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Computation of a Morse-Smale Complex of a Discrete Morse function using Minimum Spanning Forest from Mathematical Morphology

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1 Mathematical background

Let us denote the vertices (the edges) of a graph G by $V(G)$ ($E(G)$).

1.1 Simplicial complexes and pseudomanifolds

We call *(abstract) simplex* any finite nonempty set of arbitrary elements. The *dimension* of a simplex x, denoted by $\dim(x)$, is the number of its elements minus one. In the following, a simplex of dimension d will also be called a *d-simplex*. If x is a simplex, we set $\mathrm{Clo}(x) = \{y | y \subseteq x, y \neq \emptyset\}$. A finite set X of simplices is a *cell* if there exists $x \in X$ such that $X = \mathrm{Clo}(x)$.

If X is a finite set of simplices, we write $\mathrm{Clo}(X) = \{\mathrm{Clo}(x) | x \in X\}$, the set $\mathrm{Clo}(X)$ is called the *(simplicial) closure* of X. A finite set X of simplices is a *(simplicial) complex* if $X = \mathrm{Clo}(X)$. Let X be a complex. Any element in X is a face of X and we call *d-face* of X any face of X whose dimension is d. Any d-face of X that is not included in any $(d+1)$-face of X is called a *$(d-)$facet* of X or a *maximal face* of X. The dimension of X, written $\dim(X)$, is the largest dimension of its faces: $\dim(X) = \max\{\dim(x) | x \in X\}$. If d is the dimension of X, we say that X is *pure* whenever the dimension of all its facets equals d.

Let X be a set of simplices, and let $d \in \mathbb{N}$. Let $\pi = \langle x_0, \ldots, x_l \rangle$ be an ordered sequence of d-simplices in X. The sequence π is a *d-path* from x_0 to x_l in X if $x_{i-1} \cap x_i$ is a $(d-1)$-simplex in X, for any $i \in \{1, \ldots, l\}$. Two d-simplices x and y in X are said to be *d-linked* for X if there exists a d-path from x to y in X. We say that the set X is *d-connected* if any two d-simplices in X are d-linked for X.

Let X be a set of simplices, and let $\pi = \langle x_0, \ldots, x_l \rangle$ be d-path in X. The d-path π is said *simple* if for any two distinct i and j in $\{0, \ldots, l\}$, $x_i \neq x_j$. It can

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be easily seen that X is d-connected if and only if, for any two d-simplices x and y of X, there exists a simple d-path from x to y in X.

A complex X of dimension d is said to be a *$d-$pseudomanifold* if:

1. X is pure; and
2. any $(d-1)-$face of X is included in exactly two $d-$faces of X; and
3. X is $d-$connected.

Let $\mathbb{M}$ be a d-pseudomanifold. Let $x \in \mathbb{M}$, the star of X (in $\mathbb{M}$), denoted by $\mathrm{St}(x)$, is the set of all simplices of $\mathbb{M}$ that include x, i.e., $\mathrm{St}(x) = \{y \in \mathbb{M} | x \subseteq y\}$. If A is a subset of $\mathbb{M}$, the set $\mathrm{St}(A) = \cup_{x \in A} star(x)$ is called the *star* of A (in $\mathbb{M}$). A set A of simplices of $\mathbb{M}$ is a star (in $\mathbb{M}$) if $A = star(A)$.

1.2 Simplicial stacks [2,3]

Let $\mathbb{M}$ be a n-pseudomanifold. Let F be a mapping $\mathbb{M} \to \mathbb{Z}$. For any face h of $\mathbb{M}$, the value $F(h)$ is called the *altitude* of F at h. For $k \in \mathbb{Z}$, the *k-section* of F, denoted by $[F \geq k]$ is equal to $\{h \in \mathbb{M}\ ;\ F(h) \geq k\}$.

We say that a subset A of $\mathbb{M}$ is a *minimum* of F at altitude $k \in \mathbb{Z}$ when A is a connected component of $[F \leq k] := \{h \in \mathbb{M}\ ;\ F(h) \leq k\}$ and $A \cap [F \leq k-1] = \emptyset$. In the following, we denote by $M_-(F)$ the union of all minima of F.

A *simplicial stack* F on $\mathbb{M}$ is a map from $\mathbb{M}$ to $\mathbb{Z}$ which satisfies that any of its k-section is a (possibly empty) simplicial complex. In other words, a map F is a simplicial stack if, for any two faces σ and τ of $\mathbb{M}$ such that $\sigma \subseteq \tau$, $F(\sigma) \geq F(\tau)$.

Let σ be any face of $\mathbb{M}$. When σ is a free face for $[F \geq F(\sigma)]$, we say that σ is a *free face for* F. If σ is a free face for F, there exists a unique face τ in $[F \geq F(\sigma)]$ such that (σ, τ) is a free pair for $[F \geq F(\sigma)]$ and we say that (σ, τ) is a *free pair* for F.

Let (σ, τ) be a free pair for F, then it is also a free pair for $[F \geq F(\sigma)]$. Thus, τ is a face of $[F \geq F(\sigma)]$, and we have $\sigma \subseteq \tau$. Therefore, we fave $F(\tau) \geq F(\sigma)$ and $F(\tau) \leq F(\sigma)$ (since F is a stack), which imply that $F(\tau) = F(\sigma)$.

Let $\mathbb{M}_{\mathrm{sub}} \subseteq \mathbb{M}$, the *indicator function* of $\mathbb{M}_{\mathrm{sub}}$, denoted by $1_{\mathbb{M}_{\mathrm{sub}}}$, is the map from $\mathbb{M}$ into $\mathbb{Z}$ such that $1_{\mathbb{M}_{\mathrm{sub}}}(\sigma) = 1$ when σ belongs to $\mathbb{M}_{\mathrm{sub}}$ and 0 otherwise. The *lowering* of F at $\mathbb{M}_{\mathrm{sub}}$ is the map $F - 1_{\mathbb{M}_{\mathrm{sub}}}$ from $\mathbb{M}$ into $\mathbb{Z}$.

Let $(\sigma^{(d-1)}, \tau^{(d)})$ be a free pair for F. The map $F - 1_{\sigma,\tau}$ is called an *elementary d-collapse of* F. Thus, this elementary d-collapse is obtained by removing 1 to the values of F at σ and τ. Note that the obtained mapping is still a simplicial stack.

If a simplicial stack F_2 is the result of a series of elementary d-collapses on F, then we say that F_2 is a *d-collapse* of F. If furthermore, there is no free pair $(\sigma^{(d-1)}, \tau^d)$ for F_2, then F_2 is an *ultimate d-collapse* of F.

Proposition 1 (Sets and ultimate collapses [2]). *Let X be a subcomplex of the pseudomanifold $\mathbb{M}$. If the dimension of X is equal to $n \geq 0$, then necessarily there exists a free n-pair for X. In other words, the ultimate n-collapse of X is of dimension lower than the one of $\mathbb{M}$. Consequently, we say that an ultimate n-collapse is* thin.

The *support* or *divide* of a simplicial stack F is the set of all faces of $\mathbb{M}$ which do not belong to any minimum of F. Note that since the union of all minima of F, $M_{-}(F)$, is a star (and then open), the divide is a simplicial complex.

1.3 Discrete Morse functions [5]

We recall that a function $F : A \to B$ is said to be $2-1$ when for every $b \in B$, there exist at most two values $a_1, a_2 \in A$ such that $f(a_1) = f(a_2) = b$.

Let $\mathbb{K}$ be a simplicial complex. A function $f : \mathbb{K} \to \mathbb{Z}$ is called *weakly increasing* if $F(\sigma) \leq F(\tau)$ whenever the two faces σ, τ of $\mathbb{K}$ satisfy $\sigma \subseteq \tau$. A *basic discrete Morse function* $F : \mathbb{K} \to \mathbb{Z}$ is a weakly discrete Morse function which is at most $2-1$ and satisfies the property that if $f(\sigma) = f(\tau)$, then $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.

