Linear Algebra Basics

2IMM20 Foundations of data mining
2017 Semester B Quartile 3

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Linear Algebra Basics

A **vector** is a tuple of values

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_d \end{pmatrix}$$

Linear algebra provides a way to do calculations with vectors

Linear Algebra Basics

A **vector space** over $\mathbb{R}$ is a set of vectors V closed under the operations vector addition and scalar multiplication.

vector addition:

$\forall \mathbf{x}, \mathbf{y} \in V$:

$$\mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \end{pmatrix} \in V$$

scalar multiplication:

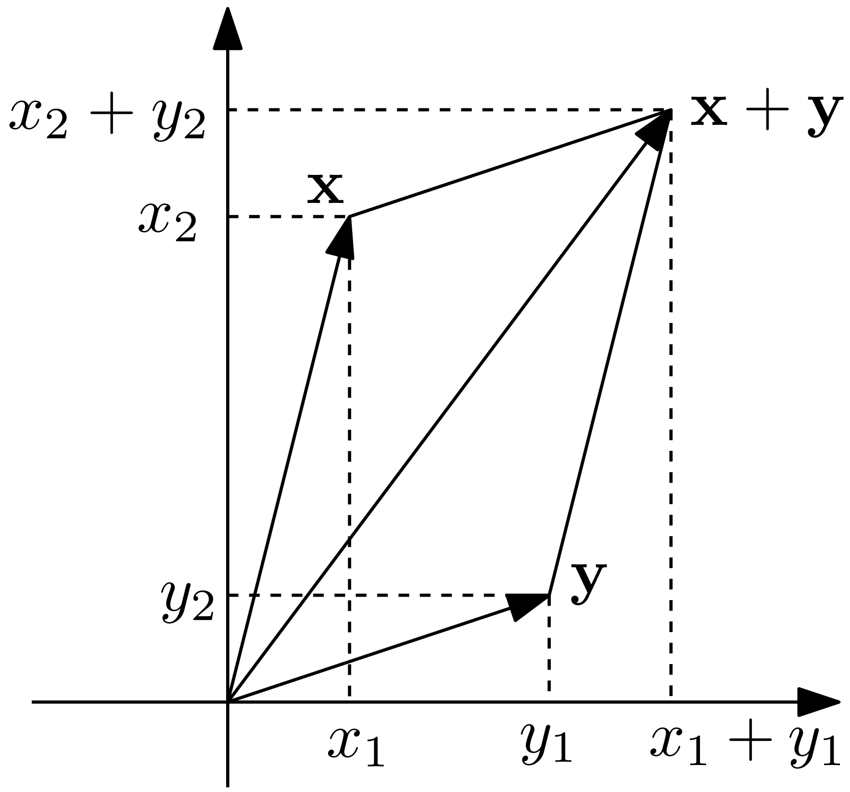
$\forall \lambda \in \mathbb{R}, \mathbf{x} \in V$:

$$\lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \end{pmatrix} \in V$$

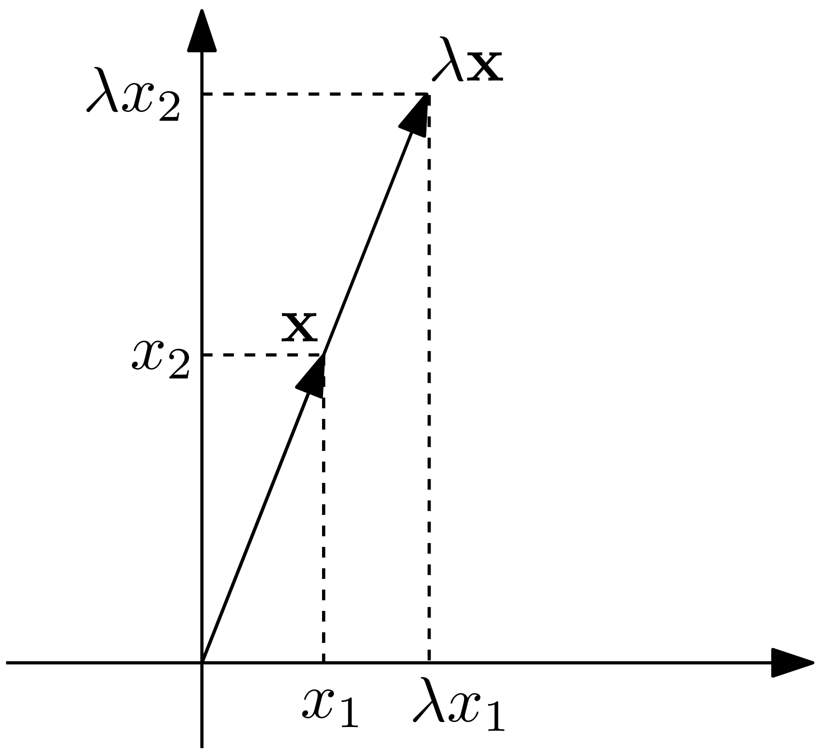
Linear Algebra Basics

Geometric interpretation

vector addition:



scalar multiplication:



Linear Algebra Basics

The **inner product** of two vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \textstyle\sum_{i=1}^{d} x_i y_i$$

The **outer product** of two vectors:

$$\mathbf{x} \otimes \mathbf{y} := \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_d \\ \vdots & \ddots & & \\ x_d y_1 & x_d y_2 & \cdots & x_d y_d \end{pmatrix}$$

Linear Algebra Basics

A **matrix** is a tuple of vectors

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}$$

Linear Algebra Basics

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A can be viewed as tuple of row vectors

$$\mathbf{a_i} = \begin{pmatrix} a_{i,1} & a_{i,2} & a_{i,3} \end{pmatrix}$$

Linear Algebra Basics

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A can be viewed as tuple of row vectors

$$\mathbf{a_i} = \begin{pmatrix} a_{i,1} & a_{i,2} & a_{i,3} \end{pmatrix}$$

... or it can be viewed as tuple of column vectors

$$\mathbf{a_i} = \begin{pmatrix} a_{1,i} \\ a_{2,i} \\ a_{3,i} \end{pmatrix}$$

Linear Algebra Basics

Transpose of a matrix switches row and column indices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \Rightarrow \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} \end{pmatrix}$$

$\mathbf{A}$ $\qquad\qquad$ $\mathbf{A^T}$

It holds that $(\mathbf{A} \cdot \mathbf{B})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}} \cdot \mathbf{A}^{\mathbf{T}}$

Linear Algebra Basics

Matrix Multiplication $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$

Each entry c_{ij} of $\mathbf{C}$ is defined as

$$c_{ij} = \sum_{\ell} a_{i,\ell} \cdot b_{\ell,j} = \langle \mathbf{a_i}, \mathbf{b_j} \rangle$$

$\mathbf{a_i}$ is the i'th row vector of $\mathbf{A}$
$\mathbf{b_j}$ is the j'th column vector of $\mathbf{B}$

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \cdot \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix} = \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}$$

Linear Algebra Basics

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$$\begin{array}{cc} & \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix} \\ \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} & \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix} \end{array}$$

Linear Algebra Basics

The **span** of a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_k}$ is the set of all possible linear combinations of of these vectors

$$\left\{ \sum_{i=0}^{k} \lambda_i \mathbf{v_i} \ : \ \lambda_i \in \mathbb{R} \right\}$$

The **rank** of a matrix is the dimension of the space spanned by its column vectors (or row vectors)

Linear Algebra Basics

A **basis** of the vector space is a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_d}\}$, such that every $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors:

$$\mathbf{x} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_d\mathbf{v_d}$$

Standard basis: $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ldots \mathbf{v_d} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Linear Algebra Basics

A **linear map** is a mapping between two vector spaces $f : V \to W$ that satisfies the following two conditions:

(1) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
(2) $\forall \gamma \in \mathbb{R} : f(\gamma \mathbf{x}) = \gamma f(\mathbf{x})$

Any matrix $\mathbf{A}$ defines a **linear map**:

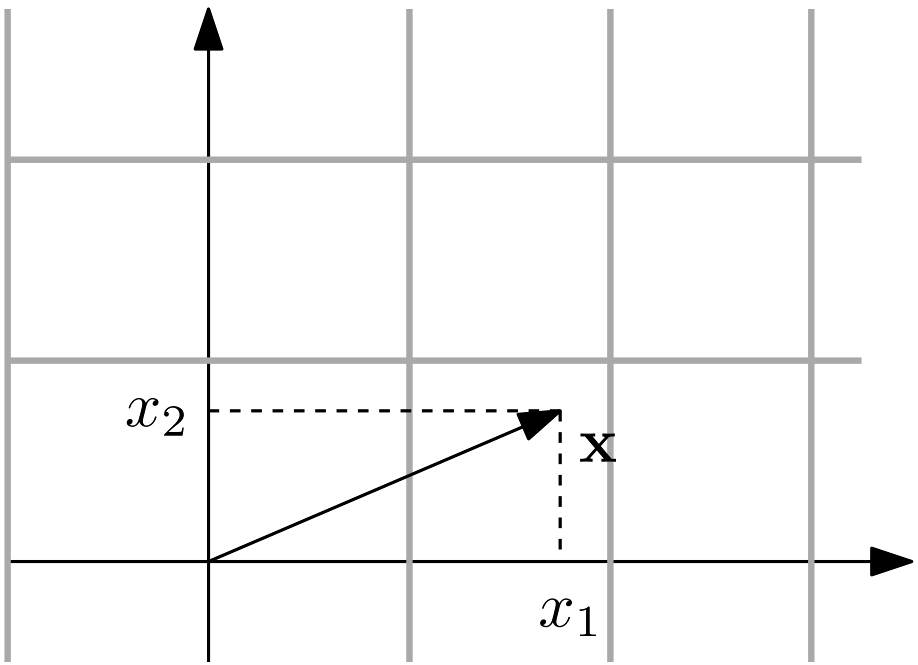
$$f : V \to W$$
$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$$

Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

$$\mathbf{Ax} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{Av_1} + x_2\mathbf{Av_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



Linear Algebra Basics

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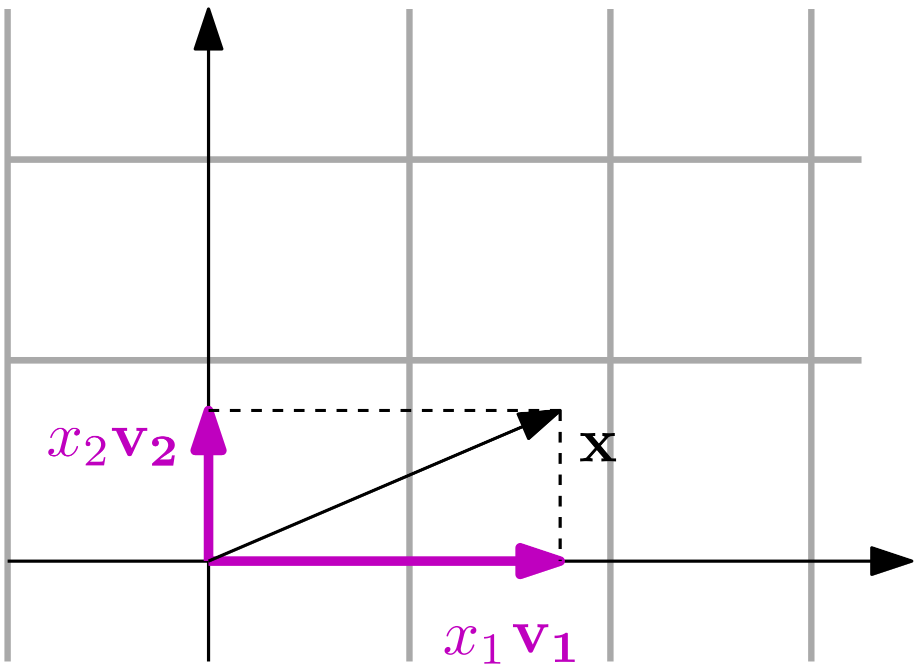
$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{A}\mathbf{v_1} + x_2\mathbf{A}\mathbf{v_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



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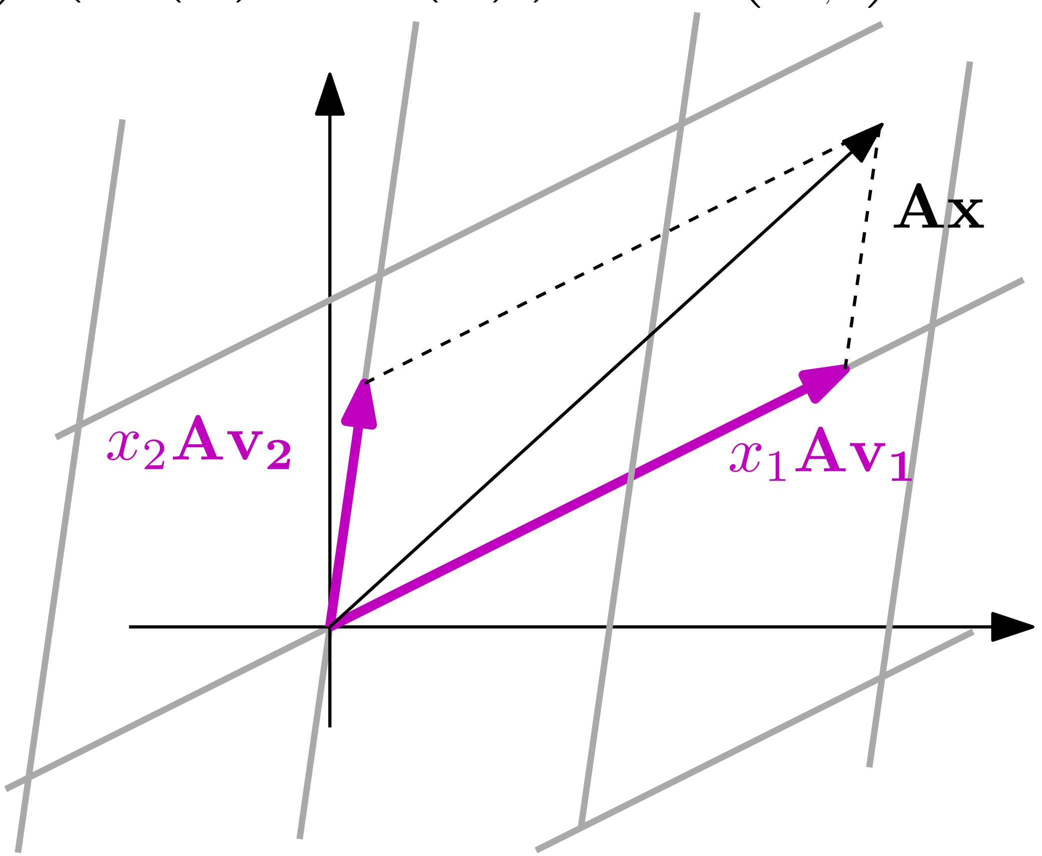
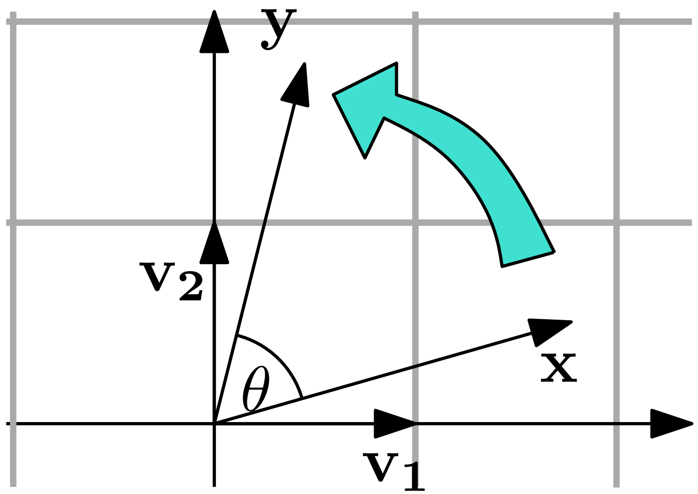
$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



