**A Study of Flatness**

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**A Thesis**

**presented to**

**the Faculty of the Graduate School**

**University of Missouri**

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In Partial Fulfillment

of the Requirements for the Degree

Master of Arts

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by

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May 2024

The undersigned, appointed by the Dean of the Graduate School, have examined the thesis entitled

A Study of Flatness

presented by Eliot Bitting, a candidate for the degree of Master of Arts, and hereby certify that in their opinion it is worthy of acceptance.

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ACKNOWLEDGEMENTS

This thesis would not have been possible without the love and support of my family and friends for which there are to many to list. Their continual understanding of my absence has made this process as painless as possible.

I am also appreciative of my advisor, Dr. Rankeya Datta, who has helped me grow immensely as a mathematician. His attention for detail and patience with often times naive questions has been extraordinary and appreciated. Without his guidance, this paper would be a hollow shell of its current form.

Next, I am extremely grateful to my committee members Dr. Ian Aberbach and Dr. Calin Chindris for their continued support and mentorship.

I would also like to thank many of my colleagues for their continued help and support. Specifically I would like to give thanks to Benjamin Ashlock, Saptarshi Banerjee, Aidan Donahue, Chris Hirni, Stephen Landsittel, Jon Moore, and Arun Suresh. Their input into my work has been invaluable and they have helped me provide better proofs than I otherwise would have.

Finally, I would like to thank many of the professors here at the University of Missouri who have helped me become the person I am today. I would like to make special mention of Eric Lancaster. Eric's guidance has made me into a more patient and understanding person. Without him, I likely would not have pursued higher education in the first place.

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ABSTRACT

Descent theory, in the sense of descending various properties of modules, is an indispensable tool in commutative algebra and algebraic geometry. In [RG71, Part II], Gruson and Raynaud delve into the subject by introducing several descent properties of modules with an eye toward showing that these properties also descend flatness in commonly occurring cases. In this paper, we focus on some of these properties and study their connection with other important questions in commutative algebra such as the direct summand theorem.

# Chapter 1. Introduction and Notation

Let $A$ be a commutative ring with identity, $M$ an $A$-module, and $\varphi : A \to B$ be a ring homomorphism that preserves identity. We say that $\varphi$ descends flatness if whenever $M \otimes_A B$ is $B$-flat, we have that $M$ is $A$-flat. Some examples of such maps are faithfully flat maps, universally injective (aka pure) ring maps, and finite injective maps. Many of these maps will descend other properties as well.

Chapter 2 contains the bulk of this paper. In this chapter, we expand Sections 1.1 and 1.2 of [RG71, Part II] of M. Raynaud and L. Gruson. We introduce what it means to descend flatness (Definition 2.1.0.1), nullity (Definition 2.2.0.1), flat universal short exactness (Definition 2.6.1.1), Hom-nullity (Definition 2.7.1.1) and Hom-left-invertibility for injectives (Definition 2.7.2.1). We then exhibit various relationships between these properties.

In Chapter 3, we discuss Melvin Hochster's direct summand conjecture and prove that it is equivalent to saying that if $\varphi\colon A \hookrightarrow B$ is an integral extension, where $A$ is Noetherian, then $\varphi$ descends flatness. This result was originally shown in [Ohi96]. Actually, we show that the direct summand conjecture is equivalent to $\varphi$ descending nullity. It just so happens that $\varphi$ descending nullity is equivalent to $\varphi$ descending flatness in aforementioned setup by a deep result in [RG71, Part II].

Throughout, $A$ is assumed to be a commutative ring with identity and ring maps are assumed to preserve identity. We denote the category of $A$-modules with $\mathrm{Mod}_A$.

# Chapter 2. Some descent properties for modules

In this chapter, we expand upon Chapter 1 of [RG71, Part II]. Our goal is to provide careful and detailed proofs of the results in loc. cit.. We then explore some properties of the examples given.

## 2.1. Descending flatness

**Definition 2.1.0.1.** *Let $\varphi : A \to B$ be a ring homomorphism. We say $\varphi$* descends flatness *if whenever $P \in \mathrm{Mod}_A$ is such that $P \otimes_A B$ is $B$-flat, then $P$ is $A$-flat.*

In [RG71, Part II], this definition is introduced as condition (P).

**Lemma 2.1.0.2.** *Let $\varphi\colon A \to B$ be a ring homomorphism. Suppose $A \to C$ is a ring homomorphism that descends flatness (for example, if $A \to C$ is faithfully flat or universally injective). If $\mathrm{id}_C \otimes_A \varphi\colon C \to C \otimes_A B$ descends flatness, then $\varphi$ descends flatness.*

*Proof.* Let $M$ be an $A$-module such that $B \otimes_A M$ is a flat $B$-module. Since $A \to C$ descends flatness by assumption, it suffices to show that $C \otimes_A M$ is a flat $C$-module. Now,

$$(C \otimes_A B) \otimes_B (B \otimes_A M) \cong C \otimes_A (B \otimes_A M) \cong (C \otimes_A B) \otimes_C (C \otimes_A M).$$

Note that $(C \otimes_A B) \otimes_B (B \otimes_A M)$ is a flat $C \otimes_A B$-module by base change since $B \otimes_A M$ is a flat $B$-module. Thus, by the above isomorphisms, $(C \otimes_A B) \otimes_C (C \otimes_A M)$ is a flat $C \otimes_A B$-module. However, we assumed that $\mathrm{id}_C \otimes_A \varphi\colon C \to C \otimes_A B$ descends flatness. This shows that $C \otimes_A M$ is a flat $C$-module, as desired. $\square$

**Lemma 2.1.0.3.** *Let $\varphi : A \to B$ be a ring map. Then $\varphi$ descends flatness if and only if for all $\mathfrak{p} \in \operatorname{Spec} A$ we have $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ descends flatness.*

*Proof.* Assume that $\varphi_{\mathfrak{p}}$ descends flatness for all $\mathfrak{p} \in \operatorname{Spec}(A)$. Let $M \in \operatorname{Mod}_A$ be such that $M \otimes_A B$ is $B$-flat. This is true if and only if for all $\mathfrak{p} \in P$ we have $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ is $B_{\mathfrak{p}}$-flat. But this means for every $\mathfrak{p} \in \operatorname{Spec}(A)$ we have $M_{\mathfrak{p}}$ is $A_{\mathfrak{p}}$-flat. But since this is true for every $\mathfrak{p} \in \operatorname{Spec}(A)$ we have that $M$ is $A$-flat as desired [Sta24, Tag 00HT(3)].

We next prove the forward implication. So assume $\varphi$ descends flatness and let $\mathfrak{p} \in \operatorname{Spec}(A)$. We will show that $\varphi_{\mathfrak{p}}\colon A_{\mathfrak{p}} \to B_{\mathfrak{p}}$-descends flatness. Let $M$ be an $A_{\mathfrak{p}}$-module such that $M \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ is a flat $B_{\mathfrak{p}}$-module. Since $B \to B_{\mathfrak{p}}$ is a flat ring homomorphism, it follows that $M \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}}$ is also a flat $B$-module by A.1.0.3. Note

$$M \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} \cong M \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \otimes_A B) \cong M \otimes_A B.$$

Thus, $M \otimes_A B$ is a flat $B$-module. But $\varphi\colon A \to B$ descends flatness. So we can conclude that $M$ is a flat $A$-module. However, since $M$ was an $A_{\mathfrak{p}}$-module to begin with, one has

$$M \otimes_A A_{\mathfrak{p}} \cong M.$$

Since flatness is preserved under base change, it follows that $M$ is also a flat $A_{\mathfrak{p}}$-module. □

The following are some properties of such maps.

**Proposition 2.1.0.4.** *Suppose $\varphi : A \to B$ be a ring homomorphism that descends flatness. Then we have the following:*

1. *Let $I$ be an ideal of $A$. If $I \subseteq \ker(\varphi)$, then $A/I$ is $A$-flat.*
2. *Any principal ideal $J \subseteq \ker(\varphi)$ is generated by an idempotent*
3. *If $I \subseteq \ker(\varphi)$ is an ideal of $A$ then the induced map $A/I \to B$ descends flatness*
4. *If $\operatorname{Spec}(A)$ is connected and $B \neq 0$, then $\varphi$ is injective.*

*Proof.* For 1. it suffices to show that $B \otimes_A A/I$ is $B$-flat. Note $B \otimes_A A/I = B/IB$. Since $I \subseteq \ker(\varphi)$, $IB = 0$. Thus $B \otimes_A A/I = B/IB = B$ which is B-flat as desired.

By 1, if $J \subseteq \ker(\varphi)$, then $A/J$ is $A$-flat. By Lemma A.1.0.2, since $A/J$ is flat, $J$ is generated by an idempotent proving 2.

Now, to see 3., let $M$ be a $A/I$ module such that $M \otimes_{A/I} B$ is $B$-flat. By Lemma A.1.0.4, we have that $M = M \otimes_A A/I$. Consequently, $(M \otimes_A A/I) \otimes_{A/I} B = M \otimes_A B$ is $B$-flat. As such, $M$ is $A$-flat. But this means that $M = M \otimes_A A/I$ is $A/I$ flat as desired.

Finally, to show 4., recall that $\mathrm{Spec}(A)$ being connected implies that the only idempotent elements are 0 and 1. Thus, if we take any principal ideal contained in the kernel, it must be generated by 0. As such the kernel itself must be 0, meaning that $\varphi$ is injective. $\square$

**Corollary 2.1.0.5.** *If $A$ is a domain, $B \neq 0$, and $\varphi$ descends flatness, then $\varphi$ is injective.*

*Proof.* Note that since $A$ is a domain, $\mathrm{Spec}(A)$ is connected. Thus by Proposition 2.1.0.4 part 4, we have that $\varphi$ is injective if it descends flatness. $\square$

## 2.2. Descending nullity

**Definition 2.2.0.1.** *We say that an $A$-module $M$* descends nullity *if whenever $P \in \mathrm{Mod}_A$ is such that $P \otimes_A M = 0$ then $P = 0$.*

In other words $M$ descends nullity as an $A$-module if the functor $\bullet \otimes_A M$ is faithful. We say that a ring homomorphism $\varphi : A \to B$ descends nullity if $B$ descends nullity as an $A$-module. This is condition (O) in [RG71]. The following are the analogs of

Lemma 2.1.0.2 and Lemma 2.1.0.3 for descent of nullity.

**Lemma 2.2.0.2.** *Let $\varphi\colon A \to B$ be a ring homomorphism. Suppose $A \to C$ is a ring homomorphism that descends nullity. If $\mathrm{id}_C \otimes_A \varphi\colon C \to C \otimes_A B$ descends nullity, then $\varphi$ descends nullity.*

*Proof.* Assume that $\mathrm{id}_C \otimes_A \varphi$ descends nullity and let $M$ be an $A$-module such that

$$M \otimes_A B = 0.$$

Then $0 = C \otimes_A (B \otimes_A M) = (C \otimes_A B) \otimes_C (C \otimes_A M)$. But $(C \otimes_A B)$ descends nullity as a $C$-module, thus $(C \otimes_A M) = 0$. But $C$ descends nullity as an $A$-module, thus $M = 0$ as desired. □

The next result says that the property of descending nullity is local.

**Lemma 2.2.0.3.** *Let $\varphi : A \to B$ be a ring map. Then $\varphi$ descends nullity if and only if for all $\mathfrak{p} \in \mathrm{Spec}(A)$ we have $\varphi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ descends nullity.*

*Proof.* Assume that $\varphi_{\mathfrak{p}}$ descends nullity for all $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $M \in \mathrm{Mod}_A$ be such that $M \otimes_a B = 0$. Then we have that $M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} B_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. But by assumption, this means that $M_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(A)$. As such, $M = 0$ as desired.

On the other hand assume that $\varphi : A \to B$ descends nullity. Fix a $\mathfrak{p} \in \mathrm{Spec}(A)$. Let $M \in \mathrm{Mod}_{A_{\mathfrak{p}}}$ be such that $B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} M = 0$. But $B_{\mathfrak{p}} = B \otimes_A A_{\mathfrak{p}}$. Thus $B \otimes_A M = 0$. But this mean $M = 0$ as an $A$-module. Then $M$ is also zero as an $A_{\mathfrak{p}}$-module. □

The following proposition illustrates that many maps that descend flatness also descend nullity.

**Proposition 2.2.0.4.** *If $\varphi : A \to B$ is an injective ring homomorphism that descends flatness, then it descends nullity.*

*Proof.* Let $M$ be an $A$-module such that $B \otimes_A M = 0$. Note that 0 is a flat module. Since $\varphi$ descends flatness, we have that $M$ is a flat A-module. Thus the mapping $\varphi \otimes_A id_M : A \otimes_A M \to B \otimes_A M$ is injective. Consequently $M = A \otimes_A M = 0$ □

**Lemma 2.2.0.5.** *Suppose $f : M \to N$ is a surjective $A$-linear map. If $N$ descends nullity, then $M$ descends nullity*

*Proof.* Let $P \in \mathrm{Mod}_A$ be such that $M \otimes_A P = 0$. Consider the exact sequence

$$0 \longrightarrow \ker(f) \longrightarrow M \xrightarrow{f} N \longrightarrow 0.$$

By right exactness of $\bullet \otimes_A P$ we have an exact sequence

$$\ker(f) \otimes_A P \longrightarrow M \otimes_A P \xrightarrow{f} N \otimes_A P \longrightarrow 0.$$

But since $M \otimes_A P = 0$ we have $N \otimes_A P = 0$. By hypothesis, this means that $P = 0$ as desired. □

**Lemma 2.2.0.6.** *Let $\varphi : A \to B$ be a ring map that descends nullity. If $M \in Mod_B$ descends nullity as a $B$-module, then $M$ descends nullity as an $A$-module.*

*Proof.* Let $N$ be an $A$-module such that $M \otimes_A N = 0$. Then $M \otimes_A (N \otimes_A B) = 0$. Now, we claim there is a natural surjection

$$M \otimes_A (N \otimes_A B) \to M \otimes_B (N \otimes_A B).$$

Indeed, note that the canonical map $f : M \times (N \otimes_A B) \to M \otimes_B (N \otimes_A B)$ is $A$-bilinear surjective. Thus by the universal property of tensor products, this map factors through $M \otimes_A (N \otimes_A B)$. Thus

$$M \otimes_A (N \otimes_A B) \to M \otimes_B (N \otimes_A B)$$

is surjective as desired. Thus $M \otimes_B (N \otimes_A B) = 0$. But $M$ descends nullity. Hence $N \otimes_A B = 0$. Finally, since $\varphi$ descends nullity, we have that $N = 0$ as desired. $\square$

We will next give examples of classes of ring homomorphisms that descend flatness and nullity.

## 2.3. Descent of flatness for faithfully flat ring maps

One of the primary examples of a map that descend flatness is faithfully flat maps. It turns out that faithfully flat maps descend more than flatness and nullity. In particular, faithfully flat maps also descend the ascending chain condition in rings, finite generation, and finite presentation.

**Lemma 2.3.0.1.** *Let $\varphi : A \to B$ be a faithfully flat ring map. Then for all $A$-modules $M$, the map $\mathrm{id}_M \otimes_A \varphi : M \to M \otimes_A B$ is injective.*

*Proof.* Since $B$ is faithfully flat, it suffices to show that $M \otimes_A B \to M \otimes_A B \otimes_A B$ is injective. However, it is clear that there is a natural section of this map. Namely $m \otimes_A b_1 \otimes_A b_2 \mapsto m \otimes_A b_1 b_2$. Thus the map is injective. $\square$

**Corollary 2.3.0.2.** *Let $\varphi : A \to B$ be a faithfully flat ring map. Then $\varphi$ is injective.*

*Proof.* Take $M = A$ in Lemma 2.3.0.1. $\square$

**Proposition 2.3.0.3.** *Let $\varphi : A \to B$ be a faithfully flat ring map. Then*

1. *$\varphi$ descends flatness;*
2. *If $M \otimes_A B$ is finitely generated as a $B$-module then $M$ is finitely generated as an $A$-module;*
3. *If $M \otimes_A B$ is finitely presented as a $B$-module then $M$ is finitely presented as an $A$-module;*

*4. If $B$ is Noetherian, then $A$ is Noetherian.*

We adopt proofs from chapter four of [Bos22] and [Sta24, Lemma 033E]

*Proof.* 1. Assume that $M \otimes_A B$ is $B$-flat and consider an injective morphism $E' \to E$. It suffices to show that $E' \otimes_A M \to E \otimes_A M$ is an $A$-linear injection. Since $B$ is $A$-flat, we have that $E' \otimes_A B \to E \otimes_A B$ injective. Now, observe that

$$E' \otimes_A B \otimes_B (M \otimes_A B) \to E \otimes_A B \otimes_B (M \otimes_A B)$$

is injective as $M \otimes_A B$ is B-flat. However, note $E' \otimes_A B \otimes_B (M \otimes_A B) \cong E' \otimes_A (M \otimes_A B)$ and similarly $E \otimes_A B \otimes_B (M \otimes_A B) \cong E \otimes_A (M \otimes_A B)$. Thus we get the injection

$$E' \otimes_A M \otimes_A B \to E \otimes_A M \otimes_A B.$$

But since $B$ is faithfully flat as an $A$-module, this is an injection if and only if $E' \otimes_A M \to E \otimes_A M$ is an injection. As such, $M$ is $A$-flat as desired.

2. Assume that $M \otimes_A B$ is finite generated over $B$. Then there exists a generating system of $M \otimes_A B$ that is of the form of $m_i \otimes_A 1$ for $i = 1, \ldots n$. We claim that these $m_i$ generate $M$ as an $A$-module. Indeed, consider the module $M' = \sum_{i=1}^n x_i A \subseteq M$. It suffices to show that the sequence of $A$-modules

$$M' \to M \to 0$$

is exact. This sequence is exact if and only if the sequence of $A$-modules

$$M' \otimes_A B \to M \otimes_A B \to 0$$

is exact as B is faithfully flat. However, the later sequence is exact due to the definition of M'. Thus $M$ is the image of a finitely generated $A$-module and must be finitely generated over $A$ itself.

3. Assume that $M \otimes_A B$ is finitely presented over $B$. From 2. we know that $M$ is finitely generated with respect to $A$. Hence, we have an exact sequence

$$A^n \xrightarrow{\varphi} M \longrightarrow 0.$$

It suffices to show that the kernel of $\varphi$ is a finitely generated $A$-module. Consider the exact sequence of $B$-modules

$$B^n \xrightarrow{\varphi \otimes_A \mathrm{id}_B} M \otimes_A B \longrightarrow 0.$$

Note $\varphi \otimes_A \mathrm{id}_B$ must have a kernel that is finitely generated over $B$ as $M \otimes_A B$ is finitely presented over $B$. Now, recall that $\ker(\varphi \otimes_A \mathrm{id}_B) \cong \ker(\varphi) \otimes_A B$. But, by part two, this means that $\ker(\varphi)$ is finitely generated over $A$ as desired.

4. Let $I_0 \subseteq I_1 \subseteq \dots$ be a chain of ideals in $A$. Since $B$ is Noetherian, there exists an $N \in \mathbb{Z}_{\geq 0}$ such that $I_N B = I_{N+1} B = \dots$. Now since $\varphi : A \to B$ is flat, we have $I_i B = I_i \otimes_A B$. Thus since $\varphi : A \to B$ is faithfully flat, we get that $I_N = I_{N+1} = \dots$ and $A$ is Noetherian as desired. □

**Corollary 2.3.0.4.** *Let $\varphi : A \to B$ be a faithfully flat ring map. Then $\varphi$ descends nullity.*

*Proof.* By Proposition 2.3.0.3, $\varphi$ descends flatness and by Corollary 2.3.0.2, it is injective. Thus, by Proposition 2.2.0.4, it descends nullity. □

## 2.4. Universally injective maps

We next introduce a class of ring homomorphisms, more general than faithfully flat maps, that also descend flatness.

**Definition 2.4.0.1.** *Let $M$ and $N$ be $A$-modules. We say that $\varphi : M \to N$ is* universally injective *or* pure *if for all $A$-modules $P$, we have*

$$\varphi \otimes_A \mathrm{id}_P : M \otimes_A P \to N \otimes_A P$$

*is injective.*

If we take $P = A$, we see that a universally injective map is injective. Moreover, if $\varphi \colon M \to N$ is left-invertible in $\mathrm{Mod}_A$, then $\varphi$ is universally injective.

We say that a sequence

$$0 \longrightarrow Q \xrightarrow{\ u\ } L \longrightarrow P \longrightarrow 0$$

is *universally exact* if $u : Q \to L$ is universally injective. A key way of detecting such sequences is by use of the following functor.

**Definition 2.4.0.2.** *The functor $T(\bullet)\colon \mathrm{Mod}_A \to \mathrm{Mod}_A$ will be defined as follows:*

$$T(\bullet) := \mathrm{Hom}_{\mathbb{Z}}(\bullet, \mathbb{Q}/\mathbb{Z}).$$

Recall $T$ is an exact and faithful functor. Indeed, it is exact because $\mathbb{Q}/\mathbb{Z}$ is an injective $\mathbb{Z}$-module. To see faithful, observe the following:

Suppose $M \neq 0$ is an $A$-module. Let $x \in M$ be a non-zero element. Consider the $\mathbb{Z}$-submodule $\mathbb{Z}x$ of $M$ generated by $x$. Since $x \neq 0$,

$$\mathbb{Z}x \cong \mathbb{Z}/a\mathbb{Z},$$

for some integer $a \neq 1, -1$. Since $\mathbb{Q}/\mathbb{Z}$ is $\mathbb{Z}$-injective, it then suffices to show that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/a\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \neq 0.$$

Note if $a = 0$, this is clear because $\mathbb{Q}/\mathbb{Z} \neq 0$. If $a \neq 0$, then consider the unique $\mathbb{Z}$-linear map

$$\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}.$$
$$1 \mapsto \frac{1}{a} + \mathbb{Z}$$

This map is non-zero because $1/a \notin \mathbb{Z}$ and it contains $a\mathbb{Z}$ in its kernel. Therefore it induces a non-zero $\mathbb{Z}$-linear map $\mathbb{Z}/a\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$, as desired.