Let $F : \mathbb{K} \to \mathbb{Z}$ be a basic discrete Morse function. A simplex σ of $\mathbb{K}$ is said to be *critical* when F is injective on σ. Otherwise, σ is called *regular*. When σ is a critical simplex, $F(\sigma)$ is called a *critical value*. If σ is a regular simplex, $F(\sigma)$ is called a *regular value*.

Two basic discrete Morse functions f, g defined on a same simplicial complex $\mathbb{K}$ are said to be *Forman-equivalent* when for any two faces $\sigma, \tau \in \mathbb{K}$ satisfying $\sigma \prec \tau$, $f(\sigma) < f(\tau)$ if and only if $g(\sigma) < g(\tau)$.

Let F be a basic discrete Morse function on $\mathbb{K}$. The *induced gradient vector field* $\overrightarrow{\mathrm{grad}}$ of F is defined by:

$$\overrightarrow{\mathrm{grad}}(F) := \{(\sigma, \tau)\ ;\ \sigma, \tau \in \mathbb{K}\ ,\ \sigma \prec \tau\ ,\ f(\sigma) = f(\tau)\}\,. \tag{1}$$

If (σ, τ) belongs to $\overrightarrow{\mathrm{grad}}(F)$, then it is called a *vector* or *arrow* whose σ is the *tail* and τ is the *head*. The vector (σ, τ) will be sometimes denoted by $\overrightarrow{\sigma\tau}$.

Let $\mathbb{K}$ be a simplicial complex. A *discrete vector field* on $\mathbb{K}$ is a set of arrows in $\mathbb{K}$ satisfying that every simplex of $\mathbb{K}$ is in at most in one of its elements. Naturally, every gradient vector field is a discrete vector field.

Let V be a discrete vector field on a simplicial complex $\mathbb{K}$. A *gradient path* is a sequence of simplices[1]:

$$(\tau_{-1}^{(p+1)},)\sigma_0^{(p)}, \tau_0^{(p+1)}, \sigma_1^{(p)}, \tau_1^{(p+1)}, \ldots, \sigma_{k-1}^{(p)}, \tau_{k-1}^{(p+1)}, \sigma_k^{(p)}, \tag{2}$$

of $\mathbb{K}$, beginning at either a critical simplex $\tau_{-1}^{(p+1)}$ or a regular simplex $\sigma_0^{(p)}$, such that $(\sigma_\ell^{(p)}, \tau_\ell^{(p+1)})$ belongs to V and $\tau_{\ell-1}^{(p+1)} \succ \sigma_\ell^{(p)}$ for $0 \leq \ell \leq k-1$. If $k \neq 0$, then this path is said to be *non-trivial*. Note that the last simplex does not need to be in a pair in V. A gradient path is said to be *closed* if $\sigma_k^{(p)} = \sigma_0^{(p)}$.

Theorem 1 (Theorem 2.51 p.61 of [5]). *A discrete vector field V is the gradient vector field of a discrete Morse function (using the Forman definition) iff the discrete vector field V contains no non-trivial closed paths.*

[1] The superscripts correspond to the dimensions of the faces.

In other words, there exists no basic discrete Morse function whose discrete vector field is V when V contains a non-trivial closed path.

Theorem 2 (Theorem 2.53 p.62 of [5]). *Two discrete Morse functions f and g defined on a complex $\mathbb{K}$ are Forman-equivalent iff f and g induce the same gradient vector field. The consequence is that any two Forman-equivalent discrete Morse functions defined on a simplicial complex have the same critical simplices.*

Let $\mathbb{K}$ be a simplicial complex and suppose that there is a pair of simplices $(\sigma, \tau$ of $\mathbb{K}$ with $\sigma \prec \tau$ such that the only coface of σ is τ. Then $\mathbb{K} \setminus \{\sigma, \tau\}$ is a simplicial complex called *an elementary collapse* of $\mathbb{K}$. The action of collapsing is denoted by $\mathbb{K} \searrow \mathbb{K} \setminus \{\sigma, \tau\}$. For an elementary collapse, such a pair $\{\sigma, \tau\}$ is called a *free pair*. Note that elementary collapses preserve simple homotopy type.

1.4 Watersheds of simplicial stacks

Let A and B be two empty open sets in $\mathbb{M}$. We say that B is an extension of A if $A \subseteq B$, and if each connected component of B includes exactly one connected component of A. We also say that B is an extension of A if $A = B = \emptyset$.

Let X be a subcomplex of the pseudomanifold $\mathbb{M}$ and let Y be a collapse of X, then the complementary of Y in $\mathbb{M}$ is an extension of the complementary of X in $\mathbb{M}$.

Let A be a nonempty open set in a pseudomanifold $\mathbb{M}$ and let X be a subcomplex of $\mathbb{M}$. We say that X is a *cut* for A if the complementary of X is an extension of A and if X is minimal for this property. Observe that there can be several distinct cuts for a same open set A and, in this case, these distinct cuts do not necessarily contain the same number of faces.

Let $\pi = \langle x_0, \dots, x_\ell \rangle$ be a path in $\mathbb{M}$. We say that the path π is *descending* for F if; for any $i \in \{1, \dots, \ell\}$,$F(x_i) \leq F(x_{i-1})$.

Let X be a subcomplex of the pseudomanifold $\mathbb{M}$. We assume that X is a cut for $M_-(F)$. We say that X is a *watershed* of F if for any $x \in X$, there exists two descending paths $\pi_1 = \langle x, x_0, \dots, x_\ell \rangle$ and $\pi_2 = \langle x, y_0, \dots, y_m \rangle$ such that:

- x_ℓ and y_m are simplices of two distinct minima of F; and
- $x_i \notin X$, $y_j \notin X$, for any $i \in \{0, \dots, \ell\}$ and $j \in \{0, \dots, m\}$.

Let X be a subset of a pseudomanifold $\mathbb{M}$. We say that X is a *cut by collapse* for F if there exist an ultimate n-collapse H of F such that X is the set of all multi-connected elements for the complementary of the minima of H.

Theorem 3 (Theorem 12 in [2]). *Let X be a subset of a pseudomanifold $\mathbb{M}$. The set X is a watershed of F iff X is a cut by collapse.*

Computation of the dual graph $(V, E, F_{\mathcal{G}})$: starting from a subset A of $\mathbb{M}$, we define the dual edge-weighted graph $\mathcal{G}$ such that its vertex set is composed of all n-simplices of A and its edge set is composed of all the pairs (x, y) such that x, y are n-faces of A and $x \cap y$ is a $(n-1)$-face of A. The valuation of the edges of

$\mathcal{G}$ is made as follow: for two distinct n-faces x, y in A sharing a (n-1)-face in A, $F_{\mathcal{G}}(\{x,y\}) = F(x \cap y)$.

Let A and B be two non-empty subgraphs of the dual graph $\mathcal{G}_{\mathbb{M}}$ of $\mathbb{M}$. We say that B is a forest relative to A when:

- B is an extension of A; and
- for any extension $C \subseteq B$ of A, we have $C = B$ whenever B and C share the same vertices.

We say that B is a *spanning forest relative to* A for $\mathcal{G}_{\mathbb{M}}$ if B is a forest relative to A and if B and $\mathcal{G}_{\mathbb{M}}$ share the same vertices. Informally speaking, the second condition imposes that we cannot remove any edge from B while keeping an extension of A that has the same vertex set as B. The *weight* of A is defined as:

$$F_{\mathcal{G}}(A) := \sum_{u \in E(\mathcal{G}(A))} F_{\mathcal{G}}(u). \tag{3}$$

Let A and B be two subgraphs of $\mathcal{G}_{\mathbb{M}}$. We say that B is a *minimum spanning forest (MSF)* relative to A for $F_{\mathcal{G}}$ in $\mathcal{G}_{\mathbb{M}}$ if B is a spanning forest relative to A and if the weight of B is less than or equal to the weight of any other spanning forest relative to A.

