Linear Algebra Basics

A **vector space** over $\mathbb{R}$ is a set of vectors V closed under the operations vector addition and scalar multiplication.

vector addition:

$\forall \mathbf{x}, \mathbf{y} \in V$:

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \end{pmatrix} \in V$$

scalar multiplication:

$\forall \lambda \in \mathbb{R}, \mathbf{x} \in V$:

$$\lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \end{pmatrix} \in V$$

Linear Algebra Basics

A **basis** of the vector space is a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_d}\}$, such that every $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors:

$$\mathbf{x} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_d\mathbf{v_d}$$

Standard basis: $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ldots \mathbf{v_d} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Linear Algebra Basics

The **inner product** of two vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{i=1}^{d} x_i y_i$$

The **outer product** of two vectors:

$$\mathbf{x} \otimes \mathbf{y} := \begin{pmatrix} x_1y_1 & x_1y_2 & \cdots & x_1y_d \\ x_2y_1 & x_2y_2 & \cdots & x_2y_d \\ \vdots & \ddots & & \\ x_dy_1 & x_dy_2 & \cdots & x_dy_d \end{pmatrix}$$

Linear Algebra Basics

Matrix Multiplication $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\mathbf{B} \quad \begin{array}{cc} & \quad\quad\quad\quad m \\ \ell & \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix} \end{array}$$

$$\mathbf{A} \quad \begin{array}{cc} & \ell \\ \\ \\ k \\ \end{array} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix} \quad \mathbf{C}$$

Each entry of $\mathbf{C}$ is defined as an inner product

$$c_{ij} = \sum_{1 \leq p \leq \ell} a_{i,p} \cdot b_{p,j} = \langle \mathbf{a_i}, \mathbf{b_j} \rangle$$

$\mathbf{a_i}$ is the i'th row vector of $\mathbf{A}$ and $\mathbf{b_j}$ is the j'th column vector of $\mathbf{B}$

Linear Algebra Basics

Matrix Multiplication $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$

$$\mathbf{B} \quad \begin{array}{cc} & m \\ \ell & \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix} \end{array}$$

$$\mathbf{A} \quad \begin{array}{cc} & \ell \\ k & \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \end{array} \quad \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix} \quad \mathbf{C}$$

We can also write $\mathbf{C}$ as a sum of outer products

$$\mathbf{C} = \sum_{\mathbf{1} \leq \mathbf{i} \leq \ell} \mathbf{a_i} \otimes \mathbf{b_i}$$

$\mathbf{a_i}$ is the i'th column vector of $\mathbf{A}$ and $\mathbf{b_j}$ is the j'th row vector of $\mathbf{B}$

Linear Algebra Basics

Transpose of a matrix switches row and column indices

$$
\begin{pmatrix}
a_{1,1} & a_{1,2} & a_{1,3} \\
a_{2,1} & a_{2,2} & a_{2,3} \\
a_{3,1} & a_{3,2} & a_{3,3} \\
a_{4,1} & a_{4,2} & a_{4,3}
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\
a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\
a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3}
\end{pmatrix}
$$

$\mathbf{A}$ $\qquad\qquad$ $\mathbf{A^T}$

It holds that $(\mathbf{A} \cdot \mathbf{B})^{\mathbf{T}} = \mathbf{B^T} \cdot \mathbf{A^T}$

Linear Algebra Basics

The **length** of a vector:

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^{d} x_i^2}$$

Vector $\mathbf{x}$ has **unit length** if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$

The **law of cosine**:

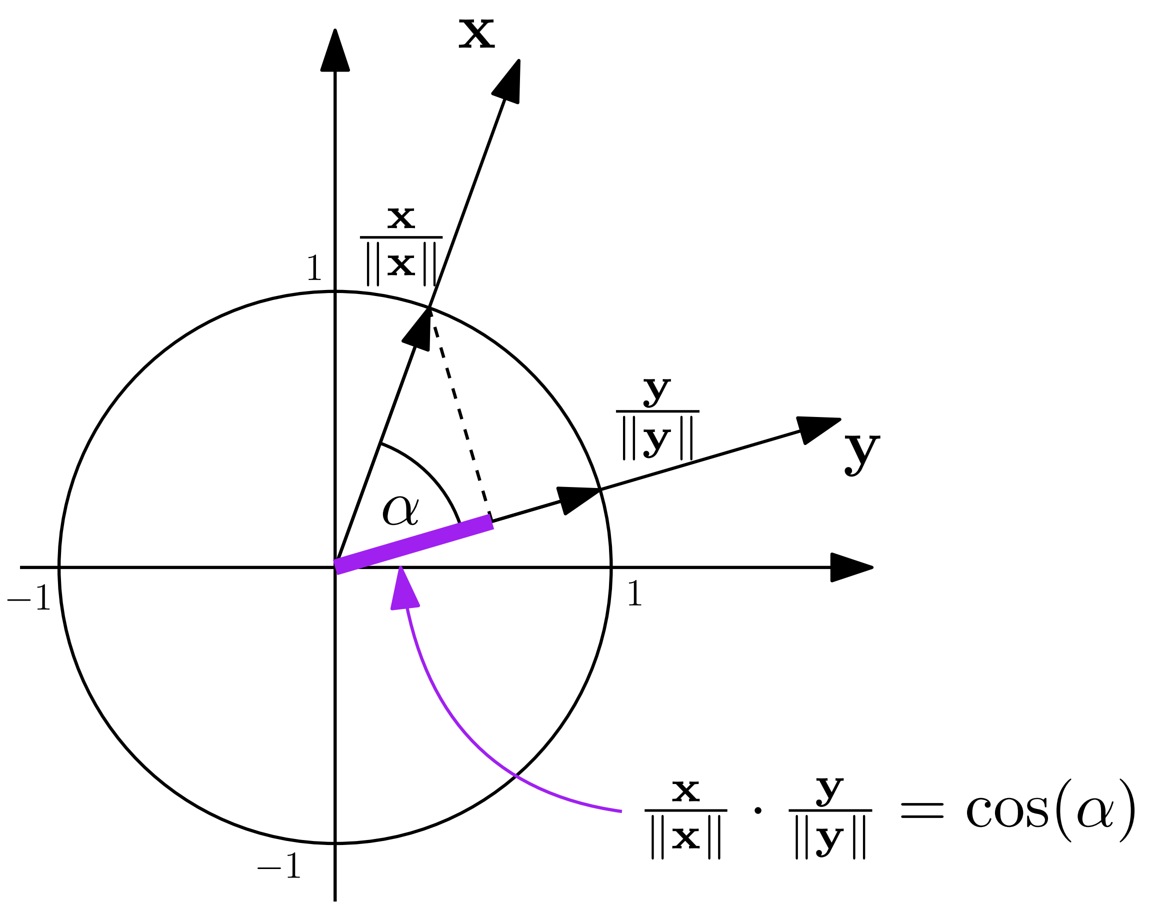
$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\alpha)$$

α is the smaller angle spanned by the two vectors

Vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Linear Algebra Basics

The law of cosine: $\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\alpha)$



Linear Algebra Basics

The **span** of a set of vectors $\mathbf{v_1}, \dots, \mathbf{v_k}$ is the set of all possible linear combinations of of these vectors

$$\left\{ \sum_{i=0}^{k} \lambda_i \mathbf{v_i} \;:\; \lambda_i \in \mathbb{R} \right\}$$

The **rank** of a matrix is the dimension of the space spanned by its column vectors (or row vectors)

Linear Algebra Basics

A **linear map** is a mapping between two vector spaces $f : V \to W$ that satisfies the following two conditions:

(1) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
(2) $\forall \gamma \in \mathbb{R} : f(\gamma \mathbf{x}) = \gamma f(\mathbf{x})$

Any matrix $\mathbf{A}$ defines a **linear map**:

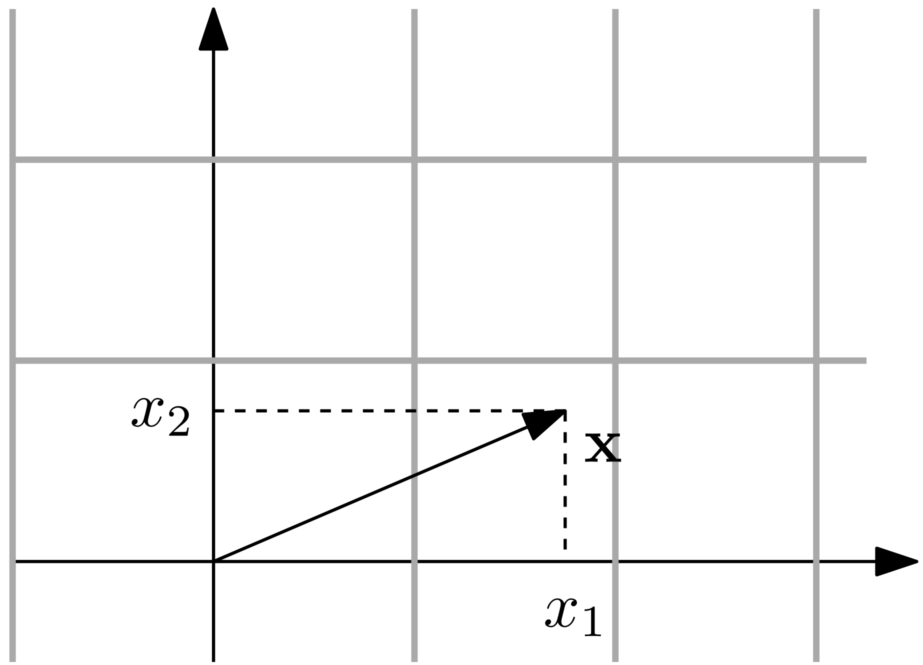
$$f : V \to W$$
$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$$

Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

$$\mathbf{Ax} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{Av_1} + x_2\mathbf{Av_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$

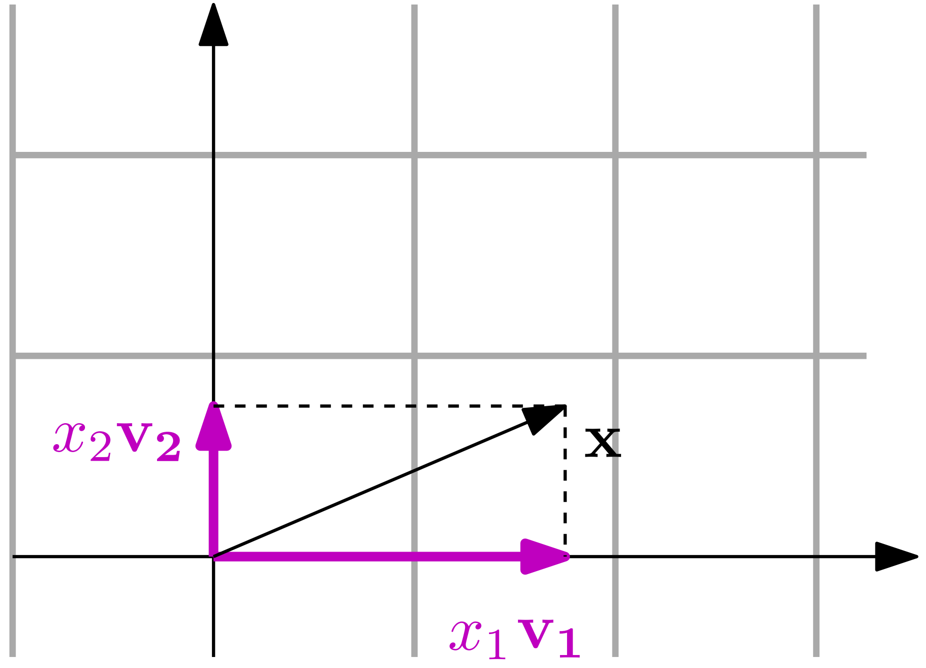


Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{A}\mathbf{v_1} + x_2\mathbf{A}\mathbf{v_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$

