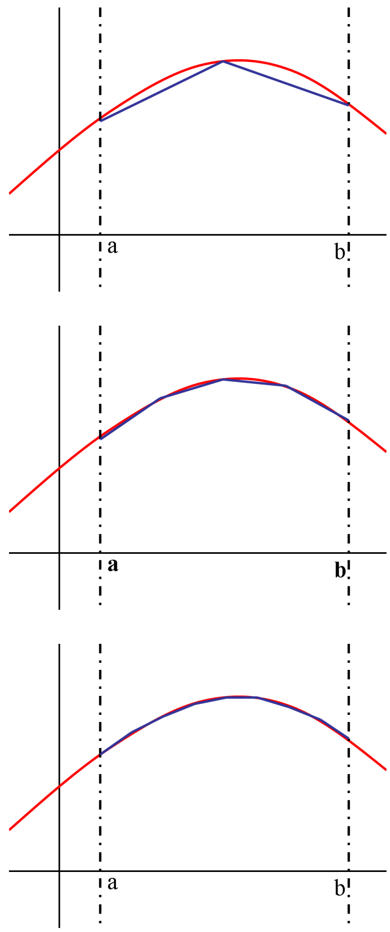
6.5: Lengths of Curves

Given two points, it is fairly easy to compute the distance between them. However, if we take a non-straight curve, then the total distance traveled is not as easily computed.

We figure out the arc length over an interval $[a, b]$ given by following the path defined by the graph of $y = f(x)$.

Given a curve defined by $y = f(x)$ as below, we can estimate the arc length by taking line segments.

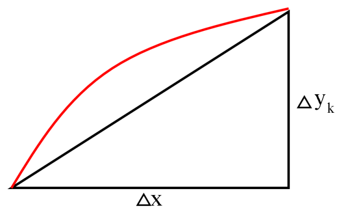


As it can be seen, by using more and more of shorter line segments, the sum approximates the actual arc length more accurately.

Suppose that we are using n line segments to approximate the arc length, then we have

$$\Delta x = \frac{b-a}{n}.$$

Then we look at the length of the kth line segment.



Here, we have the length of the segment is

$$\sqrt{(\Delta x)^2 + (\Delta y_k)^2}$$

We can rewrite this expression.

$$\sqrt{(\Delta x)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x)^2 \left[1 + \left(\frac{\Delta y}{\Delta x}\right)^2\right]}$$

$$= \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x = \sqrt{1 + f'(\bar{x_k})^2}\Delta x$$

Then we have that the arc length L is

$$L = \lim_{n \to \infty} \sum_{k=1}^{n} \sqrt{1 + f'(\bar{x_k})^2}\Delta x = \int_a^b \sqrt{1 + f'(x)^2} dx$$

Definition: Arc Length for $y = f(x)$

Let f have a continuous first derivative on the interval $[a, b]$. The length of the curve from $(a, f(a))$ to $(b, f(b))$ is

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx.$$

Ex. Compute the arc length of $y = \frac{1}{3}x^{3/2}$ on $[-1, 1]$.

Here, we have $f(x) = \frac{1}{3}x^{3/2}$. Hence the arc length is

$$\int_a^b \sqrt{1 + f'(x)^2} dx = \int_{-1}^{1} \sqrt{1 + \left(\frac{1}{2}x^{1/2}\right)^2} dx$$

$$= \int_{-1}^{1} \sqrt{1 + \frac{1}{4}x}\, dx$$

We let $u = 1 + \frac{1}{4}x$, then we have

$$du = \frac{1}{4}dx$$

and

$$u(-1) = 1 - \frac{1}{4} = \frac{3}{4} \qquad u(1) = 1 + \frac{1}{4} = \frac{5}{4}$$

Thus we have

$$\int_{-1}^{1} \sqrt{1 + \frac{1}{4}x}\, dx = \int_{3/4}^{5/4} 4\sqrt{u}\, du$$

$$= \frac{8}{3}\, u^{3/2}\Big|_{3/4}^{5/4} = \frac{8}{3}\left(\left(\frac{5}{4}\right)^{3/2} - \left(\frac{3}{4}\right)^{3/2}\right)$$

$$= \frac{8}{3}\left(\frac{5\sqrt{5}}{8} - \frac{3\sqrt{3}}{8}\right) = \frac{5\sqrt{5}}{3} - \sqrt{3}.$$

Ex. Compute the arc length of the curve

$$y = \frac{x^4}{4} + \frac{1}{8x^2}$$

over $[1, 2]$.

Here, we have that $f(x) = \frac{x^4}{4} + \frac{1}{8x^2}$. Thus

$$f'(x) = x^3 - \frac{1}{4}x^{-3}$$

Hence the arc length is

Just like we can compute the arc length for $y = f(x)$, we can compute the arc length for $x = g(y)$.

Definition: Arc Length for $x = g(y)$

Let $x = g(y)$ have a continuous first derivative on the interval $[c, d]$. The length of the curve from $(g(c), c)$ to $(g(d), d)$ is

$$L = \int_c^d \sqrt{1 + g'(y)^2}\, dy.$$

Ex. Find the arc length of the curve $x = \frac{1}{2}\left(e^y + e^{-y}\right)$ on $[-\ln(2), \ln(2)]$.

Here, we have that

$$g(y) = \frac{1}{2}\left(e^y + e^{-y}\right),$$

so

$$g'(y) = \frac{1}{2}\left(e^y - e^{-y}\right)$$

Hence the arc length is

$$L = \int_{-\ln(2)}^{\ln(2)} \sqrt{1 + \frac{1}{4}\left(e^y - e^{-y}\right)^2} dy$$

$$= \int_{-\ln(2)}^{\ln(2)} \sqrt{1 + \frac{1}{4}\left(e^{2y} - 2 + e^{-2y}\right)} dy$$

$$= \int_{-\ln(2)}^{\ln(2)} \sqrt{\frac{1}{4}e^{2y} + \frac{1}{2} + \frac{1}{4}e^{-2y}} dy$$

$$= \int_{-\ln(2)}^{\ln(2)} \sqrt{\frac{1}{4}\left(e^y + e^{-y}\right)^2} dy$$

$$= \int_{-\ln(2)}^{\ln(2)} \frac{1}{2}\left(e^y + e^{-y}\right) dy$$

$$= \frac{1}{2}\left(e^y - e^{-y}\right)\Big|_{-\ln(2)}^{\ln(2)}$$

$$= \frac{1}{2}\left(\left(2 - \frac{1}{2}\right) - \left(\frac{1}{2} - 2\right)\right) = \frac{3}{2}$$