Theorem 4 (Theorem 15 p. 10 [2])**.** *Let A be a subgraph of $\mathcal{G}_{\mathbb{M}}$. The graph A is an MSF relative to the dual graph of the minima of F iff there exists an ultimate n-collapse H of F such that A equals the dual graph of the minima of H.*

Let A be a subgraph of $\mathcal{G}_{\mathbb{M}}$ and let X be a set of edges of $\mathcal{G}_{\mathbb{M}}$. We say that X is an *MSF cut* for A if there exists an MSF B relative to A such that X is the set of all edges of $\mathcal{G}_{\mathbb{M}}$ adjacent to two distinct connected components of B.

If X is a set of $(n-1)$-faces of $\mathbb{M}$, we set

$$\mathrm{Edges}(X) = \{\{x,y\} \in E(\mathcal{G}_{\mathbb{M}}) \mid x \cap y \in X\}. \tag{4}$$

Theorem 5 (Theorem 16 p. 10 [2])**.** *Let X be a set of $(n-1)$-faces of $\mathbb{M}$. The complex resulting from the closure of X is a watershed of F iff* $\mathrm{Edges}(X)$ *is an MSF cut for the dual graph of the minima of F. In other words, to compute the watershed of a stack F, it is sufficient to compute in $\mathcal{G}_{\mathbb{M}}$ an MSF cut relative to the graph associated with the minima of F.*

The different possible algorithms able to compute the MSF cut are detailed in p.11 of [2].

1.5 Dual functions

Let $(\mathbb{M}, \subseteq)$ be a pseudomanifold with no boundary, possibly a simplicial complex which is also a discrete n-surface. Then we define its *dual complex* as the partial order $(\mathbb{M}, \supseteq)$, that is, made of the same set of faces but with the inverse relation order. For example, the dual complex of a regular simplicial grid is a regular hexagonal grid, and so on. Since the Hasse diagrams are reverted from a complex

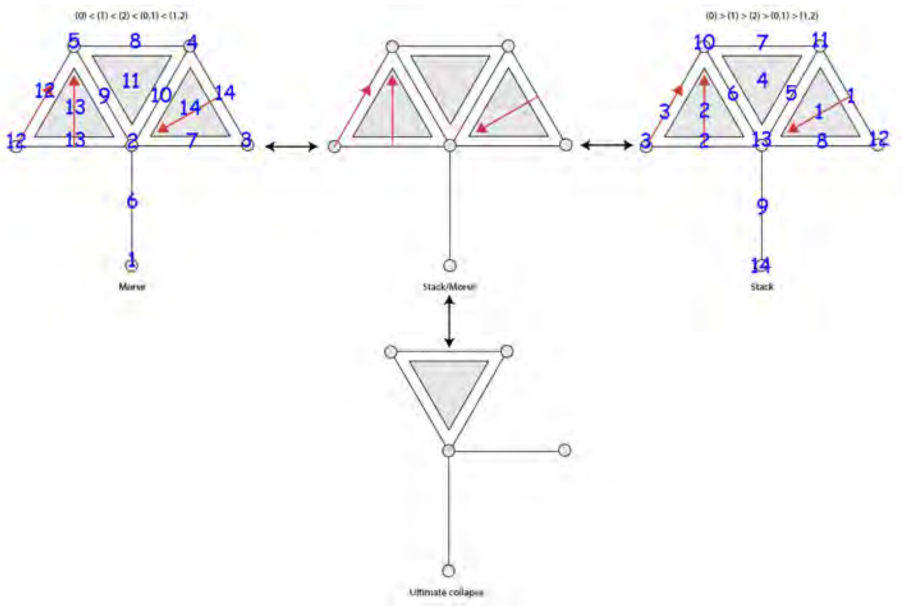

Fig. 1: Gradient vector field is the same underlying information about possible collapses in discrete Morse functions and in simplicial stacks.

to is dual, faces of dimension $k \in [0, n]$ become in the new complex faces of dimension $(n - k)$, where $n \geq 0$ is the rank of the initial complex.

Now that we have the definition of a dual complex, we can define *dual functions*: for any function $F : \mathbb{M} \to \mathbb{Z}$, we define its dual as the function $F^* : \mathbb{M}^* \to \mathbb{Z}$ satisfying $F^*(\sigma^*) = F(\sigma)$. As the reader will have observed, the valuation is then the same from F to F^*, the representation of the valued manifold $\mathbb{M}^*$ is the only thing which changes.

From now on, we will denote the dual of a face $h \in \mathbb{K}$ by h^* or equivalently $\mathrm{Dual}(h)$.

2 Equivalences between Morse functions and Stacks

2.1 Using the gradient vector field

A simple valuation method to obtain a discrete Morse function starting from a gradient vector field defined on a simplicial complex is the following:

$$\mathbb{V}(0) \lhd \mathbb{V}(1) \lhd \cdots \lhd \mathbb{V}(n) \lhd \mathbb{V}(0,1) \lhd \cdots \lhd \mathbb{V}(n-1,n) \tag{5}$$

where $\mathbb{V}(k)$ is the set of values described by the faces of dimensions k and $\mathbb{V}(k, k+1)$ is the set of values described by the matched faces of dimensions k and $k+1$ (the matching corresponding to the gradient vector field).

Concerning the same approach but for the simplicial stacks, it is simply the opposite:

$$\mathbb{V}(n-1,n) \lhd \cdots \lhd \mathbb{V}(0,1) \lhd \mathbb{V}(n) \lhd \cdots \lhd \mathbb{V}(0). \tag{6}$$

A remarkable property is the following:

{1 class of DMF's } $\leftrightarrow$ {1 discrete vector field without loop} $\leftrightarrow$ {1 class of stacks}

2.2 Fundamental propositions for Morse functions and stacks

Proposition 2. *Let F be a $2-1$ simplicial stack defined on a simplicial complex $\mathbb{K}$ of rank $n \geq 0$. Then $F_2 = -F$ is a $2-1$ Morse.*

Proof: For any two faces σ, τ of $\mathbb{K}$ with $\sigma \prec \tau$, we know that $F(\sigma) \geq F(\tau)$, thus $F_2 := -F$ satisfies:

$$F_2(\sigma) \leq F_2(\tau).$$

Furthermore, F is $2-1$ and thus F_2 is $2-1$ too by bijection of the mapping $v \to -v$. So, F_2 is a $2-1$ Morse. □

Proposition 3. *Let F be a $2-1$ discrete Morse function defined on a simplicial complex $\mathbb{K}$ of rank $n \geq 0$. Then $F_2 = -F$ is a $2-1$ stack.*

Proof: We follow exactly the same reasoning as described in the proof of Proposition 2. □

Proposition 4. *Let F be a $2-1$ simplicial stack defined on a simplicial complex $\mathbb{M}$ which is also a discrete n-surface. Then, the dual function of F is a $2-1$ discrete Morse function.*

Proof: Let σ, τ be two faces of Λ with $\sigma \prec \tau$, thus $F(\sigma) \geq F(\tau)$. Two cases are then possible. When we have an equality, τ admits an exception, the face σ, and:

$$F^*(\sigma^*) = F(\sigma) = F(\tau) = F^*(\tau^*).$$

When we have a strict inequality,

$$F^*(\sigma^*) = F(\sigma) > F(\tau) = F^*(\tau^*).$$

Since $\sigma \prec \tau$, we have $\sigma^* \succ \tau^*$, $F2 := F^*$ is weakly increasing. Furthermore, F_2 is $2-1$ since this property is preserved by duality from F. The proof is done. □

Note: we use discrete surfaces so we are ensured that the dual space is still a discrete surface, and furthermore we ensure that the domain has no boundary, so, even if the initial space is closed under inclusion, that is, is a complex, the dual is still a complex.