Linear Algebra Basics

Rotate a vector $\mathbf{x}$ by angle θ:



$$\mathbf{R} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{R} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Linear Algebra Basics

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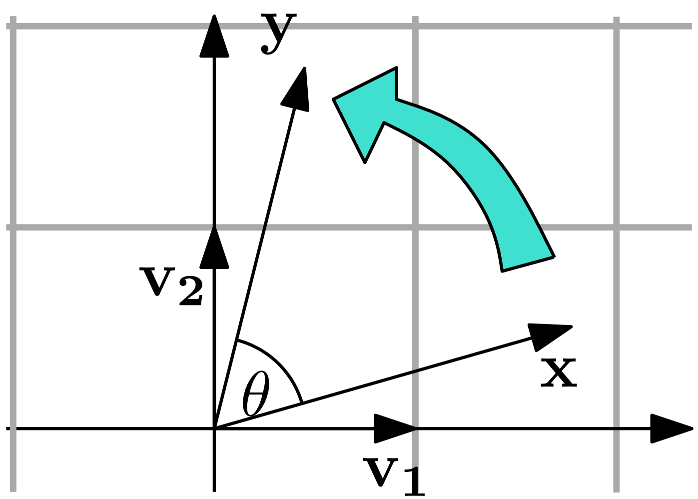

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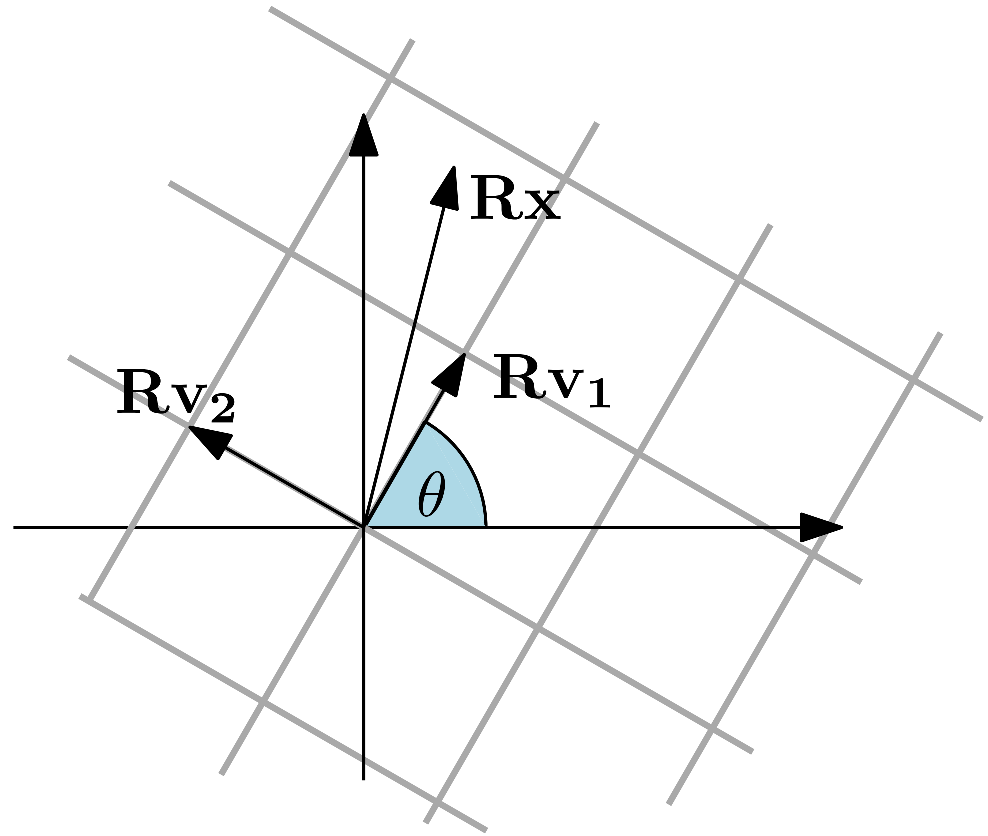