The following is proposition 1.1.1. in [RG71] and is an amalgamation of well known results.

**Proposition 2.4.0.3.** *[RG71, Part II, Prop. 1.1.1] Let*

$$\mathcal{L} \coloneqq 0 \longrightarrow M \xrightarrow{u} N \xrightarrow{v} P \longrightarrow 0$$

*be a short exact sequence of A modules. The following are equivalent:*

1. *$u$ is A-universally injective.*
2. $\mathrm{id}_{T(M)} \otimes_A u$ *is injective.*
3. *$T(u)$ is split surjective.*
4. *$T(v)$ is A-universal injective.*
5. *for an A-module of finite presentation $F$,* $\mathrm{Hom}_A(F, v)$ *is surjective.*
6. *There exists a filtered system of split short exact sequence whose colimit is $\mathcal{L}$.*

*Proof.* ($1 \implies 2$) For all $P' \in \mathrm{Mod}_A$, we must have the diagram

$$\begin{array}{ccc} \mathrm{Hom}_A(P', T(N)) & \xrightarrow{\psi} & \mathrm{Hom}_A(P', T(M)) \\ \downarrow & & \downarrow \\ T(P' \otimes_A N) & \xrightarrow{T(1_{P'} \otimes_A u)} & T(P' \otimes_A M) \end{array}$$

where $\psi = \mathrm{Hom}_A(P', T(u))$ and the columns are isomorphism by Tensor-hom adjunction. Now since $u$ is universally injective, we have

$$P' \otimes_A M \to P' \otimes_A N$$

is injective. Since $T$ is exact and faithful, we get that

$$T(P' \otimes_A N) \to T(P' \otimes_A M)$$

is surjective. Taking $P' = T(M)$ yields statement (2).

($2 \implies 3$) Consider the same diagram as above. Since our bottom row is surjective and our columns are isomorphisms, we have that

$$\psi : \mathrm{Hom}_A(T(M), T(N)) \to \mathrm{Hom}_A(T(M), T(M))$$

is surjective. Thus, there exists a $g \in \mathrm{Hom}_A(T(M), T(N))$ such that

$$\psi(g) = T(u) \circ g = \mathrm{id}_{T(M)}$$

giving us that $T(u)$ is split surjective.

($3 \implies 4$) By exactness of $T$, we have that

$$\begin{tikzcd} T(\mathcal{L}) = 0 \arrow[r] & T(P) \arrow[r, "T(v)"] & T(N) \arrow[r, "T(u)"] & T(M) \arrow[l, bend left, "g"] \arrow[r] & 0 \end{tikzcd}$$

is exact, and $g$ is the right inverse of $T(u)$. Thus, by Lemma A.1.0.7, there exists a $k$ such that $k \circ T(v) = \mathrm{id}_{T(P)}$. That is to say, our sequence is split exact. But split exact sequences remain exact after arbitrary tensors. Thus we have that $T(v) \otimes_A \mathrm{id}_Q$ is injective and $T(v)$ is universal injective.

($4 \implies 5$) Since $T(v)$ is universally injective, we have that for all $F$ finitely presented modules $A$-modules that $T(P) \otimes_A F \to T(N) \otimes_A F$ is injective. We now claim that $T(P) \otimes_A F \cong T(\mathrm{Hom}(F, P))$ and similarly $T(N) \otimes_A F \cong T(\mathrm{Hom}(F, N))$.

Indeed, since $F$ is finitely presented as an $A$-module , we have the exact sequence

$$A^m \to A^n \to F \to 0.$$

We can thus consider the exact sequence

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})^m \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})^n \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \otimes_A F \to 0.$$

We also consider

$$\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})^m \to \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})^n \to \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_A(F, M), \mathbb{Q}/\mathbb{Z}) \to 0$$

which is exact as $\mathbb{Q}/\mathbb{Z}$ is injective as a $\mathbb{Z}$-module. Thus, by the Fives Lemma, we get that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_A(F, M), \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \otimes_A F$$

as desired.

Now, this produces the commutative diagram

$$\begin{tikzcd}
T(P) \otimes_A F \arrow[r, "T(v)\otimes 1_F"] \arrow[d] & T(N) \otimes F \arrow[d] \\
T(\mathrm{Hom}_A(F, P)) \arrow[r, "T(\psi)"] & T(\mathrm{Hom}_A(F, N))
\end{tikzcd}$$

where $\psi = \mathrm{Hom}_A(F, v)$. The top row is injective, and the columns are isomorphism. Thus the bottom must be injective as well. But since T is faithful, this implies that $\psi$ is surjective as desired.

($5 \implies 6$) Note, by Lemma A.2.0.2 we can write $P$ as the filtrated colimit of finitely presented $A$-modules $P_i$. Let $N_i$ be the fiber product of $P_i$ and $N$ over $P$. By definition, thus is a submodule of $N \times P_i$ consisting of elements who's two projections on $P$ are equal. Finally, let $M_i$ be the kernel of the projection $N_i \to P_i$. Then we have a filtered system of exact sequences

$$0 \longrightarrow M_i \longrightarrow N_i \longrightarrow P_i \longrightarrow 0$$

Note, for all $i \in I$ there exists a compatible map of sequences such that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & M_i \arrow[r] \arrow[d] & N_i \arrow[r] \arrow[d] & P_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & N \arrow[r] & P \arrow[r] & 0
\end{tikzcd}$$

We claim that $M_i \to M$ is an isomorphism. Let $x \in \ker(N \to P)$. Since $P_i \to P$ is an injection, the only preimage of 0 in $P_i$ is 0. Thus the corresponding element in $N_i$ $(x, 0)$. However, clearly $(x, ) \in \ker(M_i \to P_i)$. Next, note everything in the kernel of $N_i \to P_i$ is of the form $(y, 0)$ as this is an projection map. But $(y, 0) \mapsto 0$ in $P$ by the top path. Thus $y \mapsto 0$ and the kernels are isomorphic as desired. By 5. there is a map $P_i \to N$ lifting $P_i \to P$. By the universal property of fiber products we have that there is a section of $N_i \to P_i$. Thus the sequence

$$0 \longrightarrow M_i \longrightarrow N_i \longrightarrow P_i \longrightarrow 0$$

splits. We thus finish by passing by to the colimit and using the Short Fives Lemma to get $\varinjlim_{i \in I} N_i \cong N$.

(6 $\implies$ 1) This is the direct result of the sequences being split and how tensors commute with colimits. $\square$

The following are some examples of universally exact sequences:

**Example 2.4.0.4.**

1. *Any split exact sequence is clearly universally exact.*

2. *Let* $A = \mathbb{Z}$. *The sequence*

$$0 \longrightarrow \oplus_{n\in\mathbb{N}}\mathbb{Z} \longrightarrow \prod_{n\in\mathbb{N}}\mathbb{Z} \longrightarrow C \longrightarrow 0$$

*where* $C$ *is the cokernel of the former map. All of these modules are* $\mathbb{Z}$*-flat, and we will see later that this means the sequence is universally exact. Note that this sequence does not split.*

3. *By Lemma* 2.3.0.1, *faithfully flat maps are universally injective maps.*

We make the following convenient definition for future use.

**Definition 2.4.0.5.** *We will say a submodule* $N$ *of* $M$ *is* pure *in* $M$ *if the inclusion* $N \hookrightarrow M$ *is universally injective. Equivalently,* $N$ *is pure in* $M$ *if*

$$0 \to N \to M \to M/N \to 0$$

*is universally exact.*

## 2.5. Relatively projective and relatively injective modules

**Definition 2.5.0.1.** *We say an* $A$*-module* $P$ *is* relatively projective *if the functor* $\mathrm{Hom}_A(P, \bullet)$ *takes a universally exact sequence to an exact sequence.*

We wish to characterize relatively projective modules in a method that is easily checked. To do so, we use the following prove the following Lemmas:

**Lemma 2.5.0.2.** *If* $(I, \leq)$ *is a filtered poset and* $\{(M_i, \varphi_{ij} : i \leq j, i, j \in I\}$ *is a system of* $A$*-modules with* $M = \varinjlim_{i\in I} M_i$ *then*

$$0 \longrightarrow k \longrightarrow \oplus M_i \xrightarrow{\pi} M \longrightarrow 0$$

*is universally exact.*

*Proof.* By Proposition 2.4.0.3.5 it suffices to show that for any finitely presented A-module $Q$ the homomorphism $\mathrm{Hom}_A(Q, \pi)$ is surjective. To do this, we introduce the following Lemma.

**Lemma 2.5.0.3.** *If $(I, \leq)$ is a filtered poset and $\{(M_i, \varphi_{ij} : i \leq j, i, j \in I\}$ is a system of A-modules with $M = \varinjlim_{i\in I} M_i$ then for any A-module Q, we have the following:*

1. *There exists a canonical A-linear map*

$$\varinjlim_{i\in I} \mathrm{Hom}_A(Q, M_i) \to \mathrm{Hom}_A(Q, M).$$

2. *If Q is finitely generated as an A-module, the map from 1. is injective.*

3. *If Q is finitely presented as an A-module, the map from 1. is an isomorphism.*

*Proof of Lemma* 2.5.0.3 For part 1, we claim there is a canonical mapping

$$\varinjlim_{i\in I} \mathrm{Hom}_A(Q, M_i) \to \mathrm{Hom}_A(Q, M).$$

Indeed, let $\psi \in \varinjlim_{i\in I} \mathrm{Hom}_A(Q, M_i)$. Then there exists $j \in I$ such that $\psi_j \in \mathrm{Hom}_A(Q, M_j)$ maps to $\psi$ in the colimit. We thus define

$$\varinjlim_{i\in I} \mathrm{Hom}_A(Q, M_i) \to \mathrm{Hom}_A(Q, M)$$

by $\psi_j \mapsto \varphi_j \circ \psi_j$ which is clearly in $\mathrm{Hom}(Q, M)$. Now, assume $\psi_j$ and $\psi_k$ both maps to $\psi$ in the colimit. Then there exists $\alpha > k, j$ such that there exists $\varphi_{j\alpha}$ and $\varphi_{k\alpha}$ such that the diagrams

$$\begin{tikzcd} M_j \arrow[r] \arrow[dr] & M_\alpha \arrow[d] \\ & M \end{tikzcd} \qquad \begin{tikzcd} M_k \arrow[r] \arrow[dr] & M_\alpha \arrow[d] \\ & M \end{tikzcd}$$

commute. Thus $\varphi_j \circ \psi_j = \varphi_\alpha \circ \varphi_{j\alpha} \circ \psi_j = \varphi_\alpha \circ \varphi_{k\alpha} \circ \psi_k = \varphi_k \circ \psi_k$ and our mapping is well defined as desired.

Now, for 2. suppose $A^{\oplus n} \to Q \to 0$ is exact. Let $\phi \in \varinjlim_{i \in I} \mathrm{Hom}_A(Q, M_i)$ be such that $\phi \mapsto 0 \in \mathrm{Hom}_A(Q, M)$. Choose $i$ and $\phi_i \in \mathrm{Hom}_A(M, M_i)$ such that $\phi_i$ maps to $\phi$ in $\varinjlim_{i \in I} \mathrm{Hom}_A(Q, M_i)$.

Since $Q$ is finitely generated as an $A$-module, every map is uniquely determined by the action on a set of generators. Thus, we can define $\phi_i$ as such

$$\begin{array}{ccccc} Q & \xrightarrow{\phi_i} & M_i & \longrightarrow & M \\ & & & & \\ x_1, \ldots, x_n & \longmapsto & m_{1i}, \ldots, m_{ni} & \longmapsto & m_1, \ldots, m_n. \end{array}$$

But this composition is 0 and since $M = \varinjlim_{i \in I} M_i$ there exists $j \geq i$ such that

$$\varphi_{ij}(m_{1j}) = \cdots = \varphi_{ij}(m_{ni}) = 0.$$

Choose such a $j$ and note that $\varphi_{ij} \circ \phi_i = 0$ and that both of these map to $\phi$ in the colimit. Thus $\phi = 0$ and the kernel is trivial as desired.

Finally, for 3 let $\mathcal{L} \coloneqq A^m \to A^n \to Q \to 0$ be a finite presentation of Q. Consider the functor $\mathrm{Hom}_A(\bullet, M_i)$. Applying this to $\mathcal{L}$ yields the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(Q, M_i) \arrow[r] & \mathrm{Hom}_A(A^n, M_i) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_A(A^m, M_i) \arrow[d, equal] \\ & & M_i^n & M_i^m. \end{tikzcd}$$

Taking the colimit of this gives

$$0 \longrightarrow \varinjlim_{i \in I} \mathrm{Hom}_A(Q, M_i) \longrightarrow M^n \longrightarrow M^m.$$

Now, if we apply the functor $\mathrm{Hom}_A(\bullet, M)$ to $\mathcal{L}$, we have the following

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & \varinjlim_{i \in I} \mathrm{Hom}_A(Q, M_i) \arrow[r] \arrow[d] & M^n \arrow[r] \arrow[d, equal] & M^m \arrow[d, equal] \\ 0 \arrow[r] & \mathrm{Hom}_A(Q, M) \arrow[r] & \mathrm{Hom}_A(A^n, M) \arrow[r] & \mathrm{Hom}_A(A^m, M). \end{tikzcd}$$

Thus we win by the Fives Lemma. $\square$

Now, we can finish the proof of Lemma 2.5.0.2. From Lemma 2.5.0.3 part 3 we have for any finitely presented $A$-module Q and homomorphism $f : Q \to M$ there exists an $i \in I$ and a linear map $f_i : Q \to M_i$ such that the diagram

$$\begin{tikzcd}
Q \arrow[r, "f_i"] \arrow[dr, "f"] & M_i \arrow[d] \\
 & M
\end{tikzcd}$$

commutes. Thus we can form the larger diagram:

$$\begin{tikzcd}
 & M_i \arrow[r, "i_i"] \arrow[d] & \oplus M_j \arrow[dl, "\pi"] \\
Q \arrow[ur, dashed, "f_i"] \arrow[r, "f"] & M &
\end{tikzcd}$$

We note that the right triangle commutes by the construction of the colimit. But this means that $\mathrm{Hom}_A(Q, \pi)$ is surjective as for any $f \in \mathrm{Hom}_A(Q, M)$ we have that the map $i_i \circ f_i \in \mathrm{Hom}_A(Q, \oplus M_j)$ maps to $f \in \mathrm{Hom}_A(Q, M)$ as desired. $\square$

This allows us to prove statement 1.1.2 and 1.1.3 in [RG71]

**Proposition 2.5.0.4.** *[RG71, Part II, 1.1.2] An $A$-module $P$ is relatively projective if and only if it is a direct factor of a direct sum of finitely presented modules.*

*Proof.* Let $P$ be relatively projective. We can write $P$ as the filtered colimit of a system of finitely presented $A$-modules $F_i$ by Lemma A.2.0.2. Then, by Lemma 2.5.0.2, we have the universally exact sequence

$$0 \longrightarrow K \longrightarrow \oplus F_i \longrightarrow P \longrightarrow 0$$

Since $P$ is relatively projective, we have

$$0 \longrightarrow \mathrm{Hom}_A(P, K) \longrightarrow \mathrm{Hom}_A(P, \oplus F_i) \longrightarrow \mathrm{Hom}_A(P, P) \longrightarrow 0.$$

is exact. In particular, there exists $g \in \mathrm{Hom}_A(P, \oplus F_i)$ such that $g \mapsto \mathrm{id}_P$. That is $\oplus F_i \to P$ splits and $P$ is a direct factor of $\oplus F_i$ as desired.

For the backwards direction, let

$$0 \longrightarrow Q \longrightarrow P \longrightarrow L \longrightarrow 0$$

be a universally exact sequence of $A$-modules. By Proposition 2.4.0.3 part 5, for all finitely presented $F$ we have

$$0 \longrightarrow \mathrm{Hom}_A(F,Q) \longrightarrow \mathrm{Hom}_A(F,P) \longrightarrow \mathrm{Hom}_A(F,L) \longrightarrow 0$$

is exact. That is to say all finitely presented modules are relatively projective. Further, if we take $F = \oplus F_i$ as the direct sum of finitely presented modules, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_A(F,Q) & \longrightarrow & \mathrm{Hom}_A(F,P) & \longrightarrow & \mathrm{Hom}_A(F,L) & & \\ & & \| & & \| & & \| & & \\ 0 & \longrightarrow & \prod \mathrm{Hom}_A(F_i,Q) & \longrightarrow & \prod \mathrm{Hom}_A(F_i,P) & \longrightarrow & \prod \mathrm{Hom}_A(F_i,L) & \longrightarrow & 0. \end{array}$$

Finally, it remains to show a direct factor of some direct sum of finitely presented modules is relatively projective. Let $M$ be a direct factor of $F = \oplus F_i$ where $F_i$ is finitely presented for all $i$. Then we can write $F = M \oplus N$. But

$$0 \longrightarrow \mathrm{Hom}_A(F,Q) \longrightarrow \mathrm{Hom}_A(F,P) \longrightarrow \mathrm{Hom}_A(F,L) \longrightarrow 0$$

is exact. Thus we have

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}_A(M,Q) \oplus \mathrm{Hom}_A(N,Q) &\longrightarrow \mathrm{Hom}_A(M,P) \oplus \mathrm{Hom}_A(N,P) \\ &\longrightarrow \mathrm{Hom}_A(M,L) \oplus \mathrm{Hom}_A(N,L) \longrightarrow 0 \end{aligned}$$

is exact. Thus

$$0 \longrightarrow \mathrm{Hom}_A(M,Q) \longrightarrow \mathrm{Hom}_A(M,P) \longrightarrow \mathrm{Hom}_A(M,L) \longrightarrow 0$$

is exact and $M$ is relatively projective as desired. $\square$

The following corollary provides more examples of universally exact sequences.

**Corollary 2.5.0.5.** *[RG71, Part II, 1.1.3] For all $A$-modules $M$, there exists a universal exact sequence*

$$0 \to N \to P \to M \to 0$$

*such that $P$ is relatively projective as an $A$-module.*

*Proof.* By A.2.0.2, we can write $M = \varinjlim_{i \in I} M_i$ where every $M_i$ is finitely presented as an $A$-module. By Proposition 2.5.0.2, the sequence

$$0 \longrightarrow \ker(\pi) \longrightarrow \bigoplus M_i \xrightarrow{\pi} M \longrightarrow 0$$

is universally exact. Further, by 2.5.0.4 $\oplus M_i$ is relatively projective as desired. $\square$

Similarly, we have a notion of relatively injective modules.

**Definition 2.5.0.6.** *We say an $A$-modules $E$ is* relatively injective *if the functor $\mathrm{Hom}_A(\bullet, E)$ takes a universally exact sequence to an exact sequence.*

We can characterize relatively injective modules as follows.

**Proposition 2.5.0.7.** *[RG71, Part II, 1.1.5] For all $A$-modules $M$, there exists a $A$-universal exact sequence*

$$0 \to M \to E \to N \to 0$$

*such that $E$ is relatively injective.*

*Proof.* Note that for all $M \in \mathrm{Mod}_A$, the module $T(M)$ is relatively injective. Indeed, we have the functorial isomorphism

$$T(M \otimes_A \bullet) \cong \mathrm{Hom}_A(\bullet, T(M)).$$

But the functor $T(M \otimes_A \bullet)$ takes an $A$ universally exact sequence to an exact sequence. Thus $T(T(M))$ is relatively injective as $T(M)$ is an $A$-module.

Now we claim that for any $M \in \mathrm{Mod}_A$ the the evaluation map

$$\mathrm{ev}_M : M \to T(T(M))$$

is universally injective. To see this, we note that the map

$$T(\mathrm{ev}_M) : T(T(T(M))) \to T(M)$$

has a right inverse, namely its own evalutation map $\mathrm{ev}_{T(M)} : T(M) \to T(T(T(M)))$. Thus by Proposition 2.4.0.3.3, we have the claim. Hence, for all $M \in \mathrm{Mod}_A$, the sequence

$$0 \to M \to T(T(M)) \to T(T(M))/M \to 0$$

is universally injective. □

**Corollary 2.5.0.8.** *[RG71, Part II, 1.1.4] An A-module E is relatively injective if and only if the canonical A-linear map $j_I : E \to T^2(E)$ is left-invertible.*

*Proof.* Note $T(T(I))$ is relatively injective as outlined in the proof of Proposition 2.5.0.7. But since $I \to T(T(I))$ splits, we have $T(T(I)) = I \oplus k$. Thus given a universally exact sequence

$$0 \to M \to N \to Q \to 0$$

we have

$$0 \to \mathrm{Hom}_A(Q, I \oplus k) \to \mathrm{Hom}_A(N, I \oplus k) \to \mathrm{Hom}_A(M, I \oplus k)$$

is exact. Thus we have that

$$0 \to \mathrm{Hom}_A(Q, I) \to \mathrm{Hom}_A(N, I) \to \mathrm{Hom}_A(M, I) \to 0$$

is exact and $I$ is relatively injective.