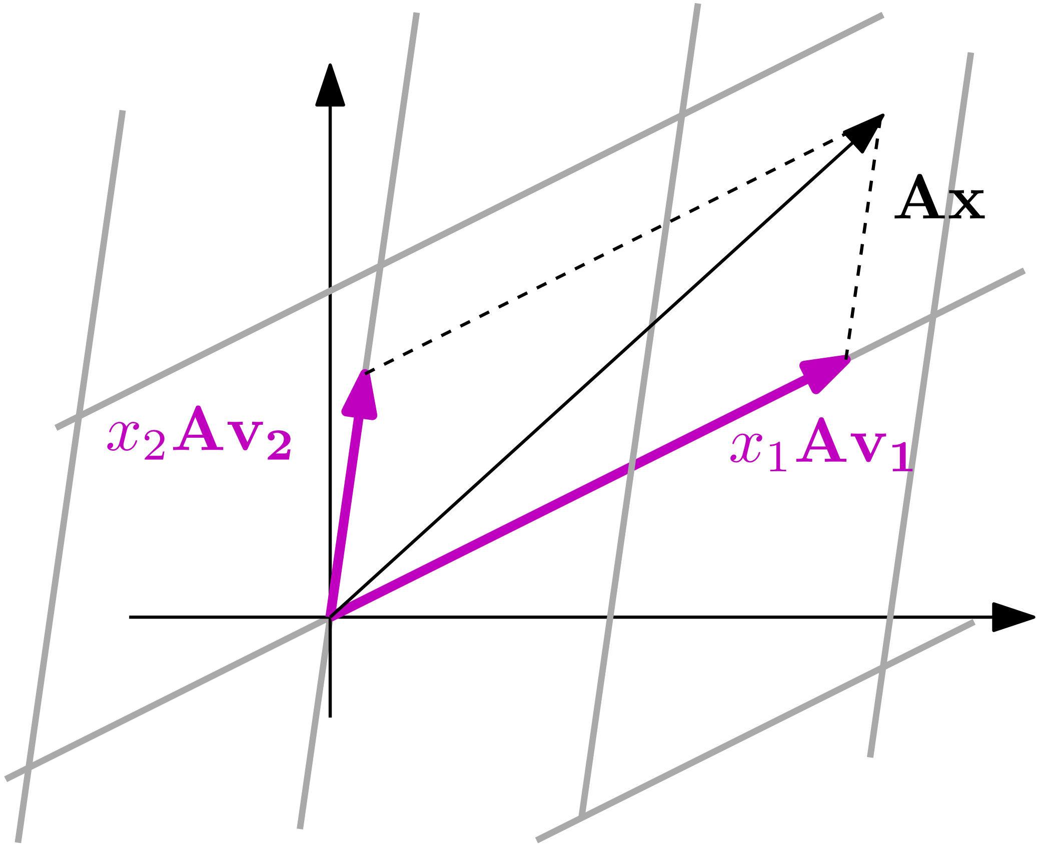


Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

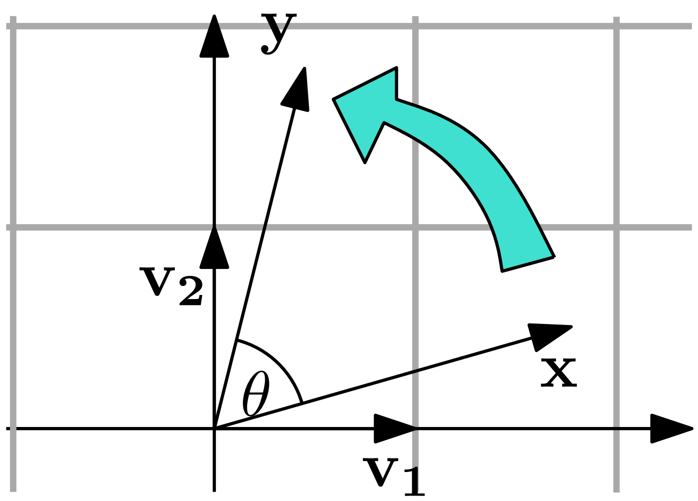
$$\mathbf{Ax} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{Av_1} + x_2\mathbf{Av_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



Linear Algebra: Rotation

Rotate a vector $\mathbf{x}$ by angle θ:



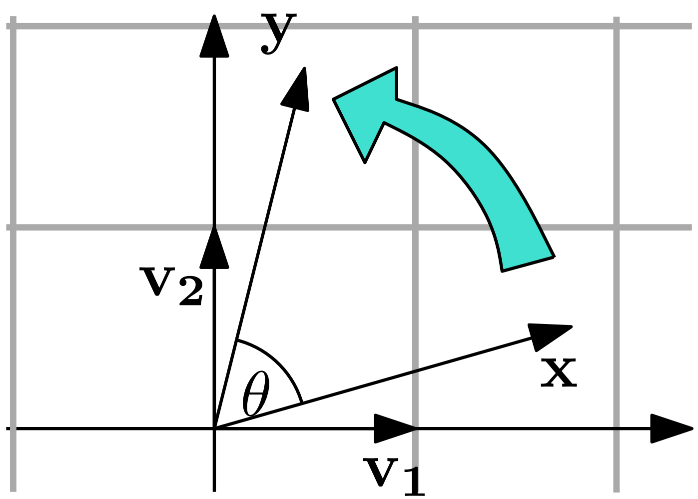
$$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

