Linear Algebra Basics

A **vector** is a tuple of values

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$$

$$\mathbf{x} = \begin{pmatrix} x_1 & x_2 & \dots & x_d \end{pmatrix}$$

Linear algebra provides a way to do calculations with vectors

Linear Algebra Basics

A **vector space** over $\mathbb{R}$ is a set of vectors V closed under the operations vector addition and scalar multiplication.

vector addition:

$$\forall \mathbf{x}, \mathbf{y} \in V: \quad \mathbf{x} + \mathbf{y} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \end{pmatrix} \in V$$

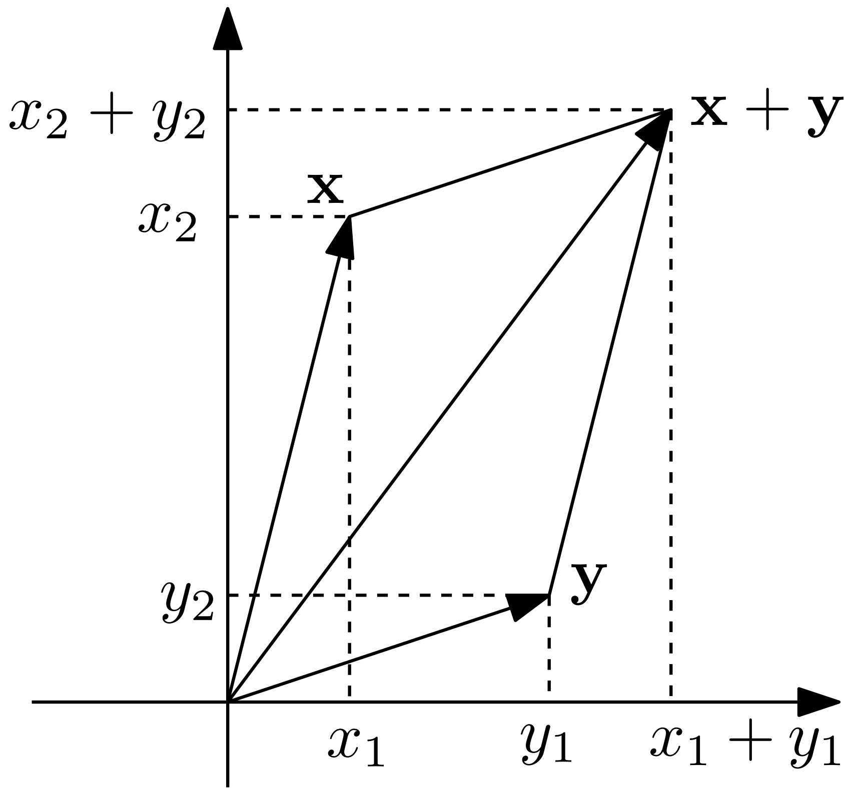
scalar multiplication:

$$\forall \lambda \in \mathbb{R}, \mathbf{x} \in V: \quad \lambda \mathbf{x} = \lambda \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \vdots \end{pmatrix} \in V$$