Now, as shown in the proof of Proposition 2.5.0.7, we can form the universally exact sequence

$$0 \to E \to T(T(E)) \to T(T(E))/E \to 0$$

Since $T(T(M))$ is relatively injective, we get that the sequence

$$0 \to \operatorname{Hom}_A(T(T(M))/M, T(T(M))) \to \operatorname{Hom}_A(T(T(M)), T(T(M))) \to$$
$$\to \operatorname{Hom}_A(M, T(T(M)) \to 0$$

is exact. This means that $M \to T(T(M))$ splits as desired. $\square$

## 2.6. Descent of flatness for universally injective maps

We would like to show that universally injective maps descend flatness (Corollary 2.6.1.4). We start by observing some relationships between flat modules and universally exact sequences.

**Lemma 2.6.0.1.** *[Sta24, Tag 058P] If $\mathcal{L} := 0 \to M' \to M \to M'' \to 0$ is a universally exact sequence of $A$-modules and $M$ is $A$-flat, then $M'$ and $M''$ are also $A$-flat.*

*Proof.* Consider a short exact sequence

$$\mathcal{S} := 0 \to N' \to N \to N'' \to 0.$$

Since $\mathcal{L}$ is universally exact, $\mathcal{L} \otimes_A N'$, $\mathcal{L} \otimes_A N$, and $\mathcal{L} \otimes_A N''$ are all exact sequences.

This gives rise to the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes_A N' & \longrightarrow & M \otimes_A N' & \longrightarrow & M'' \otimes_A N' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes_A N & \longrightarrow & M \otimes_A N & \longrightarrow & M'' \otimes_A N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M' \otimes_A N'' & \longrightarrow & M \otimes_A N'' & \longrightarrow & M'' \otimes_A N'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are exact. The middle column is also exact since $M$ is $A$-flat. Specifically $M \otimes_A N' \to M \otimes_A N$ is injective. This means that $M' \otimes_A N' \to M' \otimes_A N$ is injective, making $M'$ $A$-flat. It remains to show that $M'' \otimes_A N' \to M'' \otimes_A N$ is injective. Let $x \in \ker(M'' \otimes_A N' \to M'' \otimes_A N)$ and $y \in M \otimes_A N'$ be such that $y \mapsto x$. It suffices to show that $y \in \operatorname{im}(M' \otimes_A N' \to M \otimes_A N')$. Let $y_1$ be the image of $y$ in $M \otimes_A N$. Then $y_1 \in \ker(M \otimes_A N \to M'' \otimes_A N)$. Thus, there exists a $z \in M' \otimes_A N$ such that $z \mapsto y_1$ in $M \otimes_A N$. We claim $z \mapsto 0$ in $M' \otimes_A N''$. Indeed, since $y_1 \in \operatorname{im}(M \otimes_A N' \to M \otimes_A N)$ we have $y_1 \mapsto 0$ in $M \otimes_A N''$. But the only preimage of 0 in $M' \otimes_A N''$ is 0. Thus by commutativity of the square, $z$ maps to 0 as desired.

Hence, there exists $z_1$ in $M' \otimes_A N'$ that maps to $z$. But $z_1 \mapsto z \mapsto y_1$. Taking the top path, we have $z_1 \mapsto y \mapsto y_1$. In particular $y \in \operatorname{im}(M' \otimes_A N' \to M \otimes_A N') = \ker(M \otimes_A N' \to M'' \otimes_A N')$. But since a pre-image of $x$ is in the kernel, we have $x = 0$ as desired. $\square$

**Lemma 2.6.0.2.** *Consider a short exact sequence of $A$ modules*

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

1. *If $P$ is $A$-flat, then the sequence is universally exact*

2. *If $L$ is $A$-flat, then the sequence if universally exact if and only if $P$ is $A$-flat.*

*Proof.* To see 1, we observe the induced long exact sequence of Tor modules

$$\ldots \longrightarrow \mathrm{Tor}_1^A(M,P) \longrightarrow M \otimes_A Q \longrightarrow M \otimes_A L \longrightarrow M \otimes_A P \longrightarrow 0$$

and note that since $P$ is flat, $\mathrm{Tor}_1^A(M,P) = 0$ thus giving us a short exact sequence and that

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

is universally exact. Property (2) follows from (1) and Lemma 2.6.0.1. □

### 2.6.1 Descent of flat universal short exactness

**Definition 2.6.1.1.** *We say $M$* descends flat universal short exactness *if it satisfies the following. Let the following be short exact sequence*

$$\mathcal{L} := 0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

*where $L$ is $A$-flat. If*

$$0 \longrightarrow \mathrm{im}(\mathrm{id}_M \otimes_A u) \longrightarrow M \otimes_A L \xrightarrow{\mathrm{id}_M \otimes_A v} M \otimes_A P \longrightarrow 0$$

*is universally exact, then $\mathcal{L}$ is universally exact.*

This is condition (Q) in [RG71]. We say that a ring map $\varphi : A \to B$ *descends flat universal short exactness* if $B$ descends flat universal short exactness as an $A$-module. It is related to descending flatness in the following way.

**Lemma 2.6.1.2.** *Suppose $\varphi : A \to B$ is a ring map that descends flat universal short exactness. Then $\varphi$ descends flatness.*

*Proof.* Suppose $M$ is an $A$-module such that $B \otimes_A M$ is $B$-flat. We want to show that $M$ is $A$-flat. Choose a short exact sequence of $A$-modules

$$0 \longrightarrow K \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0$$

where $F$ is a free $A$-module. Consider the induced short exact sequence of $B$-modules

$$0 \longrightarrow \operatorname{im}(\mathrm{id}_B \otimes_A u) \longrightarrow B \otimes_A F \xrightarrow{\mathrm{id}_B \otimes_A v} B \otimes_A M \longrightarrow 0.$$

By flatness of $B \otimes_A M$ as a $B$-module, this is universally exact as a sequence of $B$-modules. But since universal exactness is preserved by restriction of scalars, this is also a universally exact sequence of $A$-modules. Thus

$$0 \longrightarrow K \xrightarrow{u} F \xrightarrow{v} M \longrightarrow 0.$$

is universally exact as $A$-modules by our assumption that $\varphi$ descends flat universal short exactness. But since $F$ is $A$-flat, it follows that both $K$ and $M$ are $A$-flat by Lemma 2.6.0.2. □

In order to show that universally injective ring maps descend flatness it therefore suffices to show that universally injective ring maps descend flat universal short exactness.

Recall that the functor $T(\bullet) = \mathrm{Hom}_{\mathbb{Z}}(\bullet, \mathbb{Q}/\mathbb{Z})$ detects universally exact sequences. That is

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

is universally exact if and only if

$$0 \longrightarrow T(P) \xrightarrow{T(v)} T(L) \xrightarrow{T(u)} T(Q) \longrightarrow 0$$

is split exact by Proposition 2.4.0.3. In particular, $Q \to L$ is universally injective, if and only if $T(L) \to T(Q)$ is split surjective.

Also recall that by Tensor-hom adjunction, for all A-modules $L$ and $M$, we have an isomorphism

$$\eta_{L,M} : T(L \otimes_A M) \to \mathrm{Hom}_A(L, T(M))$$

which is functorial in the sense that if $f : L \to P$ and $\varphi : N \to M$ are linear maps, then the following diagrams commute:

$$\begin{array}{ccc} T(L \otimes_A M) & \xrightarrow{\eta_{L,M}} & \mathrm{Hom}_A(L, T(M)) \\ \uparrow & & \uparrow \\ T(P \otimes_A M) & \xrightarrow{\eta_{P,M}} & \mathrm{Hom}_A(P, T(M)) \end{array} \qquad \begin{array}{ccc} T(L \otimes_A N) & \xrightarrow{\eta_{L,N}} & \mathrm{Hom}_A(L, T(N)) \\ \uparrow & & \uparrow \\ T(L \otimes_A M) & \xrightarrow{\eta_{L,M}} & \mathrm{Hom}_A(L, T(M)) \end{array}$$

Thus, for a short exact sequence

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

and an $A$-module $M$, we have that the induced sequence

$$0 \longrightarrow \mathrm{im}(u \otimes_A \mathrm{id}_M) \longrightarrow L \otimes_A M \xrightarrow{v \otimes_A \mathrm{id}_M} P \otimes_A M \longrightarrow 0$$

is universally exact if and only if $T(v \otimes_A \mathrm{id}_M)$ is left-invertible which is true if and only if $\mathrm{Hom}_A(v, T(M))$ is left-invertible.

We use these observations to prove the following ascent result:

**Lemma 2.6.1.3.** *Let $f : N \to M$ be a universally injective map of $A$-modules. If $N$ descends flat universal short exactness, then $M$ descends flat universal short exactness.*

*Proof.* Since $f$ is universally injective, we have $T(f) : T(M) \to T(N)$ is a split surjection. Let $\xi : T(N) \to T(M)$ be a right inverse of $T(f)$.

Now suppose

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

is a short exact sequence such that $L$ is $A$-flat and

$$0 \longrightarrow \operatorname{im}(u \otimes_A \mathrm{id}_M) \longrightarrow L \otimes_A M \xrightarrow{v \otimes_A \mathrm{id}_M} P \otimes_A M \longrightarrow 0$$

is universally exact. We need to show our original seqeunce is universally exact. By our assumption, it suffices to show that

$$0 \longrightarrow \operatorname{im}(u \otimes_A \mathrm{id}_N) \longrightarrow L \otimes_A N \xrightarrow{v \otimes_A \mathrm{id}_N} P \otimes_A N \longrightarrow 0$$

is universally exact or equivalently that $\operatorname{Hom}_A(v, T(N))$ is left-invertible. Recall that $\xi : T(N) \to T(M)$ is a split injection. This yields a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_A(L, T(N)) \arrow[r] & \operatorname{Hom}_A(L, T(M)) \\
\operatorname{Hom}_A(P, T(N)) \arrow[r] \arrow[u] & \operatorname{Hom}_A(P, T(M)) \arrow[u]
\end{tikzcd}$$

By universal exactness of

$$0 \longrightarrow \operatorname{im}(u \otimes_A \mathrm{id}_M) \longrightarrow L \otimes_A M \xrightarrow{v \otimes_A \mathrm{id}_M} P \otimes_A M \longrightarrow 0$$

we have $\operatorname{Hom}_A(v, T(M))$ is left-invertible. Let its section be denoted $\phi$. Also since $\xi$ is left-invertible, we have that $\operatorname{Hom}_A(P, \xi)$ is left-invertible with inverse $\psi$. We claim that $\psi \circ \phi \circ \operatorname{Hom}_A(L, \xi)$ is a left inverse of $\operatorname{Hom}_A(v, T(N))$. Indeed,

$$\begin{aligned}
\psi \circ \phi \circ \operatorname{Hom}_A(L, \xi) \circ \operatorname{Hom}_A(v, T(N)) =& \psi \circ \phi \circ \operatorname{Hom}_A(v, T(M)) \circ \operatorname{Hom}_A(P, \xi) \\
=& \psi \circ \operatorname{Hom}_A(P, \xi) = \mathrm{id}_{\operatorname{Hom}_A(P, T(N))} .
\end{aligned}$$

Thus we win. $\square$

**Corollary 2.6.1.4.** *Let $\varphi : A \to B$ be a universally injective ring map. Then $\varphi$ descends flatness and nullity.*

*Proof.* Note that $A$ as a module over itself clearly descends flat universal short exactness. Thus, by Lemma 2.6.1.3, $\varphi$ descends universal short exactness. Then Lemma 2.6.1.2 applies and $\varphi$ descends flatness. Finally, since $\varphi$ is injective it descends nullity by Lemma 2.2.0.4. $\square$

## 2.7. Generalization of a theorem of Ferrand

In [Fer69], Ferrand proved that an injective and finite homomorphism of Noetherian rings descends flatness. We will show that this is true regardless of the Noetherian hypothesis (Corollary 2.7.4.1). We will also show that if $A$ is a Noetherian ring and $f : A \to B$ is an injective homomorphism that descends nullity, then $f$ descends flatness (Corollary 2.7.4.11).

The proofs of the aforementioned results will rely on showing that in the setting that we are interested in, the ring homomorphisms satisfy dual, and stronger, versions of descent of nullity and descent of flatness. So we first introduce and discuss these related notions.

### 2.7.1 Descending Hom-nullity

**Definition 2.7.1.1.** *We say $M \in \mathrm{Mod}_A$* descends Hom-nullity *if $\mathrm{Hom}_A(M, N) = 0$ implies $N = 0$.*

In other words, $M$ descends Hom-nullity if the functor $\mathrm{Hom}_A(M, \bullet)$ is faithful. This is the dual statement to a module descending nullity. In [RG71] the property of a module descending Hom-nullity is called condition (O').

**Example 2.7.1.2.** *Free $A$-modules descend Hom-nullity.*

The following is a useful lemma with regards to Hom-nullity.

**Lemma 2.7.1.3.** *Let $\varphi\colon A \to B$ be a ring homomorphism. Let $M$ be an $A$-module. If $M$ descends Hom-nullity as an $A$-module, then $M \otimes_A B$ descends Hom-nullity as a $B$-module. In other words, the property of descending Hom-nullity is preserved under arbitrary base change.*

*Proof.* Let $M$ descend Hom-nullity as an $A$-module and $N$ be a $B$-module such that $\mathrm{Hom}_B(M \otimes_A B, N) = 0$. Note by [Sta24, Lemma 05DQ]

$$\mathrm{Hom}_B(M \otimes_A B, N) = \mathrm{Hom}_A(M, N_A) = 0$$

where $N_A$ is the restriction of viewing $N$ as an $A$-module. But $M$ descends Hom-nullity as an $A$-module. Thus $N_A = 0$. But this means that $N = 0$ as desired. $\square$

**Lemma 2.7.1.4.** *Let $M \in \mathrm{Mod}_A$. Suppose one has that for all cyclic $A$-modules $N$, if $\mathrm{Hom}_A(M, N) = 0$ then $N = 0$. Then $M$ descends hom-nullity.*

*Proof.* Let $P \in \mathrm{Mod}_A$ such that $\mathrm{Hom}_A(M, P) = 0$. Let $x \in P$ and $Ax \subseteq P$ be the cyclic module generated by $x$. Applying $\mathrm{Hom}_A(M, \bullet)$ to the inclusion $Ax \hookrightarrow P$ yields an inclusion

$$\mathrm{Hom}_A(M, Ax) \hookrightarrow \mathrm{Hom}_A(M, P)$$

as $\mathrm{Hom}_A(M, \bullet)$ is left exact. But $\mathrm{Hom}_A(M, P) = 0$. Thus $\mathrm{Hom}_A(M, Ax) = 0$. By assumption, this means $Ax = 0$. Since $x$ is arbitrary, we get that $P = 0$. $\square$

We have the analog of Lemma 2.2.0.5 for descent of Hom-nullity.

**Lemma 2.7.1.5.** *Let $M$ and $N$ be $A$-modules. $f : M \to N$ be a surjective $A$-linear map. If $N$ descends Hom-nullity, then $M$ descends Hom-nullity*

*Proof.* Let $P$ be an $A$-module such that $\mathrm{Hom}_A(M, P) = 0$. Note, since $f$ is a surjection, we have an exact sequence

$$M \to N \to 0$$

Applying $\mathrm{Hom}_A(\bullet, P)$ to this sequence yields an exact sequence

$$0 \to \mathrm{Hom}_A(N, P) \to \mathrm{Hom}_A(M, P) = 0$$

Thus $\mathrm{Hom}_A(N, P)$ injects into the zero module. Thus we must have $\mathrm{Hom}_A(N, P) = 0$. But since $N$ descends Hom-nullity, we have $P = 0$ as desired. $\square$

**Lemma 2.7.1.6.** *Let $M$ and $J$ be $A$-modules. Consider the following submodule of $J$:*

$$J' \coloneqq \sum_{u \in \mathrm{Hom}_A(M,J)} \mathrm{im}(u).$$

*If $M$ descends Hom-nullity, then $J' \hookrightarrow J$ is an essential extension.*

*Proof.* Let $N \subseteq J$ be such that $N \cap J' = 0$. We show that $\mathrm{Hom}_A(M, N) = 0$. Let $\varphi \in \mathrm{Hom}_A(M, N)$ and $i : N \to J$ be the canonical inclusion map. Then $i \circ \varphi \in \mathrm{Hom}_A(M, J)$. Thus $\mathrm{im}(i \circ \varphi) = \mathrm{im}(\varphi) \subseteq J'$. But also $\mathrm{im}(\varphi) \subseteq N$ and $N \cap J' = 0$. The only way this can be true is if $\mathrm{im}(\varphi) = 0$ and thus $\varphi = 0$. But $\varphi$ was chosen arbitrarily. Thus $\mathrm{Hom}_A(M, N) = 0$, and $N = 0$ as desired. $\square$

### 2.7.2 Descending Hom-left-invertibility for injectives

**Definition 2.7.2.1.** *We say $M \in \mathrm{Mod}_A$* descends Hom-left-invertibility for injectives *if whenever $E$ is an $A$-injective module and $u : J \to E$ is an injective $A$-linear map such that $\mathrm{Hom}_A(M, u)$ is left-invertible, then $u$ is left-invertible (or, equivalently, $J$ is injective).*

This condition is dual to the property of descending flat universal short exactness and is called condition (Q') in [RG71]. The following is a Lemma originally proved by Jean-Pierre Olivier in [Oli].

**Lemma 2.7.2.2.** *[RG71, Part II, Lem. 1.2.1] Let $M$ be an $A$-module that descends Hom-left-invertibility for injectives. Then for all $A$-modules $J$, if $\operatorname{Hom}_A(M, J)$ is an injective $A$-module, then $J$ is an injective $A$-module. Thus, if $\operatorname{Hom}_A(M, J) = 0$, then $J$ is an injective $A$-module.*

*Proof.* Let $J$ be an $A$-module such that $\operatorname{Hom}_A(M, J)$ is injective as an $A$-module. Let $E$ be the injective hull of $J$. Then $\operatorname{Hom}_A(M, J) \to \operatorname{Hom}_A(M, E)$ is an injective morphism of injective $A$-modules. Thus it splits. But $M$ descends Hom-left-invertibility for injectives. Thus $J \to E$ splits. But $J \to E$ was assumed to be an injective envelope. Thus $J \cong E$ and $J$ is an injective $A$-module as desired.

The second assertion follows from the first because the zero module is injective. $\square$

**Lemma 2.7.2.3.** *Let $\varphi : A \to B$ be a ring map such that $B$ descends Hom-left-invertibility for injectives as an $A$-module. Then for an $A$-module $J$, $\operatorname{Hom}_A(B, J)$ is an injective $B$-module if and only if $J$ is an injective $A$-module.*

*Proof.* The backward implication follows by Lemma A.1.0.17. For the forward implication, let $u : J \hookrightarrow E$ be an injective $A$-linear map, where $E$ is an injective $A$-module. By assumption, we have that $\operatorname{Hom}_A(B, J)$ is an injective $B$-module. Thus, $\operatorname{Hom}_A(B, u)$ has a $B$-linear left-inverse. By restriction of scalars, this left-inverse is also $A$-linear. Since $B$ descends Hom-left-invertibility for injectives, we then get that $u$ has a left-inverse, and so, $J$ is $A$-injective as desired. $\square$

The next result is not surprising, but it demonstrates that in order to check descent of Hom-left-invertibility for injectives, we only need to check the property against extensions $J \to E$, where $E$ is the injective hull of $J$.

**Lemma 2.7.2.4.** *An $A$-module $M$ descends Hom-left-invertibility for injectives if and only if the following holds: Let $L$ be an $A$-module and $u : L \to K$ be an injective envelope of $L$. If $\operatorname{Hom}_A(M, u)$ is left-invertible, then $u$ is an isomorphism.*

*Proof.* The forward implication is clear from the definition of descending Hom-left-invertibility for injectives. For the backwards implication, assume the above criterion holds. Now suppose $u : J \to E$ is an injective $A$-linear map such that $E$ is an injective $A$-module. Also suppose that $\operatorname{Hom}_A(M, u)$ is left-invertible. We wish to show that $u$ is left-invertible. Let $i\colon L \to I$ be an injective envelope. Then there exists $\tilde{u} : I \to E$ such that the following diagram commutes:

$$\begin{tikzcd} J \arrow[r, "i"] \arrow[d, "u"] & I \arrow[dl, dotted, "\tilde{u}"] \\ E. & \end{tikzcd}$$

Since $i$ is an essential extension and $u$ is injective, we have that $\tilde{u}$ is injective too. Then we have

$$\begin{aligned} \operatorname{Hom}_A(M, u) &= \operatorname{Hom}_A(M, \tilde{u} \circ i) \\ &= \operatorname{Hom}_A(M, \tilde{u}) \circ \operatorname{Hom}_A(M, i). \end{aligned}$$

But since $\operatorname{Hom}(M, u)$ is left-invertible, we have that $\operatorname{Hom}_A(M, i)$ is left-invertible as well. By hypothesis, this means that $i$ is an isomorphism and $L$ is injective. This means that $u$ is left-invertible. $\square$

### 2.7.3 Relationships between various descent properties

The following result is a first stab at relating the various descent properties that have been introduced so far.

**Lemma 2.7.3.1.** *[RG71, Part II, Lem. 1.2.2] Let $A$ be a ring and $M$ be an $A$-module.*

1. *If $M$ descends Hom-nullity, it descends nullity. The converse is true when $(A, \mathfrak{m})$ is local Noetherian and $\mathfrak{m}$-adically complete.*
2. *If $M$ descends Hom-left-invertibility for injectives, it descends flat universal short exactness.*
3. *If $A$ is an integral domain and $M$ is a non-zero $A$-module that descends Hom-left-invertibility for injectives, then $M$ descends Hom-nullity.*

*Proof.* 1) Let $N \in \mathrm{Mod}_A$ be such that $M \otimes_A N = 0$. Then $\mathrm{Hom}_A(M \otimes_A N, N) = 0$. But $\mathrm{Hom}_A(M \otimes_A N, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(N, N))$. Since $M$ descends Hom-nullity, it follows $\mathrm{Hom}_A(N, N) = 0$. But $\mathrm{id}_N \in \mathrm{Hom}_A(N, N)$. The only way that $\mathrm{id}_N = 0$ is that $N = 0$ as desired.

