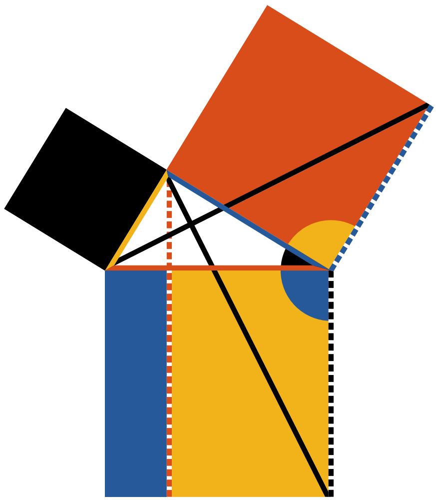
LOS PRIMEROS SEIS LIBROS DE

LOS ELEMENTOS DE EUCLIDES

EN EL QUE SE UTILIZAN DIAGRAMAS Y SÍMBOLOS COLOREADOS EN LUGAR DE LETRAS PARA FACILITAR EL APRENDIZAJE

POR OLIVER BYRNE



github.com/jemmybutton

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Introducción

Las artes y ciencias se han vuelto tan extensas, que facilitar su adquisición es de tanta importancia como extender sus límites. La ilustración, si no acorta el tiempo de estudio, al menos lo hará más agradable. Este trabajo tiene un objetivo mayor que la mera ilustración; no introducimos colores con el propósito de entretener, o para entretener *con ciertas combinaciones de matiz y forma*, pero para ayudar a la mente en sus investigaciones de la verdad, para aumentar las facilidades de introducción y para difundir el conocimiento permanente. Si quisiéramos autoridades para probar la importancia y utilidad de la geometría, podríamos citar a todos los filósofos desde el día de Platón. Entre los griegos, en la antigüedad, como en la escuela de Pestalozzi y otros en tiempos recientes, la geometría fue adoptada como el mejor gimnasio de la mente. De hecho, los Elementos de Euclides se han convertido, por consentimiento común, en la base de la ciencia matemática en todo el mundo civilizado. Pero esto no parecerá extraordinario, si consideramos que esta ciencia sublime no solo está mejor calculada que cualquier otra para despertar el espíritu de investigación, elevar la mente y fortalecer las facultades de razonamiento, sino que también constituye la mejor introducción a la mayoría de las vocaciones útiles e importantes de la vida humana. La aritmética, la topografía, la hidrostática, la neumática, la óptica, la astronomía física, etc. dependen todas de las proposiciones de la geometría.

Mucho sin embargo depende de la primera comunicación de cualquier ciencia a un alumno, aunque los métodos mejores y más fáciles rara vez se adoptan. Se presentan proposiciones a un estudiante, a quien, aunque tenga una comprensión suficiente, se le dice tan poco sobre ellas al entrar en el umbral mismo de la ciencia, que se le da una predisposición muy desfavorable para su futuro estudio de este tema delicioso; o "las formalidades y parafernalia del rigor se presentan de manera tan ostentosa, que casi ocultan la realidad. Las repeticiones interminables y desconcertantes, que no confieren mayor exactitud al razonamiento, hacen que las demostraciones sean complejas y oscuras, y ocultan a la vista del estudiante la sucesión de la evidencia." Así se crea una aversión en la mente del alumno, y un tema tan calculado para mejorar las facultades de razonamiento y dar el hábito de pensar con atención se degrada por un curso de instrucción seco y rígido en un ejercicio poco interesante de la memoria. Despertar la curiosidad y despertar las facultades letárgicas y latentes de las mentes jóvenes debería ser el objetivo de todo maestro; pero donde faltan ejemplos de excelencia, los intentos de alcanzarla son pocos, mientras que la eminencia excita la atención y produce imitación. El objetivo de esta obra es introducir un método de enseñanza de la geometría que ha sido muy aprobado por muchos hombres de ciencia en este país, así como en Francia y América. El plan aquí adoptado apela fuertemente al ojo, el más sensible y el más completo de nuestros órganos externos, y su preeminencia para grabar su tema en la mente está respaldada por el axioma incontrovertible expresado en las conocidas palabras de Horacio:—

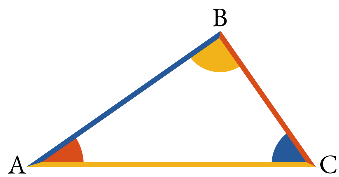
Segnius irritant animos demissa per aurem
Quam quae sunt oculis subjecta fidelibus

La impresión que llega al oído es más débil
que la que transmite el ojo fiel.

Todo el lenguaje consiste en signos representativos, y los mejores signos son aquellos que cumplen sus propósitos con la mayor precisión y rapidez. Tales, para todos los propósitos comunes, son los signos audibles llamados palabras, que todavía se consideran audibles, ya sea que se dirijan inmediatamente al oído o a través del medio de las letras al ojo. Los diagramas geométricos no son signos, sino los materiales de la ciencia geométrica, cuyo objeto es mostrar las cantidades relativas de sus partes mediante un proceso de razonamiento llamado Demostración. Este razonamiento se ha llevado a cabo generalmente con palabras, letras y diagramas negros o incoloros; pero como el uso de símbolos, signos y diagramas coloreados en las artes y ciencias lineales hace que el proceso de razonamiento sea más preciso y la adquisición más rápida, se han adoptado en este caso en consecuencia.

La expedición de este modo tan atractivo de comunicar conocimientos es tal, que los Elementos de Euclides pueden adquirirse en menos de un tercio del tiempo habitualmente empleado, y la retención por la memoria es mucho más duradera; estos hechos han sido comprobados por numerosos experimentos realizados por el inventor y varios otros que han adoptado sus planes. Los detalles de los cuales son pocos y obvios; las letras anexas a puntos, líneas u otras partes de un diagrama son en realidad meros nombres arbitrarios y los representan en la demostración; en lugar de estos, las partes, al estar coloreadas de manera diferente, se nombran a sí mismas, ya que sus formas en colores correspondientes las representan en la demostración.

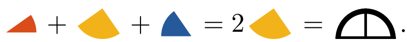
Para dar una mejor idea de este sistema y de las ventajas que se obtienen con su adopción, tomemos un triángulo rectángulo y expresemos algunas de sus propiedades tanto con colores como con el método generalmente empleado.



Algunas de las propiedades del triángulo rectángulo ABC, expresadas por el método generalmente empleado:

1. El angulo BAC, junto con los angulos BCA y ABC son iguales a dos angulos rectos, o al doble del angulo ABC.
2. El angulo CAB sumado al angulo ACB sera igual al angulo ABC.
3. El angulo ABC es mayor que cualquiera de los angulos BAC o BCA.
4. El angulo BCA o el angulo CAB es menor que el angulo ABC.
5. Si del angulo ABC se toma el angulo BAC, el residuo sera igual al angulo ACB.
6. The square of AC is equal to the sum of the squares of AB and BC.

Las mismas propiedades expresadas coloreando las diferentes partes:

1. 
   Es decir, el ángulo rojo sumado al ángulo amarillo sumado al ángulo azul, es igual al doble del ángulo amarillo, igual a dos ángulos rectos.
2. 
   O en otras palabras, el ángulo rojo sumado al ángulo azul, es igual al ángulo amarillo.
3. 
   El ángulo amarillo es mayor que cualquiera de los ángulos rojo o azul.
4. 
   El ángulo rojo o azul es menor que el ángulo amarillo.

5. 
   En otros términos, el ángulo amarillo menos el ángulo azul es igual al ángulo rojo.

6. 
   Es decir, el cuadrado de la línea amarilla es igual a la suma de los cuadrados de las líneas azul y roja.

En las demostraciones orales obtenemos con los colores esta importante ventaja: el ojo y el oído pueden ser atendidos al mismo tiempo, por lo que para enseñar geometría y otras artes y ciencias lineales en clases, el sistema es el mejor que se ha propuesto jamás, esto es evidente por los ejemplos dados.

De donde se deduce que una referencia del texto al diagrama es más rápida y segura, al dar las formas y colores de las partes, o al nombrar las partes y sus colores, que al nombrar las partes y las letras en el diagrama. Además de la simplicidad superior, este sistema también se destaca por su concentración y excluye por completo la práctica perjudicial, aunque frecuente, de permitir que el alumno memorice la demostración; hasta que la razón, el hecho y la prueba solamente dejen impresiones en el entendimiento.

De nuevo, al dar una conferencia sobre los principios o propiedades de las figuras, si mencionamos el color de la parte o partes a las que se hace referencia, como al decir, el ángulo rojo, la línea azul o líneas, etc., la parte o partes así nombradas serán vistas inmediatamente por toda la clase al mismo instante; no así si decimos el ángulo ABC, el triángulo PFQ, la figura EG Kt, y así sucesivamente; porque las letras deben trazarse una por una antes de que los estudiantes organicen en sus mentes la magnitud particular a la que se refieren, lo que a menudo ocasiona confusión y error, así como pérdida de tiempo. Además, si las partes que se dan como iguales tienen los mismos colores en cualquier diagrama, la mente no se desviará del objeto que tiene ante sí; es decir, tal disposición presenta una demostración ocular de las partes que deben probarse como iguales, y el alumno retiene los datos durante todo el razonamiento. Pero cualesquiera que sean las ventajas del plan actual, si no se sustituye, siempre puede ser un poderoso auxiliar de los otros métodos, con el propósito de introducción, o de una reminiscencia más rápida, o de una retención más permanente por la memoria.

La experiencia de todos los que han formado sistemas para impresionar hechos en la comprensión, concuerda en probar que las representaciones coloreadas, como imágenes, grabados, diagramas, etc. son más fáciles de fijar en la mente que las meras oraciones sin ninguna peculiaridad. Curioso como pueda parecer, los poetas parecen ser más conscientes de este hecho que los matemáticos; muchos poetas modernos aluden a este sistema visible de comunicar conocimiento, uno de ellos se ha expresado así:

Los sonidos que se dirigen al oído se pierden y mueren
En una hora corta, pero estos que golpean el ojo,
Viven mucho en la mente, la vista fiel
Graba el conocimiento con un rayo de luz.

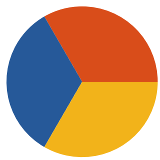
Esto quizás pueda considerarse la única mejora que ha recibido la geometría plana desde los días de Euclides, y si hubo algún geómetra de renombre antes de esa época, el éxito de Euclides ha eclipsado por completo su memoria, e incluso ha hecho que todas las cosas buenas de ese tipo se le atribuyan a él; como Æsop entre los escritores de fábulas. También cabe señalar que, dado que los diagramas tangibles proporcionan el único medio a través del cual la geometría y otras artes lineales pueden enseñarse a los ciegos, el sistema visible es igualmente adecuado para las exigencias de los sordomudos.

Se debe tener cuidado de mostrar que el color no tiene nada que ver con las líneas, ángulos o magnitudes, excepto meramente para nombrarlos. Una línea matemática, que es longitud sin anchura, no puede poseer color, sin embargo, la unión de dos colores en el mismo plano da una buena idea de lo que se entiende por una línea matemática; recordemos que estamos hablando familiarmente, dicha unión debe entenderse y no el color, cuando decimos la línea negra, la línea roja o las líneas, etc.

Los colores y los diagramas coloreados pueden parecer al principio un método torpe para transmitir nociones adecuadas de las propiedades y partes de las figuras y magnitudes matemáticas, sin embargo, se encontrarán que ofrecen un medio más refinado y extenso que cualquiera que se haya propuesto hasta ahora.

Aquí definiremos un punto, una línea y una superficie, y demostraremos una proposición para mostrar la verdad de esta afirmación.

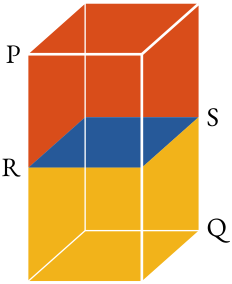
Un punto es aquello que tiene posición, pero no magnitud; o un punto es solo posición, abstraído de la consideración de longitud, anchura y grosor. Quizás la siguiente descripción esté mejor calculada para explicar la naturaleza del punto matemático a aquellos que no han adquirido la idea, que la definición anterior, aunque especiosa.



Que tres colores se encuentren y cubran una porción del papel, donde se encuentran no es azul, ni es amarillo, ni es rojo, ya que no ocupa ninguna porción del plano, porque si lo hiciera, pertenecería a la parte azul, roja o amarilla; sin embargo, existe y tiene posición sin magnitud, de modo que con un poco de reflexión, esta unión de tres colores en un plano da una buena idea de un punto matemático.

Una línea es longitud sin anchura. Con la ayuda de colores, casi de la misma manera que antes, se puede dar así una idea de una línea:—

Que dos colores se encuentren y cubran una porción del papel; donde se encuentran no es rojo, ni es azul; por lo tanto, la unión no ocupa ninguna porción del plano, y por lo tanto no puede tener anchura, sino solo longitud: de lo que podemos formarnos fácilmente una idea de lo que se entiende por una línea matemática. A efectos de ilustración, un color que difiera del color del papel, o del plano sobre el que está dibujado, habría sido suficiente; de ahí en adelante, si decimos la línea roja, la línea azul o las líneas, etc. se entiende que son las uniones con el plano sobre el que están dibujadas.

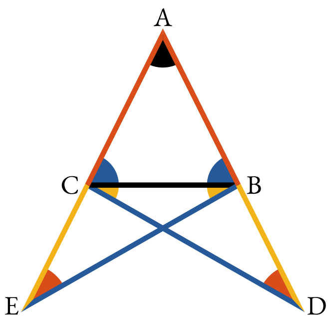


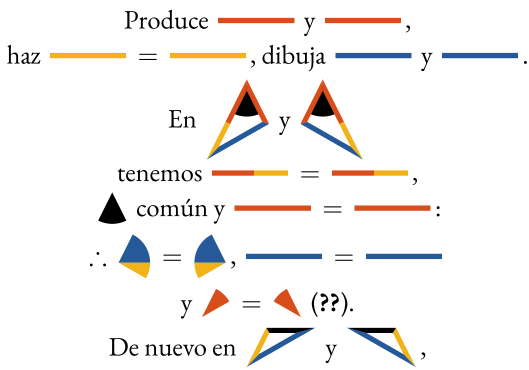
Una superficie es aquello que tiene longitud y anchura sin grosor.