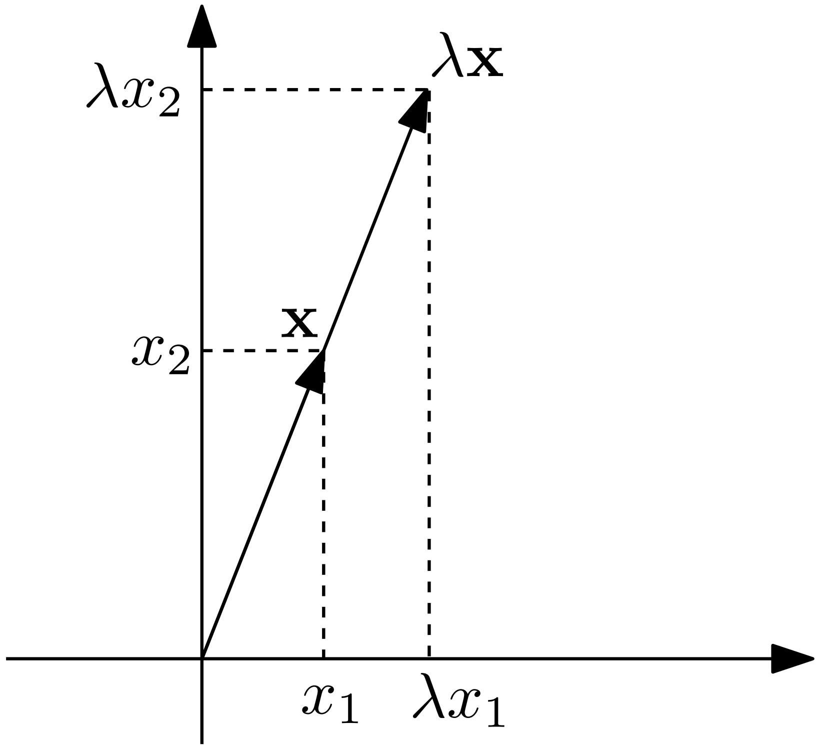
Linear Algebra Basics

Geometric interpretation

vector addition:



scalar multiplication:



Linear Algebra Basics

A **basis** of the vector space is a set of vectors $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_d}\}$, such that every $\mathbf{x} \in V$ can be expressed as a linear combination of the basis vectors:

$$\mathbf{x} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_d\mathbf{v_d}$$

Standard basis: $\mathbf{v_1} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \mathbf{v_2} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} \ldots \mathbf{v_d} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$

Linear Algebra Basics

The **inner product** of two vectors:

$$\langle \mathbf{x}, \mathbf{y} \rangle := \textstyle\sum_{i=1}^{d} x_i y_i$$

The **outer product** of two vectors:

$$\mathbf{x} \otimes \mathbf{y} := \begin{pmatrix} x_1 y_1 & x_1 y_2 & \cdots & x_1 y_d \\ x_2 y_1 & x_2 y_2 & \cdots & x_2 y_d \\ \vdots & \ddots & & \\ x_d y_1 & x_d y_2 & \cdots & x_d y_d \end{pmatrix}$$

Linear Algebra Basics

Transpose of a matrix switches row and column indices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \Longrightarrow \begin{pmatrix} a_{1,1} & a_{2,1} & a_{3,1} & a_{4,1} \\ a_{1,2} & a_{2,2} & a_{3,2} & a_{4,2} \\ a_{1,3} & a_{2,3} & a_{3,3} & a_{4,3} \end{pmatrix}$$

$\mathbf{A}$ $\qquad\qquad$ $\mathbf{A}^{\mathbf{T}}$

It holds that $(\mathbf{A} \cdot \mathbf{B})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}} \cdot \mathbf{A}^{\mathbf{T}}$

Linear Algebra Basics

Matrix Multiplication $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$

Each entry c_{ij} of $\mathbf{C}$ is defined as

$$c_{ij} = \sum_{\ell} a_{i,\ell} \cdot b_{\ell,j} = \langle \mathbf{a_i}, \mathbf{b_j} \rangle$$

$\mathbf{a_i}$ is the i'th row vector of $\mathbf{A}$
$\mathbf{b_j}$ is the j'th column vector of $\mathbf{B}$

$$\overset{\mathbf{A}}{\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix}} \cdot \overset{\mathbf{B}}{\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix}} = \overset{\mathbf{C}}{\begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}}$$

Linear Algebra Basics

Matrix Multiplication $\mathbf{A} \cdot \mathbf{B} = \mathbf{C}$

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$$\mathbf{B} \begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & b_{1,4} \\ b_{2,1} & b_{2,2} & b_{2,3} & b_{2,4} \\ b_{3,1} & b_{3,2} & b_{3,3} & b_{3,4} \end{pmatrix}$$

$$\mathbf{A} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \\ a_{4,1} & a_{4,2} & a_{4,3} \end{pmatrix} \mathbf{C} \begin{pmatrix} c_{1,1} & c_{1,2} & c_{1,3} & c_{1,4} \\ c_{2,1} & c_{2,2} & c_{2,3} & c_{2,4} \\ c_{3,1} & c_{3,2} & c_{3,3} & c_{3,4} \\ c_{4,1} & c_{4,2} & c_{4,3} & c_{4,4} \end{pmatrix}$$

Linear Algebra Basics

The **span** of a set of vectors $\mathbf{v_1}, \ldots, \mathbf{v_k}$ is the set of all possible linear combinations of of these vectors

$$\left\{ \sum_{i=0}^{k} \lambda_i \mathbf{v_i} \;:\; \lambda_i \in \mathbb{R} \right\}$$

The **rank** of a matrix is the dimension of the space spanned by its column vectors (or row vectors)