Proposition 5. *Let F be a $2-1$ discrete Morse function defined on a simplicial complex which is also a discrete surface of rank $n \geq 0$. Then $F_2 := F^*$ is a $2-1$ simplicial stack.*

Proof: The proof follows the same reasoning as Proposition 4. □

From these propositions, two corollaries follow naturally.

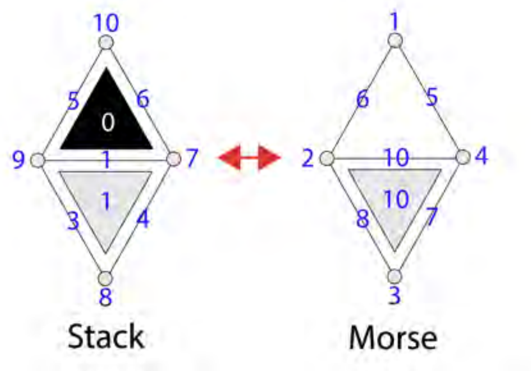

Fig. 2: Switching between stacks and discrete Morse functions.

Corollary 1. *Let F be a $2-1$ discrete Morse function defined on a simplicial complex which is also a discrete surface of rank $n \geq 0$. The mapping $-F^*$ is still a discrete Morse function.*

Corollary 2. *Let F be a $2-1$ simplicial stack defined on a simplicial complex $\mathbb{M}$ which is also a discrete n-surface. The mapping $-F^*$ is still a simplicial $2-1$ stack.*

Proposition 6. *The gradient vector field of any stack is the same as the one of its corresponding discrete Morse function.*

2.3 Practical uses

2.3.1 Conventions for discrete Morse functions

In practice, we will always consider that discrete Morse functions are defined on a subcomplex $\mathbb{K}$ of a pseudomanifold $\mathbb{M}$. This way, it will be easy to convert it into a simplicial stack. Indeed, starting from a discrete Morse function $F : \mathbb{K} \subseteq \mathbb{M} \to \mathbb{N}^*$, we define the new function $F_{\mathbb{M}}$ which is set at $F(\sigma)$ when $\sigma \in \mathbb{K}$ and at 0 when $\sigma \in \mathbb{M} \setminus \mathbb{K}$. Then, we define the new function $F_2 := \max(F_{\mathbb{M}}) + 1 - F_{\mathbb{M}}$, which is a stack and defined all over the pseudomanifold $\mathbb{M}$.

2.3.2 Conventions for simplicial stacks

Starting from a simplicial stack $F : \mathbb{M} \to \mathbb{N}$ defined on a pseudomanifold $\mathbb{M}$ where minima are set at 0, we can easily remove the minima from the pseudomanifold to obtain $\mathbb{K} := \mathbb{M} \setminus M_-(F)$ which is a complex since minima of a stack are stars. Then we define $F_{\mathbb{K}}$ as the restriction of F to $\mathbb{K}$, and we deduce easily the following discrete Morse function: $F_2 := \max(F_{\mathbb{K}}) + 1 - F_{\mathbb{K}}$.

2.3.3 Stacks are $2-1$ except on minima

The reader will have noticed that when we assume that minima are set at 0, we can lose the property that stacks are $2-1$. However, in this paper, we will consider that stacks are $2-1$ except on the minima which are plateaus of any size. We tolerate this property since it does not change the property that we can switch between stacks and discrete Morse functions, and furthermore the gradient is not defined on them since minima are critical faces.

3 Computing a Morse-Smale complex in the discrete setting

Let F be a stack. We cannot use the classical formula (known to be satisfied in the continuous case):

$$\mathcal{MS}(F) = \mathcal{WS}(F) \cup \mathcal{WS}(-F),$$

where $\mathcal{WS}$ denotes the watershed of F and $\mathcal{MS}$ denotes its Morse-Smale complex. Indeed, we cannot compute the watershed of $-F$ since it is not a stack (except in the constant case).

3.1 Watershed computation

3.1.1 Computation of the watershed of a stack F

Let us denote by $<>$ the join operator between two sequences.

So, the computation of the watershed of some given stack is as following (see Alg. 1):

- We start from a $2-1$ simplicial stack $F : \mathbb{K} \to \mathbb{R}$,
- We deduce its dual graph $\mathcal{G} = (V, E, W)$ where $V = ((\mathbb{K})_n)^*$, $E = ((\mathbb{K})_{n-1})^*$, and whose valuation is obtained using the mapping $W : E \to \mathbb{R} : \{a, b\} \to F(\text{Dual}(\{a, b\}))$. In fact, we can remark that $\text{Dual}(\{a, b\}) = a \wedge b \in (\mathbb{K})_{n-1}$ where the $\wedge$ is the infimum operator.
- Then, we compute the minima $M_-(F)$ of F in $\mathbb{K}$ to obtain the seeds for the Prim algorithm (or any other MST computation algorithm), we obtain then the MSF, denoted by $\mathcal{MSF}$, of $\mathcal{G}$.
- We deduce the MSF cut, denoted by $\mathcal{MSF}_{cut}$, whose set of edges is equal to $E_{cut} = E \setminus E(\mathcal{MSF})$, whose set of vertices is equal to:

$$V_{cut} = \{v \in e \; ; \; e \in E_{cut}\},$$

  leading finally to $\mathcal{MSF}_{cut} = (V_{cut}, E_{cut})$.
- We come back by duality to the primal space $\mathbb{K}$ to obtain the $(n-1)$-faces of the watershed, which is finally equal to $\mathcal{WS} := \text{Clo}(\text{Dual}(\mathcal{MSF}_{cut}))$.

Algorithm 1: Operator $\mathcal{WS}$ computing the watershed of a simplicial $2-1$ stack $F : \mathbb{K} \to \mathbb{N}$.

begin
/* Computation of the minima of F */
$M_-(F) := \text{ComputeMinima}(\mathbb{K}, F)$
/* Computation of the edge-valued dual graph */
$\mathcal{G} := (V, E, W) = \text{DualGraph}(\mathbb{K}, F)$
/* Extending the dual graph with an artificial node n_∞ */
$V_{artif} = \{n_\infty\}$
$E_{artif} = \{\{n_\infty, m\} \; ; \; m \in M_-(F)\}$
$W_{artif} = (-1)_{m \in M_-(F)}$
$V_{ext} = V_{artif} <> V$
$E_{ext} = E_{artif} <> E$
$W_{ext} = W_{artif} <> W$
/* Applying the Prim algorithm */
$(V_{prim}, E_{prim}, W_{prim}) = \text{Prim}(M_-(F), (V_{ext}, E_{ext}, W_{ext}))$
/* Reduction of the new graph to obtain the MSF of $\mathcal{G}$ */
$V_{final} = V_{prim} \setminus V_{artif}$
$E_{final} = E_{prim} \setminus E_{artif}$
$W_{final} = E_{prim} \setminus W_{artif}$
$\mathcal{MSF} = (V_{final}, E_{final}, W_{final})$
/* Deduction of the MSF cut (not paired, so critical dual faces) */
$\mathcal{MSF}_{cut} = E \setminus E_{final}$
/* We compute the $(n-1)$-faces whose dual belong to the MSF cut */
$\text{Critical}_{n-1} = \{h \in (\mathbb{K})_{n-1} \; ; \; dual(h) \in \mathcal{MSF}_{cut})$
/* We return the watershed connecting critical n-faces to critical $(n-1)$-faces of F */
return $\text{Clo}(\text{Critical}_{n-1})$

3.1.2 Computation of the watershed of a stack F^* on the dual complex

Let us now assume that we work on complexes which are discrete surfaces, this way we obtain $\text{Dual}(\text{Dual}(\mathbb{K})) = \mathbb{K}$.