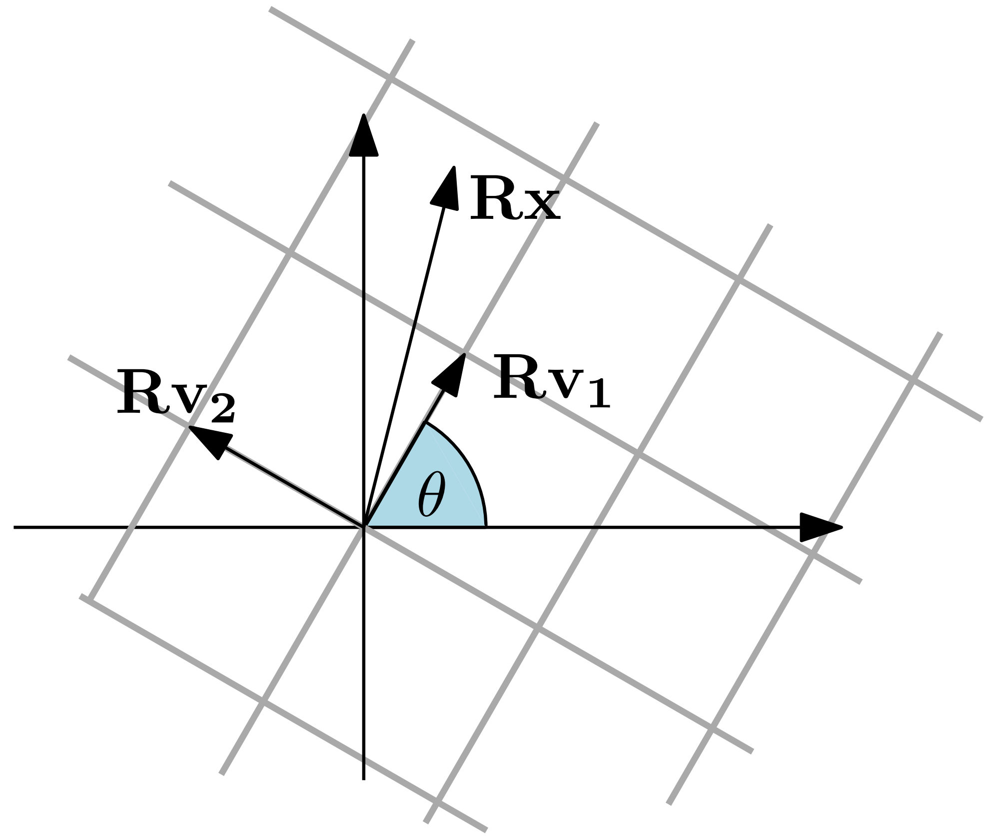
$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Linear Algebra: Rotation

Rotate a vector $\mathbf{x}$ by angle θ:



$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$



Transformation matrix:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{Rv_1} = (\cos\theta, \sin\theta)$$
$$\mathbf{Rv_2} = (-\sin\theta, \cos\theta)$$

Linear Algebra: Rotation

In general:
A matrix is a rotation iff it is orthogonal

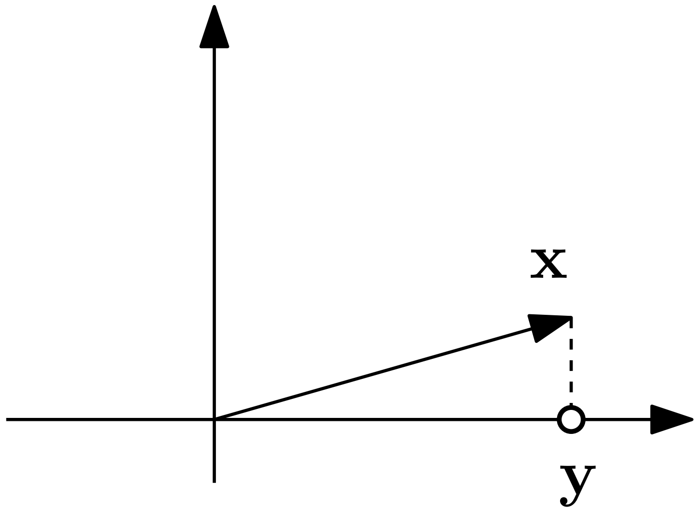
$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} = \begin{pmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} \end{pmatrix}$$