Linear Algebra Basics

A **linear map** is a mapping between two vector spaces $f : V \to W$ that satisfies the following two conditions:

(1) $f(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + f(\mathbf{y})$
(2) $\forall \gamma \in \mathbb{R} : f(\gamma \mathbf{x}) = \gamma f(\mathbf{x})$

Any matrix $\mathbf{A}$ defines a **linear map**:

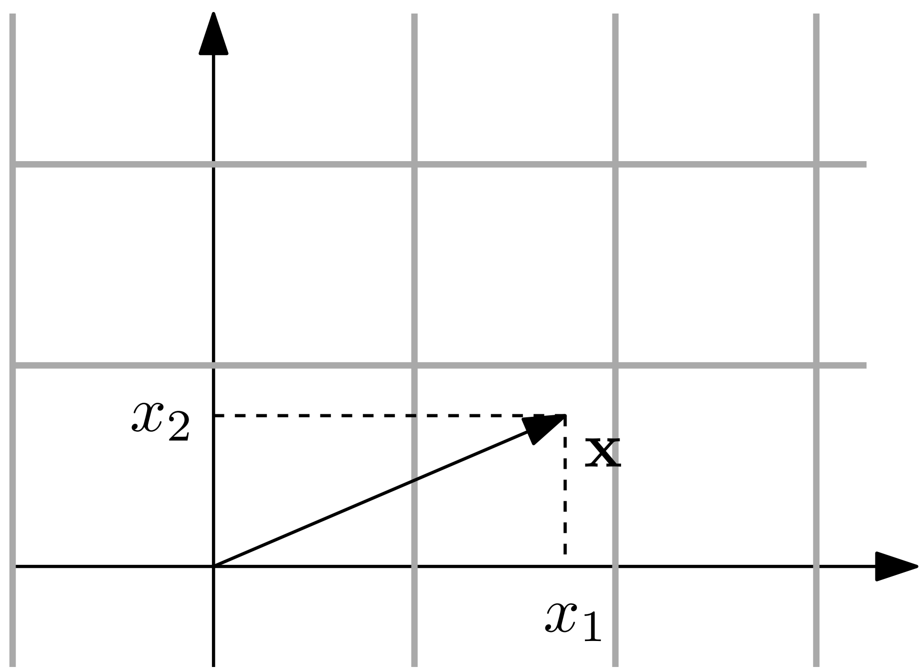
$$f : V \to W$$
$$f(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$$

Linear Algebra Basics

Using linearity, we can expand $\mathbf{A} \cdot \mathbf{x}$ using the standard basis:

$$\mathbf{Ax} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{Av_1} + x_2\mathbf{Av_2}$$

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \left(x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = x_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \end{pmatrix} + x_2 \begin{pmatrix} a_{1,2} \\ a_{2,2} \end{pmatrix}$$



Linear Algebra Basics

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$$\mathbf{A}\mathbf{x} = \mathbf{A}(x_1\mathbf{v_1} + x_2\mathbf{v_2}) = x_1\mathbf{A}\mathbf{v_1} + x_2\mathbf{A}\mathbf{v_2}$$

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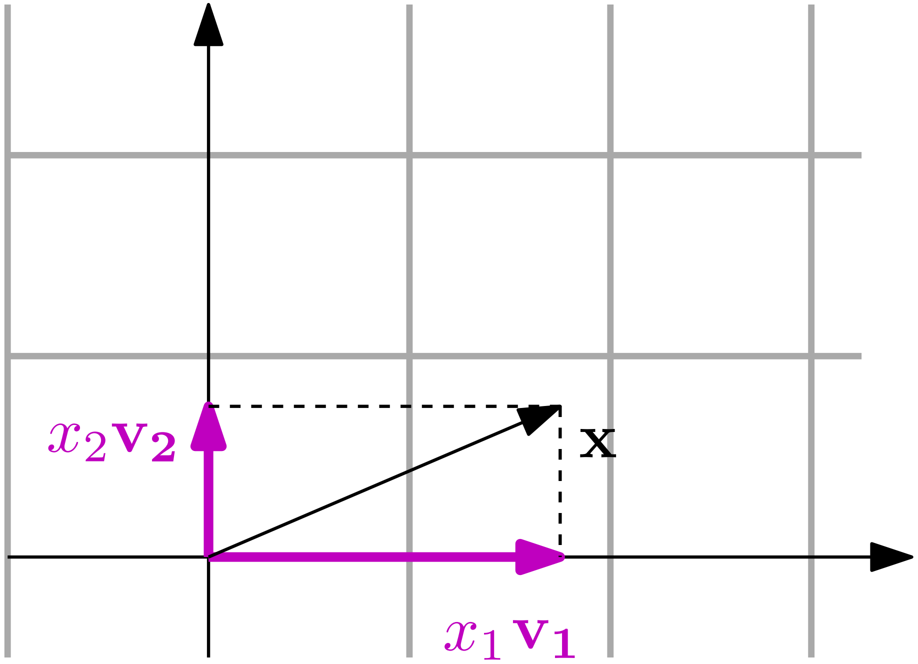