When we start from some valued complex $(\mathbb{K}, F)$, we can easily deduce its dual valued complex $(\mathbb{K}^*, F^*)$ with for any $h \in \mathbb{K}$:

$$F^*(dual(h)) = F(h).$$

Now, if we assume that F^* is a $2-1$ simplicial stack, we can compute its watershed:

- We start from the $2-1$ simplicial stack $F^* : \mathbb{K}^* \to \mathbb{R}$,
- We deduce the dual graph $\mathcal{G}^* = (V*, E*, W*)$ of $(\mathbb{K}*, F^*)$ which is in fact the 1-skeleton $(\mathbb{K})_0 \cup (\mathbb{K})_1$ of $(\mathbb{K}^*, F^*)^* = (\mathbb{K}, F)$; we call $\mathcal{G}^*$ the *primal graph* of $(\mathbb{K}^*, F^*)$. Naturally, the set V^* is equal to $(\mathbb{K})_0$, the set E^* is equal to $(\mathbb{K})_1$, and the mapping W^* is defined as $W^* : \mathbb{K} \to \mathbb{R} : h \to F(h)$ where F is obtained thanks to the property $(F^*)^* = F$.
- We deduce then the MSF relative to the minima $M_-(F^*)$ using some MSF computation algorithm to obtain the MSF of $\mathcal{G}^*$; we denote it $\mathcal{MSF}^*$.
- We deduce the MSF cut, denoted by $\mathcal{MSF}^*_{cut}$, whose set of edges is equal to $E^*_{cut} = E^* \setminus E(\mathcal{MSF}^*)$, whose set of vertices is equal to:

  $$V^*_{cut} = \{v \in e \; ; \; e \in E^*_{cut}\},$$

  leading finally to $\mathcal{MSF}^*_{cut} = (V^*_{cut}, E^*_{cut})$.

– We come back by duality to the dual space $\mathbb{K}^*$ to obtain the $(n-1)$-faces of the watershed, which is finally equal to $\mathcal{WS}^* := \text{Clo}(\text{Dual}(\mathcal{MSF}^*_{cut}))$.

Remark 1. *For any* $2-1$ *simplicial stack* $F : \mathbb{K} \to \mathbb{R}$*, the watershed of* F *is a subset of the primal graph of* F*. Additionally, the watershed of* $-F^*$ *is a subset of the dual graph of* F*. In other words, these two graphs are orthogonal to each other where they cross each other.*

3.2 Morse-Smale complex of a valued pseudo-manifold

Let $F : \mathbb{K} \to \mathbb{N}$ be a given valued complex with the faces of the minima are set at zero by convention. We consider what we call the *reordering operator* ReOrd (see Alg. 3) which outputs a $2-1$ stack when we feed the algorithm with such a valued complex. This way, we can define its *discrete Morse-Smale complex* of F as:

$$\mathcal{MS}(F) = \mathcal{WS}(F') \cup \mathcal{WS}(-(F')^*),$$

where $F' := \text{ReOrd}(F)$ is the stack corresponding to F and where the watershed computation is depicted in Alg. 1.

Note that we can also define the alternative version:

$$\mathcal{MS}(F) = \mathcal{WS}(\text{ReOrd}(F)) \cup \mathcal{WS}(\text{ReOrd}(-F^*)) \tag{7}$$

whose results will be shown hereafter.

When the given valued complex $F : \mathbb{K} \to \mathbb{N}$ is already a stack, the ReOrd operator is not needed anymore, and we can define the *discrete Morse-Smale complex* of F as:

$$\mathcal{MS}(F) = \mathcal{WS}(F) \cup \mathcal{WS}(-F^*),$$

Note that all the Morse-Smale complexes defined above are naturally extended to discrete Morse functions by switching F with $\max(F) - F$ (see Alg. 2).

Algorithm 2: Operator $\mathcal{MS}$ computing the Morse-Smale complex of a simplicial $2-1$ discrete Morse function $F : \mathbb{K} \to \mathbb{N}$.

---

begin
- /* We change the sign of F to obtain a simplicial $2-1$ stack */
- $F' = \max(F) - F$
- $F^* = Dual(F)$
- **return** $\mathcal{WS}(F') \cup \mathcal{WS}(F^*)$

We start from the computation of $\mathcal{WS}(\text{ReOrd}(F))$ (see Figure 3): since F is a valued complex but not a $2-1$ simplicial stack, we have to reorder its n-faces. For this aim, we use the ReOrd operator which transforms F into a $2-1$ simplicial stack, on which we can compute the gradient vector field $\overrightarrow{\text{grad}}$ (thanks to the paired faces). By computing the complementary of $\overrightarrow{\text{grad}}$ in $\mathbb{K}$, we obtain the critical faces

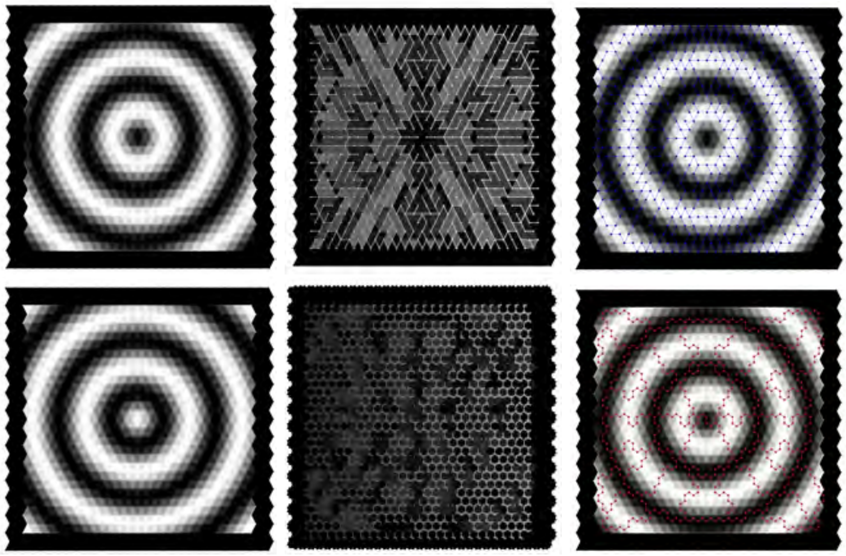

Fig. 3: Computation of the watershed and of the thalweg of a valued complex. From the left side to the right side (upper row): the initial valued complex with the border set at zero, its corresponding simplicial stack (using the ReOrd operator), and its corresponding watershed. From the left side to the right side (lower row): $-F$ with the border set at zero, the simplicial stack using ReOrd on $(-F^*)$, and its corresponding watershed.

of $\mathbb{K}$, that is, its watershed. Note that this last is made of 0-faces and 1-faces connecting these 0-faces.

We continue with the computation of the watershed of $(-F^*)$. Starting from F, we want to extract the topological information of $-F$, so we compute the opposite of F. We could then apply directly the operator ReOrd on it, which would lead to a simplicial stack. However, the resulting watershed would be represented using the 0-faces and the 1-faces of $\mathbb{K}$ when we want that the faces representing the watershed of $-F$ are *orthogonal* to the ones of the watershed of F. Furthermore, the maxima of $-F$ would be n-faces since it is a discrete Morse function, and we do not want to represent a watershed using n-faces and $(n-1)$-faces. For these two reasons, we compute the dual $-F^*$ from $-F$. Then, we just have to apply one more time the ReOrd operator, whom follows the gradient vector field of $-F^*$, whose we deduce the watershed of $-F$, sometimes called *thalweg* (see Fig. 3).

By unifying the two previous watersheds, we finally what we can the *discrete Morse-Same complex* of our valued complex F (see Figure 4); this approach uses Equation 7.