On the other hand, assume $(A, \mathfrak{m})$ is a complete local Noetherian ring. By Lemma 2.7.1.4 it suffices to show that for all finitely generated $A$-modules if $\mathrm{Hom}_A(M, N) = 0$ then $N = 0$.

Suppose $N$ is a finitely generated $A$-module such that $\mathrm{Hom}_A(M, N) = 0$. By Proposition A.3.0.13 we have that $N \cong \mathrm{Hom}_A(\mathrm{Hom}_A(N, E), E)$. Therefore we have

$$\begin{aligned} \mathrm{Hom}_A(M, N) &\cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(\mathrm{Hom}_A(N, E), E)) \\ &\cong \mathrm{Hom}_A(M \otimes_A \mathrm{Hom}_A(N, E), E) = 0. \end{aligned}$$

But $\mathrm{Hom}_A(\bullet, E)$ is a faithful functor by A.3.0.14. Thus $M \otimes_A \mathrm{Hom}_A(N, E) = 0$. But

$M$ descends nullity. Thus $\mathrm{Hom}_A(N, E) = 0$. Finally, by A.3.0.14, we have $N = 0$.

2) Let $L$ be a flat $A$-module. Clearly then $T(L)$ is as an $A$-injective by Lemma A.1.0.17. Now, let

$$0 \longrightarrow Q \xrightarrow{u} L \xrightarrow{v} P \longrightarrow 0$$

be a short exact sequence such that $\mathrm{im}(\mathrm{id}_M \otimes_A u)$ is pure in $M \otimes_A L$. Applying $T(\bullet)$ to the induced universally exact sequence yields the split exact sequence

$$0 \longrightarrow T(P \otimes_A M) \xrightarrow{T(v \otimes_A \mathrm{id}_M)} T(L \otimes_A M) \longrightarrow T(\mathrm{im}(\mathrm{id}_M \otimes_A u)) \longrightarrow 0.$$

But since $T(v \otimes_A \mathrm{id}_M)$ is left-invertible, we get that

$$\mathrm{Hom}_A(M, T(v))\colon \mathrm{Hom}_A(M, T(P)) \to \mathrm{Hom}_A(M, T(L))$$

is left invetible by Hom-tensor adjunction. Since $T(L)$ is an injective $A$-module and $M$ descends Hom-left-invertibility for injectives, we then get that $T(v)$ is left-invertible. This means that our original sequence is universally exact by Proposition 2.4.0.3.

3) If $A$ is a field then every module is free. Thus every non-zero module over a field descends Hom-nullity.

Now if $A$ is not a field, by Lemma 2.7.1.4 it suffices to show that if $I$ is an ideal of $A$ such that $\mathrm{Hom}_A(M, A/I) = 0$, then $I = A$. Let $E \in \mathrm{Mod}_A$ be an injective hull of $A/I$ and $i : A/I \to E$ the inclusion map. Then

$$\mathrm{Hom}_A(M, i) : \mathrm{Hom}_A(M, A/I) \to \mathrm{Hom}_A(A, E)$$

is left-invertible as the domain is 0. By our hypothesis, it follows that $i : A/I \to E$ is left-invertible. Then $A/I$ is also a finitely generated $A$-injective module. Since injective modules over integral domains are divisible, an application of Nakayama's

Lemma on the local rings of $A$ shows that $A/I = 0$ (see Lemma A.1.0.12), as desired.

$\square$

**Remark 2.7.3.2.** *Lemma* 2.7.3.1 *part 3. differs from [RG71, Part II, Lem. 1.2.2(iii)]. We can get by with weaker hypotheses on $M$ at the expense of assuming that $A$ is a domain. We will only need to apply Lemma* 2.7.3.1 *part 3. in what follows when $A$ is a domain.*

Our next goal is to show that finitely generated faithful modules descend Hom-left-invertibility for injectives (Theorem 2.7.3.5). We will need some preparatory results in order to accomplish this.

**Lemma 2.7.3.3.** *[RG71, Part II, Lem. 1.2.3] Let $M$ be an $A$-module, $N$ a submodule of $M$, $u : J \to E$ an injective $A$-linear map such that $E$ is injective as an $A$-module. If* $\mathrm{Hom}_A(M, u)$ *is bijective, then* $\mathrm{Hom}(N, u)$ *is also bijective.*

*Proof.* We form the commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}_A(M/N, J) \arrow[r] \arrow[d] & \mathrm{Hom}_A(M, J) \arrow[r] \arrow[d, "\mathrm{Hom}_A(M,u)"] & \mathrm{Hom}_A(N, J) \arrow[d, "\mathrm{Hom}_A(N,u)"] & \\ 0 \arrow[r] & \mathrm{Hom}_A(M/N, E) \arrow[r] & \mathrm{Hom}_A(M, E) \arrow[r] & \mathrm{Hom}_A(N, E) \arrow[r] & 0 \end{tikzcd}$$

in which the columns are injective and the central column is a bijection. It remains to show that $\mathrm{Hom}_A(N, u)$ is surjective. But this follows from the commutativity of the right square and the fact that the composition $\mathrm{Hom}_A(M, J) \xrightarrow{\mathrm{Hom}_A(M,u)} \mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(N, E)$ is surjective. $\square$

**Lemma 2.7.3.4.** *Let $A$ be a ring. Then we have the following:*

1. *Let $u \colon J \hookrightarrow E$ be an injective $A$-linear map and $\pi \colon L \twoheadrightarrow F$ be a surjective*

*A-linear map. Then the square*

$$\begin{array}{ccc} \mathrm{Hom}_A(F,J) & \xrightarrow{\mathrm{Hom}_A(F,u)} & \mathrm{Hom}_A(F,E) \\ \Big\downarrow{\scriptstyle \mathrm{Hom}_A(\pi,J)} & & \Big\downarrow{\scriptstyle \mathrm{Hom}_A(\pi,E)} \\ \mathrm{Hom}_A(L,J) & \xrightarrow[\mathrm{Hom}_A(L,u)]{} & \mathrm{Hom}_A(L,E). \end{array}$$

*is Cartesian.*

2. *Let*

$$\begin{array}{ccc} M & \xrightarrow{\phi} & N \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle g} \\ P & \xrightarrow{\varphi} & Q \end{array}$$

*be a Cartesian square of A-modules where the vertical maps are injective. If $\varphi$ is an essential extension, then $\phi$ is an essential extension.*

3. *For all $i = 1, \ldots, n$ let $\varphi_i \colon M_i \to N_i$ be an essential extension of A-modules. Then*

$$\bigoplus_{i=1}^{n} \varphi_i \colon \bigoplus_{i=1}^{n} M_i \to \bigoplus_{i=1}^{n} N_i$$

*is an essential extension of A-modules.*

4. *Let $u \colon J \to E$ be an essential extension of A-modules. Then for any finitely generated A-module F, $\mathrm{Hom}_A(F,u) \colon \mathrm{Hom}_A(F,J) \to \mathrm{Hom}_A(F,E)$ is also an essential extension of A-modules.*

*Proof.* To show 1, let $\varphi \in \mathrm{Hom}_A(L,J)$ and $\phi \in \mathrm{Hom}_A(F,E)$ be such that

$$u \circ \varphi = \phi \circ \pi.$$

It suffices to show that there exists unique $\psi \in \mathrm{Hom}_A(F,J)$ such that

$$\psi \circ \pi = \varphi$$

and

$$u \circ \psi = \phi.$$

To show uniqueness, let $\psi_2$ be another such map such that $\psi \circ \pi = \psi_2 \circ \pi = \varphi$. Since $\pi$ is surjective, it is an epimorphism. Thus $\psi = \psi_2$ giving uniqueness.

We now show existence. Since $\pi$ is surjective we have $\mathrm{im}(\phi \circ \pi) = \mathrm{im}(\phi)$. Similarly, since $u$ is injective, we have $\mathrm{im}(u \circ \varphi) \cong \mathrm{im}(\varphi)$. In other words,

$$\mathrm{im}(\phi) = \mathrm{im}(\phi \circ \pi) = \mathrm{im}(u \circ \varphi) \cong \mathrm{im}(\varphi).$$

Thus we have the following composition mapping

$$\psi : F \xrightarrow{\phi} \mathrm{im}(\phi) \cong \mathrm{im}(\varphi) \hookrightarrow J,$$

where $\mathrm{im}(\varphi) \hookrightarrow J$ is the inclusion mapping. We claim $u \circ \psi = \phi$. Indeed, let $a \in F$ and $x \in L$ be such that $\phi(a) = u \circ \varphi(x)$. Then under the isomorphism $\mathrm{im}(\varphi) \cong \mathrm{im}(\phi)$ we have $\psi(a) = \varphi(x)$. Thus $u \circ \psi(a) = u \circ \varphi(x) = \phi(a)$. Hence $u \circ \psi = \phi$

It remains to show that $\psi \circ \pi = \varphi$. Since $\phi \circ \pi = u \circ \varphi$, by definition of $\psi$ we get

$$\psi \circ \pi = L \xrightarrow{u \circ \varphi} \mathrm{im}(u \circ \varphi) \cong \mathrm{im}(\varphi) \hookrightarrow J.$$

But if $z \in L$, then under the isomorphism $\mathrm{im}(u \circ \varphi) \cong \mathrm{im}(\varphi)$, the element $u \circ \varphi(z)$ maps to $\varphi(z)$. Thus, $L \xrightarrow{u \circ \varphi} \mathrm{im}(u \circ \varphi) \cong \mathrm{im}(\varphi) \hookrightarrow J$ equals $\varphi$.

For 2. let $x$ be a non-zero element in $N$. Then $g(x)$ is non-zero in $Q$ as $g$ is injective. Since $\varphi$ is an essential extension, there exists a non-zero $y \in P$ such that $\varphi(y) = ag(x) = g(ax)$. That is, $y$ and $ax$ map to the same element in $Q$. Note that $ax \neq 0$ because $y \neq 0$ and $\varphi$ is injective. Since the diagram is Cartesian, there exists $z \in M$ such that $z$ maps to $ax$ and $y$. Since $x$ was chosen arbitrarily, this means for all elements of $N$ there is a non-zero multiple that lies in $\phi(M)$. As such $\phi$ is essential.

For 3. we proceed by induction on $n$. But first we need the following well-known fact. Let $\phi\colon M \to N$ and $\varphi\colon N \to P$ be essential extensions. We claim the composition $\varphi \circ \phi$ is an essential extension. Indeed, let $x \in P$ be non-zero. Since $\varphi$ is essential, there exists $y \in N$ such that $\varphi(y) = a_1 x$, where $a_1 \in R$ and $a_1 x$ is some non-zero element of $P$. Note $y \neq 0$ since $a_1 x \neq 0$. Thus, similarly, there exists $z \in M$ such that $\phi(z) = a_2 y$, where $a_2 \in A$ and $a_2 y \neq 0$. Then

$$\varphi \circ \phi(z) = \varphi(a_2 y) = a_2 \varphi(y) = (a_2 a_1)x.$$

Moreover, $(a_2 a_1)x \neq 0$ because $a_2 y \neq 0$ and $\varphi$ is injective. This shows $\varphi \circ \phi$ is essential.

Now, in the $n = 2$ case, let $\varphi_1 : M_1 \to N_1$ and $\varphi_2 : M_2 \to N_2$ be essential extensions. We wish to show that $\varphi_1 \bigoplus \varphi_2$ is an essential extension by showing that $\varphi_1 \bigoplus \mathrm{id}_{M_2}$ and $\mathrm{id}_{N_1} \oplus \varphi_2$ are essential extensions and observing that

$$\varphi_1 \bigoplus \varphi_2 = (\mathrm{id}_{N_1} \oplus \varphi_2) \circ (\varphi_1 \bigoplus \mathrm{id}_{M_2}).$$

We first show that $\varphi_1 \bigoplus \mathrm{id}_{M_2}$ is essential. Indeed, let $(x, m) \in N_1 \oplus M_1$. First, assume $x \neq 0$. Since $\varphi_1$ is an essential extension, there exists an $y \in M_1$ such that $y$ maps to a non-zero multiple $ax$. Then $(y, am) \in M_1 \bigoplus M_2$ and

$$\varphi_1 \oplus \mathrm{id}_{M_2}(y, am) = (ax, am),$$

which is a non-zero multiple of $(x, m)$.

Now consider $(x, m)$ when $x = 0$. Then $m \neq 0$ because $(x, m)$ is a nonzero element. Then clearly, the element $(0, m) \in M_1 \bigoplus M_2$ maps to $(0, m)$ in $N_1 \oplus M_2$.

This shows that $\varphi_1 \bigoplus \mathrm{id}_{M_2}$ is an essential extension. The argument for $\mathrm{id}_{N_1} \oplus \varphi_2$ being an essential extension is similar.

Note, we can represent the map $\bigoplus_{i=1}^n \varphi_i$ as $\bigoplus_{i=1}^{n-1} \varphi_i \oplus \varphi_n$ and use the same argument to win by induction.

Now for 4. let $F$ be finitely generated. Then we have an exact sequence

$$A^{\oplus n} \xrightarrow{\pi} F \to 0.$$

This yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(F, J) \arrow[r, "\mathrm{Hom}_A(F,u)"] \arrow[d, "\mathrm{Hom}_A(\pi,J)"'] & \mathrm{Hom}_A(F, E) \arrow[d, "\mathrm{Hom}_A(\pi,E)"] \\
\mathrm{Hom}_A(A^{\oplus n}, J) \arrow[r, "\mathrm{Hom}_A(A^{\oplus n},u)"'] & \mathrm{Hom}_A(A^{\oplus n}, E).
\end{tikzcd}$$

By 1. this diagram is Cartesian and by 3. the bottom row is an essential extension. Thus by 2. the top row is an essential extension, as desired. □

Building on the previous results, we can now show that finitely generated faithful modules descend Hom-left-invertibility for injectives.

**Theorem 2.7.3.5.** *[RG71, Part II, Thm. 1.2.4] A finitely generated faithful module over a ring $A$ descends Hom-left-invertibility for injectives.*

*Proof.* Let $u : J \to E$ be an injective envelope. Let $F$ be a finitely generated faithful $A$-module. By Lemma 2.7.2.4, it is enough to show that if $\mathrm{Hom}_A(F, u)$ is left-invertible in $\mathrm{Mod}_A$, then $u$ is an isomorphism. So we will assume from now on that $\mathrm{Hom}_A(F, u)$ is left-invertible.

Since $F$ is a finitely generated, by Lemma 2.7.3.4 we have that $\mathrm{Hom}_A(F, u)$ is an essential extension, and so, it is an isomorphism. Since $F$ is faithful, its annihilator is 0. Let $F$ be generated by the elements $x_1, x_2, \ldots, x_n$. Then the $A$-linear map $A \to F^{\oplus n}$ defined by $a \mapsto (ax_1, ax_2, \ldots, ax_n)$ is injective. This yields an exact sequence

$$0 \to A \to F^{\bigoplus n} \to F^{\bigoplus n}/A \to 0.$$

This gives us the diagram of sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_A(F^{\bigoplus n}/A, J) & \longrightarrow & \operatorname{Hom}_A(F^{\bigoplus n}, J) & \longrightarrow & \operatorname{Hom}_A(A, J) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Hom}_A(F^{\bigoplus n}/A, E) & \longrightarrow & \operatorname{Hom}_A(F^{\bigoplus n}, E) & \longrightarrow & \operatorname{Hom}_A(A, E) & \longrightarrow & 0
\end{array}$$

where the bottom row is exact as $E$ is an injective $A$-module and the top row is exact by left-exactness of $\operatorname{Hom}_A(\bullet, J)$. The middle vertical map can be identified with $\operatorname{Hom}_A(F, u)^{\bigoplus n}$, and hence is an isomorphism. Thus Lemma 2.7.3.3 applies, and $\operatorname{Hom}_A(A, u) = u$ is an isomorphism as desired. □

### 2.7.4 Deducing consequences

A first consequence of the results of the previous subsection is the following:

**Corollary 2.7.4.1.** *A finite injective ring homomorphism descends flatness.*

*Proof.* Let $f : A \to B$ be a finite injective ring homomorphism. Then $B$ is a faithful $A$-module. Thus, by Theorem 2.7.3.5, we have that $f$ descends Hom-left-invertibility for injectives. By Lemma 2.7.3.1 part 2., $f$ descends flat universal short exactness. Then by Lemma 2.6.1.2, $f$ descends flatness as desired. □

It is worth noting that since we are assuming $f$ is an injective morphism, a finite injective ring homomorphism also descends nullity. We now show such homomorphisms descend the ascending chain condition in rings. The result is originally due to Mollier.

**Corollary 2.7.4.2.** *[RG71, Part II, Cor. 1.2.5] Let $f : A \to B$ be an injective and finite morphism. If $B$ is Noetherian, then so is $A$.*

*Proof.* Let $(E_\lambda)_{\lambda\in\Lambda}$ be a family of $A$-injective modules. By Bass [Bas62] it suffices to show that $\bigoplus_{\lambda\in\Lambda} E_\lambda$ is $A$-injective. We claim that

$$\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_A(B, E_\lambda) \cong \mathrm{Hom}_A(B, \bigoplus_{\lambda\in\Lambda} E_\lambda).$$

Indeed, it is clear that $\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_A(B, E_\lambda) \to \mathrm{Hom}_A(B, \bigoplus_{\lambda\in\Lambda} E_\lambda)$ is an injection. Now let $f \in \mathrm{Hom}_A(B, \bigoplus_{\lambda\in\Lambda} E_\lambda)$. Let $b_1, \ldots, b_n$ be a generating set of $B$. Then $f$ is uniquely determined by its actions on $b_i$. Note that $f(b_i)$ is equal to 0 in all but finitely many coordinates of $\bigoplus_{\lambda\in\Lambda} E_\lambda$. Thus there is a finite subset of $\Lambda$, say $\Delta$, such that $\mathrm{im}(f) \subseteq \bigoplus_{\lambda\in\Delta} E_\lambda$. Let $f' : B \to \bigoplus_{j\in\Delta} E_j$ be the induced map. Then

$$f' \in \mathrm{Hom}_A(B, \bigoplus_{j\in\Delta} E_j) = \bigoplus_{j\in\Delta} \mathrm{Hom}_A(B, E_j) \subseteq \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_A(B, E_\lambda).$$

But clearly, $f' \mapsto f$ in $\mathrm{Hom}_A(B, \oplus_{\lambda\in\Lambda} E_\lambda)$ giving us the desired isomorphism.

Now $\bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_A(B, E_\lambda)$ is the direct sum of injective $B$-modules by Lemma A.1.0.17. Thus, by [Bas62] we have $\mathrm{Hom}_A(B, \bigoplus_{\lambda\in\Lambda} E_\lambda)$ is an injective $B$-module as $B$ is Noetherian. Now note that $B$ descends Hom-left-invertibility for injectives by 2.7.3.5. Thus Lemma 2.7.2.3 applies, and $\bigoplus_{\lambda\in\Lambda} E_\lambda$ is an injective $A$-module as desired. □

**Example 2.7.4.3.** *In light of Corollary* 2.7.4.2, *it is natural to ask if integral extensions descend the Noetherian property. That is, if $A \hookrightarrow B$ is an integral extension and $B$ is Noetherian, then one can ask if $A$ is Noetherian as well. We show that this is false*[1]*. Let $K \subset L$ be an algebraic field extension that is not finite (for example, $K = \mathbb{F}_p$ and $L = \overline{\mathbb{F}_p}$). Consider the subring*

$$K \oplus XL[X]$$

[1]The following example was provided by Karl Schwede and Linquan Ma.

*of the polynomial ring $L[X]$. Since $L/K$ is algebraic, one can check that*

$$K \oplus XL[X] \hookrightarrow L[X]$$

*is an integral extension. The ring $L[X]$ is a principal ideal domain, and is hence Noetherian. But we claim that $K \oplus XL[X]$ is* not *Noetherian. Indeed, the ideal $XL[X]$ is not a finitely generated ideal of $K \oplus XL[X]$. For the sake of contradiction, suppose $f_1, \ldots, f_n \in XL[X]$ generates $XL[X]$ as an ideal of $K \oplus XL[X]$. Then if we write*

$$f_i = a_i X + \textit{terms with higher powers of } X,$$

*for $a_i \in L$, it follows that the coefficient of the $X$ term for any $K \oplus XL[X]$-linear combination of the $f_i$'s lies in the subfield*

$$K(a_1, \ldots, a_n)$$

*of $L$. Since $K(a_1, \ldots, a_n)$ is a finite extension of $K$, it then follows that for any $\ell \in L \setminus K(a_1, \ldots, a_n)$, the element $\ell X \in XL[X]$ is not in the ideal $(f_1, \ldots, f_n)$ of $K \oplus XL[X]$. This contradicts the assumption that $f_1, \ldots, f_n$ generate $XL[X]$.*

Let us next indicate a result that generalizes [Gro66, Thm. 11.4.1]. We first need the following result is due to Bass in [Bas60]

**Lemma 2.7.4.4.** *[RG71, Part II, Lem. 1.2.6] Let $A$ be a ring and $I$ an ideal of $A$. Then the following conditions are equivalent:*

*i. The $A$-module $A/I$ descends Hom-nullity.*

*i'. If $M$ is an $A$-module then $M$ is an essential extension of* $\mathrm{Ann}_M(I)$.