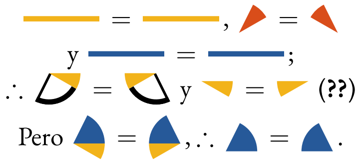
Cuando consideramos un cuerpo sólido (PQ), percibimos de inmediato que tiene tres dimensiones, a saber: longitud, anchura y grosor; supongamos que una parte de este sólido (PS) es roja, y la otra parte (QR) amarilla, y que los colores son distintos sin mezclarse, la superficie azul (RS) que separa estas partes, o que es lo mismo, que divide el sólido sin pérdida de material, debe carecer de grosor, y solo posee longitud y anchura; esto aparece claramente por un razonamiento similar al que se acaba de emplear para definir, o más bien describir, un punto y una línea.

La proposición que hemos seleccionado para elucidar la manera en que se aplican los principios, es la quinta del primer Libro.

En un triángulo isósceles ABC, los ángulos internos en la base ABC, ACB son iguales, y cuando los lados AB, AC se prolongan, los ángulos externos en la base BCE, CBD también son iguales.







Q. E. D.

Anexando las letras al diagrama.

Sean los lados iguales AB y AC producidos a través de los extremos BC, del tercer lado, y en la parte producida BD de cualquiera, sea asumido cualquier punto D, y del otro sea cortado AE igual a AD (??). Sean los puntos E y D, así tomados en los lados producidos, conectados por líneas rectas DC y BE con los extremos alternos del tercer lado del triángulo.

En los triángulos DAC y EAB los lados DA y AC son respectivamente iguales a EA y AB, y el ángulo incluido A es común a ambos triángulos. Por lo tanto (??) la línea DC es igual a BE, el ángulo ADC al ángulo AEB, y el ángulo ACD al ángulo ABE; si de las líneas iguales AD y AE se toman los lados iguales AB y AC, los residuos BD y CE serán iguales. Por lo tanto en los triángulos BDC y CEB, los lados BD y DC son respectivamente iguales a CE y EB, y los ángulos D y E incluidos por esos lados también son iguales. Por lo tanto (??) los ángulos DBC y ECB, que son los incluidos por el tercer lado BC y las producciones de los lados iguales AB y AC son iguales. También los ángulos DCB y EBC son iguales si esos iguales se toman de los ángulos DCA y EBA antes probados iguales, los residuos, que son los ángulos ABC y ACB opuestos a los lados iguales, serán iguales.

Por lo tanto en un triángulo isósceles, etc.

Q. E. D.

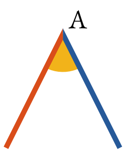
Nuestro objeto en este lugar es introducir el sistema en lugar de enseñar cualquier conjunto particular de proposiciones, por lo tanto hemos seleccionado las anteriores del curso regular. Para escuelas y otros lugares públicos de instrucción, las tizas teñidas servirán para describir los diagramas, etc. para uso privado los lápices de colores resultarán muy convenientes.

Estamos felices de encontrar que los Elementos de las Matemáticas ahora forman una parte considerable de toda educación femenina sólida, por lo tanto llamamos la atención de aquellos interesados o comprometidos en la educación de las damas a este modo muy atractivo de comunicar conocimiento, y al trabajo sucesivo para su futuro desarrollo.

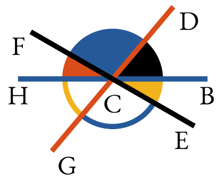
Nosotros concluiremos por el momento observando, ya que los sentidos de la vista y el oído pueden ser tan fuertemente e instantáneamente abordados por igual con mil como con uno, *el millón* podría ser enseñado geometría y otras ramas de las matemáticas con gran facilidad, esto avanzaría el propósito de la educación más que cualquier cosa *podría* ser nombrada, porque enseñaría a la gente cómo pensar, y no qué pensar; es en este particular donde se origina el gran error de la educación.

Elucidaciones

La geometría tiene como objetos principales la exposición y explicación de las propiedades de la *figura*, y la figura se define como la relación que existe entre los límites del espacio. El espacio o la magnitud es de tres clases, *lineal*, *superficial* y *sólido*.



Los ángulos podrían considerarse apropiadamente como una cuarta especie de magnitud. La magnitud angular evidentemente consta de partes, y por lo tanto debe admitirse que es una especie de cantidad. El estudiante no debe suponer que la magnitud de un ángulo es afectada por la longitud de las líneas rectas que lo incluyen y de cuya divergencia mutua es la medida. El *vértice* de un



ángulo es el punto donde se encuentran los *lados* de las *patas* del ángulo, como A.

Un ángulo se designa a menudo por una sola letra cuando sus lados son las únicas líneas que se encuentran en su vértice. Así, las líneas roja y azul forman el ángulo amarillo, que en otros sistemas se llamaría ángulo A. Pero cuando más de dos líneas se encuentran en el mismo punto, era necesario por métodos anteriores, para evitar confusiones, emplear tres letras para designar un ángulo alrededor de ese punto, la letra que marcaba el vértice del ángulo se colocaba siempre en el medio. Así, las líneas negra y roja que se encuentran en C, forman el ángulo azul, y ha sido usualmente denominado el ángulo FCD o DCF. Las líneas FC y CD son los lados del ángulo; el punto C es su vértice. De manera similar, el ángulo negro se designaría el ángulo DCB o BCD. Los ángulos rojo y azul sumados, o el ángulo HCF sumado a FCD, forman el ángulo HCD; y así de otros ángulos.

Cuando los lados de un ángulo se producen o prolongan más allá de su vértice, los ángulos formados por ellos en ambos lados del vértice se dicen que son *opuestos por el vértice* entre sí: así, los ángulos rojo y amarillo se dicen que son ángulos opuestos por el vértice.

La superposición es el proceso por el cual una magnitud puede concebirse colocada sobre otra, de modo que la cubra exactamente, o de modo que cada parte de cada una coincida exactamente.

Una línea se dice que es *producida*, cuando es extendida, prolongada, o tiene su longitud aumentada, y el aumento de longitud que recibe se llama *parte producida*, o su *producción*.

La longitud total de la línea o líneas que encierran una figura, se llama su *perímetro*. Los primeros seis libros de Euclides tratan solo de figuras planas. Una línea trazada desde el centro de un círculo hasta su circunferencia, se llama *radio*. El lado de un triángulo rectángulo, que es opuesto al ángulo recto, se llama *hipotenusa*. Un oblongo se define en el segundo libro, y se llama *rectángulo*. Se supone que todas las líneas consideradas en los primeros seis libros de los Elementos están en el mismo plano.

La *regla* y el *compás* son los únicos instrumentos cuyo uso está permitido en Euclides, o en la Geometría plana. Declarar esta restricción es el objeto de los *postulados.*

Los *axiomas* de la geometría son ciertas proposiciones generales, cuya verdad se toma como autoevidente e incapaz de ser establecida por demostración.

Las *proposiciones* son aquellos resultados que se obtienen en geometría por un proceso de razonamiento. Hay dos especies de proposiciones en geometría, *problemas* y *teoremas*.

Un *problema* es una proposición en la que se propone hacer algo; como una línea que se va a trazar bajo algunas condiciones dadas, un círculo que se va a describir, alguna figura que se va a construir, etc.

La *solución* del problema consiste en mostrar cómo se puede hacer la cosa requerida con la ayuda de la regla o straight-edge y el compás.

La *demostración* consiste en probar que el proceso indicado en la solución alcanza el fin requerido.

Un *teorema* es una proposición en la que se afirma la verdad de algún principio. Este principio debe deducirse de los axiomas y definiciones, u otras verdades previamente y de forma independiente establecidas. Mostrar esto es el objeto de la demostración.

Un *problema* es análogo a un postulado.

Un *teorema* se asemeja a un axioma.

Un *postulado* es un problema, cuya solución se asume.

Un *axioma* es un teorema, cuya verdad se concede sin demostración.

Un *corolario* es una inferencia deducida inmediatamente de una proposición.

Un *escolio* es una nota u observación sobre una proposición que no contiene una inferencia de importancia suficiente para merecer el nombre de *corolario*.

Un *lema* es una proposición introducida meramente con el propósito de establecer una proposición más importante.

Símbolos y abreviaturas

$\therefore$ expresa la palabra *por lo tanto*.

$\because$ expresa la palabra *porque*.

$=$ expresa la palabra *igual*. Este signo de igualdad puede leerse *igual a*, o *es igual a*, o *son iguales a*; pero la discrepancia con respecto a la introducción de los verbos auxiliares *es*, *son*, etc. no puede afectar el rigor geométrico.

$\neq$ significa lo mismo que si se escribieran las palabras *'not equal'*.

$>$ significa *mayor que*.

$<$ significa *menor que*.

$\not>$ significa *no mayor que*.

$\not<$ significa *no menor que*.

$+$ se lee *más*, el signo de adición; cuando se interpone entre dos o más magnitudes, significa su suma.

$-$ se lee *menos*, significa la resta; y cuando se coloca entre dos cantidades, denota que la última se toma de la primera.

$\times$ este signo expresa el producto de dos o más números cuando se coloca entre ellos en aritmética y álgebra; pero en geometría se usa generalmente para expresar un *rectángulo*, cuando se coloca entre "dos líneas rectas que contienen uno de sus ángulos rectos." Un *rectángulo* también puede representarse colocando un punto entre dos de sus lados contiguos.

$: :: :$ expresa una *analogía* o *proporción*; así, si A, B, C y D representan cuatro magnitudes, y A tiene con B la misma razón que C tiene con D, la proporción se escribe brevemente así

$A : B :: C : D, A : B = C : D$, or $\dfrac{A}{B} = \dfrac{C}{D}$.

Esta igualdad o semejanza de razón se lee,
como A es a B, así es C a D;
o A es a B, como C es a D.

$\parallel$ significa *paralelo a*.

$\perp$ significa *perpendicular a*.

 significa *ángulo*.

 significa *ángulo recto*.

 significa *dos ángulos rectos*.

 or  briefly designates a *punto*.

The square described on a line is concisely written thus, $\rule{1cm}{2pt}^2$.

In the same manner twice the square of, is expressed by $2 \cdot \rule{1cm}{2pt}^2$.

def. significa *definición*.

post. significa *postulado*.

ax. significa *axioma*.

hyp. significa *hipótesis*. Puede ser necesario aquí señalar que *hipótesis* es la condición asumida o dada por sentada. Así, la hipótesis de la proposición dada en la Introducción, es que el triángulo es isósceles, o que sus lados son iguales.

const. significa *construcción*. La *construcción* es el cambio realizado en la figura original, dibujando líneas, haciendo ángulos, describiendo círculos, etc. para adaptarla al argumento de la demostración o a la solución del problema. Las condiciones bajo las cuales se realizan estos cambios, son tan indiscutibles como las contenidas en la hipótesis. Por ejemplo, si hacemos un ángulo igual a un ángulo dado, estos dos ángulos son iguales por construcción.

Q. E. D. significa *Quod erat demonstrandum*. Lo que se quería demostrar.

Book I

Definiciones

I.1

Un *punto* es el que no tiene partes.

I.2

Una *línea* es longitud sin ancho.

I.3

Los extremos de una línea son puntos.

I.4

Una línea recta o recta es la que se encuentra uniformemente entre sus extremos.

I.5

Una superficie es aquella que sólo tiene longitud y ancho.

I.6

Los extremos de una superficie son líneas.

I.7

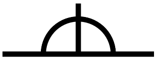
Una superficie plana es la que se encuentra uniformemente entre sus extremos.

I.8

Un ángulo plano es la inclinación de dos líneas entre sí, que se encuentran en un plano, pero no están en la misma dirección.

I.9

Un ángulo rectilíneo plano es la inclinación de dos líneas rectas entre sí, que se encuentran, pero no están en la misma línea recta.



I.10

Cuando una línea recta parada sobre otra línea recta hace iguales los ángulos adyacentes, cada uno de estos ángulos es llamado *ángulo recto*, y se dice que cada una de estas líneas es *perpendicular* a la otra.

I.11

Un ángulo obtuso es un ángulo mayor que un ángulo recto.

I.12

Un ángulo agudo es menor que un ángulo recto.

I.13

Un borde o límite es el extremo de cualquier cosa.

I.14

Una figura es una superficie encerrada en todos los lados por una línea o líneas.

I.15

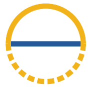
Un es una figura plana, delimitada por una línea continua, llamada circunferencia o periferia; y tiene un cierto punto dentro de él, desde el cual todas las líneas rectas dibujadas a su circunferencia son iguales.

I.16

Este punto (desde el cual se dibujan las líneas iguales) se llama el centro del círculo.

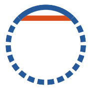
I.17

El es una línea recta dibujada a través del centro, terminada en ambos sentidos en la circunferencia.



I.18

Un semicírculo es la figura contenida por el diámetro, y la parte del círculo cortada por el diámetro.



I.19

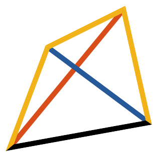
Un segmento circular es una figura contenida por una línea recta y la parte de la circunferencia que corta.

I.20

Una figura contenida solo por líneas rectas, se llama figura rectilínea.

I.21

Un triángulo es una figura rectilínea contenida por tres lados.



I.22

Una figura cuadrilátera es aquella que está limitada por cuatro lados. Las líneas rectas y que conectan los vértices de los ángulos opuestos de una figura cuadrilátera, son denominadas sus diagonales.

I.23

Un polígono es una figura rectilínea delimitada por más de cuatro lados.

I.24

Un polígono es una figura rectilínea delimitada por más de cuatro lados.

I.25

Un triángulo cuyos tres lados son iguales, se dice ser equilátero.

I.26

Un triángulo escaleno es uno que no tiene dos lados iguales.

I.27

Un triángulo escaleno es uno que no tiene dos lados iguales.

Un triángulo rectángulo es el que tiene un .

I.28

Un triángulo obtusángulo es aquel que tiene un .

I.29

Un triángulo acutángulo es aquel que tiene tres ángulos agudos.

I.30

De las figuras de cuatro lados, un cuadrado es aquel que tiene todos sus lados iguales y todos sus ángulos, ángulos rectos.

I.31

Un rombo es aquel que tiene todos sus lados iguales, pero sus ángulos no son ángulos rectos.

I.32

Un oblongo es aquel que tiene todos sus ángulos, ángulos rectos, pero no tiene todos sus lados iguales.

I.33

Un romboide es aquel que tiene sus lados opuestos iguales entre sí, pero todos sus lados no son iguales, ni sus ángulos son ángulos rectos.

I.34

Todas las demás figuras cuadriláteras se llaman trapecios.

I.35

Las líneas rectas paralelas son como las que se encuentran en el mismo plano, y que prolongadas continuamente en ambas direcciones, nunca se encontrarán.

Postulados

I.1

Deje que sea aceptado que una línea recta puede ser dibujada desde cualquier punto a cualquier otro punto.

