Predictive Systems

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Outline:

Chapter 4 : Non Linear Stochatsic Models

- ARCH
- GARCH

Volatility

Definition: Volatility (heteroscedasticity**)**

There are several definitions of the volatility. In the context of time series, it is defined as a period in the evolution of time series that is associated with high variability (high variance).

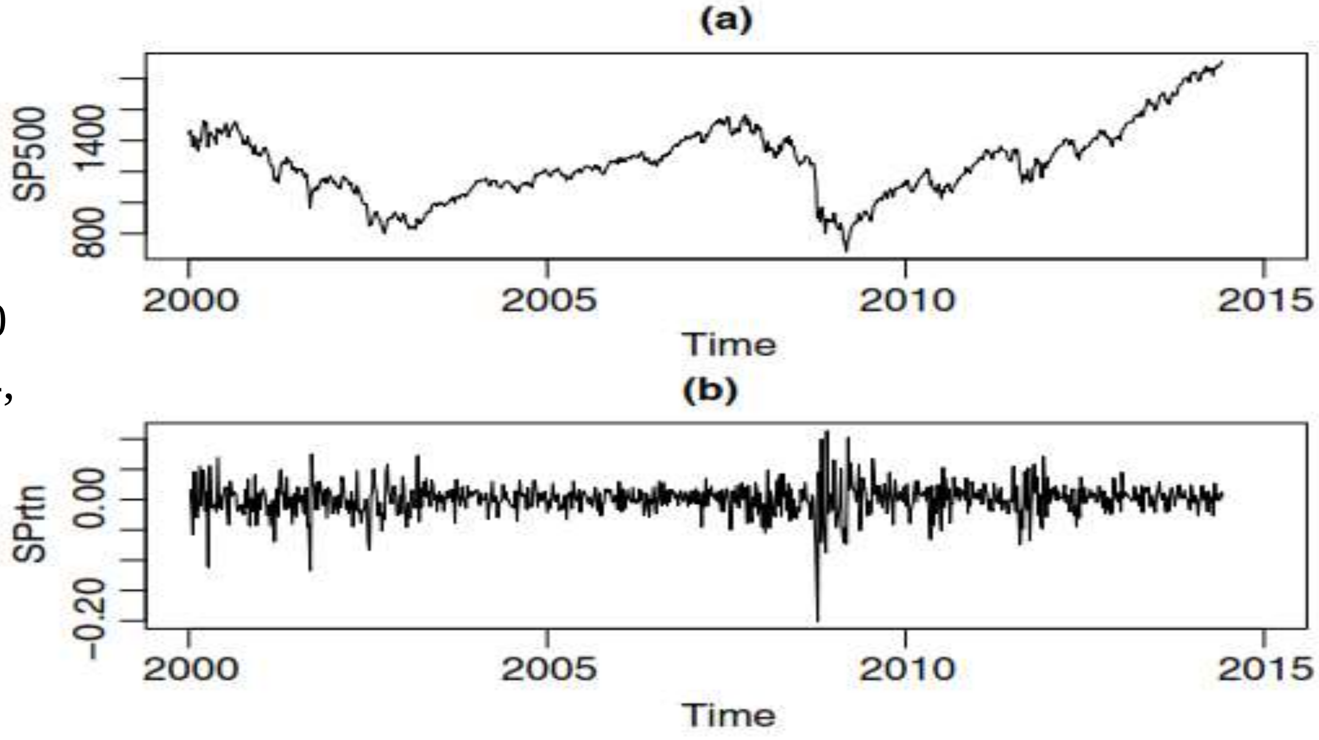
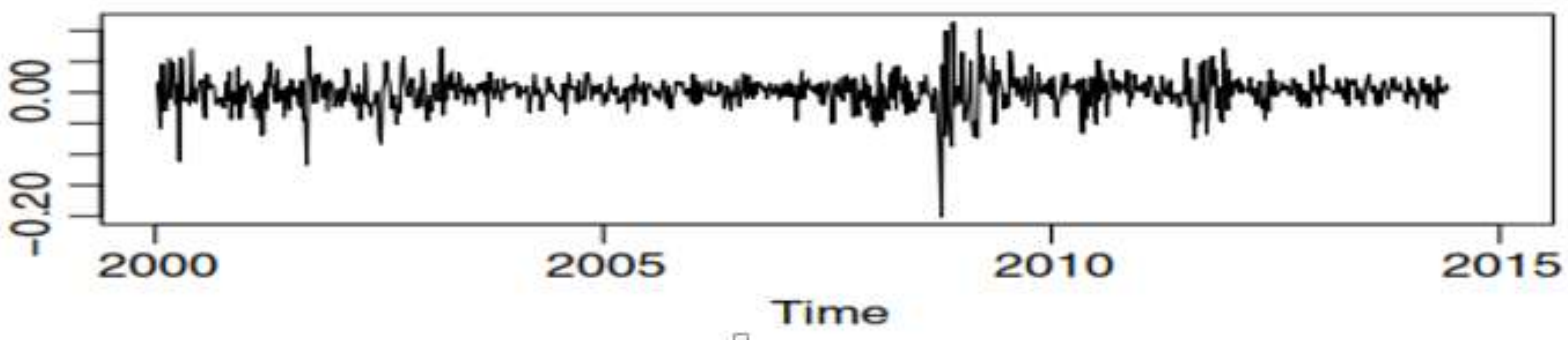