*ii. The $A$-module $A/I$ descends nullity.*

*iii. $I$ is $T$-nilpotent. In other words, for all sequence of elements $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ of $I$, there exists an integer $N \geq 0$ such that $\prod_{i=0}^{N} a_n = 0$.*

*Proof.* ($i' \implies i$) Let $M$ be an $A$-module. Suppose $\mathrm{Hom}_A(A/I, M) = 0$. Note that $\mathrm{Ann}_M(I)$ can be identified with $\mathrm{Hom}_A(A/I, M)$. Thus, since we are assuming that $\mathrm{Hom}_A(A/I, M) \hookrightarrow \mathrm{Hom}_A(A, M)$ is an essential extension, we must have $M \cong \mathrm{Hom}_A(A, M) = 0$ because the only essential extension of 0 is 0. This shows that assuming $i'$, $A/I$ descends nullity.

($i \implies i'$) Assume that $A/I$ descends Hom-nullity as an $A$-module. Let $N$ be a submodule of $M$ such that

$$\mathrm{Ann}_I(M) \cap N = 0.$$

But $\mathrm{Ann}_I(N) = \mathrm{Ann}_I(M) \cap N$. Thus $\mathrm{Ann}_I(N) = 0$. But $\mathrm{Ann}_I(N) \cong \mathrm{Hom}_A(A/I, N)$. Since $A/I$ descends Hom-nullity, we get that $N = 0$. This shows that $\mathrm{Ann}_I(M) \subseteq M$ is essential as desired.

($i \implies ii$) This is the direct result of Lemma 2.7.3.1, part 1.

($ii \implies iii$) Let $(a_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a sequence of elements in $I$, we form the $A$-module $P$ that is the filtered colimit of the following sequence of $A$-linear maps:

$$A_0 \xrightarrow{a_o} A_1 \xrightarrow{a_1} \dots \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} A_{n+2} \dots,$$

where for all $i \in \mathbb{Z}_{\geq 0}$, $A_i = A$ and $a_i \colon A_i \to A_{i+1}$ is given by left-multiplication by $a_i$. We claim that $P/IP$ is 0. Indeed, if $x \in P$, there exists $y \in A_i$ such that $y \mapsto x$. Then $a_i y \in A_{i+1}$ also maps to $x$. Since $A_{i+1} \to P$ is $A$-linear, this gives us that $x \in a_i P \subseteq IP$. Since $x$ was chosen arbitrarily we have $P/IP = 0$. But since $A/I$ descends nullity, and we have $P/IP = P \otimes_A A/I = 0$, we get that $P = 0$. Now consider $1 \in A_0$. Since $P = 0$, we have $1 \mapsto 0$ in $P$. Thus there exists $N \in \mathbb{Z}_{\geq 0}$ such that $\prod_{i=0}^{N} a_i = 0$ as desired.

($iii \implies i'$) Assume for the sake of contradiction that there exists $M$ which is not an essential extension of $\operatorname{Ann}_M(I)$. We claim without loss of generality we can let $\operatorname{Ann}_M(I) = 0$. Indeed, if $M$ is not an essential extension of $\operatorname{Ann}_M(I)$, then $M/\operatorname{Ann}_M(I) \neq 0$ and $M/\operatorname{Ann}_M(I)$ is also not an essential extension of

$$\operatorname{Ann}_{M/\operatorname{Ann}_M(I)}(I) = (0).$$

So we can replace $M$ by $M/\operatorname{Ann}_M(I)$ to assume $\operatorname{Ann}_M(I) = 0$.

Now, pick a non-zero $x_0 \in M$. Then there exists an $a_0 \in I$ such that $a_0 x_0$ is non-zero in $M$. Then one can recursively pick $a_n \in I$ such that $a_n(\prod_{i=0}^{n-1} a_i x_0) \neq 0$. Then, for all $N \in \mathbb{Z}_{\geq 0}$, we must have $\prod_{i=0}^{N} a_i \neq 0$. But this is a clear contradiction of $iii$. □

**Example 2.7.4.5.** *If $A$ is a ring, and $I$ is a ideal of $A$ such that $I^n = 0$, then $I$ is T-nilpotent. Indeed, let $(a_m)_{m \in \mathbb{Z}_{\geq 0}}$ be some elements in $I$. Then $\prod_{m=1}^{n} a_m \in I^n$. But this means that $\prod_{m=0}^{n} a_m = 0$ showing that $I$ is T-nilpotent.*

*Specifically, if $A$ is noetherian, and $I$ is the nilradical of $A$ then $I$ is T-nilpotent.*

**Proposition 2.7.4.6.** *[RG71, Part II, Prop. 1.2.7] Let $A$ be a ring and $I$ a T-nilpotent ideal of $A$. Let $M$ be an $A$-module with a submodule $F$ that descends Hom-left-invertibility for injectives (for example, if $F$ is a finitely generated faithful $A$-module). Then the $A$-module $M \bigoplus A/I$ descends Hom-left-invertibility for injectives.*

*Proof.* Let $u : J \to E$ be an injective envelope and let

$$\operatorname{Hom}_A(M \bigoplus A/I, u) = \operatorname{Hom}_A(M, u) \bigoplus \operatorname{Hom}_A(A/I, u)$$

be left-invertible. We wish to show that $u$ is an isomorphism. Note that both $\mathrm{Hom}_A(M, u)$ and $\mathrm{Hom}_A(A/I, u)$ are left-invertible. But $\mathrm{Hom}_A(A/I, u)$ is an essential extension by Lemma 2.7.3.4, part 4. Thus, $\mathrm{Hom}_A(A/I, u)$ is an isomorphism. We claim that it is enough to show that $\mathrm{Hom}_A(M, u)$ is also an isomorphism.

Indeed, we have a commutative diagram of exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_A(M/F, J) \arrow[r] \arrow[d] & \mathrm{Hom}_A(M, J) \arrow[r] \arrow[d, "\mathrm{Hom}_A(M,u)"] & \mathrm{Hom}_A(F, J) \arrow[d, "\mathrm{Hom}_A(F,u)"] & \\
0 \arrow[r] & \mathrm{Hom}_A(M/F, E) \arrow[r] & \mathrm{Hom}_A(M, E) \arrow[r] & \mathrm{Hom}_A(F, E) \arrow[r] & 0.
\end{tikzcd}$$

By Lemma 2.7.3.3, if $\mathrm{Hom}_A(M, u)$ is an isomorphism, then $\mathrm{Hom}_A(F, u)$ is an isomorphism. Since $F$ descends Hom-left-invertibility for injectives, it follows that $u$ is left-invertible, and hence that $u$ is an isomorphism since it is an essential extension. This implies $M$ descends Hom-left-invertibility for injectives by Lemma 2.7.2.4.

So we are reduced to showing that $\mathrm{Hom}_A(M, u)$ is an isomorphism. Since we already know that $\mathrm{Hom}_A(M, u)$ is left-invertible, it suffices to show that $\mathrm{Hom}_A(M, u)$ is an essential extension. To this end, we first show that $\mathrm{Hom}_A(M/IM, u)$ is an isomorphism. Note the following:

$$\begin{aligned}
\mathrm{Hom}_A(M/IM, J) &\cong \mathrm{Hom}_A(M \otimes_A A/I, J) \\
&\cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(A/I, J)) \\
&\cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(A/I, E)) \\
&\cong \mathrm{Hom}_A(M \otimes_A A/I, E) \\
&\cong \mathrm{Hom}_A(M/IM, E)
\end{aligned}$$

as we established that $\mathrm{Hom}_A(A/I, J) \cong \mathrm{Hom}_A(A/I, E)$. Thus we form the diagram

$$\begin{tikzcd}
\mathrm{Hom}_A(M/IM, J) \arrow[r, "\mathrm{Hom}_A(M/IM,u)"] \arrow[d, "f"] & \mathrm{Hom}_A(M/IM, E) \arrow[d, "g"] \\
\mathrm{Hom}_A(M, J) \arrow[r, "\mathrm{Hom}_A(M,u)"] & \mathrm{Hom}_A(M, E)
\end{tikzcd}$$

where the top row is an isomorphism and the columns are essential extensions by Lemma 2.7.4.4.i' (note that this is the only place where we use that $I$ is a $T$-nilpotent ideal). Then the bottom map $\mathrm{Hom}_A(M, u)$ is an essential extension as well. Indeed, let $N \subseteq \mathrm{Hom}_A(M, E)$ be a non-zero submodule. Then $\mathrm{Hom}_A(M/IM, u)^{-1}(g^{-1}(N)) \neq 0$. Hence, by the commutativity of the above diagram,

$$\mathrm{Hom}_A(M, u)^{-1}(N) \neq 0.$$

This shows that $\mathrm{Hom}_A(M, u)$ is an essential extension, completing the proof. □

The proof of Proposition 2.7.4.6 actually shows the following, which we record as a separate result. However, we will omit the proof.

**Lemma 2.7.4.7.** *Let $A$ be a ring and $I$ be a $T$-nilpotent ideal of $A$. Let $u\colon J \to E$ be an essential extension such that* $\mathrm{Hom}_A(A/I, u)$ *is an isomorphism. Then for any $A$-module $M$,* $\mathrm{Hom}_A(M, u)$ *is also an essential extension.*

We can now generalize [Gro66, Thm. 11.4.1].

**Corollary 2.7.4.8.** *[RG71, Part II, Corrollary 1.2.8] Let $f : A \to B$ be an injective homomorphism. Let $I$ be a $T$-nilpotent ideal of $A$. Let $P$ be an $A$-module such that $B \otimes_A P$ is $B$-flat and such that $P/IP$ is $A/I$-flat. Then $P$ is $A$-flat.*

*Proof.* Note $(B \times A/I) \otimes_A P \cong (B \otimes_A P) \times (A/I \otimes_A P)$ is a flat $B \otimes_A A/I$-module.

We now claim that $B \times A/I$ descends Hom-left-invertibility for injectives as an $A$-module. By Proposition 2.7.4.6, it suffices to show that $B$ has a finitely generated faithful $A$-submodule. But $f : A \to B$ is injective, thus $\operatorname{im}(A)$ is a finitely generated faithful submodule of $B$ as an $A$-module, proving our claim.

Now, since $B \times A/I$ descends Hom-left-invertibility for injectives as an $A$-module, by Proposition 2.7.3.1, part 2 we have that $B \times A/I$ descends flat universal short exactness. Thus by Lemma 2.6.1.2, since the ring map $\phi : A \to B \times A/I$ descends flat universal short exactness, $\phi$ descends flatness. But $(B \times A/I) \otimes_A P$ is $B \times A/I$-flat, and thus $P$ is $A$-flat as desired. $\square$

Our next goal will be to show that when the base ring $A$ is reduced and Noetherian, then the converse of Lemma 2.7.3.1, part 3 holds; see Theorem 2.7.4.10. In order to prove Theorem 2.7.4.10, we will need the following result.

**Lemma 2.7.4.9.** *Suppose* $u\colon J \to E$ *is an injective* $A$*-linear map such that* $\operatorname{Hom}_A(M, u)$ *is left-invertible as a map of* $A$*-modules. Then* $\operatorname{Hom}_{A/I}(M/IM, \operatorname{Hom}_A(A/I, u))$ *is also left-invertible as a map of* $A/I$*-modules.*

*Proof.* First, note that

$$\begin{aligned}\operatorname{Hom}_{A/I}(M/IM, \operatorname{Hom}_A(A/I, \bullet)) &\cong \operatorname{Hom}_A(M/IM \otimes_{A/I} A/I, \bullet)\\ &\cong \operatorname{Hom}_A(M/IM, \bullet).\end{aligned}$$

Therefore, it suffices to show that $\operatorname{Hom}_A(M/IM, u)$ is left-invertible. Consider the diagram

$$\begin{tikzcd} \operatorname{Hom}_A(M/IM, J) \arrow[r, "Hom_A(M/IM,u)"] \arrow[d] & \operatorname{Hom}_A(M/IM, E) \arrow[d] \\ \operatorname{Hom}_A(M, J) \arrow[r, "\operatorname{Hom}_A(M,u)"] & \operatorname{Hom}_A(M, E) \end{tikzcd}$$

where both the columns and rows are injections. Note that the bottom row is left-invertible by assumption. Furthermore, we can identify $\mathrm{Hom}_A(M/IM, J)$ and $\mathrm{Hom}_A(M/IM, E)$ as the morphisms in $\mathrm{Hom}_A(M, J)$ and $\mathrm{Hom}_A(M, E)$ that contain $IM$ in their kernels.

Now, let $\chi : \mathrm{Hom}_A(M, E) \to \mathrm{Hom}_A(M, J)$ be an $A$-linear left-inverse of the map $\mathrm{Hom}_A(M, u)$. Let $\varphi \in \mathrm{Hom}_A(M, E)$ such that $IM \subseteq \ker(\varphi)$. Then for all $a \in I$ we have $a\varphi = \varphi(a\cdot\ ) = 0$, and so, $\chi(a\varphi) = 0$. But $\chi$ is $A$-linear, thus $a\chi(\varphi) = 0$ for all $a \in I$. This means that $\chi(\varphi)\colon M \to J$ contains $IM$ in its kernel as well. Thus, upon making the identifications from the previous paragraph, it follows $\chi$ also induces a left-inverse of $\mathrm{Hom}_A(M/IM, u)$. Moreover, this map is $A/I$-linear because both $\mathrm{Hom}_A(M/IM, J)$ and $\mathrm{Hom}_A(M/IM, E)$ are $A/I$-modules. □

We now prove the partial converse of Lemma 2.7.3.1, part 3.

**Theorem 2.7.4.10.** *[RG71, Part II, Thm. 1.2.9] Let $A$ be a reduced Noetherian ring and $M$ be an $A$-module that descends Hom-nullity. Then $M$ also descends Hom-left-invertibility for injectives.*

*Proof.* By Noetherian induction, we can assume that the statement is true for all quotient rings of $A$ that are distinct from $A$ and are reduced. Let $u : J \to E$ be an injective envelope of $A$-modules such that $\mathrm{Hom}_A(M, u)$ is left-invertible. As before (Lemma 2.7.2.4), it is enough to show that $u$ is an isomorphism, or equivalently, that $J$ is injective.

Since the proof is a bit involved, we provide a road-map for the reader. Using Noetherian induction, we adopt the following strategy:

(1) We first reduce to the case where $A$ is a domain.

(2) We next show that the torsion submodule $J_{\text{tors}}$ of $J$ must be an injective $A$-module by proving that for each non-zero element $s \in A$, $\operatorname{Ann}_J(s)$ is an injective $A/sA$-module.

(3) We then reduce to the case where $J$ itself is torsion-free and consider the submodule

$$J' := \sum_{\varphi \in \operatorname{Hom}_A(M,J)} \operatorname{im}(\varphi)$$

of $J$. Since $M$ descends Hom-nullity, $J' \hookrightarrow J$ is an essential extension.

(4) We show $J'$ is a divisible $A$-module. Since $J'$ is also a torsion-free $A$-module, it follows that $J'$ is an injective $K$-module, where $K$ is the field of fractions of $A$. But $A \to K$ is flat. Thus, $J'$ is $A$-injective, and so, $J = J'$.

Noetherian induction has the following consequence. Let $I$ be a non-zero radical ideal of $A$. Then left-invertibility of $\operatorname{Hom}_A(M,u)$ as a map of $A$-modules implies left-invertibility of $\operatorname{Hom}_{A/I}(M/IM, \operatorname{Hom}_A(A/I,u))$ as a map of $A/I$-modules by Lemma 2.7.4.9. Now $M/IM$ descends Hom-nullity as an $A/I$-module by base change because $M$ does (see Lemma 2.7.1.3). Therefore by Noetherian induction, $M/IM$ descends Hom-left-invertibility for injectives as an $A/I$-module. Since $\operatorname{Hom}_A(A/I,E)$ is an injective $A/I$-module, it then follows that $\operatorname{Hom}_A(A/I,u)$ is left-invertible. Thus,

$$\operatorname{Hom}_A(A/I,u)$$

is an isomorphism of $A/I$-modules because $\operatorname{Hom}_A(A/I,u)$ is an essential extension of $A$-modules Lemma 2.7.3.4 part 4, thus also an essential extension of $A/I$-modules. In particular, $\operatorname{Hom}_A(A/I,J)$ is an injective $A/I$-module.

Suppose first that $A$ is not an integral domain. Let $P$ be a minimal prime of $A$. By the previous paragraph, $\operatorname{Hom}_A(A/P,J)$ is an injective $A/P$-module. Now note the ring map $A \to \prod_{P \in \mathrm{Min}(A)} A/P$ is injective and finite. Indeed, since $A$ is reduced, we have $\bigcap_{P \in \mathrm{Min}(A)} P = 0$ giving injectivity. To see finiteness, recall that if A is a Noetherian ring, then it has only finitely many minimal prime ideals. Thus, $\prod_{P \in \mathrm{Min}(A)} A/P$ descends Hom-left-invertibility for injectives as an $A$-module by Theorem 2.7.3.5. Then

$$\mathrm{Hom}_A\left(\prod_{P \in \mathrm{Min}(A)} A/P, J\right) = \prod_{P \in \mathrm{Min}(A)} \mathrm{Hom}_A(A/P, J)$$

is an injective $\prod_{P \in \mathrm{Min}(A)} A/P$-module by the previous paragraph. Then $J$ is an injective $A$-module by Lemma 2.7.2.3, which is what we want.

The upshot is that we can reduce to the case where $A$ is an integral domain. We then first check that the torsion submodule of $J$ is injective as an $A$-module. But the torsion submodule of $J$ is the union

$$\bigcup_{0 \neq s \in A} \mathrm{Ann}_J(s).$$

Note that this union is filtered because for $s, t \neq 0$ one has

$$\mathrm{Ann}_J(s), \mathrm{Ann}_J(t) \subseteq \mathrm{Ann}_J(st).$$

Also note that $\mathrm{Ann}_J(s)$ is an $A/sA$-module. We claim that in order to show that the torsion submodule of $J$ is $A$-injective, it suffices to show that each $\mathrm{Ann}_J(s)$ is an injective $A/sA$-module for $s \neq 0$. We will use Baer's criterion of injectivity [Sta24, Tag 05NU] for this, that is, we will show that for any ideal $I$ of $A$, an $A$-linear map

$$\varphi \colon I \to \bigcup_{0 \neq s \in A} \mathrm{Ann}_J(s)$$

extends to an $A$-linear map

$$A \to \bigcup_{0 \neq s \in A} \mathrm{Ann}_J(s).$$

We may of course assume that $I \neq 0$. Since $A$ is Noetherian, $I$ is a finitely generated ideal. Thus, $\operatorname{im}(\varphi)$ must be contained in $\operatorname{Ann}_J(t)$, for some $t \neq 0$ because the union is filtered. Furthermore, since $I \neq 0$, upon choosing a non-zero $i \in I$ and using the fact that $\operatorname{Ann}_J(t) \subseteq \operatorname{Ann}_J(it)$, we may assume that $t \in I$. Let $\tilde{\varphi}\colon I \to \operatorname{Ann}_J(t)$ be the map obtained by restricting the codomain of $\varphi$. Then we get an induced map

$$\tilde{\varphi} \otimes_A A/tA\colon I/tA \to \operatorname{Ann}_J(t).$$

Since $\operatorname{Ann}_J(t)$ is $A/tA$-injective, $\tilde{\varphi} \otimes_A A/tA$ extends to an $A/tA$-linear map

$$A/tA \to \operatorname{Ann}_J(t).$$

One can then verify that the composition

$$A \twoheadrightarrow A/tA \to \operatorname{Ann}_J(t)$$

is an $A$-linear map that extends $\tilde{\varphi}$, and so,

$$A \to \operatorname{Ann}_J(t) \hookrightarrow \bigcup_{0 \neq s \in A} \operatorname{Ann}_J(s)$$

is an extension of $\varphi$. Thus, $\bigcup_{0 \neq s \in A} \operatorname{Ann}_J(s)$ is $A$-injective if each $\operatorname{Ann}_J(s)$ is $A/sA$-injective for $0 \neq s \in A$, proving our claim.

Since $\operatorname{Ann}_J(s) \cong \operatorname{Hom}_A(A/sA, J)$, it is enough to check that for any non-zero $s \in A$, $\operatorname{Hom}_A(A/sA, J)$ is an injective $A/sA$-module. Recall that $M$ descends Hom-nullity, and so, $\operatorname{Hom}_A(M, A) \neq 0$. Thus, there exists a non-zero linear form on $M$, say $f$. Let $F = f(M)$. Note that $F$ is a non-zero ideal of $A$. Choose a non-zero element $t \in F$ and let $I = \operatorname{Ann}_A(F/stF)$. We claim that $I \subseteq sA$. Indeed, suppose $i \in I$. Then $it \in stF$, and so, $i \in sF \subseteq sA$ since $A$ is a domain and $t$ is a non-zero element. Note that $I \neq 0$ since $st \in I$.

Note that $F/IF$ is a finitely generated faithful $A/I$-module. Also, since $M \twoheadrightarrow F$, we get an $A/I$-linear surjection

$$M/IM \twoheadrightarrow F/IF.$$

Let $x_1, \ldots, x_n \in M/IM$ be preimages of generators of $F/IF$ as an $A/I$-module and $N$ be the $A/I$-submodule of $M/IM$ generated by $x_1, \ldots, x_n$. Since $\operatorname{Ann}_{A/I}(N) \subseteq \operatorname{Ann}_{A/I}(F/IF) = 0$, it follows that $N$ is a finitely generated faithful submodule of $M/IM$. Note that $\sqrt{I}/I$ is a $T$-nilpotent ideal of $A/I$ (because $\sqrt{I}/I$ is nilpotent). Thus, by Proposition 2.7.4.6

$$M/IM \oplus A/\sqrt{I}$$

descends Hom-left-invertibility for injectives as an $A/I$-module.