This means its row vectors are..
(1) pairwise orthogonal: $\mathbf{r_i} \cdot \mathbf{r_j} = 0$
(2) unit vectors: $\|\mathbf{r_i}\| = 1$

Furthermore, it holds that $\mathbf{R}^{-1} = \mathbf{R}^{\mathbf{T}}$ and that the length of any vector is preserved under $\mathbf{R}$

Linear Algebra: Axis-orthogonal Projection

Project a vector $\mathbf{x}$ into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

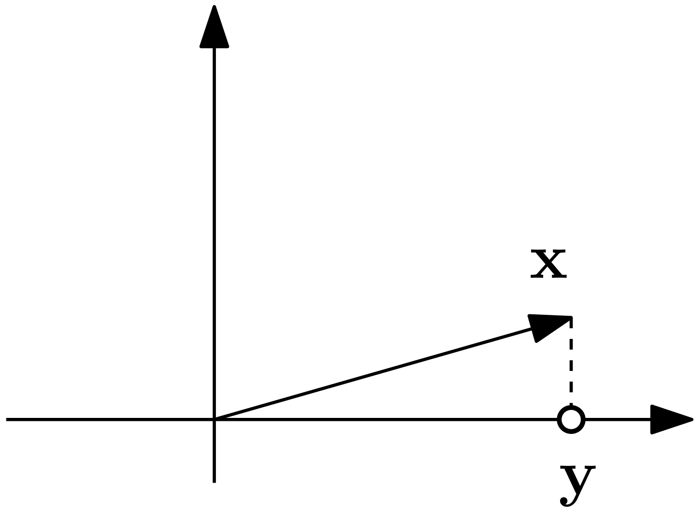
In general:

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Linear Algebra: Axis-orthogonal Projection

Project a vector $\mathbf{x}$ into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

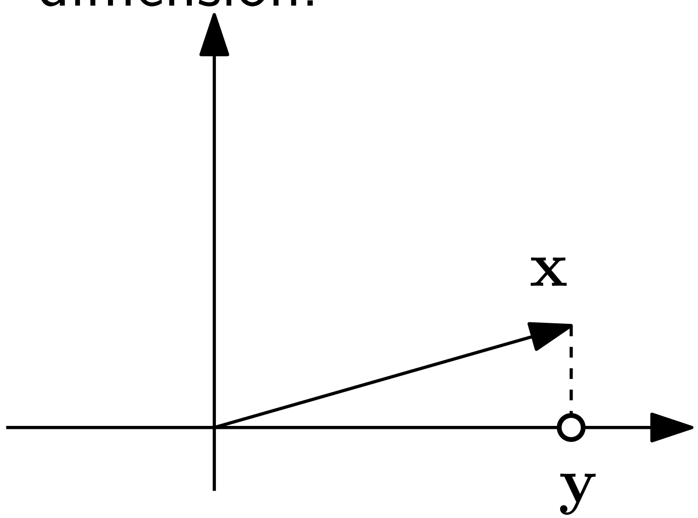
In general:

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Linear Algebra: Axis-orthogonal Projection

Project a vector $\mathbf{x}$ into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

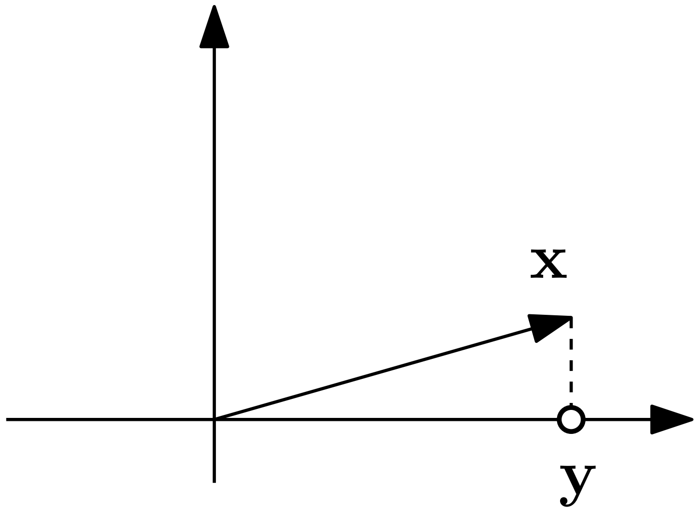
In general:

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ \cancel{0} & \cancel{0} & \cancel{0} & \cancel{0} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Linear Algebra: Axis-orthogonal Projection