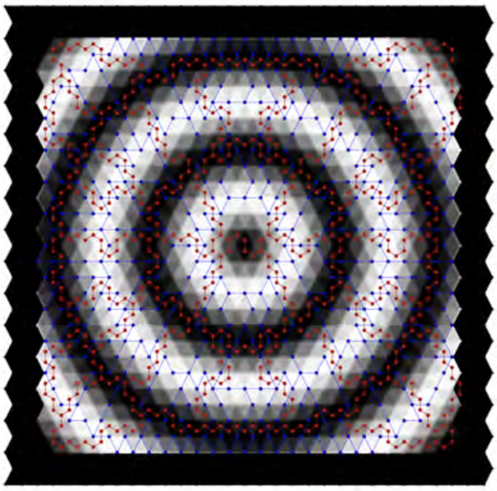

Fig. 4: A Morse-Smale complex of a map defined on a simplicial complex.

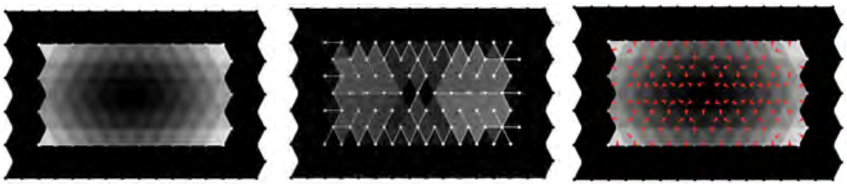

Fig. 5: An initial map, its reordered version, and its gradient vector field.

4 The minimum spanning forest of a stack is its gradient vector field

4.1 Preamble: the natural extension of a gradient vector field

Figures 5 shows an initial map on a simplicial complex which is revalued thanks to an algorithm inspired from Prim's algorithm with a priority queue. This algorithm outputs a $2-1$ simplicial stack on which we can compute the gradient vector field. Using then Algorithm 3, we can compute what we call the *(natural) extension* of the gradient vector field $\overrightarrow{\text{grad}}$, or in an equivalent manner, the *forest induced* by the gradient vector field $\overrightarrow{\text{grad}}$.

4.2 Why the extension of the gradient vector field is equal to its MSF

Proposition 7. *Let $\mathbb{K}$ be a n-dimensional simplicial complex, let $F : \mathbb{K} \to \mathbb{R}$ be a simplicial $2-1$ stack and let $\overrightarrow{\text{grad}}$ be the gradient vector field of F. Then, any path $\pi = (h^k)_{k \in [0,N]}$ following $\overrightarrow{\text{grad}}$ is* increasing*, that is, for any $k \in [0, N-1]$, $F(h^k) \leq F(h^{k+1})$.*

Algorithm 3: Computing the MSF from the gradient.

```
begin
    /* The vertices correspond to the n-faces of the pseudomanifold. */
    V = Dual((𝕄)_n)
    E = {}
    for m_1 ∩ m_2 ∈ (𝕄)_{n-1} s.t. m_1, m_2 ∈ (M_−(F))_n do
        PUSH(m_1 ∩ m_2, E)
for ab→ ∈ grad→ do
    c ← Opp_a(b)
    PUSH(cb, E)
return E
```

Proof: Let π some path following $\overrightarrow{\text{grad}}$ and let us assume without generality that $\pi(0)$ is a n-face of $\mathbb{K}$. We know that the $(n-1)$-face $\pi(2k+1)$ is paired with the n-face $\pi(2k+2)$ in $\overrightarrow{\text{grad}}$ for any $k \in [0,(N-1)/2-1]$, which means that $F(\pi(2k+1)) = F(\pi(2k+2))$. We also know that F is a stack, and then F decreases when we increase the dimension of the face, so for any $k \in [0,(N-1)/2-1]$, $F(\pi(2k)) \leq F(\pi(2k+1))$. The proof is done. $\square$

Proposition 8 (MST Lemma [4,1]). *Let $G = (V,E,F)$ be some valued graph. Let $v \in V$ be any vertex in G. The minimum spanning tree for G must contain the edge vw that is the minimum weighted edge incident on v.*

Theorem 6. *Let $\mathbb{K}$ be a n-dimensional simplicial complex, let $F : \mathbb{K} \to \mathbb{R}_+$ be a simplicial $2-1$ stack, and let $M_-(F)$ be the minima of F. Now, let $\overrightarrow{\text{grad}}$ be the gradient vector field of F starting from the minima of F and covering all the n-faces of $\mathbb{K}$. We assume furthermore that F is strictly positive on any face of $\mathbb{K} \setminus M_-(F)$. Then, the MSF relative to $M_-(F)$ of the graph G dual to $\mathbb{K}$ is equal to the spanning forest induced by its gradient vector field $\overrightarrow{\text{grad}}$.*

Proof: We assume without loss of generalization that $M_-(F)$ is equal to $\{\{m\}\}$, that is, F admits only one minimum $\{m\}$ and it is isolated.

For any v in $V(G) \setminus \{m\}$, there exists at least one vector in $\overrightarrow{\text{grad}}$ which can be written $\overrightarrow{av}$ since, by hypothesis, $\overrightarrow{\text{grad}}$ covers V. Since $\overrightarrow{\text{grad}}$ is a gradient vector field of a $2-1$ mapping, we have at most one vector in $\overrightarrow{\text{grad}}$ which is incident to v (otherwise, we could have at least three faces in $\mathbb{K}$ whose image is the same, and F would not be a $2-1$ mapping anymore). So, we have exactly one vector $\overrightarrow{av}$ incident to v in $\overrightarrow{\text{grad}}$.

The $(n-1)$-face corresponding to the edge av is paired with the n-face corresponding to the vertex v, which can be written $F(av) = F(v)$ in an equivalent manner. Since F is a simplicial stack, the other edges which can be written vb are either critical (with $F(vb) > \max(F(v), F(b))$), or not critical but with vb paired with b (thus $F(b) = F(vb) > F(v)$). Therefore, the edge av is the lowest cost edge incident to v:

$$F(av) < \{F(bv)\ ;\ bv \in E(G),\ b \neq a\}$$

and thus belongs to the MST of F by Proposition 8.

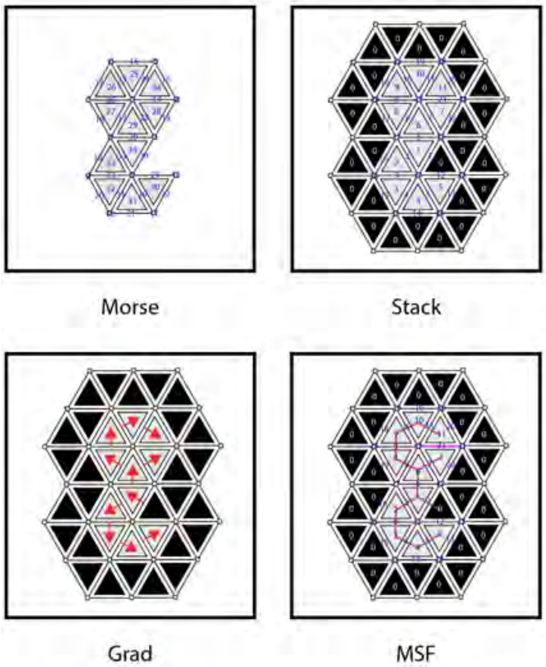

Fig. 6: Starting from a Morse function, we can compute its equivalent simplicial stack (up to the minus sign and up the the added zero used to extend the support of the function to avoid border effects). Then, we can deduce its MSF thanks to optimized algorithms coming from MM, and from this MSF, we can deduce the gradient vector field of the initial Morse function. More than a new numerical scheme used to compute gradients, it shows that **the MSF is a gradient vector field** of a Morse/stack.