We claim that $\operatorname{Hom}_A(A/I, J)$ is an injective $A/I$-module. Indeed, this would follow if we can show that $\operatorname{Hom}_A(A/I, u)\colon \operatorname{Hom}_A(A/I, J) \to \operatorname{Hom}_A(A/I, E)$ is left-invertible as a map of $A/I$-modules because $\operatorname{Hom}_A(A/I, E)$ is $A/I$-injective. For this, it suffices to show by the previous paragraph that

$$\operatorname{Hom}_{A/I}(M/IM \oplus A/\sqrt{I}, \operatorname{Hom}_A(A/I, u))$$

is left-invertible as a map of $A/I$-modules.

But

$$\begin{aligned}&\operatorname{Hom}_{A/I}(M/IM \oplus A/\sqrt{I}, \operatorname{Hom}_A(A/I, u)) = \\ &\operatorname{Hom}_{A/I}(M/IM, \operatorname{Hom}_A(A/I, u)) \oplus \operatorname{Hom}_{A/I}(A/\sqrt{I}, \operatorname{Hom}_A(A/I, u)).\end{aligned}$$

So we show $\operatorname{Hom}_{A/I}(M/IM, \operatorname{Hom}_A(A/I, u))$ and $\operatorname{Hom}_{A/I}(A/\sqrt{I}, \operatorname{Hom}_A(A/I, u))$ are both left-invertible. The left-invertibility of the first map follows by Lemma 2.7.4.8

since $\mathrm{Hom}_A(M, u)$ is left-invertible by assumption. For the second map, by adjunction, we can identify

$$\mathrm{Hom}_{A/I}(A/\sqrt{I}, \mathrm{Hom}_A(A/I, u))$$

with

$$\mathrm{Hom}_A(A/\sqrt{I}, u).$$

This last map is an isomorphism by the argument given in the second paragraph of the proof of this Theorem.

Thus, $\mathrm{Hom}_A(A/I, J)$ is an injective $A/I$-module. Note that $I \subseteq sA$, that is, $A/sA$ is a further quotient of $A/I$. Thus, $\mathrm{Hom}_{A/I}(A/sA, \mathrm{Hom}_A(A/I, J)) = \mathrm{Hom}_A(A/sA, J)$ is an injective $A/sA$-module, which is what we needed in order to show that $J_{\mathrm{tors}}$, the torsion submodule of $J$, is an injective $A$-module.

Then the short exact sequence

$$0 \to J_{\mathrm{tors}} \to J \to J/J_{\mathrm{tors}} \to 0$$

splits. Thus, there exists a torsion-free submodule $\tilde{J}$ of $J$ such that $J = J_{\mathrm{tors}} \oplus \tilde{J}$. Then $E$, which is the injective hull of $J$, decomposes as

$$E \cong J_{\mathrm{tors}} \oplus E(\tilde{J}),$$

where $E(\tilde{J})$ is the injective hull of $\tilde{J}$ by Lemma A.3.0.1. Let $\tilde{u}\colon \tilde{J} \to E(\tilde{J})$ be the corresponding extension. Left-invertibility of $\mathrm{Hom}_A(M, u)$ implies left-invertibility of $\mathrm{Hom}_A(M, \tilde{u})$. Thus, if we can show that $\tilde{u}$ is left-invertible, we would be able to conclude that $\tilde{J}$, and hence $J$, are $A$-injective. The upshot is that we can reduce to the case where $J$ is a torsion-free $A$-module.

Let $K$ denote the field of fractions of $A$. Note $E$ can be viewed as a $K$-module by A.3.0.2. Thus $\mathrm{Hom}_A(M, E)$ and its direct summand $\mathrm{Hom}_A(M, J)$ can be viewed as a $K$-module. To see the latter, note that since $\mathrm{Hom}_A(M, J)$ is a submodule of $\mathrm{Hom}_A(M, E)$ and multiplication by a nonzero element $s \in A$ on $\mathrm{Hom}_A(M, E)$ is an isomorphism, multiplication by $s$ on the submodule $\mathrm{Hom}_A(M, J)$ is injective. But, $\mathrm{Hom}_A(M, J)$ is also a quotient of $\mathrm{Hom}_A(M, E)$. Thus, multiplication by $s$ is also surjective on $\mathrm{Hom}_A(M, J)$. In other words, for all non-zero $s \in A$, $s\cdot\colon \mathrm{Hom}_A(M, J) \to \mathrm{Hom}_A(M, J)$ is an isomorphism, and so, $\mathrm{Hom}_A(M, J)$ is naturally a $K$-module.

Now let $J'$ be the $J$ submodule

$$J' := \sum_{u \in \mathrm{Hom}_A(M,J)} \mathrm{im}(u)$$

Since $M$ descends Hom-nullity, we have by Lemma 2.7.1.6 that $J'$ is an essential submodule of $J$.

We claim that since $\mathrm{Hom}_A(M, J)$ is a $K$-module, $J'$ is divisible. Indeed, let $j \in J'$. Then $j = \sum_{\lambda \in \Lambda} j_\lambda$ such that $j_\lambda \in \mathrm{im}(g_\lambda)$ for some $g_\lambda \in \mathrm{Hom}_A(M, J)$, where $\Lambda$ is a finite set. But for all $a \neq 0$ we have $\frac{1}{a} g_\lambda \in \mathrm{Hom}_A(M, J)$. Thus, we can write $j = \sum_{\lambda \in \Lambda} a j'_\lambda$, where $a j'_\lambda = j_\lambda$ and $j'_\lambda \in \mathrm{im}(\frac{1}{a} g_\lambda)$. Hence $J'$ is a divisible $A$-module, as desired.

Recall, we reduced to the case where $J$ is a torsion-free $A$-module. Thus, $J'$ is also a torsion-free $A$-module. As such, $J'$ is a divisible torsion-free $A$-module and thus by Lemma A.1.0.19 $J'$ is an injective $A$-module. Thus $J' = J$ since $J' \to J$ is essential and we win. $\square$

Our next goal will be to show that if an injective ring homomorphism $A \hookrightarrow B$ descends nullity when $A$ is Noetherian, then $A \hookrightarrow B$ also descends flatness.

**Corollary 2.7.4.11.** *[RG71, Part II, Cor. 1.2.10] Let $A$ be a Noetherian ring and $f : A \to B$ an injective homomorphism that descends nullity. Then $f$ descends flatness.*

*Proof.* By Lemma 2.1.0.2 we can assume $A$ is local. Since faithfully flat maps descend nullity, by Lemma 2.1.0.3 we can also additionally assume that $A$ is complete. Let $A_{\mathrm{red}}$ be the quotient of $A$ by its nilradical. Let $P$ be an $A$-module such that $B \otimes_A P$ is $B$-flat. We claim that $A_{\mathrm{red}} \otimes_A P$ is $A_{\mathrm{red}}$-flat. Indeed, since $f : A \to B$ descends nullity, we have $\mathrm{id}_{A_{\mathrm{red}}} \otimes_A f \colon A_{\mathrm{red}} \to A_{\mathrm{red}} \otimes_A B$ descends nullity by base change (Lemma 2.2.0.2). Thus, $\mathrm{id}_{A_{\mathrm{red}}} \otimes_A f \colon A_{\mathrm{red}} \to A_{\mathrm{red}} \otimes_A B$ descends Hom-nullity since $A_{\mathrm{red}}$ is still Noetherian complete local (Lemma 2.7.3.1 part 1). Since $A_{\mathrm{red}}$ is reduced, we then apply Theorem 2.7.4.10, Lemma 2.7.3.1 part 2 and Lemma 2.6.1.2 to conclude that $\mathrm{id}_{A_{\mathrm{red}}} \otimes_A f : A_{\mathrm{red}} \to A_{\mathrm{red}} \otimes_A B$ descends flatness. Since $B \otimes_A P$ is $B$-flat, we have

$$(A_{\mathrm{red}} \otimes_A B) \otimes_B (B \otimes_A P) = A_{\mathrm{red}} \otimes_A (B \otimes_A P) = (A_{\mathrm{red}} \otimes_A B) \otimes_{A_{\mathrm{red}}} (A_{\mathrm{red}} \otimes_A P)$$

is $A_{\mathrm{red}} \otimes_A B$-flat by base change. Thus $A_{\mathrm{red}} \otimes_A P$ is $A_{\mathrm{red}}$-flat. Now note that the nilradical of $A$ is a $T$-nilpotent ideal since $A$ is Noetherian by Example 2.7.4.5. Thus, by Corollary 2.7.4.8, we have that $P$ is a flat $A$-module. $\square$

## Chapter 3. The Direct Summand Theorem

The direct summand theorem was raised as a conjecture by Melvin Hochster in [Hoc75], where he showed that the astonishing relationship of this conjecture to many other homological conjectures in commutative algebra. The statement is as follows:

**Conjecture 3.0.0.1.** *[Hoc75] Let $A$ be a Noetherian regular ring and $\varphi : A \to B$ be a finite injective ring homomorphism. Then $A$ is a direct summand of $B$.*

Hochster proved in [Hoc73] that Conjecture 3.0.0.1 is true when $A$ is a regular ring containing a field (the equal characteristic case). More than four decades later, Yves André proved in [And18] that Conjecture 3.0.0.1 is true when $A$ contains $\mathbb{Z}$ but not necessarily a field (the mixed characteristic case), thereby settling the conjecture in full generality.

In [Hoc83], Hochster provides 8 equivalent statements to this conjecture. We black box the following equivalence:

**Theorem 3.0.0.2** (Theroem 6.1)**.** *[Hoc83] The following is equivalent to the direct summand conjecture: Let $A$ be a complete unramified regular local ring and $B$ be an integral closure of $A$ in an algebraic closure of the fraction field of $A$, that is, $B$ is an absolute integral closure of $A$. Then $\operatorname{Hom}_A(B, A) \neq 0$.*

It turns out that because of the equivalent reformulation above, Conjecture 3.0.0.1 is related intimately with descent properties discussed in the previous chapter. Takeo Ohi proves the following in [Ohi96]:

**Theorem 3.0.0.3.** *The following assertions are equivalent:*

1. *Let $R$ be a complete unramified regular local ring. Then for any integral extension $R \hookrightarrow B$, $\operatorname{Hom}_R(B, R) \neq 0$.*

*2. If $A$ is a Noetherian ring and $\varphi\colon A \hookrightarrow B$ is an integral extension, then $\varphi$ descends flatness.*

*3. If $A$ is a Noetherian ring and $\varphi\colon A \hookrightarrow B$ is an integral extension, then $\varphi$ descends nullity.*

Our goal in this Chapter is to prove this Theorem.

**Remark 3.0.0.4.** *The equivalence of* 1. *and* 2. *in Theorem* 3.0.0.3 *is due to [Ohi96, Theorem]. But Ohi really only uses the equivalence of* 1. *and* 3. *in his paper. Note that we already have that* 2. *and* 3. *are equivalent by Proposition* 2.2.0.4 *and Corollary* 2.7.4.11*. So our main goal here will be to directly show that* 1. *and* 3. *are equivalent.*

We will first need some preparatory results.

**Lemma 3.0.0.5.** *Let $A$ be a Noetherian integral domain. Let $M \in Mod_A$ be such that* $\mathrm{Hom}_A(M, A) \neq 0$*. Then $M$ descends Hom-nullity. In particular, $M$ descends nullity.*

*Proof.* The second assertion follows from the first by Lemma 2.7.3.1, part 1. Thus, it suffices to show that $M$ descends Hom-nullity. Let $\varphi : M \to A$ be a non-zero linear form. Let $I \coloneqq \mathrm{im}(\varphi)$. Then $I$ is a non-zero ideal of the Noetherian integral domain $A$. In particular, $I$ is finitely generated and faithful. Thus $I$ descends Hom-left invertibility for injectives as an $A$-module by 2.7.3.5. Since $I \neq 0$, it descends Hom-nullity by 2.7.3.1 part 3. But by construction, $M \to I$ is a surjection. Thus by Lemma 2.7.1.5 we have $M$ descends Hom-nullity. □

**Proposition 3.0.0.6.** *Let $A$ be a complete Noetherian local domain, $M$ be an $A$-module and $E$ be the injective hull of the residue field of $A$. Then the following are equivalent:*

1. *$M$ descends Hom-nullity.*

2. *$M$ descends nullity.*

3. *$M \otimes_A E \neq 0$.*

4. *$\mathrm{Hom}_A(M, A) \neq 0$.*

*Proof.* We have already seen that 1. and 2. are equivalent in Lemma 2.7.3.1 part 1. Note this equivalence does not need $A$ to be a domain.

Note $E \neq 0$. Thus assuming 2., we have $M \otimes_A E \neq 0$, that is 2. $\implies$ 3..

Now assume 3. holds. Note that by A.3.0.14 we have that

$$\mathrm{Hom}_A(M \otimes_A E, E) \neq 0.$$

Thus, by Hom-tensor adjunction, we have $\mathrm{Hom}_A(M, \mathrm{Hom}_A(E, E)) \neq 0$. But since $A$ is complete and local, we have $\mathrm{Hom}_A(E, E) \cong A$ by Lemma A.3.0.12. Thus, we have $\mathrm{Hom}_A(M, A) \neq 0$, and so, 3. $\implies$ 4..

It remains to show 4. $\implies$ 1.. But this follows by Lemma 3.0.0.5. Also, this is the only implication where the assumption that $A$ is a domain is used. $\square$

**Proposition 3.0.0.7.** *Let $\varphi : A \to B$ be a finite ring extension. Then $\varphi$ descends nullity.*

*Proof.* For a general $A$, this follows by Corollary 2.7.4.1 and Proposition 2.2.0.4.

However, when $A$ is a Noetherian domain (which is the setting in which we will need to apply this result), we can give a much simpler proof of this assertion via Lemma 3.0.0.5. Indeed, it is enough to show that $\mathrm{Hom}_A(B, A) \neq 0$. Since $A$ is Noetherian, $B$ is a finitely presented $A$-module. Let $K$ be the fraction field of $A$.

Then $A \to K$ is flat, and so,

$$K \otimes_A \operatorname{Hom}_A(B, A) \cong \operatorname{Hom}_K(K \otimes_A B, K) \neq 0$$

by [Mat86, Theorem 7.11]. Note $\operatorname{Hom}_K(K \otimes_A B, K) \neq 0$ because $K \hookrightarrow K \otimes_A B$ by flat base change, and so, $K \otimes_A B \neq 0$. Thus, $\operatorname{Hom}_A(B, A) \neq 0$ as well. $\square$

**Lemma 3.0.0.8.** *Let $\eta : (A, \mathfrak{m}) \to (B, \mathfrak{n})$ be a finite local extension of Noetherian local domains. Suppose $A$ is $\mathfrak{m}$-adically complete and $B$ is $\mathfrak{n}$-adically complete. Let $M$ be a $B$-module. Then $M$ descends nullity as an $A$-module if and only if $M$ descends nullity as a $B$-module.*

*Proof.* By Corollary 2.7.4.1 $(A, \mathfrak{m}) \to (B, \mathfrak{n})$ descends flatness and thus descends nullity. If $M$ descends nullity as a $B$-module, then $M$ descends nullity by Lemma 2.2.0.6.

Assume $M$ descends nullity as an $A$-module. Let $E$ be the injective hull of the residue field of $A$. Then $M \otimes_A E \neq 0$ and since the Matlis functor is faithful, we have

$$\operatorname{Hom}_A(M \otimes_A E, E) \cong \operatorname{Hom}_A(M, A) \neq 0$$

as $A$ is complete. We claim there exists a $B$-module $N$ such that $M \to N$ is a surjection and $N$ descends nullity as a $B$ module. Indeed, let $f : M \to A$ be a non-zero linear form on $M$.

We define a new map $\tilde{f} : M \to \operatorname{Hom}_A(B, A)$ defined by $m \mapsto \phi_m : B \to A$ where $\phi_m(b) \coloneqq f(bm)$. We show that $\tilde{f}$ is $B$-linear. Indeed, consider $\tilde{f}(m+n)$. Then this maps to $\phi_{m+n}$. But $\phi_{m+n}(b) = f(b(m+n)) = f(bm) + f(bn)$ as $f$ is $A$-linear. Now consider $\tilde{f}(b_1 m)$. This maps to $\phi_{b_1 m}$. We claim $\phi_{b_1 m} = b_1 \phi_m$. But $b_1 \phi_m$ is defined by premultiplication by $b_1$. That is $b_1\phi_m = \phi_m \circ b_1$. Thus it is clear that $b_1\phi_m = \phi_{b_1 m}$ as desired.

We next claim that since $f \neq 0$ we have $\tilde{f} \neq 0$. Indeed, suppose $f(m) \neq 0$. Then $\phi_m = \tilde{f}(m)$ is a non-zero map from $B \to A$ as $\tilde{f}(m)(1) = \phi_m(1) = f(m) \neq 0$.

Next we note that $\mathrm{Hom}_A(B, A)$ is finitely generated as a $B$-module. Indeed, we start by showing that $\mathrm{Hom}_A(B, A)$ is finitely generated as an $A$-module. Let $b_1, \dots, b_n$ be a generating set for $B$. Define $\varphi_i : B \to A$ by $b_i \mapsto 1$ and $b_j \mapsto 0$ for all $j \neq i$. We claim this is a generating set of $\mathrm{Hom}_A(B, A)$ as an $A$-module. Indeed, let $\psi \in \mathrm{Hom}_A(B, A)$. Say $\psi(b_i) = a_i$. Then $\psi = \sum_i^n a_i\varphi_i$. Now note, since $A \cong \eta(A) \subseteq B$, we have that $\varphi_1 \dots \varphi_n$ generated $\mathrm{Hom}_A(B, A)$ as a $B$-module too, giving the claim.

Now, we claim that $\mathrm{Hom}_A(B, A)$ is torsion free. Indeed, let $b$ be a non-zero element in $B$ and $\varphi \in \mathrm{Hom}_A(B, A)$. Suppose $b \cdot \varphi = 0$. That is $\varphi(b \cdot \bullet) = 0$. Since $A \to B$ is an integral extension of domains, there exists a non-zero $b' \in B$ such that $bb' \in A$. Indeed, since $b$ is integral over $A$ there exists a polynomial $p(x)$ with coefficients in $A$ such that $p(b) = 0$. Then we have

$$a_n b^n + a_{n-1} b^{n-1} + \dots + a_0 = 0.$$

Solving for $a_0$ gives

$$a_0 = -(a_n b^n + \dots + a_1 b)$$

Factoring out a $b$ gives

$$a_0 = -b(a_n b^{n-1} + \dots + a_1)$$

Thus $b(a_n b^{n-1} + \dots + a_1) \in A$. But $(a_n b^{n-1} + \dots + a_1) = b' \in B$.

Now since $b\varphi = 0$ we have $b'b\varphi = a_0\varphi = 0$. But this means that $\varphi = 0$ as $A$ is a domain.

Thus $\mathrm{im}(\tilde{f}) = N$ is a finitely generated faithful $B$-modules. Hence by Theorem 2.7.3.5 $N$ descends Hom-left invertibility for injectives. Thus by Lemma 2.7.3.1 part 3 N descends Hom-nullity. Furthermore by Lemma 2.7.3.1 part 1, N descends nullity. Now, consider the exact sequence

$$M \to N \to 0.$$

If $P$ is a $B$-module such that $M \otimes_B P = 0$, then we have

$$M \otimes_B \to N \otimes_B P \to 0$$

and $N \otimes_B P = 0$. But since $N$ descends nullity as a $B$-module, $M$ descends nullity as a $B$-module as desired.

□

**Lemma 3.0.0.9.** *Let $\varphi : A \to B$ be an integral extension where $A$ is Noetherian. Suppose for all prime ideals $P$ of $A$, the induced map*

$$\mathrm{id}_{A/P} \otimes_A \varphi : A/P \to B/PB$$

*descends nullity. Then $\varphi$ descends nullity.*

*Proof.* Let $M$ be an $A$-module such that $B \otimes_A M = 0$. Then for all prime ideals $P$ of $A$ we have that $A/P \otimes_A (B \otimes_A M) = 0$. But $A/P \otimes_A (B \otimes_A M) \cong B/PB \otimes_{A/P} M/PM$. Thus for all prime ideals $M/PM = 0$. That is $M = PM$ for all prime ideals. Then $M = 0$ by Lemma A.1.0.13 □

Note that $A/P \to B/PB$ is an integral extension by Lemma A.1.0.16. We now show the equivalence of statement 1. and 3. in Theorem 3.0.0.3

*Proof of Theorem 3.0.3* We show that $\varphi$ desending nullity implies the condition in Theorem 3.0.0.2. Indeed, let $(A, \mathfrak{m}, k)$ be a complete local ring and $E$ the injective hull of $k$. Let $B$ be the integral closure of $A$ in an algebraic closure of its fraction field. Note that the $B$-module $B \otimes_A E = 0$ if and only if the $A$-module $E = 0$. But $k \neq 0$ and since $k \subseteq E$ we have $E \neq 0$. Thus $B \otimes_A E \neq 0$. Since the Matlis functor is faithful, we get

$$\mathrm{Hom}_A(B \otimes_A E, E) = \mathrm{Hom}_A(B, A) \neq 0.$$

This is precisely the condition in 3.0.0.2, giving us the direct summand conjecture.

Now, assume that 1. holds. We can without loss of generality let $A$ be a complete local Noetherian domain by Lemma 2.2.0.2, Lemma 2.2.0.3, and Lemma 3.0.0.9. By Cohen-structure Theorem, there exists $A_0 \subseteq A$ such that $A_0 \to A$ is finite and $A_0$ is either isomorphic to $k[\![x_1, \ldots, x_d]\!]$ where $k$ is a field or $\Lambda[\![x_1, \ldots, x_d]\!]$ where $\Lambda$ is a complete discrete valuation ring. These rings are both unramified. Thus by Lemma 3.0.0.8 without loss of generality we can assume that $A$ is an unramified regular complete local Noetherian ring. By Theorem 3.0.0.2 we thus have $\mathrm{Hom}_A(B, A) \neq 0$. We thus win by Lemma 3.0.0.5. $\square$

# Chapter A. Appendix
## A.1. Properties of Modules

This part of the appendix contains proofs of well known Module properties used in the paper.