Figure: (a) Time plot of the weekly S&P 500 Index from January 3, 2000 to May 27, 2014, and (b) the weekly log returns on the S&P 500 Index.

Volatility

In 1982, Engle introduced a stochastic model named Autoregressive Conditional Heteroskedastic (ARCH) to capture the time varying variance in a time series of inflation rates.

In 1986, this model was extended by Bollerslev to the Generalized Autoregressive Conditional Heteroskedastic (GARCH).



Reminder

The process ARMA(p,q) is described by the following equation:

$$\varphi(B)z_t = \theta_0 + \theta(B)\varepsilon_t$$

This can be written as the sum of the predictable part and the prediction error as:

$$z_t = \mu_t + \varepsilon_t$$

Where:

F_{t-1} is the past information available at time $t-1$.

μ_t is the conditional mean.

ε_t is the prediction errors (innovations/noise).

Reminder

ARMA process assumes that:

- The unconditional mean of the series is constant over time: $E(z_t) = cte$
- The conditional mean varies as a function of past observations and past errors: $E(z_t|F_{t-1}) = \mu_t$

ARCH model assumes that:

- The unconditional variance of the errors is constant over time: $Var(\varepsilon_t) = \sigma_\varepsilon^2$
- The conditional variance varies as a function of past squared errors: $Var(\varepsilon_t|F_{t-1}) = \sigma_t^2$

ARCH(s) model

The ARCH(s) model can be expressed as:

$$\boldsymbol{\varepsilon_t = \sigma_t . e_t}$$

$$\boldsymbol{\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_s \varepsilon_{t-s}^2}$$

Such that $(e_t)_t$ is a sequence of iid random variables that usually follow a standard normal distribution $e_t \sim N(0,1)$ or Student t-distribution.

ARCH(s) model

Constraints

- To ensure that the conditional variance σ_t^2 is positive:

$$\alpha_0, \alpha_s > 0 \text{ and for } i = 1, \dots, s - 1 \; \alpha_i \geq 0$$

- To ensure that the errors ε_t have finite unconditional variance σ_ε^2:

$$\sum_{i=1}^{s} \alpha_i < 1$$

Properties of ARCH(1,1) model

Let's first examine the ARCH(1) model.

$$\sigma_t^2 = Var(\varepsilon_t|F_{t-1}) = E(\varepsilon_t^2|F_{t-1}) = \alpha_0 + \alpha_1\varepsilon_{t-1}^2$$

Such that $\alpha_0, \alpha_1 > 0$.

