant |A| of the formed matrix, which could be denoted by Δx_0.
3. Replace the first column of the formed matrix with the constants to find its determinant, which could be denoted by Δx_1. Then divide ${}^{\Delta x_1}/_{\Delta x_0}$ to find x_1.
4. Replace the second column of the formed matrix with the constants to find its determinant, which could be denoted by Δx_2. Then divide ${}^{\Delta x_2}/_{\Delta x_0}$ to find x_2.
5. Replace the third column of the formed matrix with the constants to find its determinant, which could be denoted by Δx_3. Then divide ${}^{\Delta x_3}/_{\Delta x_0}$ to find x_3.
6. Continue the same process depending on the matrix type (i.e. 3×3, 4×4, 5×5, etc).
7. Compute your values for x_1, x_2, x_3 ….. etc. which forms the matrix of the solution x.

Example: Solve the system of equations expressed below and determine also the matrix of the solution f x.

$x_1 + 2x_2 + 3x_3 = 4$

$3x_1 - 4x_2 - 2x_3 = 2$

$5x_1 + 3x_2 + 5x_3 = -1$

Solution

STEP 1: forming the matrix we have; $\begin{pmatrix} 1 & 2 & 3 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix}$

STEP 2: finding the Determinant |A| denoted by Δx_o we have;

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} = 1 \begin{vmatrix} -4 & -2 \\ 3 & 5 \end{vmatrix} - 2 \begin{vmatrix} 3 & -2 \\ 5 & 5 \end{vmatrix} \; 3 \begin{vmatrix} 3 & -4 \\ 5 & 3 \end{vmatrix}$$

$\Delta x_o = 1(-20 + 6) = -14;\ 2(15 + 10) = 50;\ 3(9 + 20) = 87$

$\Delta x_o = -14 - 50 + 87 = 87 - 64 = 23$

$\Delta x_o = 23$

STEP 3: Replace the first column of the formed matrix with the constants and find its determinant denoted by Δx_1 then divide Δx_1 by Δx_o to get x_1.

$$\left(\begin{pmatrix}1\\3\\5\end{pmatrix}\begin{matrix}2 & 3\\-4 & -2\\3 & 5\end{matrix}\right) \Rightarrow \begin{pmatrix}4 & 2 & 3\\2 & -4 & -2\\-1 & 3 & 5\end{pmatrix} = 4\begin{vmatrix}-4 & -2\\3 & 5\end{vmatrix} - 2\begin{vmatrix}2 & -2\\-1 & 5\end{vmatrix} + 3\begin{vmatrix}2 & -4\\-1 & 3\end{vmatrix}$$

$\Delta x_1 = 4(-20+6);\ -2(10-2);\ \ 3(6-4)$

$= 4(-14) = -56;\ -2(8) = -16;\ \ 3(2) = 6$

$\Delta x_1 = -56 - 16 + 6 = -66$

$\therefore\ x_1 = \frac{\Delta x_1}{\Delta x_0} = \frac{-66}{23} = -2.8696$

STEP 4: Replace the second column of the formed matrix with the constants and find its determinant denoted by Δx_2 then divide Δx_2 by Δx_0 to get x_2.

$$\left(\begin{matrix}1\\3\\5\end{matrix}\begin{pmatrix}2\\-4\\3\end{pmatrix}\begin{matrix}3\\-2\\5\end{matrix}\right) \Rightarrow \begin{pmatrix}1 & 4 & 3\\3 & 2 & -2\\5 & -1 & 5\end{pmatrix} = 1\begin{vmatrix}2 & -2\\-1 & 5\end{vmatrix} - 4\begin{vmatrix}3 & -2\\5 & 5\end{vmatrix} + 3\begin{vmatrix}3 & 2\\5 & -1\end{vmatrix}$$

$\Delta x_2 = 1(10-2);\ -4(15+20);\ \ 3(-3-10)$

$\Delta x_2 = 1(8) = 8;\ -4(25) = -100;\ \ 3(-13) = -39$

$\Delta x_2 = 8 - 100 - 39 = -131$

$\therefore x_2 = \frac{\Delta \mathrm{x}_2}{\Delta \mathrm{x}_0} = \frac{-131}{23} = -5.6957$

STEP 5: Replace the third column of the formed matrix with the constants and find its determinant denoted by Δx_3 then divide Δx_3 by Δx_0 to get x_3.

$$\left(\begin{matrix}1 & 2\\3 & -4\\5 & 3\end{matrix}\begin{pmatrix}3\\-2\\5\end{pmatrix}\right) \Rightarrow \begin{pmatrix}1 & 2 & 4\\3 & -4 & 2\\5 & 3 & -1\end{pmatrix} = 1\begin{vmatrix}-4 & 2\\3 & -1\end{vmatrix} - 2\begin{vmatrix}3 & 2\\5 & -1\end{vmatrix} + 4\begin{vmatrix}3 & -4\\5 & 3\end{vmatrix}$$

$\Delta x_3 = 1(4-6);\ -2(-3-10);\ \ 4(9+20)$

$\Delta x_3 = 1(-2) = -2;\ -2(-13) = 26;\ \ 4(29) = 116$

$\Delta x_3 = -2 + 26 + 116 = 140$

$\therefore x_3 = \frac{\Delta x_3}{\Delta x_0} = \frac{140}{23} = 6.0869$

STEP 6: Continue the same process if need be. However, there is no need for continuation as could be seen.

STEP 7: Complete your values of x_1, x_2, x_3 etc.

Finally x = $\frac{1}{23}\begin{pmatrix} -66 \\ -131 \\ 140 \end{pmatrix} = \begin{pmatrix} -2.8696 \\ -5.6957 \\ 6.0869 \end{pmatrix}$

Hence x = $\begin{pmatrix} \mathrm{x}_1 \\ \mathrm{x}_2 \\ \mathrm{x}_3 \end{pmatrix} = \begin{pmatrix} -2.8696 \\ -5.6957 \\ 6.0869 \end{pmatrix}$

$x_1 = -2.8696$
Finally $x_2 = -5.6957$
$x_3 = 6.0869$