**Lemma A.1.0.1.** *Let $A$ be a ring, $I$ an ideal, and $M$ an $A$-module. Then $A/I \otimes_A M \cong M/IM$.*

*Proof.* Consider the exact sequence

$$0 \to I \to A \to A/I \to 0.$$

If we tensor this sequence by $M$, we have

$$I \otimes_A M \to M \to A/I \otimes_A M \to 0.$$

That is $A/I \otimes_A M$ is the cokernel to $I \otimes_A M \to M$. But, the natural image of $I \otimes_A M \to M$ is $IM$. Thus the cokernel can also be written as $M/IM$. As such, $M/IM \cong A/I \otimes_A M$ as desired. □

**Lemma A.1.0.2.** *Let $A$ be a ring and $I$ a principal ideal of $A$. If $A/I$ is $A$-flat then $I$ is generated by an idempotent.*

*Proof.* Let $I = (x)$ and $A/I$ be a flat $A$-module. We consider the sequence of A modules

$$0 \longrightarrow (x) \longrightarrow A \longrightarrow A/(x) \longrightarrow 0.$$

Since $A/(x)$ is flat we have that

$$0 \longrightarrow (x) \otimes_A A/(x) \longrightarrow A \otimes_A A/(x) \longrightarrow A/(x) \otimes_A A/(x) \longrightarrow 0$$

is an exact sequence. But $(x) \otimes_A A/(x) \cong (x)/(x^2)$. Furthermore, by Lemma A.1.0.4, $A/(x) \cong A/(x) \otimes_A A/(x)$. Thus $(x)/(x^2) = 0$ and $(x) = (x^2)$. Thus, there exists $a \in A$ such that $x = ax^2$. We claim that $I = (ax)$ and that $ax$ is idempotent. Indeed, clearly $ax \in (x) = I$. Also, $x = x(ax) \in (ax)$. Thus $(ax) = (x) = I$. Now to see that $ax$ is idempotent, observe $(ax)^2 = a(ax^2) = ax$. $\square$

**Lemma A.1.0.3.** *Let $A \to B$ be a flat ring map and $M$ is a flat $B$-module. Then by restriction of scalars $M$ is also a flat $A$-module.*

*Proof.* Let $E \to E'$ be an $A$-linear injection. It suffices to show that

$$E \otimes_A M \to E' \otimes_A M$$

is an $A$-linear injection. Note that since $B$ is flat as an $A$-module, we have $E \otimes_A B \to E' \otimes_A B$ is an $A$-linear injection. By base change, it is also a $B$-linear injection. Thus, since $M$ is $B$-flat, we have $E \otimes_A B \otimes_B M \to E' \otimes_A B \otimes_B M$ is a $B$-linear injection. But this means that $E \otimes_A M \to E' \otimes_A M$ is a $B$-linear injection. By restriction of scalars, this gives us that $E \otimes_A M \to E' \otimes_A M$ is an $A$-linear injection as desired. $\square$

**Lemma A.1.0.4.** *If $I$ is an ideal of a ring $A$, then for $M \in \mathrm{Mod}_{A/I}$, we have $M \otimes_A A/I = M$.*

*Proof.* Note that $M \otimes_A A/I = M/IM$ by Lemma A.1.0.1. However, $IM = 0$ as $M$ is an $A/I$-module. Thus we have $M \otimes_A A/I = M/IM = M$ as desired. $\square$

Two properties that we hope to descend with our maps are finite presentation and finite generation. Recall that a module is said to be finitely generated as an $A$-module if there exists an exact sequence

$$A^n \to M \to 0$$

and is said to be finitely presented as an $A$-module if there exists an exact sequence

$$A^m \to A^n \to M \to 0.$$

These modules have implication for other exact sequences.

**Lemma A.1.0.5.** *Let*

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

*be an exact sequence of $A$-modules. If $M'$ and $M''$ are finitely generated then the same holds for $M$.*

*Proof.* Let $p' : A^m \to M'$ and $p'' : A^n \to M''$ be A-module epimorphisms. There exists a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A^m & \xrightarrow{f*} & A^{m+n} & \xrightarrow{g*} & A^n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle p'} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p''} & & \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' & \longrightarrow & 0 \end{array}$$

with exact rows. Thus, by snakes lemma. we have an exact sequence

$$\ldots \longrightarrow \mathrm{coker}(p') \longrightarrow \mathrm{coker}(p) \longrightarrow \mathrm{coker}(p'') \longrightarrow 0$$

But $\mathrm{coker}(p')$ and $\mathrm{coker}(p'')$ are both trivial. Thus so must be $\mathrm{coker}(p)$. As such $p$ is surjective and $M$ is finitely generated as desired. □

The following is a useful way to categorize finitely presented module.

**Lemma A.1.0.6.** *An $A$-module $M$ is finitely presented if and only if $M$ is finitely generated and every epimorphism of type $A^n \to M$ has a finitely generated kernel.*

The following proof was adopted from [Bos22]

*Proof.* The backwards direction is clear, as given a finitely generated kernel, we would have a sequence of type $A^m \to A^n \to M \to 0$, which means $M$ is finitely presented.

For the forwards direction, let

$$A^m \longrightarrow A^n \longrightarrow M \longrightarrow 0$$

be a finite presentation of M and let $\varphi : A^{n'} \to M$ be an epimorphism. In order to show that $\ker(\varphi)$ is of finite type, consider the sequence

$$0 \longrightarrow \ker(\varphi) \longrightarrow A^{n'} \longrightarrow M \longrightarrow 0.$$

We want to combine these sequences as follows

$$\begin{tikzcd}
 & A^m \arrow[r, "f_1"] \arrow[d, "u_1"] & A^n \arrow[r, "f^2"] \arrow[d, "u_2"] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker(\varphi) \arrow[r, "i"] & A^{n'} \arrow[r, "\varphi"] & M \arrow[r] & 0.
\end{tikzcd}$$

Let $e_1, \dots e_n$ be the standard generating set of $A^n$. Since $\varphi$ is surjective, there exists at least one set $a_1, \dots a_n$ in $A^{n'}$ such that $\varphi(a_i) = f_2(e_i)$. Choose one such set and define $u_2$ such that $e_i \mapsto a_i$. This makes the left hand square commutative. Thus $\varphi \circ u_2 \circ f_1 = f_2 \circ f_2 = 0$. It follows that $\mathrm{im}(u_2 \circ f_1)$ is contained in $\ker(\varphi) = \mathrm{im}(i)$. Thus, we can define $u_1$ by restricting the range of $u_2 \circ f_1$. Now, using snakes lemma, since $i$ is injective and $f_2$ is surjective, we have an exact sequence

$$0 = \ker(id_M) \to \mathrm{coker}(u_1) \to \mathrm{coker}(u_2) \to \mathrm{coker}(id_M) = 0.$$

As such $\mathrm{coker}(u_1) \cong \mathrm{coker}(u_2)$. But $\mathrm{coker}(u_2)$ is finitely generated, and thus so is $\mathrm{coker}(u_1)$. As such, we have the sequence

$$0 \to \mathrm{im}(u_1) \to \ker(\varphi) \to \mathrm{coker}(u_1) \to 0$$

where $\mathrm{im}(u_1)$ and $\mathrm{coker}(u_1)$ are finitely generated. Thus, by Lemma A.1.0.5, so must be $\ker(\varphi)$ as desired. $\square$

**Lemma A.1.0.7.** *Let $A$ be a ring and* $0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$ *be a short exact sequence of $A$ modules. Then the following are equivalent*

1. *There exists an $A-$module morphism $h : M'' \to M$ such that $gh = id_{M''}$;*
2. *There exists an $A-$module morphism $k : M \to M'$ such that $kf = id_{M'}$;*
3. *The given sequence is isomorphic to the sequence*

$$0 \longrightarrow M' \xrightarrow{i} M' \oplus M'' \xrightarrow{\pi} M'' \longrightarrow 0$$

*where $i$ is the canoncial inclusion map and $\pi$ is the canonical projectition map.*

The following is a proof adapted from [Hun80]

*Proof.* ($1 \implies 3$) Using morphisms $f$ and $h$, we have an induced map $\varphi : M' \oplus M'' \to M$ defined by $(a_1, a_2) \mapsto f(a_1) + h(a_2)$. This means, the following diagram is commutative

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, equal] & M' \oplus M'' \arrow[r, "\pi"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0.
\end{tikzcd}$$

Thus, by the Short fives Lemma, $\varphi$ is an isomorphism.

($2 \implies 3$) Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, equal] & M \arrow[r, "g"] \arrow[d, "\psi"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, "i"] & M' \oplus M'' \arrow[r, "\pi"] & M'' \arrow[r] & 0
\end{tikzcd}$$

where $\psi$ is defined by $\psi(m) = (k(m), g(m))$. This makes the diagram commutative and $\psi$ an isomorphism by the Short Five Lemma.

(3 $\implies$ 1 and 2) Given a commutative diagram with exact rows and $\varphi$ an isomorphism

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r, "i"] \arrow[d, equal] & M' \oplus M'' \arrow[r, "\pi"] \arrow[d, "\varphi"] & M'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

we note that $i$ has a left inverse map in the canonical projection map $\pi_1 : M' \oplus M'' \to M'$ and $\pi$ has a right inverse map in the canonical inclusion map $i_2 : M'' \to M' \oplus M''$. Thus, from we can observes that $\pi_1 \circ \varphi^{-1} \circ f = id_{M'}$ and $g \circ \varphi \circ i_2 = id_{M''}$ and thus we win. $\square$

**Lemma A.1.0.8.** *Let $A$ be a ring and $M$ an $A$-module. Then* $\mathrm{Hom}_A(A/I, M) \cong \mathrm{Ann}_M(I)$.

*Proof.* Note that by identifying $\mathrm{Hom}_A(A/I, M)$ as the induced maps of $\mathrm{Hom}_A(A, M)$ we have

$$\begin{aligned}
\mathrm{Hom}_A(A/I, M) &= \{\varphi \in \mathrm{Hom}_A(A, M) : I \subseteq \ker(\varphi)\} \\
&= \{\varphi \in \mathrm{Hom}_A(A, M) : \forall i \in I\ i\varphi = 0\}.
\end{aligned}$$

But $\mathrm{Hom}_A(A, M) \cong M$. Thus every $\phi \in \mathrm{Hom}_A(A, M)$ can be identified as some $m \in M$. Thus

$$\begin{aligned}
\mathrm{Hom}_A(A/I, M) &= \{m \in M : \forall i \in I\ im = 0\} \\
&= \mathrm{Ann}_M(I)
\end{aligned}$$

giving the desired isomorphism. $\square$

**Lemma A.1.0.9.** *Let $N$ be an $A$-module and $I$ an ideal of $A$. If $I \subseteq \mathrm{Ann}_A(N)$ then for all $M \in Mod_A$ we have* $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_A(M/IM, N)$.

*Proof.* Let $\varphi \in \mathrm{Hom}_A(M, N)$. Define $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M/IM, N)$ by $\varphi \mapsto \overline{\varphi}$ where $\overline{\varphi} : M/IM \to N$ is the canonically induced map. It suffices to verify that the map

$$\mathrm{Hom}_A(M/IM, N) \to \mathrm{Hom}_A(M, N)$$

defined by $\psi \mapsto \psi \circ \pi$ where $\pi : M \to M/IM$ is the canonical map is an inverse. Indeed, let $\phi \in \mathrm{Hom}_A(M, N)$. First note that for all $i \in I$ we have $i\phi = 0$ as $iN = 0$. Then $\phi \mapsto \overline{\phi} \mapsto \overline{\phi} \circ \pi$. However, the induced diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ M/IM \arrow[ur] & \end{tikzcd}$$

is commutative. Thus we have $\phi = \overline{\phi} \circ \pi$. Now let $\eta \in \mathrm{Hom}_A(M/IM, N)$. Then $\eta \mapsto \eta \circ \pi$. But clearly $\overline{\eta \circ \pi} = \eta$, giving us a double sided inverse and the desired isomorphism. □

**Lemma A.1.0.10.** *Let $M, N$ be two $A$-modules. Let $I$ be an ideal of $A$ such that $I \subseteq \mathrm{Ann}_A(N)$ and $I \subseteq \mathrm{Ann}_A(M)$. Then $\mathrm{Hom}_{A/I}(M, N) \cong \mathrm{Hom}_A(M, N)$ as $A/I$-modules.*

*Proof.* Note $\mathrm{Hom}_A(M, N) = \mathrm{Hom}_A(M \otimes_A A/R, N) \cong \mathrm{Hom}_{A/R}(M, \mathrm{Hom}_A(A/R, N)$. But we also have $N \cong \mathrm{Hom}_A(A, N)$. And by Lemma A.1.0.9 we have

$$\mathrm{Hom}_A(A, N) \cong \mathrm{Hom}_A(A/R, N).$$

Thus $\mathrm{Hom}_A(M, N) \cong \mathrm{Hom}_{A/R}(M, N)$ as desired. □

**Lemma A.1.0.11.** *Let $A$ be an integral domain. Let $M$ be an $A$-module. If $M$ is injective then it is divisible.*

*Proof.* To see the forward implication let $(a)$ be a principal ideal of $A$ and let $E$ be an $A$-injective module. Fix $x \in E$ and define the map $f : (a) \to E$ by $f(a_1 a) = a_1 n$. This is not the 0 map as $A$ is a domain. Since $E$ is injective, by Baer's criterion, this map expands to a map $\tilde{f} : A \to E$. Thus $\tilde{f}(a) = a\tilde{(f)} = ax'$. That is $x = ax'$. We can do this with every element of $E$, thus $E$ is divisible as desired. □

**Lemma A.1.0.12.** *Let $A$ be an integral domain that is not a field. Suppose $E$ is a finitely generated injective $A$-module. Then $E = 0$.*

*Proof.* Let $\mathfrak{m}$ be a maximal ideal of $A$. It is enough to show that $E_{\mathfrak{m}} = 0$. Indeed, if $E_{\mathfrak{m}} = 0$ then we have $\frac{e}{a} = 0$ for every non-zero $a \in A - \mathfrak{m}$ and $e \in E$. But $E$ is divisible by A.1.0.11. Thus we have $e = ae'$. Thus $\frac{e'}{1} = 0$ which is true if and only if $e' = 0$ since $A$ is an integral domain.

Now, since $A$ is not a field, we have $\mathfrak{m} \neq 0$. Let $a$ be a non-zero element of $\mathfrak{m}$. Since $E$ is a divisible $A$-module by A.1.0.11, we get $aE = E$. Hence $\mathfrak{m}E = E$. When we localize by $\mathfrak{m}$ we get $\mathfrak{m}A_{\mathfrak{m}}E_{\mathfrak{m}} = E_{\mathfrak{m}}$ But $E_{\mathfrak{m}}$ is a finitely generated $A_{\mathfrak{m}}$-module. Thus, by Nakayama's Lemma, we have $E_{\mathfrak{m}} = 0$ as desired. □

**Lemma A.1.0.13.** *Let $A$ be a Noetherian ring. Suppose $M$ is an $A$-module such that $M = \mathfrak{p}M$ for all prime ideals $\mathfrak{p}$ of $A$. Then $M = 0$.*

*Proof.* We show that if $I$ is an ideal of $A$ such that $M = IM$ then for all $n \in \mathbb{Z}_{>0}$ we have $M = I^n M$. Indeed, if $M = IM$, then we can substitute for $IM$ for $M$ in the equality to immediately get $M = IM = I(IM) = I^2 M$. We can repeat this process any finite number of times in order to get $M = I^n M$ as desired.

Now, suppose $M = I_j M$ for $j = 1, \ldots, l$. Start with $M = I_1 M$. Note $I_2 M = M$. Thus by substitution we have $M = I_2 M = I_1 I_2 M$. We can repeat this process to get

$M = I_1 I_2 \dots I_l M$.

Since $A$ is Noetherian, there are only finitely many prime ideals. Let $P_1, \dots, P_n$ be the minimal prime ideals of $A$. We claim there exists $N$ such that $P_1^N \dots P_n^N = (0)$. Indeed, $P_1 \dots P_n \subseteq \bigcap_{i=1}^n P_i = \mathfrak{n}$ where $\mathfrak{n}$ is the nilradical of $A$. Let $\mathfrak{n}$ be generated by $x_1, \dots, x_m$. Each of the of these elements are killed by a power, say $n_i$. Take $N = n_1 \dots n_m$. Then $x_i^N = 0$ for all $i \in \{1, \dots, m\}$. Thus $\mathfrak{n}^N = (0)$.

But this means that $(P_1 \dots P_n)^N \subseteq (0)$. But $M = P_i M$. Thus we have $M = P_i^N M$ and $M = P_1^N \dots P_n^N M$. But this means $M = 0M$ and $M = 0$ as desired. □

**Lemma A.1.0.14.** *Let $\varphi : A \to B$ be a ring homomorphism. Let $J$ be an ideal of $B$ and $I = \varphi^{-1}(J)$. Then the induced map*

$$\varphi \otimes_A \mathrm{id}_{A/I} : A/I \to B/IB$$

*is injective.*

*Proof.* Note $IB \subseteq J$, thus, the in injection $A/I \to B/J$ factors via

$$A/I \to B/IB \to B/J.$$

But since the composition is injective, the first map $A/I \to B/IB$ must be injective as well. □

For the following, we adapt a proof from [AM69].

**Theorem A.1.0.15** (Lying Over Theorem)**.** *Suppose that $\varphi : A \to B$ is an integral extension of rings. Let $P$ be a prime ideal of $A$. Then there exists a prime ideal $Q$ in $B$ such that $Q \cap A = P$.*

*Proof.* Note that $B_P$ is intgral of $A_P$ and we have the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\alpha"] & B \arrow[d, "\beta"] \\ A_P \arrow[r] & B_P \end{tikzcd}$$

in which the horizontal arrows are injections. Let $\mathfrak{n}$ be a maximal ideal of $B_P$. Then $\mathfrak{m} = A_P \cap \mathfrak{n}$ is maximal. Thus, $\mathfrak{m}$ is the unique maximal ideal of $A_P$. If $Q = \beta^{-1}(\mathfrak{n})$ then $Q$ is prime. Moreover, $Q \cap A = \alpha^{-1}(\mathfrak{m}) = P$. $\square$

**Lemma A.1.0.16.** *Let $\varphi : A \to B$ be an integral extension. Then for all prime ideals $\mathfrak{p}$ of $A$, the induced map*

$$\varphi \otimes_A A/\mathfrak{p} : A/\mathfrak{p} \to B/\mathfrak{p}B$$

*is also an integral extension. More over, if $\mathfrak{n}$ is the nilradical of $A$, then*

$$\varphi \otimes_A A/\mathfrak{n} : A/\mathfrak{n} \to B/\mathfrak{n}B$$

*is an integral extension.*

*Proof.* Let $I$ be a prime ideal or the nilradical of $A$. To show that $A/I \to B/IB$ is an extension, it suffices to show that there exists an ideal $J$ of $B$ such that $J \cap A = I$ by Lemma A.1.0.14. If $I$ is prime, this is obvious as the induced map

$$\varphi^\circ : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$$

is surjective by Theorem A.1.0.15. Thus let $I$ be the nilradical of $A$. Let $\{P_\alpha\}_{\alpha\in\Sigma}$ be the set of minimial primes of $A$ and let $\{J_\alpha\}_{\alpha\in\Sigma}$ be a family of prime ideals in $B$ such that $A \cap J_\alpha = P_\alpha$. Then $A \cap (\bigcap J_\alpha) = \bigcap P_\alpha = I$. But $\bigcap J_\alpha$ is an ideal in $B$. Thus we win. $\square$

**Lemma A.1.0.17.** *Let $A \to B$ be a ring homomorphism, $E$ is an injective $A$-module, and $F$ a flat $B$-module. Then* $\mathrm{Hom}_A(F, E)$ *is an injective $B$-module.*

*Proof.* Note that the functor $\mathrm{Hom}_B(\bullet, \mathrm{Hom}_A(F, E))$ can be naturally identified with the functor $\mathrm{Hom}_A(\bullet \otimes_B F, E)$ through hom-tensor adjunction. However, $\mathrm{Hom}_A(\bullet \otimes_B, E)$ can be viewed as the composition of the functors $\bullet \otimes_B F$ and $\mathrm{Hom}_A(\bullet, E)$.