Linear Algebra Basics

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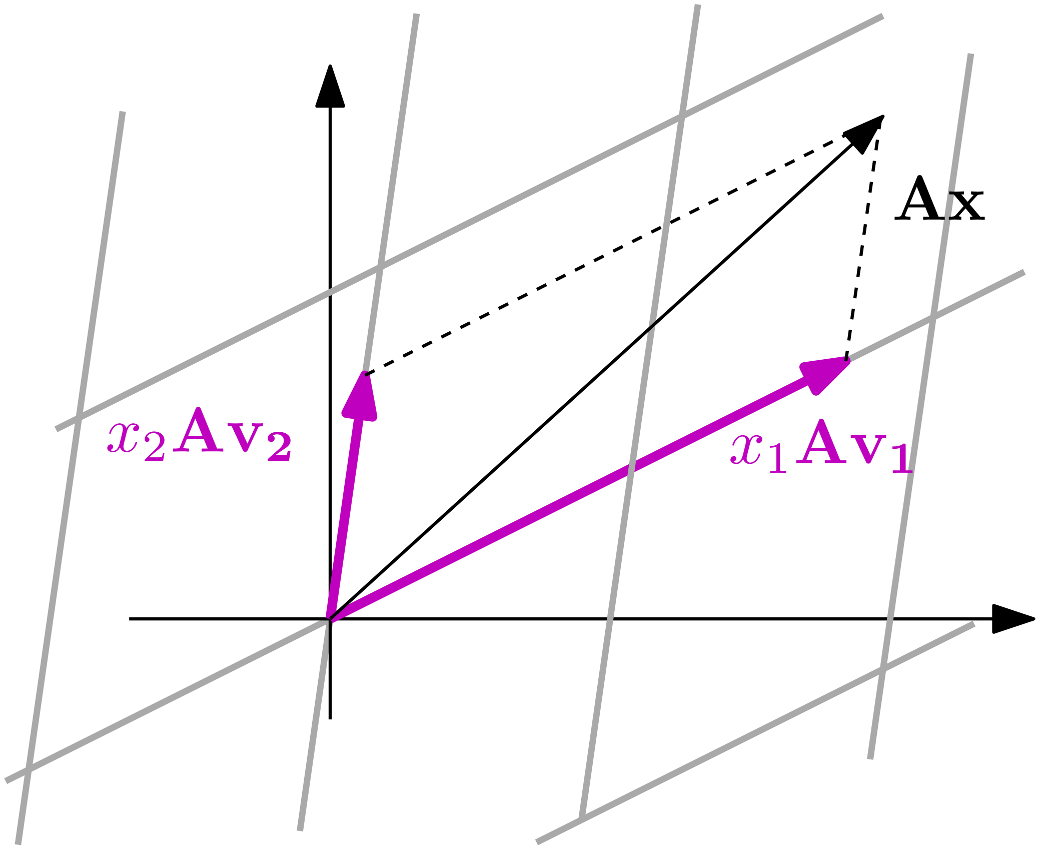

Linear Algebra Basics

The **length** of a vector:

$$\|\mathbf{x}\| := \sqrt{\textstyle\sum_{i=1}^{d} x_i^2}$$

Vector $\mathbf{x}$ has **unit length** if and only if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$

The **angle** between two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\cos(\alpha) = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

Vectors $\mathbf{x}$ and $\mathbf{y}$ are **orthogonal** if and only if $\langle \mathbf{x}, \mathbf{y} \rangle = 0$

Linear Algebra Basics

Geometric interpretation

$$\langle \mathbf{x}, \mathbf{y} \rangle = \|\mathbf{x}\| \|\mathbf{y}\| \cos(\alpha)$$