I.2

Deje que sea aceptado que una línea recta finita puede ser prolongada a cualquier longitud en una línea recta.

I.3

Deje que sea aceptado que un círculo puede ser trazado con cualquier centro a cualquier distancia de ese centro.

Axiomas

I.1

Las magnitudes que son iguales a lo mismo son iguales entre sí.

I.2

Si se suma igual a igual, las sumas serán iguales.

I.3

Si se quita igual a igual, el residuo será igual.

I.4

Si se agregan iguales a desiguales, las sumas serán desiguales.

I.5

Si se eliminan iguales de desiguales, el residuo será desigual.

I.6

Los dobles de lo mismo o magnitudes iguales son iguales.

I.7

Las mitades de lo mismo o magnitudes iguales son iguales.

I.8

Las magnitudes que coinciden entre sí, o que llenan exactamente el mismo espacio, son iguales.

I.9

El todo es mayor que su parte.

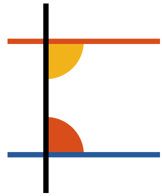
I.10

Dos líneas rectas no pueden contener un espacio.

I.11

Todos los ángulos rectos son iguales.

I.12



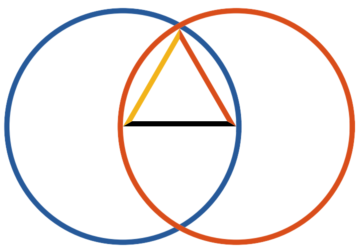
Si dos líneas rectas (y) se encuentran con una tercera línea recta ( ) para hacer que los dos ángulos interiores (y) en el mismo lado sean menores que dos ángulos rectos, estas dos líneas rectas se encontrarán si se prolongan en el lado en el que los ángulos son menores que dos ángulos rectos.

El duodécimo axioma puede ser expresado en cualquiera de las siguientes maneras:

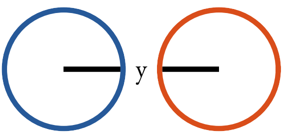
1. Dos líneas rectas divergentes no pueden ser ambas paralelas a la misma línea recta.
2. Si una línea recta corta una de dos líneas rectas paralelas, debe también cortar la otra.
3. Solo se puede trazar una línea recta a través de un punto dado, paralela a una línea recta dada.

Proposiciones

I.1 Prop. I. Prob.



En una línea recta finita dada ( ) para trazar un triángulo equilátero.

Traza  (??);

dibuja y (??).

Entonces  será equilátero.

Para = (??); y = (??),

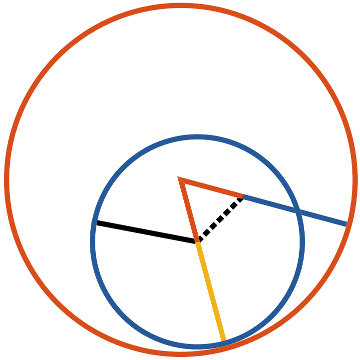
∴ = (??);

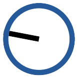
y por lo tanto es el triángulo equilátero requerido.

Q. E. D.

Q. E. D.

De un punto dado ( ), dibujar una línea recta igual a una línea recta finita dada ( ).

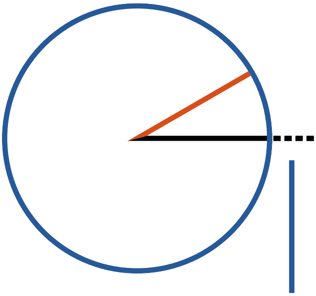


Dibuja (??), traza  (??), prolonga (??), traza  (??), y  (??); prolonga (??), entonces es la línea requerida.

Para = (??), y = (conft.), ∴ = (??), pero (??) = = ; ∴ dibujada de un punto dado ( ), es igual a la línea dada .

Q. E. D. (??)

Q. E. D.



FROM *the greater ( ) of two given straight lines, to cut off a part equal to the less ( ).*

Draw = (??);

describe  (??),

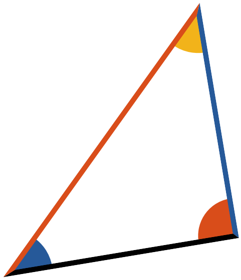
then =

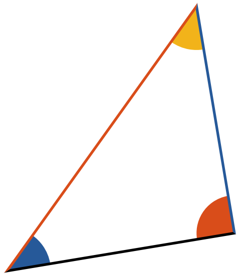
For = (??),

and = (??);

∴ = (??).

Q. E. D.

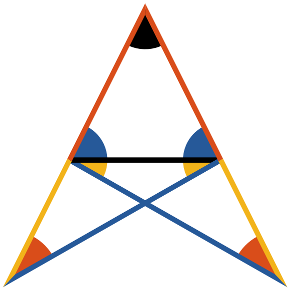




IF *two triangles have two sides of the one respectively equal to two sides of the other, (to and to) and the angles (and) contained by those equal sides also equal; then their bases or their sides (and) are also equal: and the remaining angles opposite to equal sides are respectively equal (= and =): and the triangles are equal in every respect.*

Let two triangles be conceived, to be so placed, that the vertex of the one of the equal angles, or ; shall fall upon that of the other, and to coincide with , then will coincide with if applied: consequently will coincide with , or two straight lines will enclose a space, which is impossible (**??**), therefore = , = and = , and as the triangles and coincide, when applied, they are equal in every respect.

Q. E. D.



IN *any isosceles triangle if the equal sides be produced, the external angles at the base are equal, and the internal angles at the base are also equal.*

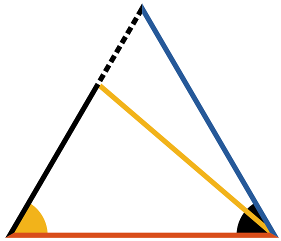
Produce and (??),
take = (??);
draw and .

Then in and
we have = (??),
common to both,
and = (??)
∴ = , = and = (??).

Again in and
we have = , = and
= ,
∴ = and = (??)
but = , ∴ = (??).

Q. E. D.

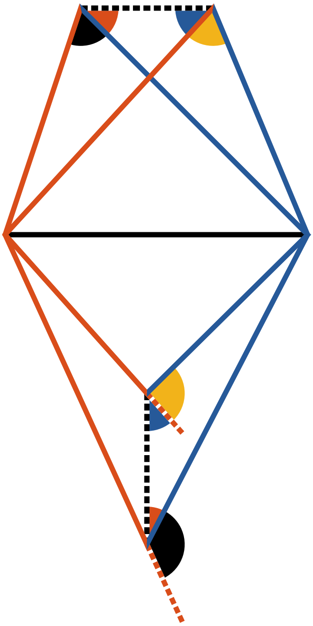
In any triangle ( ) if two angles (and) are equal, the sides (and) opposite to them are also equal.



For if the sides be not equal, let one of them be greater than the other , and from it to cut off = (??), draw .

Then in and ,
= (??),
= (??)
and common,
∴ the triangles are equal (??)
a part equal to the whole, which is absurd;
∴ neither of the sides or is greater than the other,
hence they are equal.

Q. E. D.



On the same base ( ), and on the same side of it there cannot be two triangles having their conterminous sides (and , and) at both extremities of the base, equal to each other.

When two triangles stand on the same base, and on the same side of it, the vertex of the one shall either fall outside of the other triangle, or within it; or, lastly, on one of its sides.

If it be possible let the two triangles be constructed so that $\left\{ \begin{array}{c} = \\ = \end{array} \right\}$, then draw and,

= (??)

∴ < and

∴ <

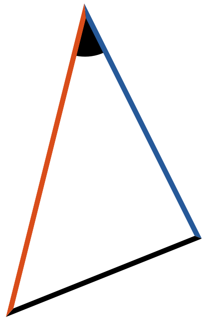
but (??) =

which is absurd,

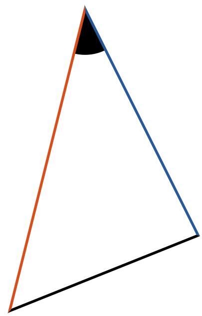
therefore the two triangles cannot have their conterminous sides equal at both extremities of the base.

Q. E. D.

If two triangles have two sides of the one respectively equal to two sides of the other (= and =) and also their bases (=), equal; then the angles (and) contained by their equal sides are also equal.

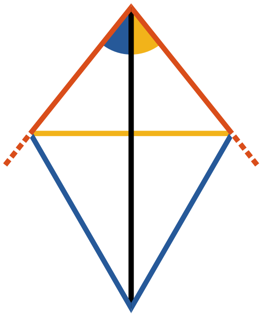


If the equal bases and be conceived to be placed one upon the other, so that the triangles shall lie at the same side of them, and that the equal sides and , and be conterminous, the vertex of the one must fall on the vertex of the other; for to suppose them not coincident would contradict the last proposition.



Therefore sides and , being coincident with and , ∴ = .

Q. E. D.



T*o bisect a given rectilinear angle* ( ).

Take = (??)

draw , upon which describe (??),

draw .

∵ = (??)

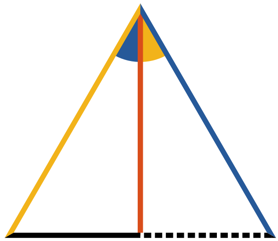
and common to the two triangles

and = (??),

∴ = (??).

Q. E. D.

T*o bisect a given finite straight line* ( ).



Construct  (??),

draw , making = (??).

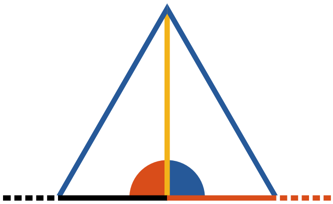
Then = by (??),

for = (??) =

and common to the two triangles.

Therefore the given line is bisected.

Q. E. D.

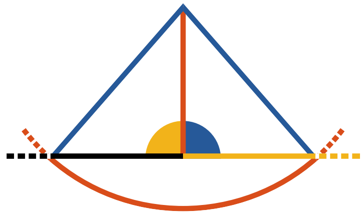


F*ROM a given point ( ), in a given straight line ( ), to draw a perpendicular.*

Take any point ( ) in the given line,
cut off = (??),
construct (??),
draw and it shall be perpendicular to the given line.
For = (??)
= (??)
and common to the two triangles.
Therefore = (??)
∴ ⊥ (??).

Q. E. D.

To *draw a straight line perpendicular to a given indefinite straight line ( ) from a given point ( ) without.*



With the given point as centre, at one side of the line, and any distance capable of extending to the other side, describe .

Make = (??),

draw , and ,

then ⊥ .

For (??) since = (??),

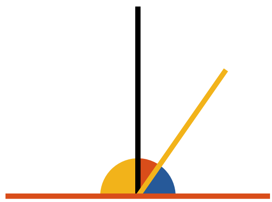
common to both,

and = (??),

∴ = , and

∴ ⊥ (??).

Q. E. D.



WHEN *a straight line ( ) standing upon another straight line ( ) makes angles with it; they are either two right angles or together equal to two right angles.*

If be ⊥ to then,

and = (??),

but if be not ⊥ to ,

draw ⊥ (??);

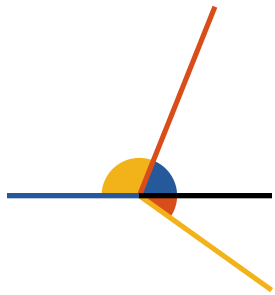
+ = (??),

= = +

∴ + = + + (??)

= + = .

Q. E. D.



I*f two straight lines (and), meeting a third straight line ( ), at the same point, and at opposite sides of it, make with it adjacent angles (and) equal to two right angles; these straight lines lie in one continuous straight line.*

For, if possible let , and not ,
be the continuation of ,

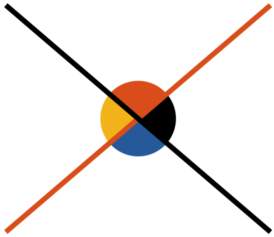
then + =

but by the hypothesis + =

∴ = (**??**); which is absurd (**??**).

∴ is not the continuation of , and the like may be demonstrated of any other straight line except , ∴ is the continuation of .

Q. E. D.



IF *two right lines (and) intersect one another, the vertical angles and , and are equal.*

$+ =$

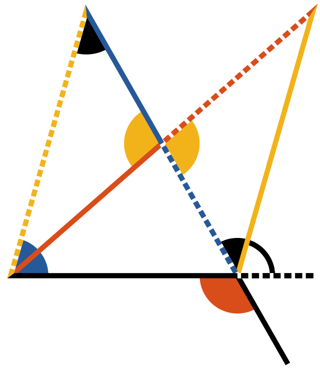
$+ =$

$\therefore =$.

In the same manner it may be shown that

$=$.

Q. E. D.

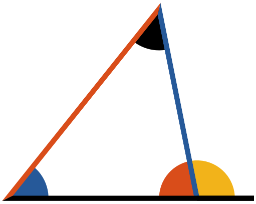


If a side of a triangle ( ) is produced, the external angle ( ) is greater than either of the internal remote angles (or).

Make = (??);
Draw and produce it until = ;
draw .
In and ;

= , = and
= (?? and ??),
∴ = (??),
∴ > .
In like manner it can be shown, that if be produced, >
and therefore which is = is > .

Q. E. D.



ANY *two angles of a triangle*  *are together less than two right angles.*

Produce , then will

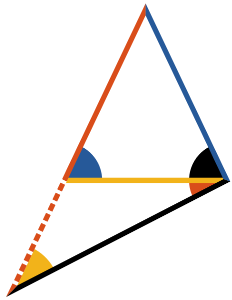
 +  = .

But  >  (??)

∴  +  < ,

and in the same manner it may be shown that any other two angles of the triangle taken together are less than two right angles.

Q. E. D.



IN *any triangle* *if one side* *be greater than another* *, the angle opposite to the greater side is greater than the angle opposite to the less. I. e.* > .

Make = (??), draw .

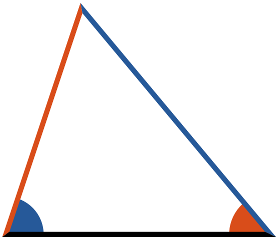
Then will = (??);

but > (??)

∴ > and much more

is > .

Q. E. D.



If in any triangle  *one angle*  *be greater than another*  *the side* *which is opposite to the greater angle, is greater than the side* *opposite the less.*

If be not greater than then must

= or < .

If = then

 =  (??);

which is contrary to the hypothesis.

is not less than ; for if it were,

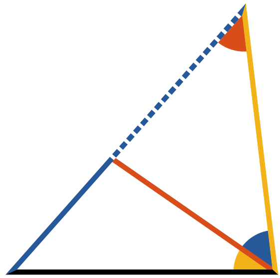
 <  (??)

which is contrary to the hypothesis:

∴ > .