Project a vector $\mathbf{x}$ into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

In general:

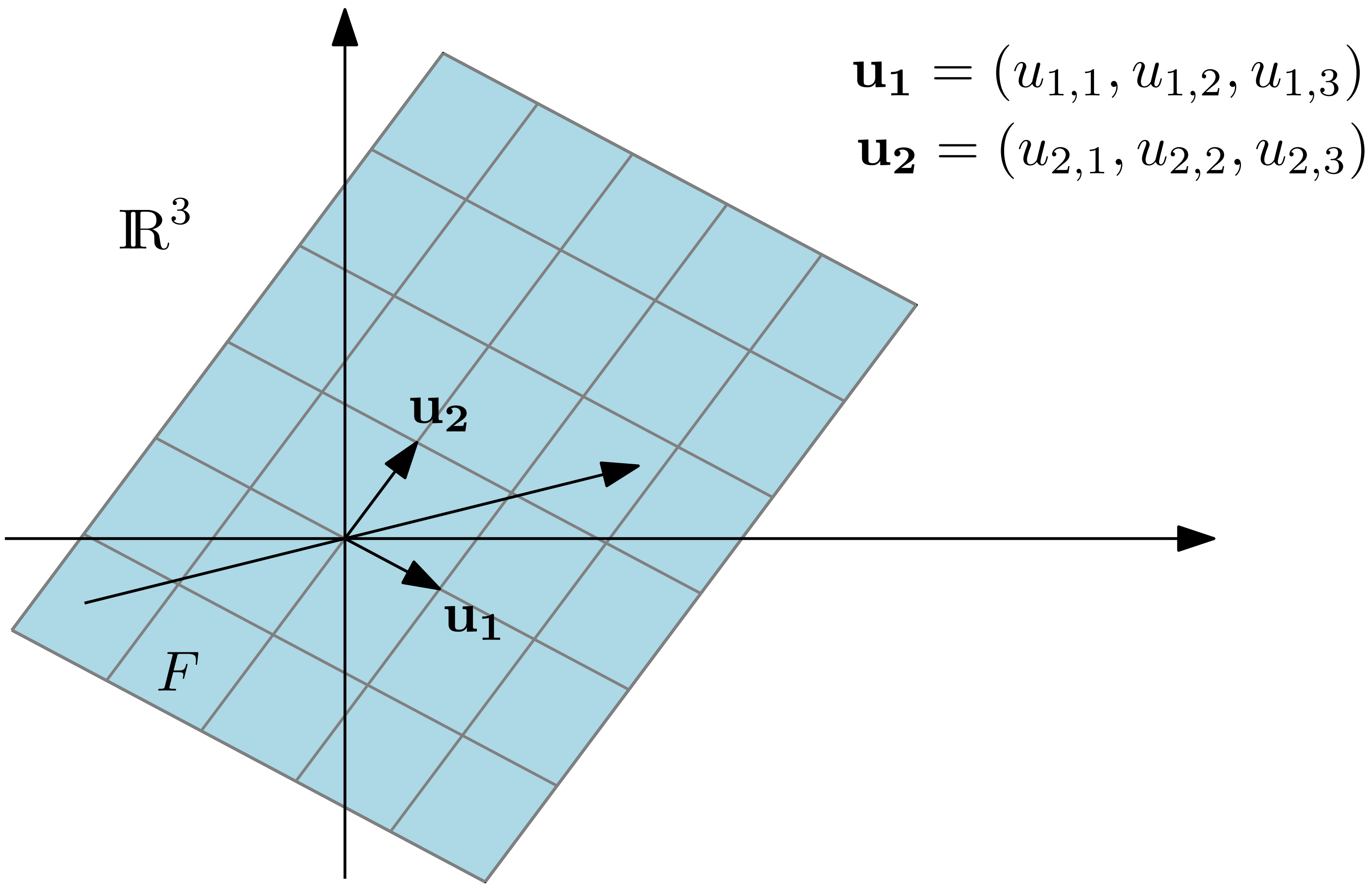
An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