We have: $\varepsilon_t = \sigma_t . e_t$

So, the conditional mean is:

$$E(\varepsilon_t|F_{t-1}) = 0$$

And the unconditional mean is also:

$$E(\varepsilon_t) = E\big(E(\varepsilon_t|F_{t-1})\big) = 0$$

Properties of ARCH(1,1) model

And ε_t are serially uncorrelated because:

$$Cov(\varepsilon_t, \varepsilon_{t-j}) = E(\varepsilon_t . \varepsilon_{t-j}) = E\left(E(\varepsilon_t . \varepsilon_{t-j} | F_{t-1})\right) = E\left(\varepsilon_{t-j} E(\varepsilon_t | F_{t-1})\right)$$
$$= 0$$

However, they are not mutually independent since they are interrelated through their conditional variance.

Further:

$$\sigma_\varepsilon^2 = Var(\varepsilon_t) = E(\varepsilon_t^2) = E\left(E(\varepsilon_t^2 | F_{t-1})\right) = E(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)$$

$$= \alpha_0 + \alpha_1 E(\varepsilon_{t-1}^2) = \alpha_0 + \alpha_1 \sigma_\varepsilon^2$$

$$\sigma_\varepsilon^2 = \frac{\alpha_0}{1 - \alpha_1}$$

Properties of ARCH(1,1) model

This implies that if $\alpha_1 < 1$, the unconditional variance exists.

Finally, we get:

$$\sigma_t^2 = \sigma_\varepsilon^2(1 - \alpha_1) + \alpha_1 \varepsilon_{t-1}^2$$

$$\sigma_t^2 = \sigma_\varepsilon^2 + \alpha_1(\varepsilon_{t-1}^2 - \sigma_\varepsilon^2)$$

Consequently, the conditional variance will be above the unconditional variance if $\varepsilon_{t-1}^2 > \sigma_\varepsilon^2$.

Alternative form of ARCH model

To derive an alternative form of ARCH(s) we will take:

$$v_t = \varepsilon_t^2 - \sigma_t^2$$

So, v_t is a sequence of random variables of zero mean. They are also serially uncorrelated:

$$Cov(v_t, v_{t-j}) = E(v_t . v_{t-j}) = E\left((\varepsilon_t^2 - \sigma_t^2).(\varepsilon_{t-j}^2 - \sigma_{t-j}^2)\right)$$

$$= E\left(E\{(\varepsilon_t^2 - \sigma_t^2).(\varepsilon_{t-j}^2 - \sigma_{t-j}^2)\}\middle|F_{t-1}\right)$$

$$= E\left((\varepsilon_{t-j}^2 - \sigma_{t-j}^2).E(\varepsilon_t^2 - \sigma_t^2)\middle|F_{t-1}\right) = 0$$

Alternative form of ARCH model

Let’s put v_t in the first equation of ARCH(1):

$$\sigma_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

$$\varepsilon_t^2 - v_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

So:

$$\boldsymbol{\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + v_t}$$

This form shows that the process of squared errors $(\boldsymbol{\varepsilon_t^2})_t$ can be viewed as an AR(1) model with uncorrelated innovations v_t.

Alternative form of ARCH model

Similarly, ARCH(s) will be formulated as:

$$\boldsymbol{\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \alpha_2 \varepsilon_{t-2}^2 + \cdots + \alpha_s \varepsilon_{t-s}^2 + v_t}$$

This form shows that the process of squared errors $(\boldsymbol{\varepsilon_t^2})_t$ can be viewed as an AR(s) model with uncorrelated innovations v_t.