Q. E. D.

ANY *two sides* and *of a triangle* taken together are greater than the third side ( ).*



Produce , and

make = (??);

draw .

Then ∵ = (??),

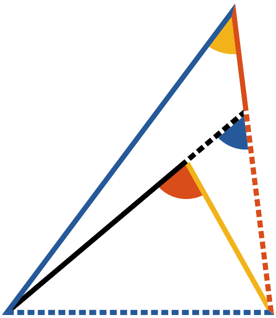
= (??)

∴ > (??)

∴ + > (??)

and ∴ + > .

Q. E. D.



If from any point ( ) within a triangle straight lines be drawn to the extremities of one side ( ), these lines must be together less than the other two sides, but must contain a greater angle.

Produce ,

+ > (??),

add to each,

+ > + (??)

in the same manner it may be shown that

+ > + ,

∴ + > + ,

which was to be proved.

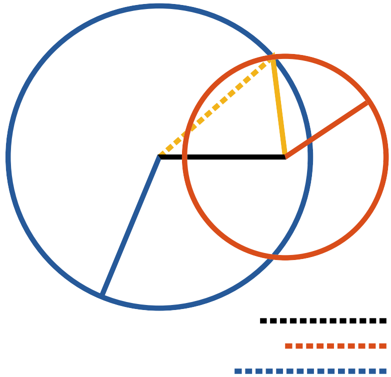
Again > (??),

and also > (??),

∴ > .

Q. E. D.

G IVEN *three right lines* { *the sum of any two greater than the third, to construct a triangle whose sides shall be respectively equal to the given lines.*

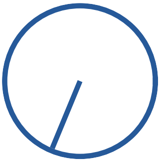


Assume = (??).

Draw = } (??).

and =

With and as radii, describe

 and  (??);

draw and ,

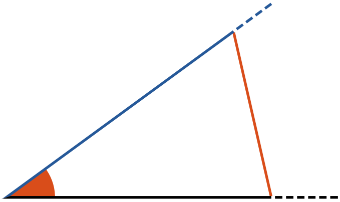
then will  be the triangle required.

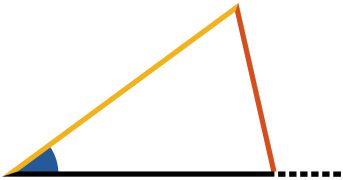
For = ,

= = , } (??)

and = = .

Q. E. D.





At a given point () *in a given straight line* ( ), *to make an angle equal to a given rectilinear angle* ().

Draw between any two points in the legs of the given angle.

Construct  (??)

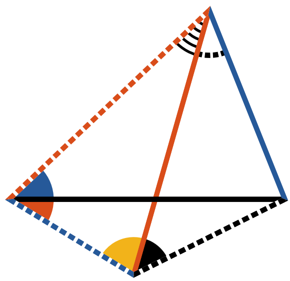
so that = , =

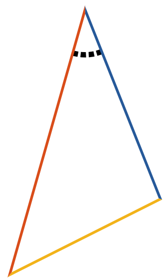
and = .

Then  =  (??).

Q. E. D.

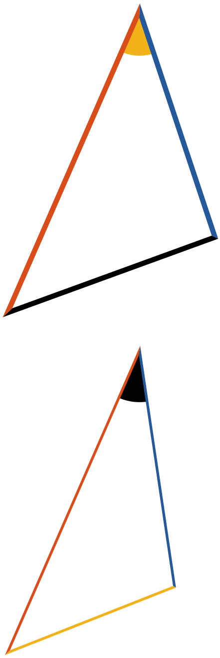
If two triangles have two sides of the one respectively equal to two sides of the other (to and to), and if one of the angles ( ) contained by the equal sides be greater than the other ( ), the side ( ) which is opposite to the greater angle is greater than the side ( ) which is opposite to the less angle.





Make = (??),
and = (??),
draw and .
∵ = (??, ?? and ??)
∴ = (??)
but < ,
and ∴ < ,
∴ > (??)
but = (??)
∴ > .

Q. E. D.



If two triangles have two sides (and) of the one respectively equal to two sides (and) of the other, but their bases unequal, the angle subtended by the greater base ( ) of the one, must be greater than the angle subtended by the less base ( ) of the other.

= , > or <

is not equal to

for if = then = (??)

which is contrary to the hypothesis;

is not less than

for if <

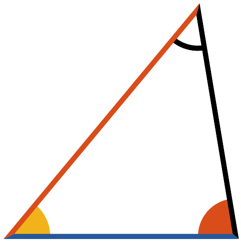
then < (??),

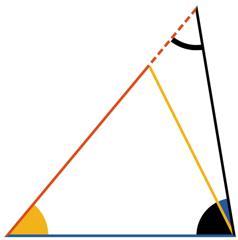
which is also contrary to the hypothesis:

∴ > .

Q. E. D.

IF *two triangles have two angles of the one respectively equal to two angles of the other (= and =), and a side of the one equal to a side of the other similarly placed with respect to the equal angles, the remaining sides and angles are respectively equal to one another.*





Case I.

Let and which lie between the equal angles be equal,

then = .

For if it be possible, let one of them be greater than the other;

make = , draw .

In  and  we have

= , = , = ;

∴ = (pr. 4.)

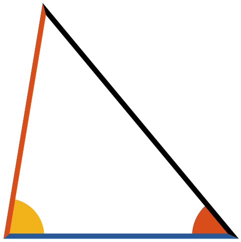
but = (??)

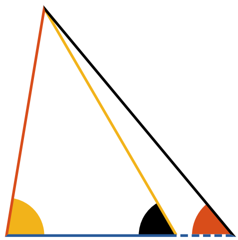
and therefore = , which is absurd;

hence neither of the sides and is greater than the other;

and ∴ they are equal;

∴ = , and = , (??).

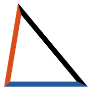
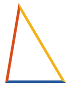




Case II.

Again, let = , which lie opposite the equal angles and .

If it be possible, let > , then take = , draw .

Then in  and  we have

= , = and

= ,

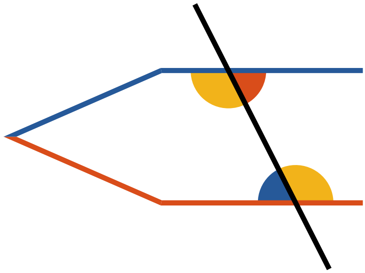
∴ = (??)

but = (??)

∴ = which is absurd (??).

Consequently, neither of the sides or is greater than the other, hence they must be equal. It follows (by (??)) that the triangles are equal in all respects.

Q. E. D.

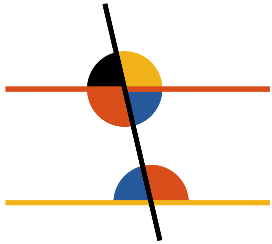


If a straight line ( ) meeting two other straight lines (and) makes with them the alternate angles (and ; and) equal, these two straight lines are parallel.

If be not parallel to they shall meet when produced.

If it be possible, let those lines be not parallel, but meet when produced; then the external angle is greater than (**??**), but they are also equal (**??**), which is absurd: in the same manner it may be shown that they cannot meet on the other side; ∴ they are parallel.

Q. E. D.



IF a straight line ( ), cutting two other straight lines (and), makes the external equal to the internal and opposite angle, at the same side of the cutting line (namely = or =), or if it makes the two internal angles at the same side (and , or and) together equal to two right angles, those two straight lines are parallel.

First, if = , then = (??),

∴ = ∴ ∥ (??).

Secondly, if + = ,

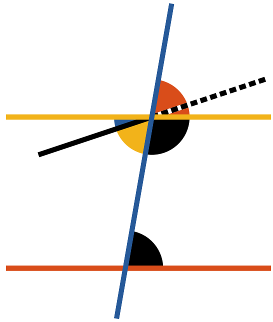
then + = (??),

∴ + = + (??)

∴ =

∴ ∥ (??).

Q. E. D.



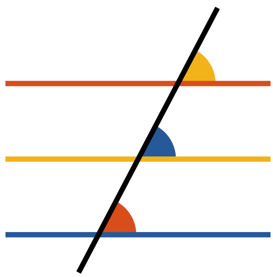
A STRAIGHT *line ( ) falling on two parallel straight lines (and), makes the alternate angles equal to one another; and also the external equal to the internal and opposite angle on the same side; and the two internal angles on the same side together equal to two right angles.*

For if the alternate angles and be not equal, draw , making = (??).

Therefore ∥ (??) and therefore two straight lines which intersect are parallel to the same straight line, which is impossible (??).

Hence and are not unequal, that is, they are equal: = (??); ∴ = , the external angle equal to the internal and opposite on the same side: if be added to both, then + = = (??). That is to say, the two internal angles at the same side of the cutting line are equal to two right angles.

Q. E. D.



STRAIGHT *lines (and) which are parallel to the same straight line ( ), are parallel to one another.*

Let intersect $\{\ \}$;

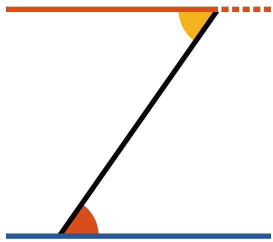
Then, $=$ $=$ (??),

$\therefore$ $=$

$\therefore$ $\parallel$ (??).

Q. E. D.

FROM *a given point* *to draw a straight line parallel to a given straight line* ( ).

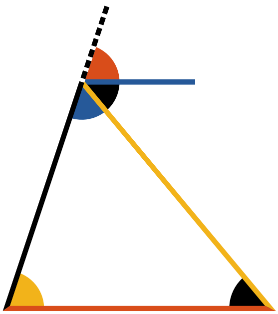


Draw from the point to any point in ,

make = (??),

then || (??).

Q. E. D.



IF any side ( ) of a triangle be produced, the external angle ( ) is equal to the sum of the two internal and opposite angles (and), and the three internal angles of any triangle taken together are equal to two right angles.

Through the point draw

$\parallel$ (??).

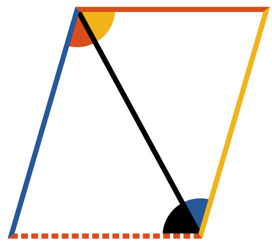
Then $\left\{ \begin{array}{c} = \\ = \end{array} \right\}$ (??),

$\therefore$ $+$ $=$ (??),

and therefore

$+$ $+$ $=$ $=$ (??).

Q. E. D.



STRAIGHT *lines (and) which join the adjacent extremities of two equal and parallel straight lines (and), are themselves equal and parallel.*

Draw the diagonal.

= (??)

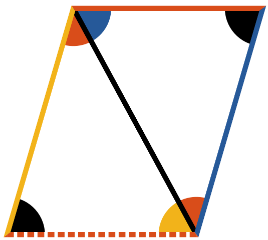
= (??)

and common to the two triangles;

∴ = , and = (??);

and ∴ ∥ (??).

Q. E. D.



THE *opposite sides and angles of any parallelogram are equal, and the diagonal ( ) divides it into two equal parts.*

Since { = ; = } (??)

and common to the two triangles.

∴ { = ; = ; = } (??)

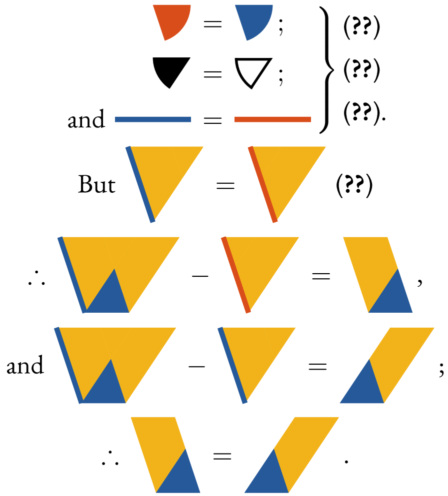
and = (??).

Therefore the opposite sides and angles of the parallelogram are equal: and as the triangles  and  are equal in every respect (??), the diagonal divides the parallelogram into two equal parts.

Q. E. D.

PARALLELOGRAMS *on the same base, and between the same parallels, are (in area) equal.*

On account of the parallels,



Q. E. D.

PARALLELOGRAMS *(*  *and*  *) on equal bases, and between the same parallels, are equal.*

Draw and

= = by (?? and ??);

∴ = and ‖ ;

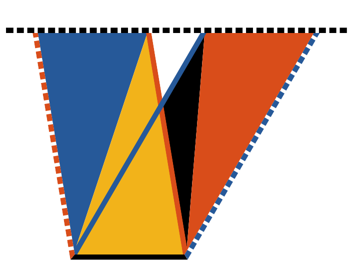
∴ = and ‖ (??)

And therefore  is a parallelogram:

but  =  =  (??)

∴  =  (??).

Q. E. D.



TRIANGLES and on the same base ( ) *and between the same parallels are equal.*

Draw || , || (??)

Produce .

and are parallelograms on the same base and between the same parallels, and therefore equal. (??)

$$\therefore \left\{ \begin{array}{l} = \text{twice} \\ = \text{twice} \end{array} \right\} \text{(??)} \therefore \; = \; .$$

Q. E. D.

TRIANGLES ( *and* ) *on equal bases and between the same parallels are equal.*

Draw $\|$
and $\|$ } (??)

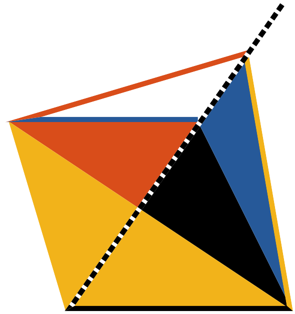
 =  (??);

but  = twice  (??),

and  = twice  (??),

$\therefore$  =  (??).

Q. E. D.



EQUAL *triangles* and *on the same base* ( ) *and on the same side of it, are between the same parallels.*

If , which joins the vertices of the triangles, be not ∥ , draw ∥ (??), meeting .

Draw .

∵ ∥ (??)

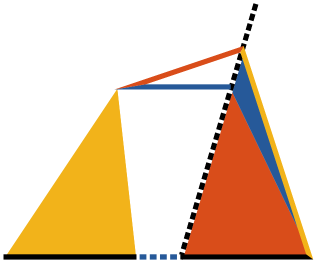
= (??);

but = (??);

∴ = , a part equal to the whole, which is absurd.

∴ ∦ ; and in the same manner it can be demonstrated, that no other line except is ∥ ; ∴ ∥ .

Q. E. D.



EQUAL *triangles (and) on equal bases, and on the same side, are between the same parallels.*

If which joins the vertices of triangles be not ∥ ,

draw ∥ (??),

meeting .