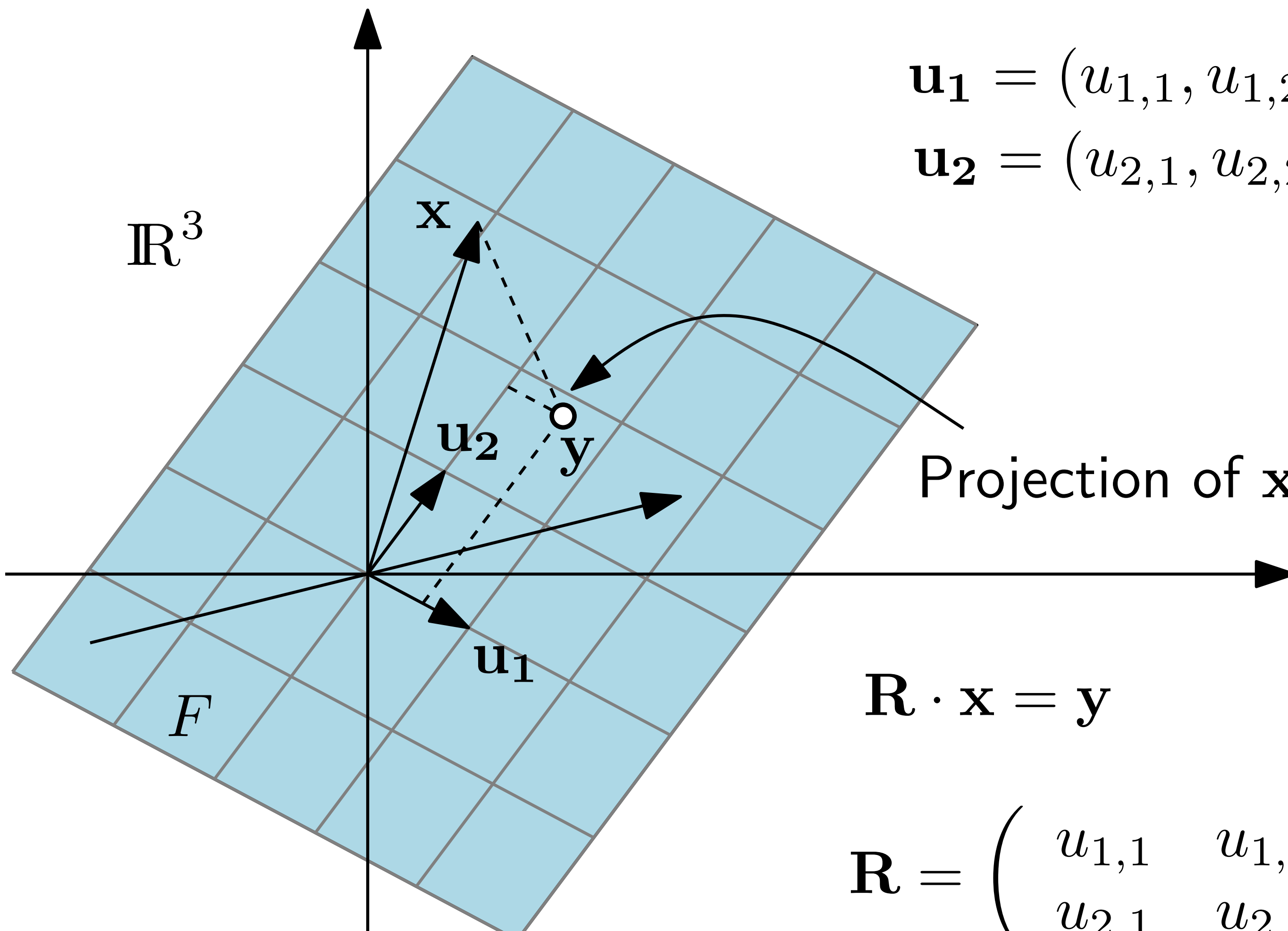
Linear Algebra: Projection onto subspace

Let F be a k-dimensional linear subspace of $\mathbb{R}^d$ spanned by orthonormal vectors $\mathbf{u_1}, \ldots, \mathbf{u_k}$ and let $\mathbf{R}$ be the projection onto F



Linear Algebra: Projection onto subspace

Let F be a k-dimensional linear subspace of $\mathbb{R}^d$ spanned by orthonormal vectors $\mathbf{u_1}, \ldots, \mathbf{u_k}$ and let $\mathbf{R}$ be the projection onto F



$$\mathbf{u_1} = (u_{1,1}, u_{1,2}, u_{1,3})$$
$$\mathbf{u_2} = (u_{2,1}, u_{2,2}, u_{2,3})$$

$$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$$

$$\mathbf{R} = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \end{pmatrix}$$