Here too if $\sum_{i=1}^{s} \alpha_i < 1$, the unconditional variance is:

$$\sigma_\varepsilon^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{s} \alpha_i}$$

GARCH(s, r) model

Likewise, GARCH(s,r) model assumes that:

$$\boldsymbol{\varepsilon_t = \sigma_t . e_t}$$

$$\boldsymbol{\sigma_t^2 = \alpha_0 + \sum_{i=1}^{s} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{r} \beta_j \sigma_{t-j}^2}$$

Such that $(e_t)_t$ is a sequence of iid random variables that usually follow a standard normal distribution $e_t \sim N(0,1)$ or Student t-distribution.

GARCH(s, r) model

Constraints

- To ensure that the conditional variance σ_t^2 is positive:

  $\alpha_0, \alpha_s, \beta_r > 0$, for $i = 1, \ldots, s-1$ $\alpha_i \geq 0$ and for $j = 1, \ldots, r-1$ $\beta_j \geq 0$

- To ensure that the errors ε_t have finite unconditional variance σ_ε^2:

$$\sum_{i=1}^{m} (\alpha_i + \beta_i) < 1$$

Where $m = \max(s, r)$ with $\alpha_i = 0$ for $i > s$ and $\beta_j = 0$ for $j > r$.

Properties of GARCH(1,1) model

The most widely used model is GARCH(1,1), where:

$$\sigma_t^2 = Var(\varepsilon_t | F_{t-1}) = E(\varepsilon_t^2 | F_{t-1}) = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 \sigma_{t-1}^2$$

Since the parameters are positive, one can notice that large values of σ_{t-1}^2 and ε_{t-1}^2 result in large values of σ_t^2.

Assuming that $\alpha_1 + \beta_1 < 1$, the unconditional variance of ε_t is:

$$\sigma_\varepsilon^2 = Var(\varepsilon_t) = \frac{\alpha_0}{1 - (\alpha_1 + \beta_1)}$$

Alternative form of GARCH model

Now let's take: $$v_t = \varepsilon_t^2 - \sigma_t^2$$

Where v_t have zero mean and are serially uncorrelated. So, the GARCH(1,1) model can be rearranged as:

$$\varepsilon_t^2 - v_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1(\varepsilon_{t-1}^2 - v_{t-1})$$

$$\boldsymbol{\varepsilon_t^2 = \alpha_0 + (\alpha_1 + \beta_1)\varepsilon_{t-1}^2 - \beta_1 v_{t-1} + v_t}$$

This form shows that the process of squared errors $(\boldsymbol{\varepsilon_t^2})_t$ can be viewed as an ARMA(1,1) model with uncorrelated innovations v_t.

Alternative form of GARCH model

Similarly, GARCH(s, r) will be formulated as:

$$\boldsymbol{\varepsilon_t^2 = \alpha_0 + \sum_{i=1}^{m}(\alpha_i + \beta_i)\varepsilon_{t-i}^2 - \sum_{i=1}^{s}\beta_i v_{t-i} + v_t}$$

This form shows that the process of squared errors $(\boldsymbol{\varepsilon_t^2})_t$ can be viewed as an ARMA(m, s) model with uncorrelated innovations v_t. Such that the AR order is $m = max(r, s)$.

Alternative form of GARCH model

Here too if $\sum_{i=1}^{m}(\alpha_i + \beta_i) < 1$, the unconditional variance is:

$$\sigma_\varepsilon^2 = \frac{\alpha_0}{1 - \sum_{i=1}^{m}(\alpha_i + \beta_i)}$$

Notice:

Numerous studies have shown that low-order models such as the GARCH(1, 1), GARCH(2, 1), and GARCH(1, 2) models are often adequate in practice, with the GARCH(1, 1) model being the most popular.

Forecasting

For ARMA process with ARCH errors:

- The one-step-ahead forecast error is:

$$e_t(h = 1) = \varepsilon_{t+1}.$$

- The h-step-ahead forecast error is :

$$e_t(h) = \sum_{j=0}^{h-1} \psi_j \varepsilon_{t+h-j}$$

With $\psi_0 = 1.$

Forecasting

However, the presence of heteroscedasticity will impact the variance of the forecast errors.