Draw .

∵ ∥ (??)

= but =

∴ = , a part equal to the whole, which is absurd.

∴ ∦ : and in the same manner it can be demonstrated, that no other line except is ∥ : ∴ ∥ .

Q. E. D.

IF a parallelogram  *and a triangle*  *are upon the same base* and *between the same parallels* and, *the parallelogram is double the triangle.*

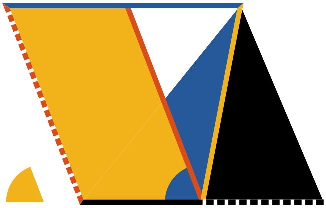
Draw the diagonal.

Then  =  (??)

 = twice  (??)

∴  = twice  .

Q. E. D.



TO *construct a parallelogram equal to a given triangle*  *and having an angle equal to a given rectilinear angle* .

Make = (??)

Draw .

Make  =  (??)

Draw $\left\{\begin{array}{c} \parallel \\ \parallel \end{array}\right\}$ (??)

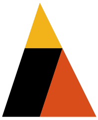
 = twice  (??)

but  =  (??)

$\therefore$  = .

Q. E. D.

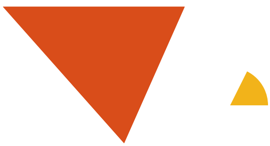
THE *complements*  *and*  *of the parallelograms which are about the diagonal of a parallelogram are equal.*

 =  (??)

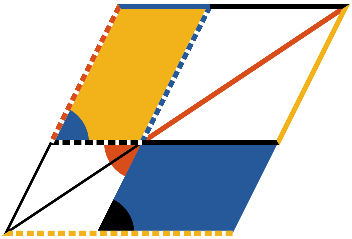
and  =  (??)

$\therefore$  =  (??).

Q. E. D.



To a given straight line ( ) to apply a parallelogram equal to a given triangle ( ), and having an angle equal to a given rectilinear angle ( ).



Make = with = (??)
and having one of its sides conterminous with
and in continuation of .
Produce till it meets ||
draw produce it till it meets continued;
draw || meeting produced and
produce .

= (??)

but = (??)

∴ = ;

and = = = (?? and ??).

Q. E. D.

To *construct a parallelogram equal to a given rectilinear figure (*  *) and having an angle equal to a given rectilinear angle (*  *).*

Draw and dividing the rectilinear figure into triangles.

Construct  = 

having  =  (??)

to apply  = 

having  =  (??)

to apply  = 

having  =  (??)

∴  = 

and  is a parallelogram. (??, ?? and ??)

having  = .

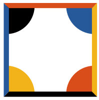
Q. E. D.

UPON *a given straight line ( ) to construct a square.*

Draw ⊥ and = (?? and ??)

Draw ∥ ,

and meeting drawn ∥ .

In  = (??)

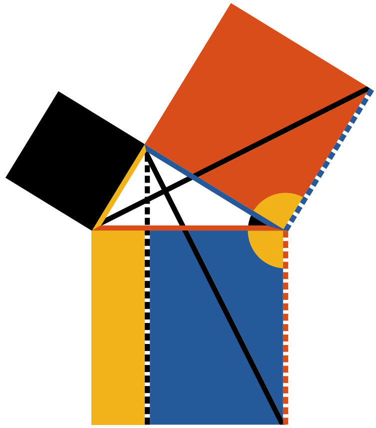
 = a right angle (??)

∴  =  = a right angle (??),

and the remaining sides and angles must be equal (??).

And ∴  is a square (??).

Q. E. D.



I*N a right angled triangle the square on the hypotenuse is equal to the sum of the squares of the sides (and).*

On , , describe squares, (??)

Draw ∥ (??)

also draw and .

= .

To each add ∴ = ,

= and = ;

∴ = .

Again, ∵ ∥

= twice ,

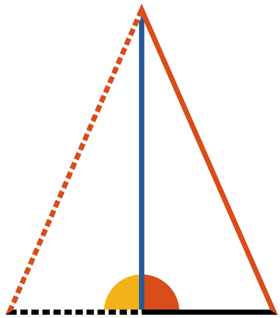
and = twice ;

∴ = .

In the same manner it may be shown that = ;

hence = .

Q. E. D.



I*f the square of one side ( ) of a triangle is equal to the squares of the other two sides (and), the angle ( ) subtended by that side is a right angle.*

Draw $\perp$ and $=$ (?? and ??)

and draw also.

Since $=$ (??)

${}^2 = {}^2$;

$\therefore {}^2 + {}^2 = {}^2 + {}^2$

but ${}^2 + {}^2 = {}^2$ (??),

and ${}^2 + {}^2 = {}^2$ (??)

$\therefore {}^2 = {}^2$,

$\therefore =$;

and $\therefore =$ (??),

consequently is a right angle.

Q. E. D.

Book II

II.1 Definition I

A RECTANGLE *or a right angled parallelogram is said to be contained by any two of its adjacent or conterminous sides.*

Thus: the right angled parallelogram  is said to be contained by the sides and ; or it may be briefly designated by · .

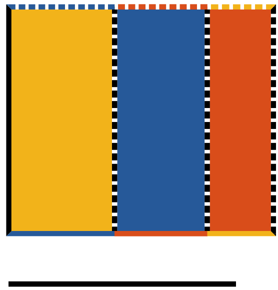
If the adjacent sides are equal; i. e. = , then · which is the expression for the rectangle under and is a square, and

is equal to $\begin{cases} \cdot \quad \text{or} \quad {}^2 \\ \cdot \quad \text{or} \quad {}^2 \end{cases}$

II.2 Definition II

In *a parallelogram, the figure composed of one of the parallelograms about the diagonal, together with the two complements, is called a Gnomon.*

Thus  and  are called Gnomons.



THE *rectangle contained by two straight lines, one of which is divided into any number of parts,* $\cdot = \left\{ \begin{array}{l} \cdot \\ + \cdot \\ + \cdot \end{array} \right.$

is equal to the sum of the rectangles contained by the undivided line, and the several parts of the divided line.

Draw $\perp$ and $=$ (?? and ??);

complete the parallelograms, that is to say,

draw $\left\{ \begin{array}{c} \parallel \\ \parallel \end{array} \right\}$ (??)

$= + +$

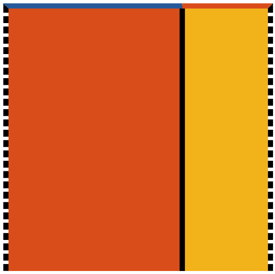
$= \cdot$

$= \cdot$, $= \cdot$,

$= \cdot$

$\therefore \cdot = \cdot + \cdot + \cdot$.

Q. E. D.



IF a straight line be divided into any two parts , the square of the whole line is equal to the sum of the rectangles contained by the whole line and each of its parts.

Describe (??)

Draw parallel to (??)

Q. E. D.

If a straight line be divided into any two parts , the rectangle contained by the whole line and either of its parts, is equal to the square of that part, together with the rectangle under the parts.

$\cdot = {}^2 + \cdot$, *or*

$\cdot = {}^2 + \cdot$.

Describe (??)

Describe (??)

Then $=$ $+$, but

$= \cdot$ and

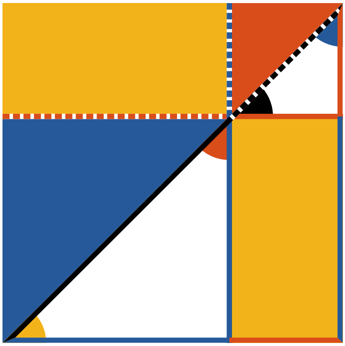
$= {}^2$, $= \cdot$,

$\therefore \cdot = {}^2 + \cdot$.

In a similar manner it may be readily shown that

$\cdot = {}^2 + \cdot$.

Q. E. D.



IF *a straight line be divided into any two parts, the square of the whole line is equal to the squares of the parts, together with twice the rectangle contained by the parts.*

$^2 = {}^2 + {}^2 +$ *twice* $\cdot$

Describe (??)

draw (??)

and $\left\{ \begin{array}{c} \parallel \\ \parallel \end{array} \right\}$ (??)

$=$ (??),

$=$ (??),

$\therefore$ $=$

$\therefore$ by (??, ?? and ??)

is a square $=$ ^{2}.

For the same reasons is a square $=$ ^{2},

$=$ $=$ $\cdot$ (??)

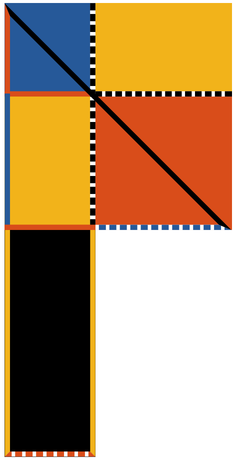
But $=$ $+$ $+$ $+$,

$\therefore$ $^2 =$ $^2 +$ $^2 +$ twice $\cdot$.

Q. E. D.

If a straight line be divided into two equal parts and also into two unequal parts, the rectangle contained by the unequal parts, together with the square of the line between the points of section, is equal to the square of half that line

$\cdot \quad + \quad {}^2 = \quad {}^2 = \quad {}^2.$



Describe (??),

draw

and $\left\{ \begin{array}{c} \parallel \\ \parallel \\ \parallel \end{array} \right\}$ (??)

$=$ (??)

$=$ (??)

$\therefore$ (??) $= \quad = \quad \cdot$ but $= \quad {}^2$

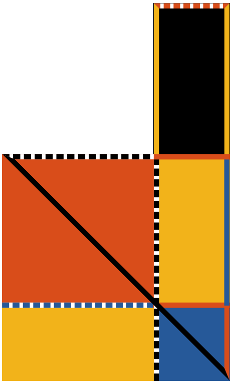
(??)

and $= \quad {}^2$ (??)

$\therefore$ (??) $=$

$\therefore \quad \cdot \quad + \quad {}^2 = \quad {}^2 = \quad {}^2.$

Q. E. D.



If a straight line be bisected and produced to any point , the rectangle contained by the whole line so increased, and the part produced, together with the square of half the line, is equal to the square of the line made up of the half, and the produced part.

$\cdot \; + \; {}^2 = {}^2$

Describe (??), draw

and { ‖ ; ‖ ; ‖ } (??)

$=$ $=$ (?? and ??)

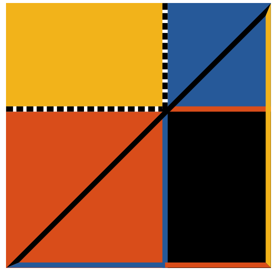
$\therefore$ $=$ $=$ $\cdot$;

but $=$ ^{2} (??)

$\therefore$ $=$ $^2 =$ (?? and ??)

$\therefore \; \cdot \; + \; {}^2 = {}^2.$

Q. E. D.



If a straight line be divided into any two parts, the squares of the whole line and one of the parts are equal to twice the rectangle contained by the whole line and that part, together with the square of the other parts.

$^2 + {}^2 = 2 \cdot + {}^2$

Describe (??), draw (??),

and $\left\{ \begin{array}{c} \parallel \\ \parallel \end{array} \right\}$

$=$ (??),

add $=$ ^{2} to both (??)

$= = \cdot$

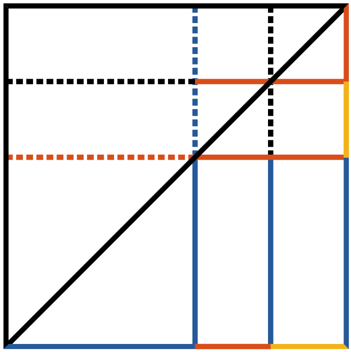
$= {}^2$ (??)

$+ + =$

$2 \cdot + {}^2 = + ;$

$^2 + {}^2 = 2 \cdot + {}^2.$

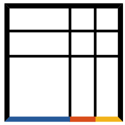
Q. E. D.



I*f a straight line be divided into any two parts* , *the square of the sum of the whole line and any one of its parts is equal to four times the rectangle contained by the whole line, and that part together with the square of the other part.*

$$^{2} = 4 \cdot \cdot + ^{2}$$

Produce and make $=$

Construct  (??);

draw ,

} ∥ } (??)

} ∥

$$^{2} = ^{2} + ^{2} + 2 \cdot \cdot$$

(??)

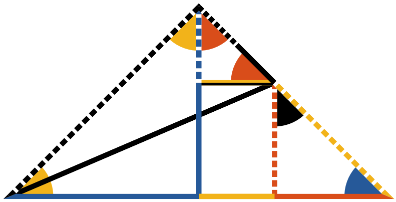
but $^{2} + ^{2} = 2 \cdot \cdot + ^{2}$

(??)

$$\therefore \; ^{2} = 4 \cdot \cdot + ^{2}.$$

Q. E. D.

If a straight line be divided into two equal parts and also into two unequal parts, the squares of the unequal parts are together double the squares of half the line, and of the part between the points of section.



$^2 + {}^2 = 2 \cdot {}^2 + 2 \cdot {}^2$

Make ⊥ and = or ,
Draw and ,
∥ , ∥ and draw .

= (??) = half a right angle (??)

= (??) = half a right angle (??)

∴ = a right angle.

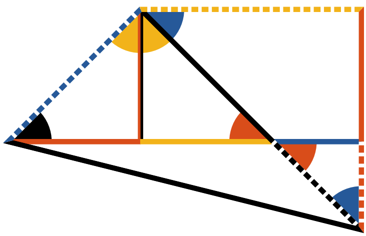
= = = (?? and ??).

hence = , = = (?? and ??)

$^2 = \begin{cases} {}^2 + {}^2, \text{ or } + {}^2 \\ = \begin{cases} = {}^2 + {}^2 \\ = 2 \cdot {}^2 + 2 \cdot {}^2 \end{cases} \end{cases}$ (??)

$\therefore {}^2 + {}^2 = 2 \cdot {}^2 + 2 \cdot {}^2.$

Q. E. D.



If a straight line be bisected and produced to any point, the squares of the whole produced line, and of the produced part, are together double of the squares of the half line, and of the line made up of the half and produced part.

$$\;^2 + \;^2 = 2\cdot\;^2 + 2\cdot\;^2$$

Make $\perp$ and $=$ to or ,
draw and ,
and $\left\{ \begin{array}{c} \parallel \\ \parallel \end{array} \right\}$ (??)
draw also.

$=$ (??) $=$ half a right angle (??)

$=$ (??) $=$ half a right angle (??)

$\therefore$ $=$ a right angle.

$= = = =$
half a right angle (??, ??, ?? and ??),
and $=$, $=$ $=$, (?? and ??).

Hence by (??)