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$$\mathbf{Rv_1} = (\cos\theta, \sin\theta)$$
$$\mathbf{Rv_2} = (-\sin\theta, \cos\theta)$$

Linear Algebra Basics

The **length** of a vector:

$$\|\mathbf{x}\| := \sqrt{\sum_{i=1}^{d} x_i^2}$$

Vector $\mathbf{x}$ has **unit length** if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$

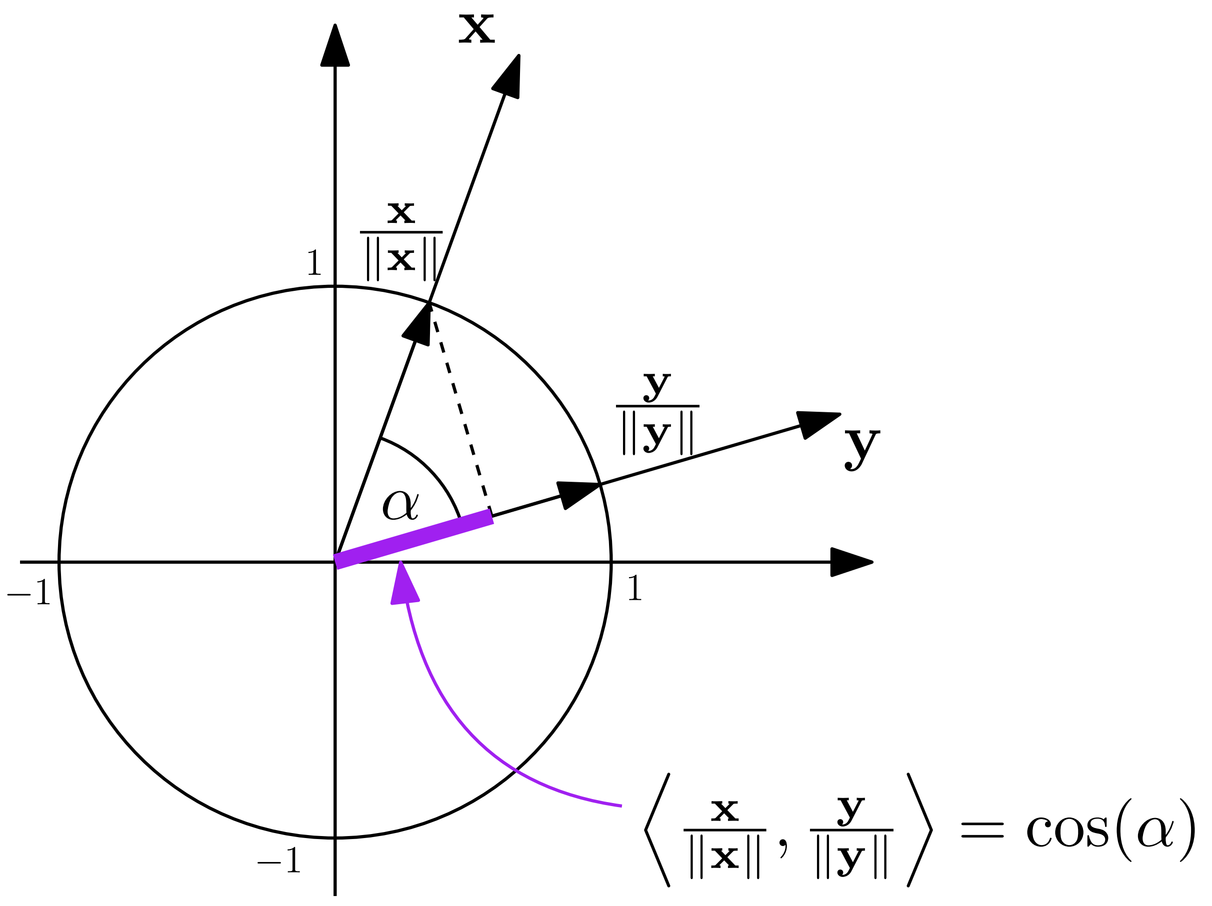
The **angle** between two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\cos(\alpha) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

Vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Linear Algebra Basics

Geometric interpretation



Linear Algebra Basics

A matrix is a **rotation** iff it is orthogonal

$$\mathbf{R} = \begin{pmatrix} r_{1,1} & r_{1,2} & r_{1,3} \\ r_{2,1} & r_{2,2} & r_{2,3} \\ r_{3,1} & r_{3,2} & r_{3,3} \end{pmatrix} = \begin{pmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} \end{pmatrix}$$

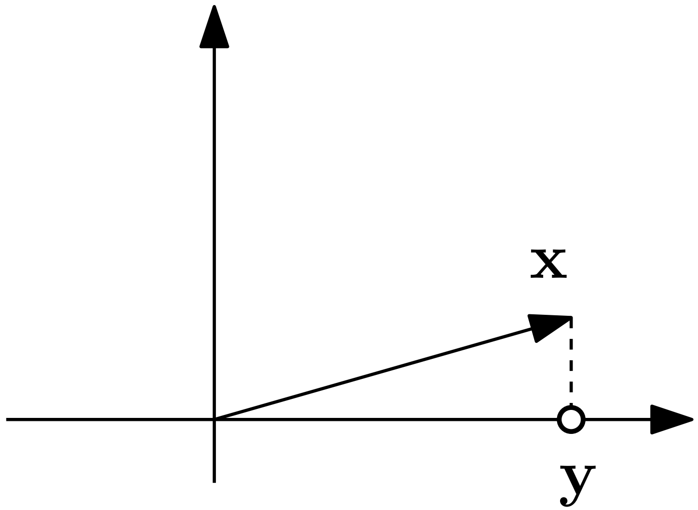
A matrix is orthogonal iff its row vectors are..

(1) pairwise orthogonal: $\mathbf{r_i} \cdot \mathbf{r_j} = 0$
(2) unit vectors: $\|\mathbf{r_i}\| = 1$

It holds that $\mathbf{R}^{-\mathbf{1}} = \mathbf{R}^{\mathbf{T}}$ and that the length of any vector is preserved under $\mathbf{R}$

Linear Algebra Basics

Project a vector $\mathbf{x}$ into first dimension:



$$\mathbf{P} \cdot \mathbf{x} = \mathbf{y}$$

Transformation matrix:

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \end{pmatrix}$$

In general:

An axis-orthogonal projection is an identity matrix with some 'collapsed' dimensions

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Linear Algebra Basics

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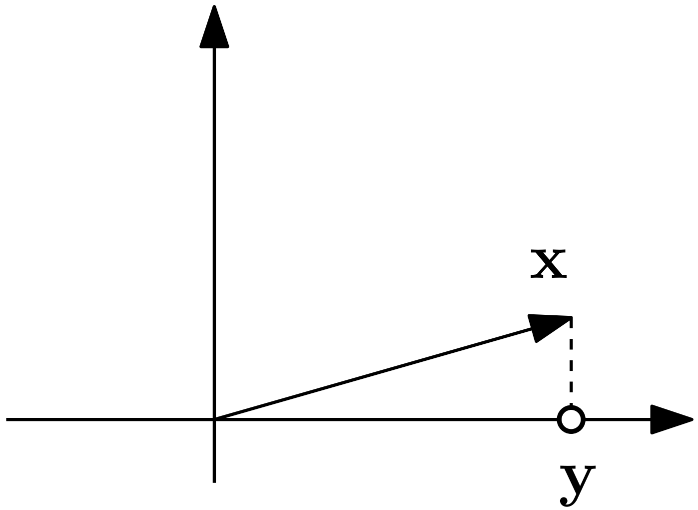

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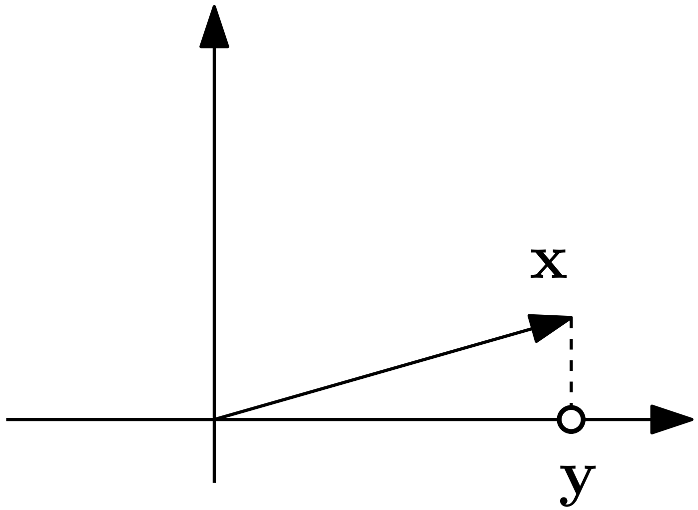

