Support Vector machine
with Random variables
Robust Support Vector Regression

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Robust Support Vector Regression

Using the definition of the Super-sphere sets, where the input data is defined by: $x_i \in \{x \mid \| x - \overline{x_i} \| \leq \delta_i\}$

Meaning that $x_i = \overline{x_i} + \Delta x_i$ and the noise is bounded by $\|\Delta x_i\| \leq \delta_i$.

The original problem is defined as follows

$$
\begin{cases}
Min \quad \dfrac{1}{2}\|w\|^2 + C\displaystyle\sum_{i=1}^{m}\left(\xi_i + \overline{\xi_i}\right) \\
s.t \quad w^T\overline{x_i} + b - y_i + \delta_i\|w\| \leq \varepsilon + \xi_i, \qquad i = 1, \ldots, m \\
\quad y_i - (w^T\overline{x_i} + b) + \delta_i\|w\| \leq \varepsilon + \overline{\xi_i}, \qquad i = 1, \ldots, m \\
\quad w \in \mathbb{R}^d, b \in \mathbb{R}, \xi_i \geq 0, \overline{\xi_i} \geq 0, \qquad i = 1, \ldots, m
\end{cases}
$$

Robust SVR with Ellipsoidal Uncertainty

Following the same approach as RSVC, in order to convert the above problem into the dual form for a simpler solution, we first introduce a variable t with $\|w\| \leq t$, and write is as :

$$
\begin{cases}
Min \quad \frac{1}{2}t^2 + C\sum_{i=1}^{m}\left(\xi_i + \overline{\xi_i}\right) \\
s.t \quad w^T\overline{x_i} + b - y_i + \delta_i t \leq \varepsilon + \xi_i, \qquad i = 1, \dots, m \\
\quad y_i - (w^T\overline{x_i} + b) + \delta_i t \leq \varepsilon + \overline{\xi_i}, \qquad i = 1, \dots, m \\
\quad \|w\| \leq t \\
\quad w \in \mathbb{R}^d, b \in \mathbb{R}, \xi_i \geq 0, \overline{\xi_i} \geq 0, \qquad i = 1, \dots, m
\end{cases}
$$

Robust SVR with Ellipsoidal Uncertainty

We also introduce two variables u and v with the constraints $u + v = 1$ and $\sqrt{t^2 + v^2} \leq u$. Therefore, we have $t^2 = u^2 - v^2 = (u - v)(u + v) = u - v$:

$$\begin{cases} Min \quad \frac{1}{2}t^2 + C\sum_{i=1}^{m}\left(\xi_i + \overline{\xi_i}\right) \\ s.t \quad w^T\overline{x_i} + b - y_i + \delta_i t \leq \varepsilon + \xi_i, \qquad i = 1, \dots, m \\ y_i - (w^T\overline{x_i} + b) + \delta_i t \leq \varepsilon + \overline{\xi_i}, \qquad i = 1, \dots, m \\ u + v = 1 \\ \sqrt{t^2 + v^2} \leq u \\ \|w\| \leq t \\ w \in \mathbb{R}^d, b \in \mathbb{R}, \xi_i \geq 0, \overline{\xi_i} \geq 0, \qquad i = 1, \dots, m \end{cases}$$

Robust SVR with Ellipsoidal Uncertainty

The Lagrange function for this problem is as follows:

$$L = \frac{1}{2}(u - v) + C\sum_{i=1}^{m}(\xi_i + \overline{\xi_i})$$
$$+ \sum_{i=1}^{m}\alpha_i(w^T\overline{x_i} + b - y_i + \delta_i t - \varepsilon - \xi_i) + \sum_{i=1}^{m}\beta_i\big(y_i - (w^T\overline{x_i} + b) + \delta_i t - \varepsilon - \overline{\xi_i}\big)$$
$$- \sum_{i=1}^{m}\eta_i\xi_i - \sum_{i=1}^{m}\theta_i\xi_i - \varphi(u + v - 1) - z_u u - z_v v - \gamma t - z_t t - z_w^T w,$$

Where $\alpha, \beta, \eta, \theta \in \mathbb{R}^m$, $\varphi, z_u, z_v, \gamma, z_t \in \mathbb{R}$, $z_w \in \mathbb{R}^n$ are the multiplier vectors.

Robust SVR with Ellipsoidal Uncertainty

Thus, the dual problem can be obtained:

$$\max_{\alpha,\beta,\gamma,z_u,z_v} -\sum_{i=1}^{m} \alpha_i \,(y_i+\varepsilon)+\sum_{i=1}^{m} \beta_i \,(y_i-\varepsilon)-\varphi,$$

$$s.t. \gamma \le \sum_{i=1}^{m} \delta_i(\alpha_i+\beta_i)-\sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}(\alpha_i-\beta_i)(\alpha_j-\beta_j)\left(x_i\cdot x_j\right)},$$

$$z_u-\varphi=\frac{1}{2}, \qquad z_v-\varphi=-\frac{1}{2},$$

$$\sum_{i=1}^{m}\alpha_i-\sum_{i=1}^{m}\beta_i=0,$$

$$0\le\alpha_i\le C, \qquad 0\le\beta_i\le C, \qquad i=1,\cdots,m,$$

$$\sqrt{\gamma^2+z_v^2}\le z_u.$$

Robust SVR with Ellipsoidal Uncertainty

Theorem:

Suppose that $(\alpha^{*T}, \gamma^*) = ((\alpha_1^*, \ldots, \alpha_m^*), \gamma^*)$,is a solution to the dual problem. If there exists a component of α^*, $\alpha_j^* \in \ (0, C)$, then a solution $(w^*, b^*\ )$ to the problem can be obtained by:

- $w^* = \frac{\gamma^*}{(\sum_{i=1}^m (\alpha_i^* + \beta_i^*)\delta_i - \gamma^*)} \sum_{i=1}^m (\alpha_i^* - \beta_i^*) y_i x_i$,
- $b^* = \frac{y_j - \gamma^*}{(\sum_{i=1}^m (\alpha_i^* + \beta_i^*)\delta_i - \gamma^*) + \varepsilon + \delta_i \gamma^*} \sum_{i=1}^m (\alpha_i^* - \beta_i^*)(x_i . x_j)$