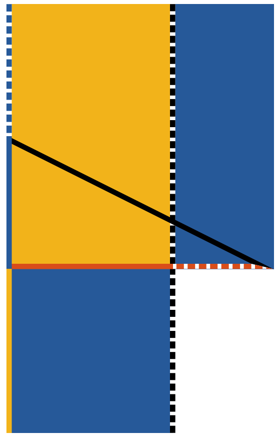
$$\;^2 = \left\{ \begin{array}{l} \;^2 + \;^2 \text{ or } \;^2 \\ \left\{ \begin{array}{l} + \;^2 = 2\cdot\;^2 \\ + \;^2 = 2\cdot\;^2 \end{array} \right. \end{array} \right.$$

$$\therefore\; \;^2 + \;^2 = 2\cdot\;^2 + 2\cdot\;^2.$$

Q. E. D.

To *divide a given straight line in such a manner, that the rectangle contained by the whole line and one of its parts may be equal to the square of the other.*

$\cdot \quad = \quad {}^2$



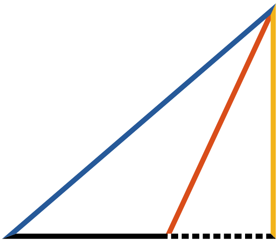
Describe  (??),
make $=$ (??),
draw ,
take $=$ (??),
on describe  (??).
Produce (??).

Then, (??) $\cdot \quad + \quad {}^2 =$
${}^2 = {}^2 = {}^2 + {}^2 \;\therefore$
$\cdot \quad = \quad {}^2$, or,

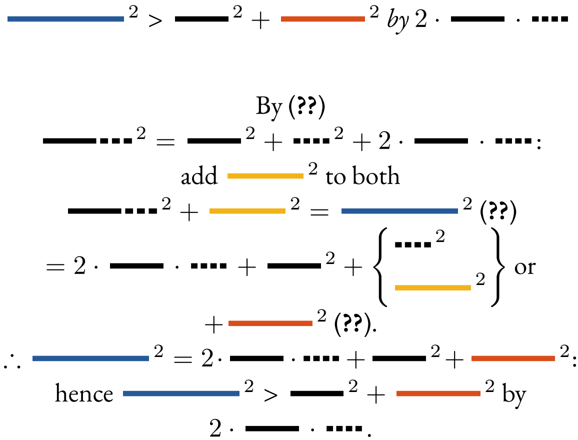
 $=$  $\therefore$  $=$ 

$\therefore \quad \cdot \quad = \quad {}^2$.

Q. E. D.



IN *any obtuse angled triangle, the square of the side subtending the obtuse angle exceeds the sum of the squares of the sides containing the obtuse angle, by twice the rectangle contained by either of these sides and the produced parts of the same from the obtuse angle to the perpendicular let fall on it from the opposite acute angle.*

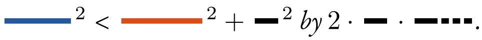


Q. E. D.

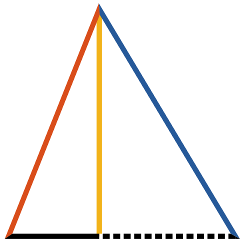
IN *any triangle, the square of the side subtending an acute angle, is less than the sum of the squares of the sides containing that angle, by twice the rectangle contained by either of these sides, and the part of it intercepted between the foot of the perpendicular let fall on it from the opposite angle, and the angular point of the acute angle.*

First.

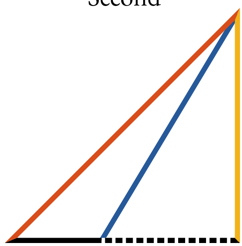
Second.



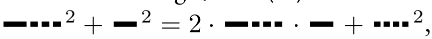
First



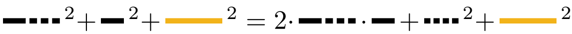
Second



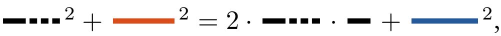
First, suppose the perpendicular to fall within the triangle, then (??)



add to each  then,

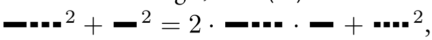


$\therefore$ (??)

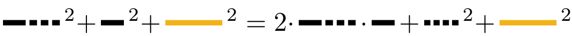


and $\therefore$ 

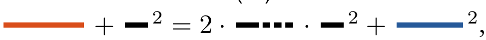
Next suppose the perpendicular to fall without the triangle, then (??)

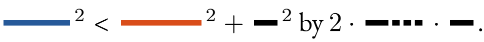


add to each  then

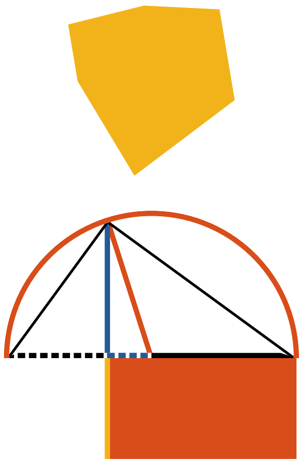


$\therefore$ (??)



$\therefore$ 

Q. E. D.



To draw a right line of which the square shall be equal to a given rectilinear figure.

To draw such that, ^{2} =

Make = (??), produce until

= ;

take = (??).

Describe (??),

and produce to meet it: draw .

^{2} or ^{2} = · + ^{2} (??),

but ^{2} = ^{2} + ^{2} (??);

$\therefore$ ^{2} + ^{2} = · + ^{2}

$\therefore$ ^{2} = · , and

$\therefore$ ^{2} = = .

Q. E. D.

Book III

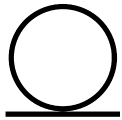
Definitions

III.1

Equal circles are those whose diameters are equal.

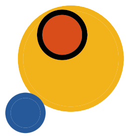
III.2

A right line is said to touch a circle when it meets the circle, and being produced does not cut it.



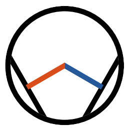
III.3

Circles are said to touch one another which meet, but do not cut one another.



III.4

Right lines are said to be equally distant from the centre of a circle when the perpendiculars drawn to them from the centre are equal.

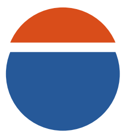


III.5

And the straight line on which the greater perpendicular falls is said to be farther from the centre.

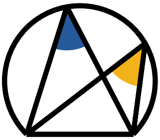
III.6

A segment of a circle is the figure contained by a straight line and the part of the circumference it cuts off.



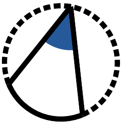
III.7

An angle of a segment is that contained by a straight line and a circumference of a circle.



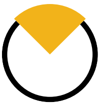
III.8

An angle in a segment is the angle contained by two straight lines drawn from any point in the circumference of the segment to the extremities of the straight line which is the base of the segment.



III.9

An angle is said to stand on the part of the circumference, or the arch, intercepted between the right lines that contain the angle.



III.10

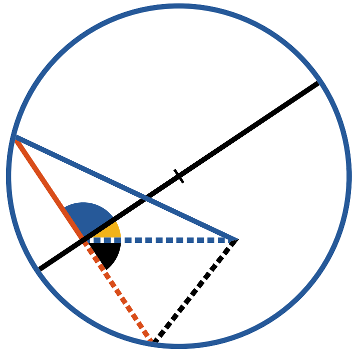
A sector of a circle is the figure contained by two radii and the arch between them.

III.11

Similar segments of circles are those which contain equal angles.

III.12

Circles which have the same centre are called *concentric circles*.



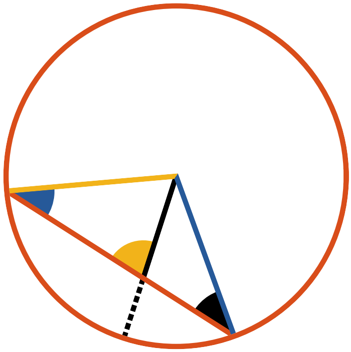
To *find the centre of a given circle* .

Draw within the circle any straight line , make = , draw ⊥ ; bisect , and the point of bisection is the centre.

For, if it be possible, let any other point as the point of concourse of , and be the centre.

Because in and = (?? and ??), = (??) and common, = (??), and are therefore right angles; but = (??) = (??) which is absurd; therefore the assumed point is not the centre of the circle; and in the same manner it can be proved that no other point which is not on is the centre, therefore the centre is in , and therefore the point where is bisected is the centre.

Q. E. D.



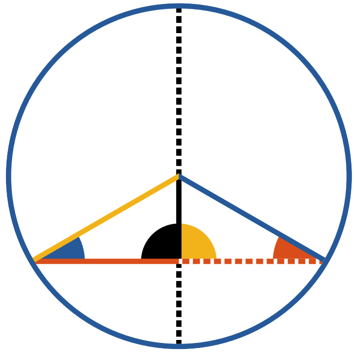
A STRAIGHT *line ( ) joining two points in the circumference of a circle , lies wholly within the circle.*

Find the centre of (??);

from the centre draw to any point in ,
meeting the circumference from the centre;
draw and .
Then = (??)
but > or > (??)
∴ > (??)
but = ,
∴ > ;
∴ < ;
∴ every point in lies within the circle.

Q. E. D.

If a straight line ( ) drawn through the centre of a circle bisect a chord ( ) which does not pass through the centre, it is perpendicular to it; or, if perpendicular to it, it bisects it.



Draw and to the centre of the circle.

In and

= , common,

and = ∴ = (??)

and ∴ ⊥ (??)

Again let ⊥

Then in and

= (??)

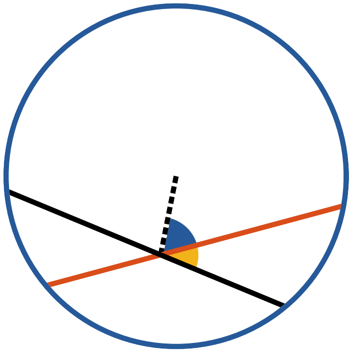
= (??)

and =

∴ = (??)

and ∴ bisects .

Q. E. D.



IF *in a circle two straight lines cut one another, which do not both pass through the centre, they do not bisect one another.*

If one of the lines pass through the centre, it is evident that it cannot be bisected by the other, which does not pass through the centre.

But if neither of the lines or pass through the centre, draw from the centre to their intersection.

If be bisected, $\perp$ to it (??)

$\therefore$ =

and if be bisected, $\perp$ (??)

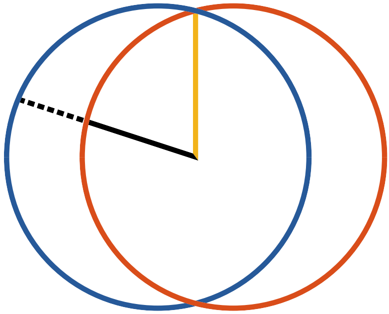
$\therefore$ = ;

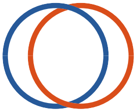
and $\therefore$ = ;

a part equal to the whole, which is absurd:

$\therefore$ and do not bisect one another.

Q. E. D.

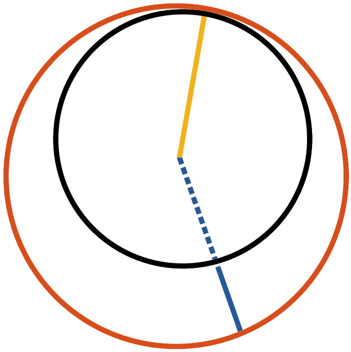


IF *two circles intersect* *, they have not the same centre.*

Suppose it possible that two intersecting circles have a common centre; from such supposed centre draw to the intersecting point, and meeting the circumferences of the circles.

Then = (??)
and = (??)
∴ =
a part equal to the whole, which is absurd:
∴ circles supposed to intersect in any point cannot have the same centre.

Q. E. D.



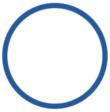
If two circles  *touch one another internally, they have not the same centre.*

For, if it be possible, let both circles have the same centre; from such a supposed centre draw , and to the point of contact.

Then = (??)
and = (??)
∴ = ;

a part equal to the whole, which is absurd; therefore the assumed point is not the centre of both circles, and in the same manner it can be demonstrated that no other point is.

Q. E. D.

I F *from any point within a circle*  *which is not the centre, lines* { *are drawn to the circumference; the greatest of those lines is that* ( ) *which passes through the centre, and the least is the remaining part* ( ) *of the diameter.*

Of the others, that ( ) *which is nearer to the line passing through the centre, is greater than that* ( ) *which is more remote.*

Fig. 2. The two lines (*and*) *which make equal angles with that passing through the centre, on opposite sides of it, are equal to each other; and there cannot be drawn a third line equal to them, from the same point to the circumference.*

Fig. 1

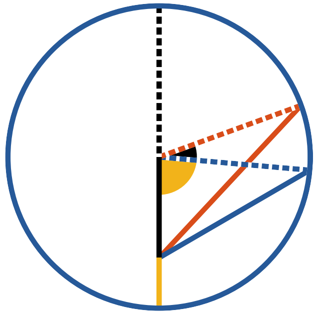


Fig. 2

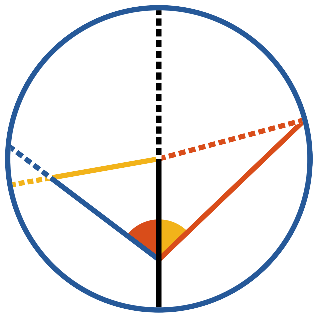


Figure I.

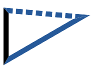
To the centre of the circle draw and ; then = (??) = + > (??) in like manner may be shown to be greater than , or any other line drawn from the same point to the circumference. Again, by (??) + > = + , take from both; ∴ > (??), and in like manner it may be shown that is less than any other line drawn from the same point to the circumference. Again, in  and  , common,  >  , and = ∴ > (??) and may in like manner to be proved greater than any other line drawn from the same point to the circumference more remote from .

Figure II.

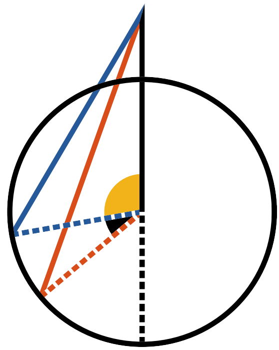
If = then = ,
if not take = draw
then in and , common,
= and =
$\therefore$ = (??)
$\therefore$ = =
a part equal to the whole, which is absurd:

$\therefore$ = ; and no other line is equal to drawn from the same point to the circumference; for if it were nearer to the one passing through the centre it would be greater, and if it were more remote it would be less.

Q. E. D.

THE *original text of this proposition is here divided into three parts.*

I.