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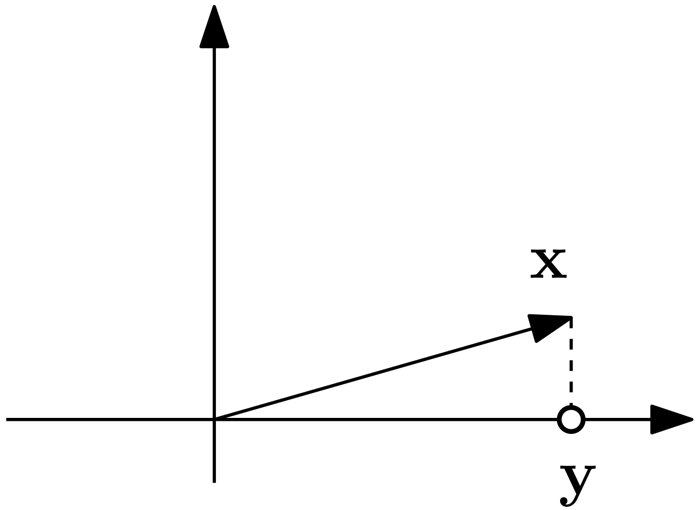

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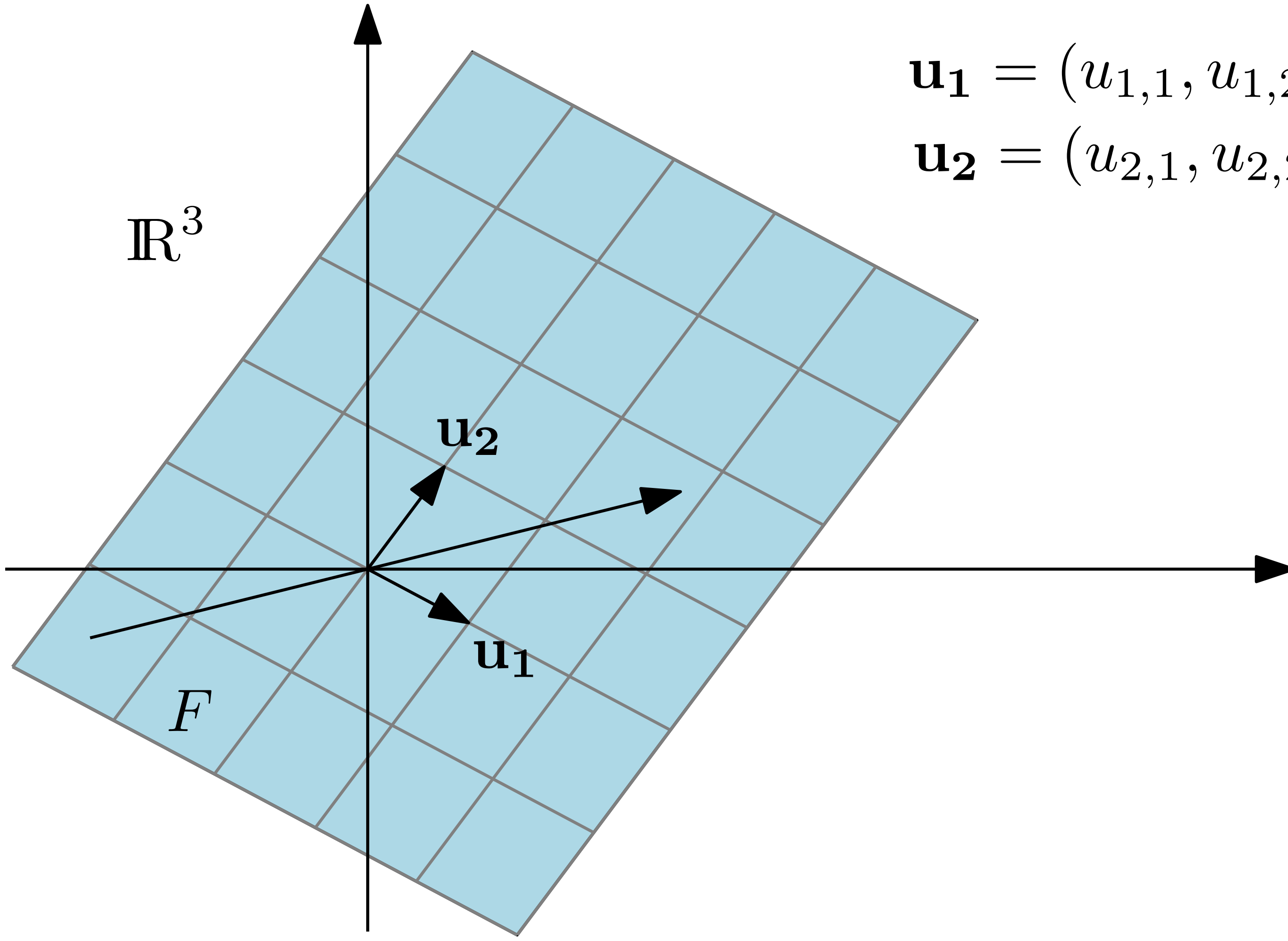
$$\begin{array}{cc} & \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \end{array}$$