Clearly $\bullet \otimes_B F : \mathrm{Mod}_B \Rightarrow \mathrm{Mod}_B$ is an exact functor as $F$ is $B$-flat. We claim that the functor $\mathrm{Hom}_A(\bullet, E)$ can viewed as a functor of $B$ modules. Indeed, it suffices to show that for all $M \in \mathrm{Mod}_B$ we can view the module $\mathrm{Hom}_A(M, E)$ as a $B$-module. Let $\varphi \in \mathrm{Hom}_A(M, E)$. For all $b \in B$ we defined $b \cdot \varphi$ as the composition map

$$M \xrightarrow{b} M \xrightarrow{\varphi} E.$$

Now, to see that $\mathrm{Hom}_A(\bullet, E)$ is exact, notice that since $E$ is an injective $A$-module, for any exact sequence of $B$-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

we have an exact sequence of $A$-modules

$$0 \longrightarrow \mathrm{Hom}_A(M_1, E) \longrightarrow \mathrm{Hom}_A(M_2, E) \longrightarrow \mathrm{Hom}_A(M_3, E) \longrightarrow 0.$$

But through the natural identification of these modules as $B$-modules, we have an exact sequence of $B$-modules as desired. $\square$

**Lemma A.1.0.18.** *Let $A \to B$ be a flat ring homomorphism. If $F$ is injective as a $B$-module, then $F$ is injective as an $A$-module*

*Proof.* It suffices to show that the functor $\mathrm{Hom}_A(\bullet, F)$ is exact on $A$-modules. Let

$$0 \to M \to N \to Q \to 0$$

be a short exact sequence of $A$-modules. Then

$$0 \to M \otimes_A B \to N \otimes_A B \to Q \otimes_A B \to 0$$

is a short exact sequence since $B$ is $A$-flat. By base change, we can consider these $B$-modules. Thus we have

$$0 \to \mathrm{Hom}_B(M \otimes_A B, F) \to \mathrm{Hom}_B(N \otimes_A B, F) \to \mathrm{Hom}_B(Q \otimes_A B, F) \to 0$$

is a short exact sequence of $B$-modules. But by Hom-tensor adjunction, we have

$$0 \to \mathrm{Hom}_A(M, F) \to \mathrm{Hom}_A(N, F) \to \mathrm{Hom}_A(Q, F) \to 0$$

is a short exact sequnece of $B$-module. By restriction of scalars, this is also a short-exact sequence of $A$-modules. Thus the functor $\mathrm{Hom}_A(\bullet, F)$ is exact and $F$ is $A$-injective. $\square$

**Lemma A.1.0.19.** *Let $A$ be a integral domain. If $E$ is a divisible torsion-free $A$-module, then it is an injective $A$-module*

*Proof.* Let $K$ be the field of fractions of $A$. Since $E$ is divisible and torsion-free, it is a $K$-module. Indeed, for all non-zero $s \in A$ we have that $s\cdot : E \to E$ is an isomorphism as E is divisible and torsion free. Thus by the universal property of localization, E is a $K$-module. But then it is an injective $K$-module since $K$ is a field. But $A \to K$ is a flat morphism. Thus by A.1.0.18 we have that $E$ is $A$-injective as desird. $\square$

## A.2. Filtered Colimits

Filtered colimits, also known as filtered direct limits, are a powerful tool in commutative algebra. We provide the construction of the colimit as described in [AM69].

A partially ordered set $I$ is said to be a filtered set if for each pair $i, j \in I$ there exists $k \in I$ such that $i \leq k$ and $j \leq k$. Let $A$ be a ring and let $I$ be a filtered set. Let $(M_i)_{i \in I}$ be a family of $A$-modules indexed by $I$. For every pair $i \leq j$ let

$\varphi_{ij} : M_i \to M_j$ be a module homomorphism. If $\varphi_{ii} = \mathrm{id}_{M_i}$ and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ then we say that the modules $M_i$ form a filtered system over $I$.

Now, let $M = \oplus_{i \in I} M_i$ and let $M^*$ be the module generated by elements of the form $x_i - \varphi_{ij}$. Then we say that $\varinjlim_{i \in I} M_i = M/M^*$. We thus have the canonical morphism $\varphi : M \to \varinjlim_{i \in I} M_i$. Restricting $\varphi$ to $M_i$, we create a morphism

$$\varphi_i : M_i \to \varinjlim_{i \in I} M_i$$

Note, $\varphi_i = \varphi_j \circ \varphi_{ij}$. We thus have a diagram

$$\begin{tikzcd} M_i \arrow[rr, "\varphi_{ij}"] \arrow[dr, "\varphi_i"'] & & M_j \arrow[dl, "\varphi_j"] \\ & \varinjlim_{i \in I} M_i & \end{tikzcd}$$

**Lemma A.2.0.1.** *Every element in $\varinjlim_{i \in I} M_i$ can be written of the form $\varphi_i(m_i)$ where $m_i \in M_i$.*

*Proof.* Let $m \in M$. Then $\varphi(m) = m + M^*$. Now $m = \sum_{i \in I} m_i$ where all but finitely $m_i$ are 0. Thus $\varphi(m) = \sum_{i=1}^n m_i + M^*$. Now, note that $\varphi_i(m_i) = \varphi_n(\varphi_{in}(m_i))$. Thus $m_i + M^* \sim \varphi_{in}(m_i) + M^*$ Thus $m + M^* \sim \sum_{i=1}^n \varphi_{in}(m_i) + M^*$ which is in one coordinate. $\square$

**Lemma A.2.0.2.** *Every $A-$module can be written as the colimit of a filtered system of finitely presented modules*

*Proof.* For all $M$, we have an exact sequence

$$0 \to K \to A^{(I)} \to M \to 0$$

where K is the kernel of the $A^{(I)} \to M$ map. Without loss of generality we can say $K \subseteq A^{(I)}$ and $M = A^{(I)}/K$. Now consider the set

$$\mathcal{S} = \{(N, S) : S \subseteq I, |S| < \infty, N \subseteq K \cap A^{(S)}, \text{ N finitely generated}\}$$

ordered by $(N_1, S_1) \leq (N_2, S_2)$ if $N_1 \subseteq N_2$ and $S_1 \leq S_2$. This is a filtered poset and we have that if $N_1$ and $N_2$ are finitely generated, then $N_1 + N_2$ is finitely generated. Further, if $S_1$ and $S_2$ are finite, then $S_1 \cup S_2$ is finite. By construction, for all $(N, S) \in \mathcal{S}$, $A^{(S)}/N$ is finitely presented. We now verify that $\varinjlim_{(N,S)\in\mathcal{S}} A^{(S)}/N = A^{(I)}/K$.

Notice, we have the exact sequence

$$0 \to N \to A^{(S)} \to A^{(S)}/N \to 0.$$

We thus apply the colimit to this sequence and notice that since its an exact functor, we have an exact sequence

$$0 \to K \to A^{(I)} \to \varinjlim_{(N,S)\in\mathcal{S}} A^{(S)}/N \to 0$$

as clearly the colimit of the $N$'s is $K$ and the colimit of $A^{(S)}$ is $A^{(I)}$. Thus by the Fives Lemma, we win. $\square$

## A.3. Injective Hulls

An injective hull is a useful tool many as often we can reduce arbitrary injective modules to injective hulls. Our goal is to show that for a complete local Noetherian ring $A$ we have $\mathrm{Hom}_A(E, E) = A$ where $E$ is the injective hull of the residue field. We use notes taken from [Abe23]

**Lemma A.3.0.1.** *Let $A$ be a ring. Let $M, N \in Mod_A$. Let $E_A(M)$ and $E_A(N)$ be the injective hulls of $M$ and $N$. Then the injective hull of $M \oplus N$ $E_A(M \oplus N)$ is isomorphic to $E_A(M) \oplus E_A(N)$.*

*Proof.* First note that $M \oplus N \to E_A(M) \oplus E_A(N)$ is an essential extension as we showed in Lemma 2.7.3.4 that the direct sum of essential extensions is an essential extension. Furthermore, $E_A(M) \oplus E_A(M)$ is an injective module as the finite direct sum of injective modules is injective. Now, since $M \oplus N \to E_A(M) \oplus E_A(N)$ is an essential extension, the map $E_A(M) \oplus E_A(N) \to E_A(M \oplus N)$ is an essential extension. But this is an injection of two injective modules. Thus the map splits. But since its an essential extension, this means its an isomorphism. $\square$

**Lemma A.3.0.2.** *Let $A$ be a domain and let $u : J \to E$ be an essential envelope. If $J$ is torsion free, then $E$ is torsion free. In particular if $E$ is an injective hull of a torsion free module, then $E$ is also a module of the fraction field of $K$.*

*Proof.* Note that since $J$ is torsion free, we have that $s\cdot : J \to J$ is injective. We claim that $s\cdot : E \to E$ is also injective. Indeed, note that $\ker(s\cdot) \cap J = \ker(s\cdot|_J) = 0$. But since $J \to E$ is an essential envelope, we have that $\ker(s\cdot) = 0$ and $s\cdot : E \to E$ is an injective $A$-module homomorphism.

Now, if $E$ is an injective $A$-module, then it is divisible by Lemma A.1.0.11. That is, for any non-zero $s \in A$ we have $s\cdot : E \to E$ is surjective.

Thus if $E$ is the injective hull, we have that $s\cdot$ is an isomorphism. That is to say that the map $A \to \mathrm{Hom}_A(E, E)$ sends every non-zero element of $A$ to a unit in $\mathrm{Hom}_A(E, E)$. Thus, by the universal property of localization, there is a unique homomorphism such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathrm{Hom}_A(E,E) \\ K \arrow[ur] & \end{tikzcd}$$

As such $E$ is a $K$-module. $\square$

**Definition A.3.0.3.** *Let $A$ be a Noetherian local ring and let $k$ be the residue field of $A$. Throughout let $\bullet^\vee$ denote the functor*

$$\mathrm{Hom}_A(\bullet, E)$$

*where $E$ is the injective hull of $k$ over $A$.*

**Lemma A.3.0.4.** *Let $A$ be Noetherian and $E$ be an $A$-injective module. Then $E = \oplus_{\lambda\in\Lambda} E_A(A/P_\lambda)$ where every $P_\lambda$ is some prime ideal.*

*Proof.* Choose a collection $\{E_i\}$ of submodules of $E$ that is maximal with respect to the following:

1. $E_\lambda \cong E_A(A/P_\lambda)$
2. $\sum_\lambda E_\lambda$ is an internal direct sum

This is possible via Zorn's Lemma. Now, let $E_1 = \sum_\lambda E_\lambda = \oplus_\lambda E_\lambda$. By [Bas62], this is injective as $A$ is Noetherian. Thus $E = E_1 \oplus E/E_1$. If $E/E_1 \neq 0$ then, since $A$ is Noetherian, we have that $\mathrm{Ass}_A(E/E_1) \neq 0$. Thus there is a prime ideal $P$ such that $A/P$ injects into $E/E_1$. Thus we have the following chain of injections

$$A/P \to E_A(A/P) \to E/E_1.$$

This means that $E_1 \oplus E_A(A/P)$ would be a larger module than $E_1$ with respect to the above conditions. This contradicts $E_1$'s maximality. Thus $E/E_1 = 0$ and $E = E_1$ as desired. $\square$

**Lemma A.3.0.5.** *Let $A$ be a Noetherian ring, $P$ a prime ideal of $A$ and $E = E_A(A/P)$. Let $F = A_p/PA_p$. Then $(0 :_E P) \cong F$*

*Proof.* We know that $A/P \to F$ is Since $E$ is an $A_p$ module, we have

$$(0 :_E P) \cong \operatorname{Hom}_A(A/P, E) = \operatorname{Hom}_{A_p}(A_p/PA_p, E) = \operatorname{Hom}_{A_P}(F, E) \subseteq E.$$

Thus $\operatorname{Hom}_{A_P}(F, E)$ is an $F$ vector-space and is essential over $F$. Thus it is not a proper direct sum and $\dim_F(\operatorname{Hom}_{A_P}(F, E)) = 1$, i.e. $\operatorname{Hom}_{A_P}(F, E) \cong F$ and we win. □

**Lemma A.3.0.6.** *Let $A$ be noetherian and $M$ be finitely generated. Let*

$$\mathcal{C} := 0 \to E^0 \to E^1 \to \dots$$

*be a minimal injective resolution of $M$. Fix a prime ideal $P$. Then the number of $E_A(A/P)$ summands in $E^i$ is equal to $\mu_i(P, M)$.*

*Proof.* We claim that $\mu_i(P, M) = \dim_k(\operatorname{Ext}^i_A(k, M))$. Indeed, we have $\operatorname{Hom}_A(k, E^i) = k^{t_i}$ where $t_i$ is the number of direct summands of $E_A(K)$ in $E^i$.

Now, define

$$\mathcal{K} := 0 \longrightarrow k^{t_0} \longrightarrow k^{t_1} \longrightarrow \dots$$

Then, we have the following

$$\operatorname{Ext}^i_A(k, M) = H^i(\operatorname{Hom}_A(k, E^i)) = H^i(\mathcal{K})$$

We win if all the maps are 0. Apply the functor $\operatorname{Hom}_A(k, \bullet)$ and get

$$\dots \longrightarrow k^{t_{i-1}} \longrightarrow k^{t_i} \longrightarrow k^{t_{i+1}} \longrightarrow \dots$$

Let $x \in k^{t_i} \subseteq E^i$ be a non-zero element. Since we have $\operatorname{coker}(d^{i-1}) \subseteq E^i$ is essential, $x$ has non-zero multiples in $\operatorname{coker}(d^{i-1})$. But $k^{t_i}$ is a $k$-vector space. Thus all non-zero multiples of $x$ are in $\operatorname{coker}(d^{i-1})$, i.e. $x \in \operatorname{coker}(d^{i-1}) = \ker(d^i)$. Thus $d^{i+1}(x) = 0$ and $d^i = 0$ for all $i$. Thus $\operatorname{Ext}^i_A(k, M) = k^{t_i}$ and $t_i = \mu_i(P, M)$. $\square$

**Remark A.3.0.7.** $0 = \dim(k) = \dim(B/\mathfrak{m}B)$ *gives us that* $\mathfrak{n}$ *is nilpotent mod* $\mathfrak{m}B$*, i.e. there exists a* $t$ *such that* $\mathfrak{n}^t \subseteq \mathfrak{m}B$ *and* $\sqrt{\mathfrak{m}B} = \mathfrak{n}$*.*

**Theorem A.3.0.8.** *Suppose* $(A, \mathfrak{m}, k) \to (B, \mathfrak{n}, l)$ *is a local ring homorphism such that* $B$ *is a finitely generated* $A$*-module. Let* $E = E_A(K)$*. Then* $\operatorname{Hom}_A(B, E) = E_B(l)$*.*

*Proof.* From the remark, we have $\mathfrak{n} = \sqrt{\mathfrak{m}B}$. Now, since $B$ is finitely generated, consifed a sequence

$$A^t \to B \to 0.$$

We use our $\bullet^\vee$ functor on this sequence to get

$$0 \to B^\vee \to E^t$$

Thus, every $x \in B^\vee$ is killed by a power of $\mathfrak{m}$. Furthermore, this means every $x$ is killed by a power of $\mathfrak{n}$. But recall that $\operatorname{Hom}_A(B, E)$ is a $B$-injective module. Since $x \in \operatorname{Hom}_A(B, E)$ is killed by a power of $\mathfrak{m}$ by Lemma A.3.0.4 and Lemma A.3.0.5 we have $\operatorname{Hom}_A(B, E)$ is a direct sum of copies of $E_B(l)$. By Lemma A.3.0.6, the number of copies is $\mu_0(\mathfrak{n}, B^\vee) = \dim_l(\operatorname{Hom}_A(l, E))$

Now, as an $A$-module we have every $x$ in $l$ is killed by $\mathfrak{m}$. So for an $A$-linear map, $\theta : l \to E$ we have $\operatorname{im}(l) \subseteq (0 :_E \mathfrak{m}) = k$. Thus $\operatorname{Hom}_A(l, E) = \operatorname{Hom}_A(l, k) = \operatorname{Hom}_k(l, k) \cong l$. Thus the number of copies of is 1 and we win. $\square$

**Lemma A.3.0.9.** *Let $(A, \mathfrak{m}, k)$ be an Artinian, local, and Noetherian ring. Let $E$ be the injective hull of $k$. Then the length of $A$ is the same as the length of $E$*

*Proof.* We induce on $l(A)$. If $l(A) = 1$ then $A = k$ and $E = A$. If $l(A) > 1$, let $x \in \operatorname{Soc}(A)$. Then $x\mathfrak{m} = 0$ and thus $xA \cong k$. Now, consider the sequence

$$0 \to xA \to A \to A/xA \to 0.$$

Note $l(A/xA) = l(A) - l(k) = l(A) - 1$. Now, if we apply the functor $\operatorname{Hom}_A(\bullet, E)$ to the sequence, we get

$$\begin{tikzcd} 0 \arrow[r] & A/xA^{\vee} \arrow[r] \arrow[d, equal] & A^{\vee} \arrow[r] \arrow[d, equal] & xA^{\vee} \arrow[r] \arrow[d, equal] & 0 \\ & E_{A/xA}(k) & E & k & \end{tikzcd}$$

Thus $l(E) = l(E_{A/xA}(K)) + l(k) = l(A/xA) + 1 = l(A)$ as desired. □

**Lemma A.3.0.10.** *Let $(A, \mathfrak{m}, k)$ be local and $E$ be the injective hull of $k$. If the length of $M$ is finite, then the length of $M^{\vee}$ is the same of the length of $M$*

*Proof.* Note that $k \cong k^{\vee}$. This takes care of the case where $l(M) = 1$. We now induce on $l(M)$. If $l(M) > 1$ then there is a sequence

$$0 \to k \to M \to M' \to 0.$$

Then $l(M') = l(M) - 1$. Using the previously defined dual yields

$$0 \to M'^{\vee} \to M^{\vee} \to k^{\vee} \to 0.$$

Thus $l(M^{\vee}) = l(M'^{\vee}) + l(\operatorname{Hom}_A(k, E)) = l(M') + 1 = l(M)$ by induction hypothesis.

□

**Lemma A.3.0.11.** *Let $(A, \mathfrak{m}, k)$ be a local Artinian Noetherian ring. Let $E$ be the injective hull of $k$. Then $A \to \operatorname{Hom}_A(E, E)$ is an isomorphism*

*Proof.* First note that $l(E) = l(A) = l(E)$ We also have the sequence

$$0 \longrightarrow \ker(\theta) \longrightarrow A \xrightarrow{\theta} E^{\vee} \longrightarrow \operatorname{coker}(\theta) \longrightarrow 0.$$

But since $l(A) = l(E^{\vee})$ we have that $l(\operatorname{coker}(\theta)) = l(\ker(\theta))$. Thus $\theta$ is an isomorphism if and only if it is injective.

Now, let $x \in \ker(\theta)$. Then $xE = 0$. Furthermore $(A/xA)^{\vee} = (0 :_E x) = E$. But $A/xA^{\vee} = E_{A/xA}(k)$. Thus $l(A) = l(E) = l(E_{A/xA}(k)) = l(A/xA)$ which can only be true if $x = 0$. Thus $\theta$ is an injection and thus an isomorphism as desired. $\square$

**Theorem A.3.0.12.** *Let $A$ be local, Noetherian, and complete. Let $\mathfrak{m}$ be the maximal ideal of $A$ and $E$ the injective hull of $A/\mathfrak{m}$. Then* $\operatorname{Hom}_A(E, E) = A$

*Proof.* Let $A_t = A/\mathfrak{m}^t$. Then we have $A = \lim_{t \in \mathbb{N}} A_t$. Let

$$E_t \coloneqq (0 :_E \mathfrak{m}^t) = E_{A_t}(k) \subseteq E.$$

Thus we have $E = \cup_t E_t$. Let $\varphi : E \to E$ be $A$ linear. Then $\varphi\,|_{E_t}\colon E_t \to E_t$. But $\operatorname{Hom}_{A_t}(E_t, E_t) = A_t$. Thus we can identify $\varphi\,|_{E_t}$ as $r_t + \mathfrak{m}^t$ for some $r_t \in A$. Further restriction to $t_1 < t$ is coherent, thus $\varphi$ gives $\{\cdot, (a_t + \mathfrak{m}^t)\}$ which is coherent, i.e. is an element of $\varprojlim_t A_t = A$. $\square$

Furthermore, this allows us to define a dual on the category of $A$ modules. This is an especially powerful tool on finitely generated modules by the following proposition.

**Proposition A.3.0.13** (Matlis Duality)**.** *Let $A$ be a local, Noetherian, and complete ring. Ket $\mathfrak{m}$ be the maximal ideal of $A$ and let $E$ be the injective hull of $A/\mathfrak{m}$. Then* *the functor* $\mathrm{Hom}_A(\bullet, E)$ *defines a dual on* $Mod_A$ *such that for finitely generated* $N$ *we have*

$$N \cong \mathrm{Hom}_A(\mathrm{Hom}_A(N,E),E).$$

*Proof.* Indeed, since $A$ is Noetherian, we have a finite presentation of $N$

$$A^m \xrightarrow{\varphi} A^n \longrightarrow N \longrightarrow 0.$$

When we dualize into $E$, and make a choice of basis we can write the dual of $\varphi$ as a $n \times m$ matrix $a_{ij}$. This yields the sequence

$$0 \longrightarrow \mathrm{Hom}_A(N,E) \longrightarrow E^n \xrightarrow{a_{ij}} E^m.$$

Dualizing once more gives us

$$\begin{tikzcd} A^m \arrow[r] \arrow[d, equal] & A^n \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ A^m \arrow[r] & A^n \arrow[r] & \mathrm{Hom}_A(\mathrm{Hom}_A(N,E),E) \arrow[r] & 0 \end{tikzcd}$$

as $\mathrm{Hom}_A(E,E) \cong A$ by A.3.0.12. Thus thus by the fives lemma, we have the desired isomorphism. □

**Lemma A.3.0.14.** *Let* $A$ *be a complete ring. Then the functor* $\bullet^\vee$ *is faithful.*

*Proof.* Let $M$ an $A$-module such that $M^\vee = 0$. Let $x \in M$. Then, we have the sequence

$$0 \to xA \to M \to M/xA \to 0.$$

Applying $\bullet^\vee$ to the sequence yields

$$0 \to M/xA^\vee \to M^\vee \to xA^\vee \to 0.$$

But $M^\vee = 0$. Thus $xA^\vee = 0$. But this means that $xA \cong (xA^\vee)^\vee = 0$. This can only be true if $x = 0$. But $x$ was chosen arbitrarily, thus $M = 0$ as desired. □

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