Converting to discrete time

- The R–C models we have seen to date are expressed in continuous time as ordinary differential equations
- We wish to convert them to discrete-time ordinary difference equations (ODEs) for easier use in a final application
- In this lesson, you will learn how to convert generic

$$\dot{x}(t) = ax(t) + bu(t)$$

into an equivalent discrete-time

$$x[k+1] = a_d x[k] + b_d u[k]$$

- This conversion can then be applied to specific cases that we have seen

Step 1: Solve differential equation

- We start with the solution to the differential equation

$$\dot{x}(t) = ax(t) + bu(t)$$

$$x(t) = e^{at}x(0) + \underbrace{\int_0^t e^{a(t-\tau)}bu(\tau)\,\mathrm{d}\tau}_{\text{convolution}}$$

- How did we get this result?

1. $\dot{x}(t) - ax(t) = bu(t)$
2. $e^{-at}[\dot{x}(t) - ax(t)] = \dfrac{\mathrm{d}}{\mathrm{d}t}[e^{-at}x(t)] = e^{-at}bu(t)$
3. $\displaystyle\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau}[e^{-a\tau}x(\tau)]\,\mathrm{d}\tau = e^{-at}x(t) - x(0) = \int_0^t e^{-a\tau}bu(\tau)\,\mathrm{d}\tau$

Step 2: Factor out $x[k]$

- We wish to evaluate $x(t)$ at discrete times $x[k] \stackrel{\Delta}{=} x(k\Delta t)$

$$\begin{aligned} x[k+1] &= x((k+1)\Delta t) \\ &= e^{a(k+1)\Delta t}x(0) + \int_0^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau \end{aligned}$$

- Break both the exponential and integral into two pieces each: So, $x[k+1]$ is

$$\begin{aligned} &= e^{a\Delta t}e^{ak\Delta t}x(0) + \int_0^{k\Delta t} e^{a((k+1)\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau + \int_{k\Delta t}^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau \\ &= e^{a\Delta t}e^{ak\Delta t}x(0) + \int_0^{k\Delta t} e^{a\Delta t}e^{a(k\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau + \int_{k\Delta t}^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau \\ &= e^{a\Delta t}x[k] + \int_{k\Delta t}^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)}bu(\tau)\,\mathrm{d}\tau \end{aligned}$$

Step 3: For $a \neq 0$

- So far: $x[k+1] = e^{a\Delta t}x[k] + \int_{k\Delta t}^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)} bu(\tau)\,\mathrm{d}\tau$
- Assume $u(\tau)$ is constant from $k\Delta t$ to $(k+1)\Delta t$ and equal to $u(k\Delta t)$

$$\begin{aligned}
x[k+1] &= e^{a\Delta t}x[k] + e^{a(k+1)\Delta t}\left(\int_{k\Delta t}^{(k+1)\Delta t} e^{-a\tau}\,\mathrm{d}\tau\right) bu[k] \\
&= e^{a\Delta t}x[k] + e^{a(k+1)\Delta t}\left(-\frac{1}{a}e^{-a\tau}\Big|_{k\Delta t}^{(k+1)\Delta t}\right) bu[k] \\
&= e^{a\Delta t}x[k] + \frac{1}{a}e^{a(k+1)\Delta t}\left(e^{-ak\Delta t} - e^{-a(k+1)\Delta t}\right) bu[k] \\
&= e^{a\Delta t}x[k] + \frac{1}{a}\left(e^{a\Delta t} - 1\right) bu[k]
\end{aligned}$$

Application to the R–C equation

- So, we can convert $\dot{x}(t) = ax(t) + bu(t)$ into

$$x[k+1] = e^{a\Delta t}x[k] + \frac{1}{a}\left(e^{a\Delta t} - 1\right) bu[k]$$

- To use this result for the ODE describing the R–C circuit ($\tau_1 = R_1C_1$)

$$\begin{aligned}
&\mathrm{d}i_{R_1}(t)/\mathrm{d}t = (-1/\tau_1)\, i_{R_1}(t) + (1/\tau_1)\, i(t) \\
&\qquad \Rightarrow a = -1/\tau_1, \quad b = 1/\tau_1, \quad x[k] = i_{R_1}[k], \quad \text{and} \quad u[k] = i[k].
\end{aligned}$$

- Substituting these values into the generic result, we get

$$\begin{aligned}
i_{R_1}[k+1] &= \exp(-\Delta t/\tau_1)\, i_{R_1}[k] + (-\tau_1)\left(\exp(-\Delta t/\tau_1) - 1\right)(1/\tau_1)\, i[k] \\
&= \exp(-\Delta t/\tau_1)\, i_{R_1}[k] + \left(1 - \exp(-\Delta t/\tau_1)\right) i[k]
\end{aligned}$$

Step 3: For $a = 0$ and the SOC equation

- Recall: $x[k+1] = e^{a\Delta t}x[k] + \int_{k\Delta t}^{(k+1)\Delta t} e^{a((k+1)\Delta t-\tau)} bu(\tau)\,\mathrm{d}\tau$
- Now, if $a = 0$ and $u(\tau)$ is constant from $k\Delta t$ to $(k+1)\Delta t$ and equal to $u(k\Delta t)$

$$x[k+1] = x[k] + \left(\int_{k\Delta t}^{(k+1)\Delta t} 1\,\mathrm{d}\tau\right) bu[k] = x[k] + (\Delta t)\, bu[k]$$

- To use this result for the ODE describing SOC, $\dot{z}(t) = (-\eta(t)/Q)\, i(t)$, we have $a = 0$, $b = -\eta[k]/Q$, $x[k] = z[k]$, and $u[k] = i[k]$
- So, we have now proven the result that was stated earlier without proof

$$z[k+1] = z[k] - \frac{\eta[k]\Delta t}{Q} i[k]$$

Discrete-time model

- Our present model is now fully converted to discrete time:

$$\begin{aligned}
\frac{\mathrm{d}z(t)}{\mathrm{d}t} &= -\frac{\eta(t)}{Q}i(t) \\
\frac{\mathrm{d}i_{R_1}(t)}{\mathrm{d}t} &= -\frac{1}{R_1C_1}i_{R_1}(t) \\
&\quad + \frac{1}{R_1C_1}i(t) \\
v(t) &= \text{OCV}(z(t)) \\
&\quad - R_1 i_{R_1}(t) - R_0 i(t)
\end{aligned}
\quad \Longrightarrow \quad
\begin{aligned}
z[k+1] &= z[k] - \frac{\eta[k]\Delta t}{Q}i[k] \\
i_{R_1}[k+1] &= \exp\left(-\frac{\Delta t}{R_1C_1}\right)i_{R_1}[k] \\
&\quad + \left(1 - \exp\left(-\frac{\Delta t}{R_1C_1}\right)\right)i[k] \\
v[k] &= \text{OCV}(z[k]) - R_1 i_{R_1}[k] - R_0 i[k]
\end{aligned}$$

Summary

- It is easiest to derive models first in continuous time but final application will be in discrete time
- So, we have developed a process to convert first-order linear models
- Generically (except when $a = 0$),

$$\dot{x}(t) = ax(t) + bu(t) \quad \Longrightarrow \quad x[k+1] = e^{a\Delta t}x[k] + \frac{1}{a}\left(e^{a\Delta t} - 1\right)bu[k]$$

- In the special case when $a = 0$,

$$\dot{x}(t) = ax(t) + bu(t) \quad \Longrightarrow \quad x[k+1] = x[k] + (b\Delta t)\,u[k]$$

- We have applied this process to convert our present battery-cell model