Generalizing this property to every $v \in V \setminus \{m\}$, we obtain that all the lowest cost edges of these vertices, that is, $\mathrm{Ext}(\overrightarrow{\mathrm{grad}})$, belong to the MSF of F:

$$\mathrm{Ext}(\overrightarrow{\mathrm{grad}}) \subseteq MSF(F).$$

Since $\mathrm{Ext}(\overrightarrow{\mathrm{grad}})$ covers V by hypothesis, it is then equal to the MST of F: any added edge would increase the total weight of the tree. The proof is done for the MST case.

To generalize the proof to a MSF, it is sufficient to add an artificial node n_∞ to V and to connect it to all the vertices of $M_-(F)$, to apply Proposition 8 as before, and to remove the artificial node n_∞. We obtain one more time that the MSF relative to $M_-(F)$ is equal to the spanning forest induced by its gradient vector field $\overrightarrow{\mathrm{grad}}$. $\square$

A summary of this strong result is depicted in Figure 6.

Corollary 3. *The spanning forest $\mathcal{IND}$ induced by the gradient vector field of a simplicial 2-1 stack $F : \mathbb{K} \to \mathbb{R}$ computed by Algorithm 3 is equal to the MSF of its*

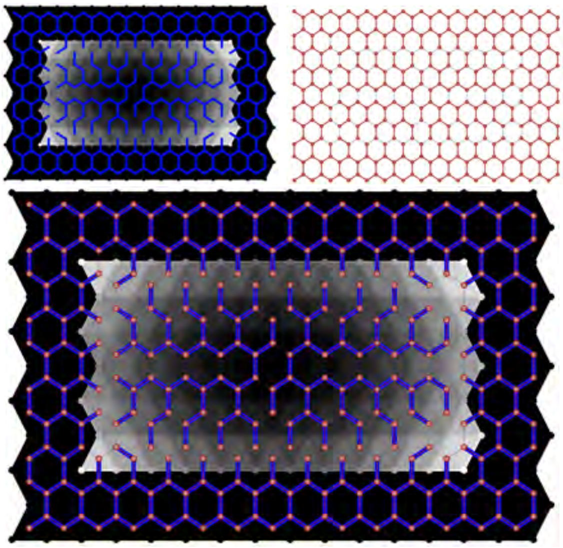

Fig. 7: Comparison between the MSF induced from the gradient vector field (in blue) and the MSF computed directly on the dual graph (in red) using the Prim algorithm: they are equal.

revaluation F' (see Algorithm 3). In other words, the minimum spanning forest of (the revaluation of) F is the gradient vector field of F.

Proof: Using Algorithm 3 and starting from a valued complex $(\mathbb{K}, F)$, we obtain a (reordered) $2-1$ simplicial stack F'. Then, we can easily compute its gradient vector field $\overrightarrow{\text{grad}}$ by pairing the faces which are neighbors and whose image by F' is the same. Thanks to Theorem 7, we know that the spanning forest induced by $\overrightarrow{\text{grad}}$ will be equal to the MSF of F'. The proof is done. □

4.3 Experimental proof

In Figure 7, we see that the extension of the gradient vector field $\overrightarrow{\text{grad}}$ of F' is equal to the MSF of the same function (see Corollary 3 for the proof of this fact).

5 How to compute the Morse-Smale complex of any 2D real function

Let us now show that our revaluation algorithm (see Algo. 4) can be used on continuous functions.

Let us assume that we have some real function $f : \mathcal{D} \to \mathbb{R}$ which is differentiable on $\mathcal{D}$, and thus admits a gradient vector field. Let us show that, thanks to a slight modification of Equation 7, we can easily obtain the Morse-Smale complex of its *simplicial discretization* (we valuate a simplicial complex covering $\mathcal{D}$ so that we can

Algorithm 4: Revaluation Algorithm ReOrd.

```
begin
    /* Initialization */
    G := the dual graph of 𝕂
    F' := F
    deja_vu := M_−(F)
    G_past := subgraph of G induced by the minima of F
    for ℓ ∈ ℕ do
        Q[ℓ] := {}
    for m ∈ M_−(F) do
        for h neighbor in G of m and h ∉ deja_vu do
            Cost = F(h) − F(m)
            if Cost ≥ 0 then
                PUSH(Q[Cost], mh)
    /* Propagation until the queue is empty */
    cpt = 1
    while Q ≠ ∅ do
        ab := PULLMAX(Q)
        /* We treat ab when adding it to G_past does not create a cycle or connect
        two different minima */
        if a ∉ deja_vu or b ∉ deja_vu then
            Cost := F(b) − F(a)
            PUSH(Q[Cost], ab)
            deja_vu := deja_vu ∪ {b}
            G_past := G_past ∪ {ab}
            /* The three next lines are for pairing */
            v := cpt + +
            F'(Dual(ab)) := v
            F'(b) := v
            /* We look for the new edges to push */
            for bc ∈ G do
                Cost := F(c) − F(b)
                if cost ≥ 0 then
                    PUSH(bc, Cost)
    /* Valuation of the remaining simplices to ensure we obtain a stack */
    for ab ∈ Edges(G) \ Edges(G_past) do
        F'(Dual(ab)) := cpt + +
    for v ∈ Vertices(𝕂) do
        F'(v) := cpt + +
    return F'
```

work on f as described before). Let us call $\triangle$ this generic discretization operator, we obtain thus:

$$\mathcal{MS}(f) = \mathcal{WS}(\text{ReOrd}(\triangle(f))) \cup \mathcal{WS}(\text{ReOrd}((\triangle(-f))^*)) \tag{8}$$

5.1 Computation of the watershed of f

Let us start from a n-D positive function $f : \mathcal{D} \to \mathbb{R}$ which is differentiable. We compute some simplicial complex $\mathbb{K}$ which covers $\mathcal{D}$ with any triangulation