For an ARCH(1) process the conditional variance of the one-step-ahead forecast error ε_{t+1} is given by:

$$Var(e_t(1)|F_t) = E(e_t^2(1)|F_t) = \sigma_{t+1}^2 = \sigma_\varepsilon^2 + \alpha_1(\varepsilon_t^2 - \sigma_\varepsilon^2)$$

Now, for an ARCH(1) process, the conditional variance of the h-step-ahead forecast error ε_{t+1} is given by:

$$Var(e_t(h)|F_t) = E(e_t^2(h)|F_t) = \sum_{j=0}^{h-1} \psi_j^2 E(\varepsilon_{t+h-j}^2|F_t)$$

Forecasting

For ARCH(1) model:

$$E\left(\varepsilon_{t+h}^2 \middle| F_t\right) = E\left(E\left(\varepsilon_{t+h}^2 \middle| F_t\right)\right) = E\left(\alpha_0 + \alpha_1 \varepsilon_{t+h-1}^2\right) = \alpha_0 + \alpha_1 E\left(\varepsilon_{t+h-1}^2 \middle| F_t\right)$$

$$\alpha_0 + \alpha_1 E\left(\varepsilon_{t+h-1}^2 \middle| F_t\right) = \alpha_0 + \alpha_1\left(\alpha_0 + \alpha_1 E\left(\varepsilon_{t+h-2}^2 | F_t\right)\right)$$

By substituting each error term we obtain for $h > 0$:

$$E\left(\varepsilon_{t+h}^2 \middle| F_t\right) = \alpha_0\left(1 + \alpha_1 + \cdots + \alpha_1^{h-1}\right) + \alpha_1^h \varepsilon_t^2$$

Forecasting

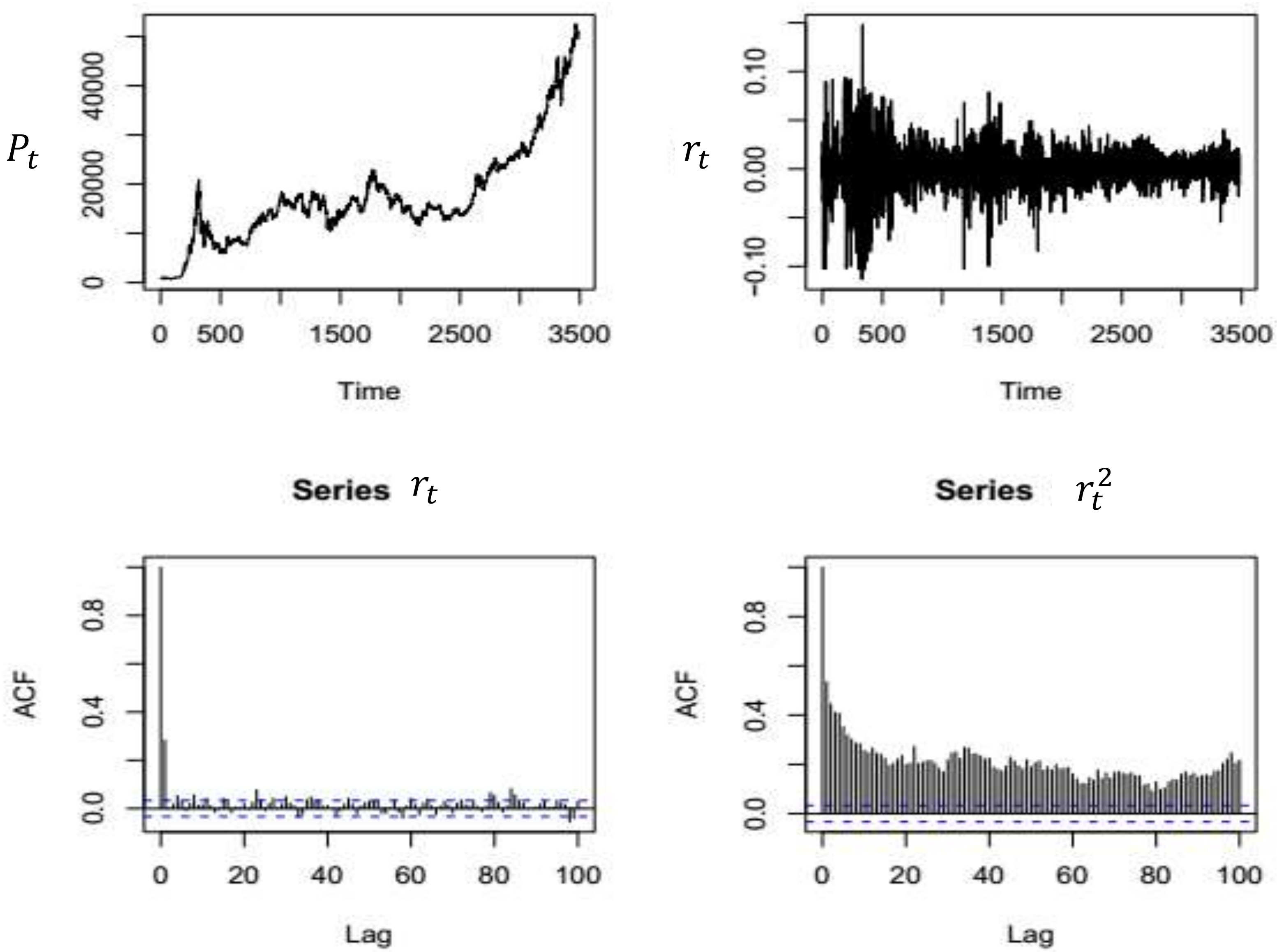
Finally, we get:

$$Var(e_t(h)|F_t) = {\sigma_\varepsilon}^2 \sum_{j=0}^{h-1} \psi_j^2 + \sum_{j=0}^{h-1} \psi_j^2\, \alpha_1^{h-j} (\varepsilon_t^2 - {\sigma_\varepsilon}^2)$$

Notice:

- For ARCH(s) model the second term of RHS of the above equation will be a function of the s past errors $\varepsilon_t^2, \varepsilon_{t-1}^2, \dots, \varepsilon_{t-s+1}^2$.
- If the original time series z_t follows an AR(1) model, $\psi_j = \varphi^{j-1}$.

Example

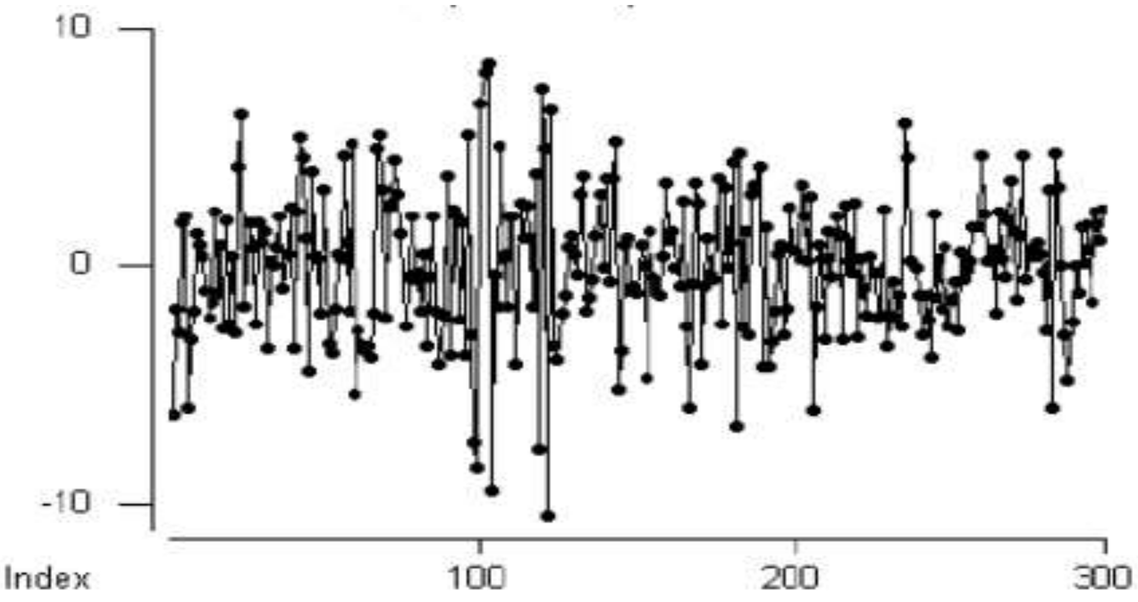


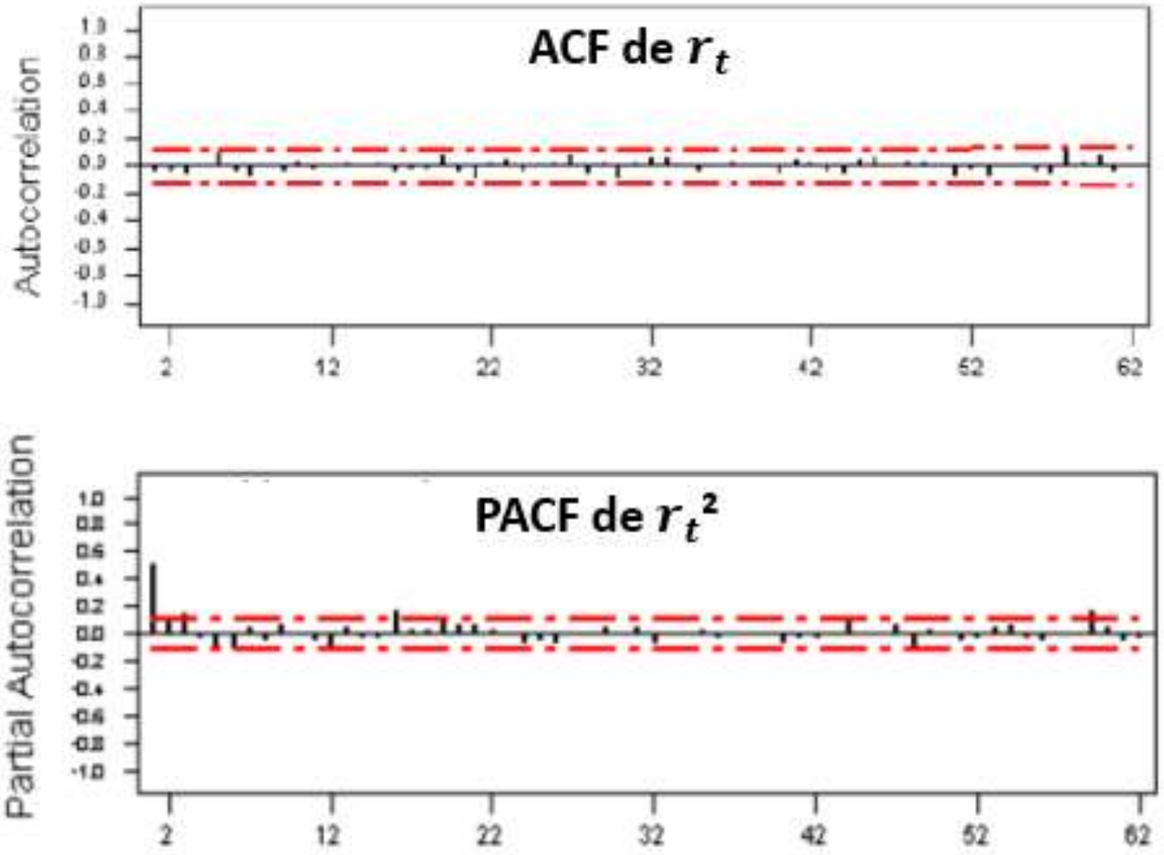
P_t
0
20000
40000
0
500
1500
2500
3500
Time
r_t
-0.10
0.00
0.10
0
500
1500
2500
3500
Time
Series r_t
ACF
0.0
0.4
0.8
0
20
40
60
80
100
Lag
Series r_t^2
ACF
0.0
0.4
0.8
0
20
40
60
80
100
Lag