If from a point without a circle, straight lines { } are drawn to the circumference; of those falling upon the concave circumference the greatest is that ( ) which passes through the centre, and the line ( ) which is nearer the greatest is greater than that ( ) which is more remote.

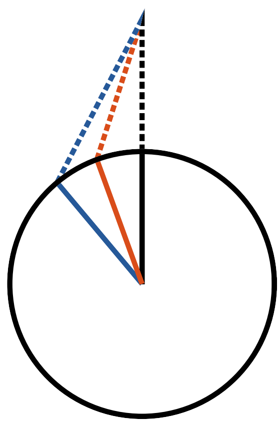
Draw and to the centre.

Then, which passes through the centre, is greatest;
for since = , if be added to both
= + ; but > (??)
∴ is greater than any other line drawn from the same point to the concave circumference.

Again in and , = , and common,

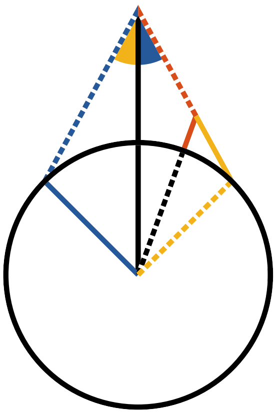
but > , ∴ > (??);
and in like manner may be shown > than any other line more remote from .

II.



Of those lines falling on the convex circumference the least is that ( ) which being produced would pass through the centre, and the line which is nearer to least is less than that which is more remote.

For, since + > (??)
and = , ∴ > (??)
And again, since + > + (??),
and = ,
∴ < . And so of others.



III.

Also the lines making equal angles with that which passes through the centre are equal, whether falling on the concave or convex circumference; and no third line can be drawn equal to them from the same point to the circumference.

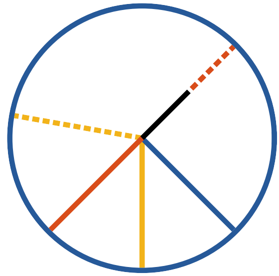
For if > , but making = ;
make = , and draw .

Then in and we have =

and common, and also = ,
∴ = (??);
but = ;
∴ = , which is absurd.
∴ is ≠ , nor to any part of ,
∴ is ≯ .
Neither is > , they are ∴= to each other.

And any other line drawn from the same point to the circumference must lie at the same side with one of these lines, and be more or less remote than it from the line passing through the centre, and cannot therefore be equal to it.

Q. E. D.



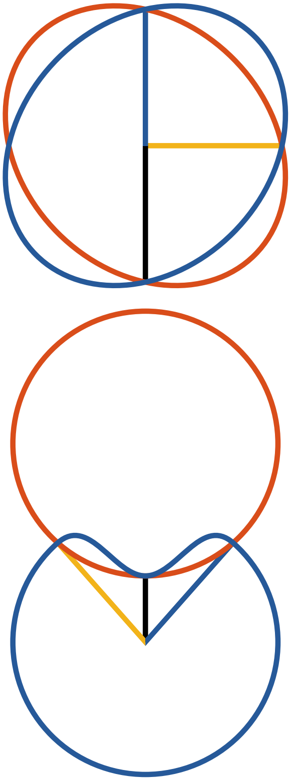
If a point be taken within a circle , from which more than two equal straight lines (, ,) can be drawn to the circumference, that point must be the centre of the circle.

For if it be supposed that the point in which more than two equal straight lines meet is not the centre, some other point must be; join these two points by and produce it both ways to the circumference.

Then since more than two equal straight lines are drawn from a point which is not the centre, to the circumference, two of them at least must lie at the same side of the diameter ; and since from a point , which is not the centre, straight lines are drawn to the circumference; the greatest is , which passes through the centre: and which is nearer to , > which is more remote (**??**); but = (**??**) which is absurd.

The same may be demonstrated of any other point, different from , which must be the centre of the circle.

Q. E. D.



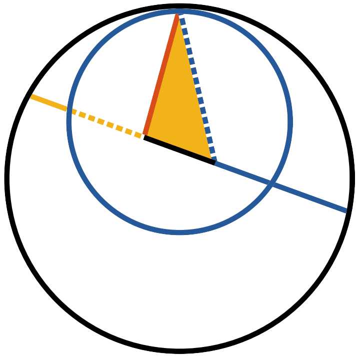
ONE *circle*  *cannot intersect another*  *in more points than two.*

For, if it be possible, let it intersect in three points; from the centre of  draw , and to the points of intersection;

∴ = = (??), but as the circles intersect, they have not the same centre (??):

∴ the assumed point is not the centre of , and ∴ as , and are drawn from a point not the centre, they are not equal (?? and ??); but it was shown before that they were equal, which is absurd; the circles therefore do not intersect in three points.

Q. E. D.



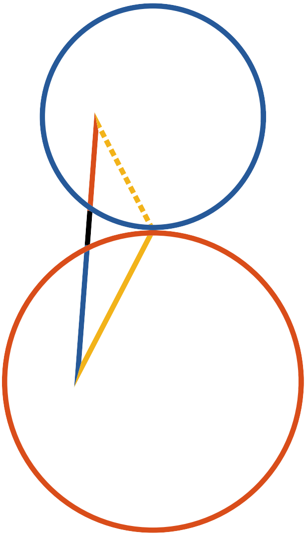
IF *two circles and touch one another internally, the right line joining their centres, being produced, shall pass through a point of contact.*

For, if it be possible, let join their centres, and produce it both ways; from a point of contact draw to the centre of , and from the same point of contact draw to the centre of .

Because in ; + > (??),
and = as they are radii of ,
but + > ;
take away which is common,
and > ;
but = ,
because they are radii of ,
and ∴ > a part greater than the whole,
which is absurd.

The centres are not therefore so placed, that a line joining them can pass through any point but a point of contact.

Q. E. D.



If two circle and touch one another externally, the straight line joining their centres, passes through the point of contact.

If it be possible, let joining the centres, and not pass through a point of contact; then from a point of contact draw and to the centres.

∵ + > (??)

and = (??),

and = (??),

∴ + > ,

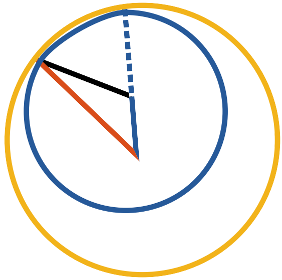
a part greater than the whole, which is absurd.

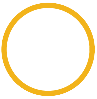
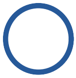
The centres are not therefore so placed, that the line joining them can pass through any point but the point of contact.

Q. E. D.

ONE *circle cannot touch another, either externally or internally, in more points than one*

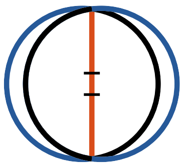
Figure I.



For, if it possible, let  and  touch one another internally in two points; draw joining their centres, and produce it until it pass through one of the points of contact (**??**);

draw and .
But = (**??**),
∴ if be added to both,
 = + ;
but = (**??**),
and ∴ + = ;
but ∴ + > (**??**),
which is absurd.

Figure II.



But if the points of contact be the extremities of the right line joining the centres, this straight line must be bisected in two different points for the two centres; because it is the diameter of both circles, which is absurd.

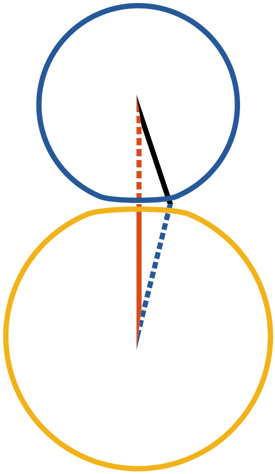


Figure III.

Next, if it be possible, let and touch externally in two points; draw joining the centres of the circles, and passing through one of the points of contact, and draw and .

= (??);

and = (??);

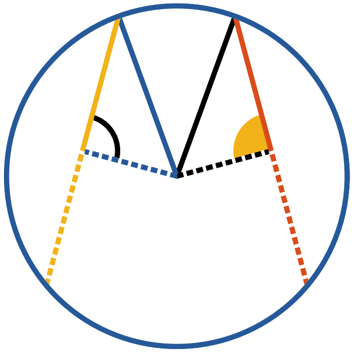
∴ + = ;

but + > (??), which is absurd.

There is therefore no case in which two circles can touch one another in two points.

Q. E. D.

EQUAL *straight lines ( ) inscribed in a circle are equally distant from the centre; and also, straight lines equally distant from the centre are equal.*



From the centre of draw ⊥ to

and ⊥ , join and .

Then = half (??)

and = $\frac{1}{2}$ (??),

since = (??)

∴ = ,

and = (??)

∴ ^{2} = ^{2};

but since is a right angle

^{2} = ^{2} + ^{2} (??)

and ^{2} = ^{2} + ^{2} for the same reason,

∴ ^{2} + ^{2} = ^{2} + ^{2}

∴ ^{2} = ^{2}

∴ =

Also, if the lines and be equally distant from the centre; that is to say, if the perpendiculars and be given equal, then = .

For, as in the preceding case,

^{2} + ^{2} = ^{2} + ^{2}

but ^{2} = ^{2}

∴ = , and the doubles of these and are also equal.

Q. E. D.

THE *diameter is the greatest straight line in a circle: and, of all others, that which is nearest to the centre is greater than the more remote.*

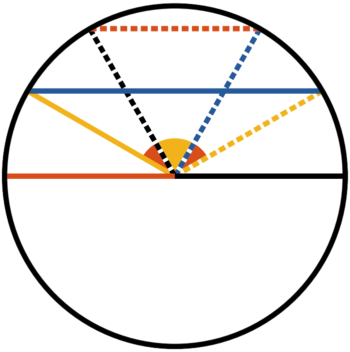


Figure I.

The diameter is > any line .

For draw and .

Then =

and = ,

∴ + =

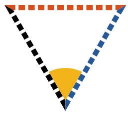
but + > (??)

∴ > .

Again, the line which is nearer the centre is greater than the one more remote.

First, let the given lines be and , which are at the same side of the centre and do not intersect;

draw ,
,
,
.

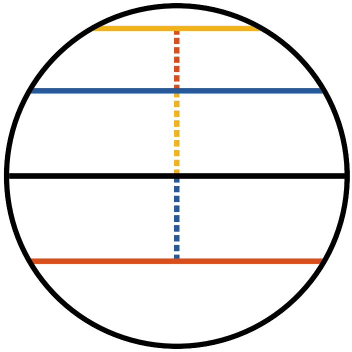
In  and  ,

and = and ;

but  >  ,

∴ > (??)

Figure II.



Let the given lines be and which either are at different sides of the centre, or intersect; from the centre draw and $\perp$ and ,

make $=$,

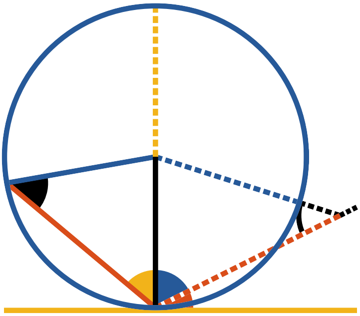
and draw $\perp$.

Since and are equally distant from the centre, $=$ (??);

but $>$ (??),

$\therefore$ $>$.

Q. E. D.



THE *straight line* drawn *from the extremity of the diameter* *of a circle perpendicular to it falls without the circle And if any straight line* *by drawn from a point within that perpendicular to the point of contact, it cuts the circle.*

Part I.

If it be possible, let , which meets the circle again, be ⊥ , and draw .

Then, ∵ = , = (??),
and ∴ each of these angles is acute (??)
but = (??), which is absurd,
therefore drawn ⊥ does not meet the circle again.

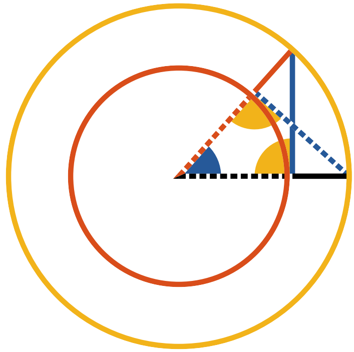
Part II.

Let be ⊥ and let be drawn from a point between and the circle, which, if it be possible, does not cut the circle.

∵ = ,
∴ is an acute angle;
suppose ⊥ , drawn from the centre of the circle, it must fall at the side of the acute angle.
∴ which is supposed to be a right angle, is > ,
∴ > ;
but = ,

and ∴ > , a part greater than the whole, which is absurd. Therefore the point does not fall outside the circle, and therefore the straight line cuts the circle.

Q. E. D.



T*o draw a tangent to a given circle from a given point, either in or outside of its circumference.*

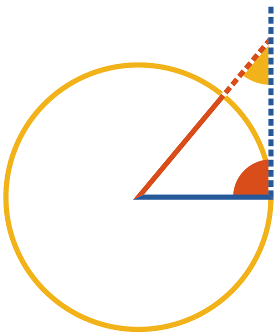
If a given point be in the circumference, as at , it is plain that the straight line ⊥ the radius, will be the required tangent (??).

But if the given point be outside of the circumference,

draw from it to the centre, cutting ;
and draw ⊥ ,
describe concentric with radius = ,
draw to the centre from the point where
falls on circumference,
draw ⊥ from the point where it cuts ,
Then will be the tangent required.

For in and = , common,
and = ,
∴ (??) = = ,
∴ is a tangent to .

Q. E. D.



If a straight line be a tangent to a circle, the straight line drawn from the centre to the point of contact, is perpendicular to it.

For, if it be possible, let be $\perp$,

then $\because$ $=$, is acute (??)

$\therefore$ $>$ (??);

but $=$,

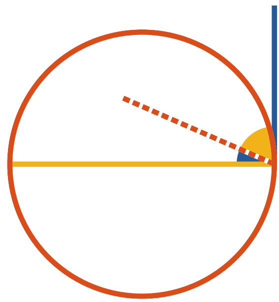
and $\therefore$ $>$, a part greater than the whole,

which is absurd.

$\therefore$ is not $\perp$;

and in the same manner it can be demonstrated, that no other line except is perpendicular to .

Q. E. D.



If a straight line be a tangent to a circle, the straight line , drawn perpendicular to it from point of the contact, passes through the centre of the circle.

For, if it be possible, let the centre be without , and draw from the supposed centre to the point of contact.

$\because$ $\perp$ (??)

$\therefore$ = , a right angle;

but = (??),

and $\therefore$ = , a part equal to the whole, which is absurd.

Therefore the assumed point is not the centre; and in the same manner it can be demonstrated, that no other point without is the centre.

Q. E. D.

THE *angle at the centre of a circle is double the angle at the circumference, when they have the same part of the circumference for their base.*

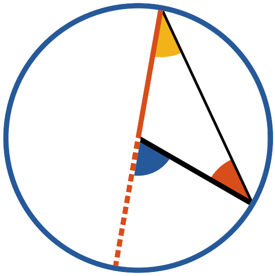


Figure I.