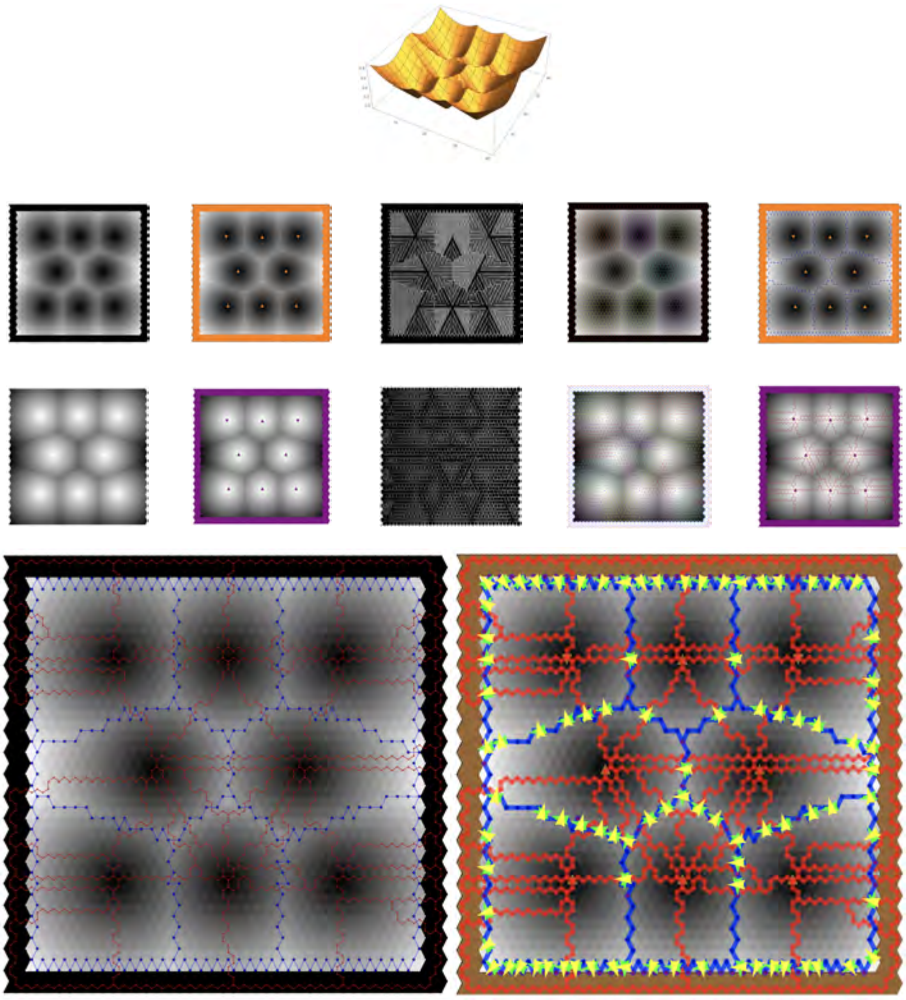

Fig. 8: The complete process from a real function to the computation of its Morse-Smale complex. At the top, the initial real function using a topographic view. The two following rows show the valued simplicial complexes, the detection of the minima (triangles) and the maxima (disks) of f, the revalued functions using ReOrd, the basins in the respective spaces (one color for each basin), then the watershed (second row) going through the maxima of f and the watershed (third row) of $-f$ going through the minima of f. The last row regroup all the topological information of the previous rows: on the left side, our proposition of Morse-Smale complex of f, and on the right side, this same complex with the principal directions of the Hessian matrices at the saddle points, intersections of the two watersheds.

method. We valuate each face of this simplicial complex using the value of f at the barycentric position of the studied face. We enforce all the faces at the border of the domain to be set at 0 to avoid side effects. Now that we have a valued simplicial complex, we look for the minima of this complex by looking for the minimal n-faces, that is, the faces $h \in \mathbb{K}$ whose adjacent n-faces $h' \in \mathbb{K}$ have greater or equal values $f(h')$ compared to $f(h)$. These minima are in fact the seeds of the propagation algorithm. The revaluation algorithm starts from the seeds and propagates along adjacent n-faces in the direction of the maximal gradient available in the priority queue, while the creation of cycles is forbidden (in the graph sense). Let us note that, at each step of the propagation, choosing the steepest gradient in the priority queue ensures the revaluation of a pair (h_{n-1}, h_n) of a $(n-1)$-face h_{n-1} and a n-face h_n where h_n covers h_{n-1} and where (h_{n-1}, h_n) is in the direction of the gradient of f . In other words, we pair h_{n-1} with h_n by setting them at the same number, and we are building the $2-1$ stack corresponding to the initial function f and with the same gradient vector field. Thus we obtain a gradient vector field, that is, the minimum spanning forest, with several connected components (one component for each seed). We recall that computing the gradient vector field can then be done using two equivalent ways: either by looking for the faces paired thanks to our revaluation algorithm or by computing the MSF using any MST algorithm on the minima of the discretization. Then, we compute the cut of this MSF, equal to the set of $(n-1)$-faces joining components of the MSF of different labels. Naturally, this cut is the set of edges which satisfy the dropping water principle and thus it is the watershed we are looking for.

5.2 Computation of the watershed of $-f$

Now, let us present the dual approach. We start one more time from f defined on $\mathcal{D}$ and we use the same simplicial complex as built before, except that this tame we value it using $\max(f)-(f-\min(f))$ or an equivalent formula ensuring that we work with a positive version of $-f$. To be coherent with the primal approach, and since some discrete function F is a stack iff $-F^*$ is a stack, we work with dual complex $\mathbb{K}^*$. If the previous complex is simplicial and regular, we obtain then an hexagonal complex (see Fig. 8) valued by $-f$. We enforce the border to be valued not at 0 but at the maximum value of $-f$ on $\mathbb{K}^*$ since the reasoning is inverted compared to the handling of f. Then, we locate the minimal n-faces of $\mathbb{K}^*$ following the same reasoning as before, based on adjacency. We have the seeds for the propagation algorithm which will start from the minima and propagate following adjacent n-faces in the steepest gradient directions without creating cycles. We obtain then a MSF on $\mathbb{K}^*$, the corresponding cut, and thus the watershed of $-f$. Note that, since the cut is computed in the dual space of $\mathbb{K}$, these faces are naturally perpendicular to the watershed of f computed before.

By superposing the two watersheds, we obtain our proposition of Morse-Smale complex.

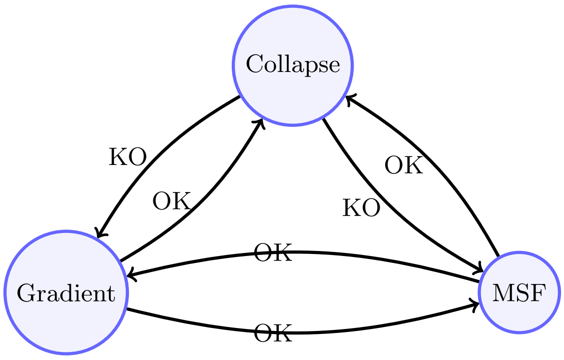

Fig. 9: Possibilities to compute the different elements when another is known. Note that we assume that the locations of the minima of F are known.

6 Switching between ultimate collapse, gradient, and MSF

From now on, we call the *gradient vector field* of a simplicial stack as the gradient vector field of its equivalent discrete Morse function.

As depicted in Figure 9, the ultimate collapse does not contain enough information to compute the gradient vector field of F or its MSF. However, the four other cases are possible, and here we describe the corresponding algorithms except the one providing the extension of a gradient vector field (since it has been described earlier).

6.1 From MSF to collapse

$$\text{UltimateCollapse} := \text{clo}\left(\mathbb{M}_{n-1} \setminus \text{Dual}(E)\right),$$

where E is the set of edges of the $\mathcal{MSF}$.

6.2 From MSF to gradient

See Algorithm 5.

6.3 From gradient to collapse

See Algorithm 6.

Figures 10 shows the result of the computation of the ultimate collapse from the gradient vector field.

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Algorithm 5: Computing the gradient from the MSF.

begin
  /* $MSF = (V, E)$ is given */
  $\overrightarrow{\text{grad}} = \{\}$
  Set $= \{ab \in E \; ; \; a \in M_{-}(F) \; ; \; b \in \Lambda(F)\}$
  /* We list all the paired faces. */
  while Set $\neq \emptyset$ **do**
    $ab =$ POP(Set)
    for $bc \in E$ **do**
      PUSH(bc, Set)
      /* The edges starting from a minima are not considered. */
      $\overrightarrow{\text{grad}} \leftarrow \overrightarrow{\text{grad}} \cup \{\overrightarrow{bc}\}$

Algorithm 6: Computing the ultimate collapse from the gradient.

begin
  NoCritical $= \{\}$
  /* We list all the paired faces. */
  for $(v_{n-1}, v_n) \in \overrightarrow{\text{grad}}$ **do**
    NoCritical $\leftarrow$ NoCritical $\cup \{v_{n-1}, v_n\}$
  /* The ultimate collapse is the closure of the critical faces minus the minima. */
  UltimateCollapse $\leftarrow$ clo($\mathbb{M} \setminus$ NoCritical) $\setminus M_{-}(F)$

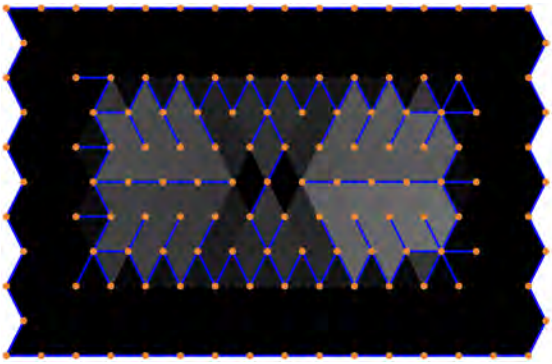

Fig. 10: The ultimate collapse (in blue) obtained from the gradient vector field of the simplicial stack (in gray).

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