Réf.

Example

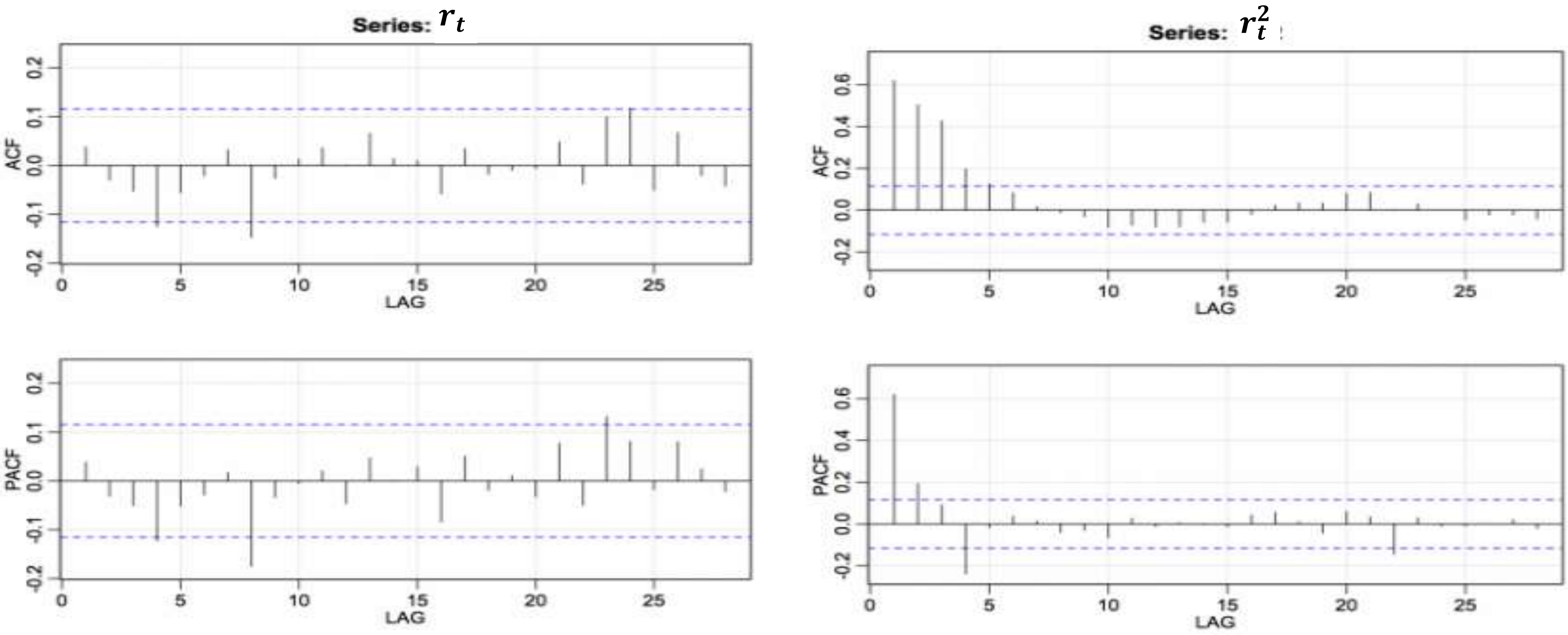
What is the type of this model?





Réf.

Example



Réf.

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