Linear Algebra Basics

Let F be a k-dimensional linear subspace of $\mathbb{R}^d$ spanned by orthonormal vectors $\mathbf{u_1}, \ldots, \mathbf{u_k}$ and let $\mathbf{R}$ be the projection onto F



$$\mathbf{u_1} = (u_{1,1}, u_{1,2}, u_{1,3})$$
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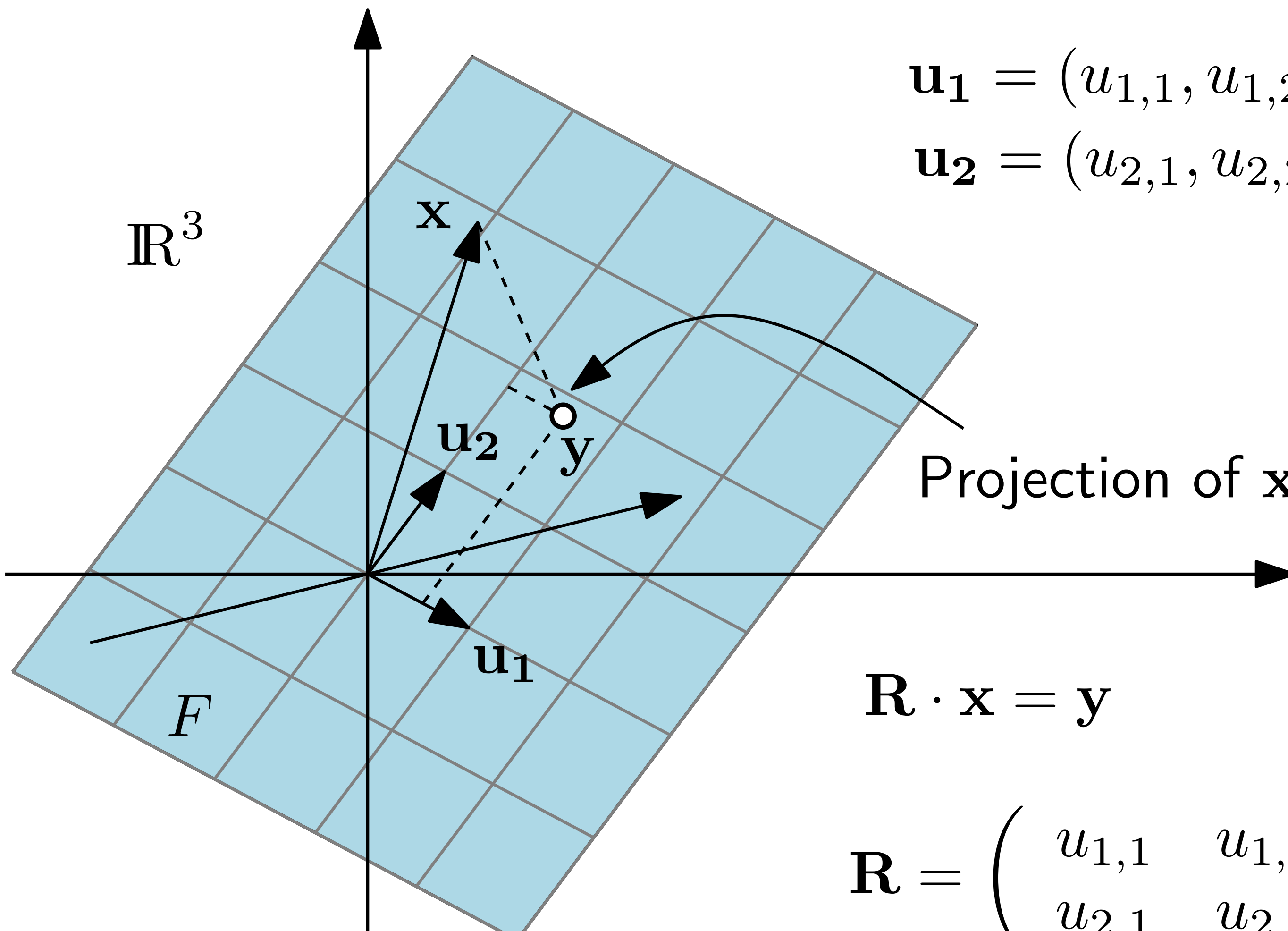

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Linear Algebra Basics

Summary

- Vectors and Matrices
- Basic operations
  - Addition
  - Scalar multiplication
  - Inner product
  - Outer product
  - Matrix multiplication
- Linear maps
- Basis of a vector space
- Geometry (length, angles)
- Rotations and Projections