Let the centre of the circle be on a side of .

∵ = ,

= (??).

But = + ,

or = twice (??).

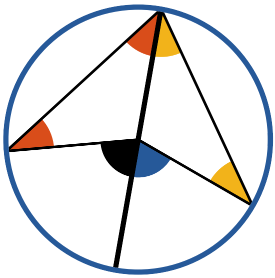


Figure II.

Let the centre be within , the angle at the circumference;

draw from the angular point through the centre of the circle;

then = , and = , because of the equality of the sides (??).

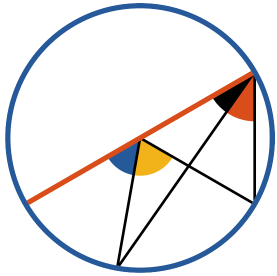
Hence + + + = twice .

But = + ,

and = + ,

∴ = twice .

Figure III.



Let the centre be without
and draw , the diameter.

∵ = twice ;

and = twice (case I.);

∴ = twice .

Q. E. D.

THE *angles in the same segment of a circle are equal.*

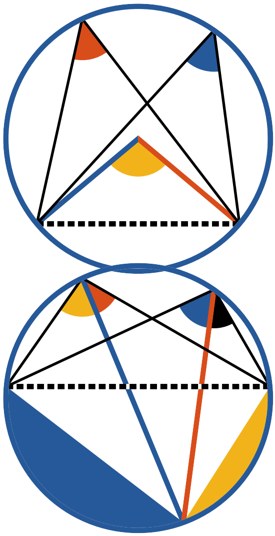


Figure I.

Let the segment be greater than a semicircle, and draw and to the centre.

= twice or twice = (??);

∴ =

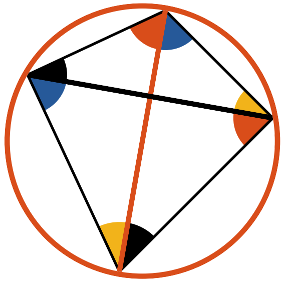
Figure II.

Let the segment be a semicircle, or less than a semicircle, draw the diameter, also draw .

= and = (case I.)

∴ = .

Q. E. D.



THE *opposite angles* and , and *of any quadrilateral figure inscribed in a circle, are together equal to two right angles.*

Draw and the diagonals;
and because angles in the same segment are equal

= ,

and = ;

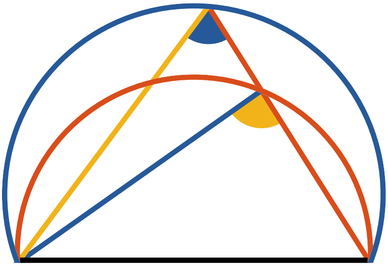
add to both.

+ = + + = two right angles (??).

In like manner it may be shown that,

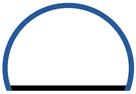
+ = .

Q. E. D.



UPON *the same straight line, and upon the same side of it, two similar segments of circles cannot be constructed, which do not coincide.*

For if it be possible, let two similar segments

 and  be constructed;

draw any right line cutting both the segments,

draw and .

Because the segments are similar,

 =  (??),

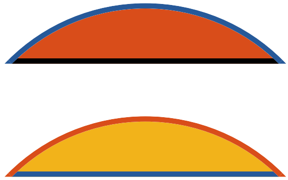
but  >  (??)

which is absurd;

therefore no point in either of the segments falls without the other, and therefore the segments coincide.

Q. E. D.

S IMILAR *segments*  *and* *, of circles upon equal straight lines (and) are each equal to the other.*

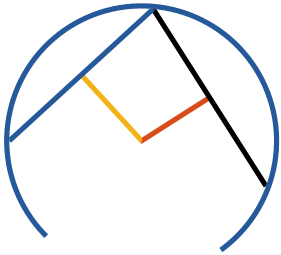


For, if  be so applied to ,
that may fall on ,
the extremities of may be on the extremities

and  at the same side as ;
∵ = ,
must wholly coincide with ;

and the similar segments being then upon the same straight line and at the same side of it, must also coincide (**??**), and are therefore equal.

Q. E. D.



A SEGMENT *of a circle being given, to describe the circle of which it is the segment.*

From any point in the segment draw and

,

bisect them, and from the points of bisection

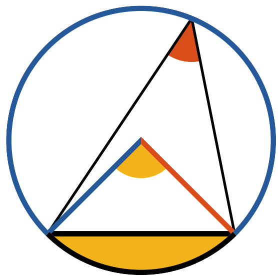
draw ⊥

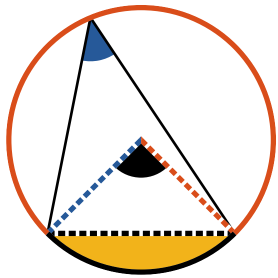
and ⊥

where they meet is the centre of the circle.

Because terminated in the circle is bisected perpendicularly by , it passes through the centre (??), likewise passes through the centre, therefore the centre is in the intersection of these perpendiculars.

Q. E. D.





IN *equal circles and , the arcs , on which stand equal angles, whether at the centre of circumference, are equal.*

For, let = at the centre,
draw and .
Then since = ,

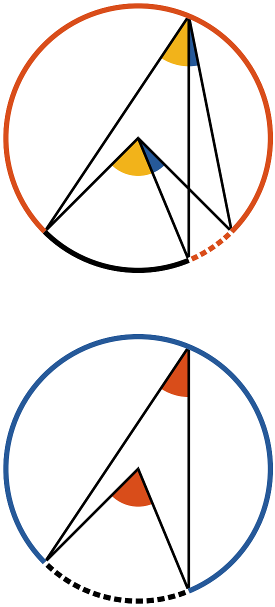
and have
= = = ,
and = ,
∴ = (??).
But = (??);
∴ and are similar (??);
they are also equal (??)

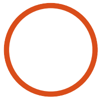
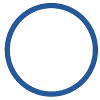
If therefore the equal segments be taken from the equal circles, the remaining segments will be equal;

hence = (??);
and ∴ = .

But if the given equal angles be at the circumference, it is evident that the angles at the centre, being double of those at the circumference, are also equal, and therefore the arcs on which they stand are equal.

Q. E. D.



IN *equal circles*  *and*  *the angles*  *and*  *which stand upon equal arches are equal, whether they be at the centres or at the circumferences.*

For if it be possible, let one of them  be greater than the other 

and make  = 

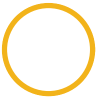
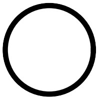
∴  =  (??)

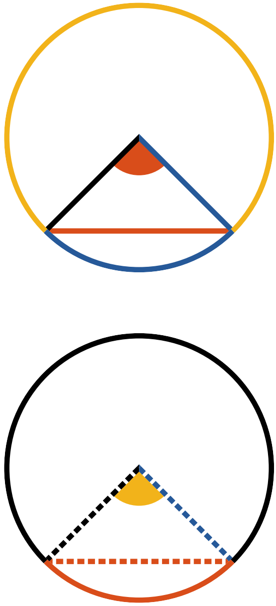
but  =  (??)

∴  =  a part equal to the whole, which is absurd;

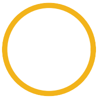
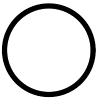
∴ neither angle is greater than the other, and ∴ they are equal.

Q. E. D.

IN *equal circles*  *and* , *equal chords* , *cut off equal arches.*



From the centres of the equal circles,
draw , and , ;

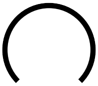
and ∵  = 

, = ,

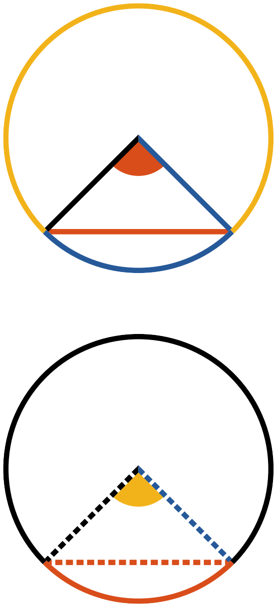
also = (??)

∴  = 

∴  =  (??)

and ∴  =  (??).

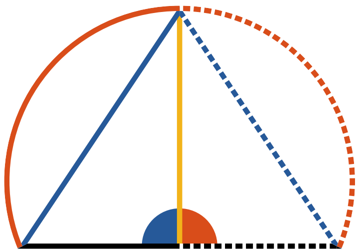
Q. E. D.



IN equal circles and the chords and which subtend equal arcs are equal.

If the equal arcs be semicircles the proposition is evident.
But if not,
let , and , be drawn to the centres;
∵ = (??)
and = (??);
but and = and
∴ = (??);
but these are the chords subtending the equal arcs.

Q. E. D.



To *bisect a given arc* .

Draw ;
make = ,
draw ⊥ , and it bisects the arc.
Draw and .
= (??)
is common,
and = (??)
∴ = (??)
= (??),
and therefore the given arc is bisected.

Q. E. D.

IN *a circle the angle in a semicircle is a right angle, the angle in a segment greater than a semicircle is acute, and the angle in a segment less than a semicircle is obtuse.*

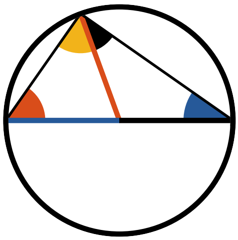


Figure I.

The angle in a semicircle is a right angle.

Draw and

= and = (??)

+ = = the half of =

(??)

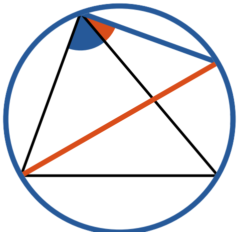


Figure II.

The angle in a segment greater than a semicircle is acute

Draw the diameter, and

∴ = a right angle, ∴ is acute.

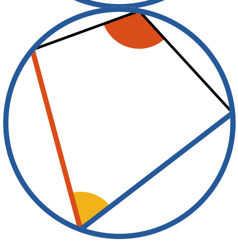


Figure III.

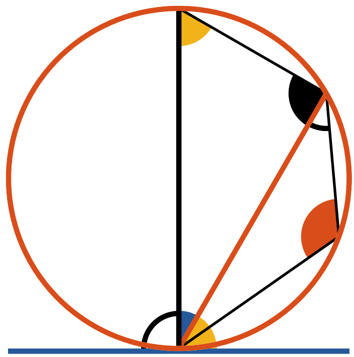
The angle in a segment less than a semicircle is obtuse.

Take in the opposite circumference any point, to which draw and .

∵ + = (??)

but < (figure II.), ∴ is obtuse.

Q. E. D.



If a right line be a tangent to a circle, and from the point of contact a right line be drawn cutting the circle, the angle made by this line with the tangent is equal to the angle in the alternate segment of the circle.

If the chord should pass through the centre, it is evident the angles are equal, for each of them is a right angle. (**??** and **??**)

But if not, draw ⊥ from the point of contact, it must pass through the centre of the circle, (**??**)

∴ = (**??**)

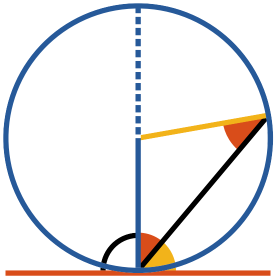
\+ = = (**??**)

∴ = (**??**).

Again = = + (**??**)

∴ = , (**??**), which is the angle in the alternate segment.

Q. E. D.



ON a given straight line to describe a segment of a circle that shall contain an angle equal to a given angle , , .

If a given angle be a right angle, bisect the given line, and describe a semicircle on it, this will evidently contain a right angle. (??)

If the given angle be acute or obtuse, make with the given line, at its extremity,

= ,

draw ⊥

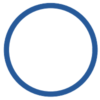
and make = ,

describe with or as radius, for they are equal.

is tangent to (??)

∴ divides the circle into two segments capable of containing angles equal to and which were made respectively equal to and (??).

Q. E. D.

T*o cut off from a given circle*  *a segment which shall contain an angle equal to a given angle*  .

Draw (??), a tangent to the circle at any point; at the point of contact make  =  the given angle;

and  contains an angle = the given angle.

Because is a tangent,
and cuts it,
the angle in  =  (??),

but  =  (??).

Q. E. D.

If two chords { } in a circle intersect each other, the rectangle contained by the segments of the one is equal to the rectangle contained by the segments of the other.

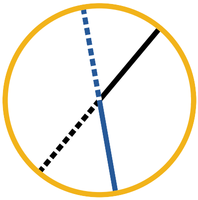


Figure I.

If the given right lines pass through the centre, they are bisected in the point of intersection, hence the rectangles under their segments are squares of their halves, and therefore equal.

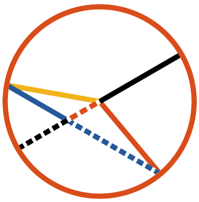


Figure II.

Let pass through the centre, and not; draw and .

Then $\times = {}^2 - {}^2$ (??), or

$\times = {}^2 - {}^2$.

$\therefore \times = \times$ (??).

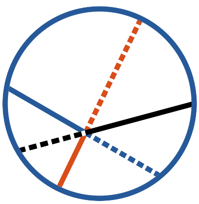


Figure III.

Let neither of the given lines pass through the centre, draw through their intersection a diameter ,

and $\times = \times$ (part 2.),

also $\times = \times$ (part 2.);

$\therefore \times = \times$.

Q. E. D.

IF *from a point without a circle two straight lines be drawn to it, one of which is a tangent to the circle, and the other cuts it; the rectangle under the whole cutting line and the external segment is equal to the square of the tangent .*

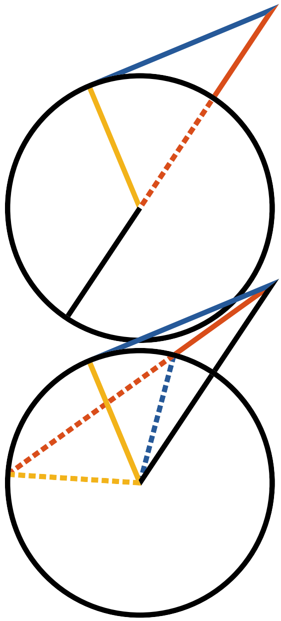


Figure I.

Let pass through the centre;

draw from the centre to the point of contact;

$$^2 = {}^2 - {}^2 \text{ (??)},$$

or $$^2 = {}^2 - {}^2,$$

$$\therefore\ {}^2 = \quad \times \quad \text{(??)}.$$

Figure II.

If do not pass through the centre,

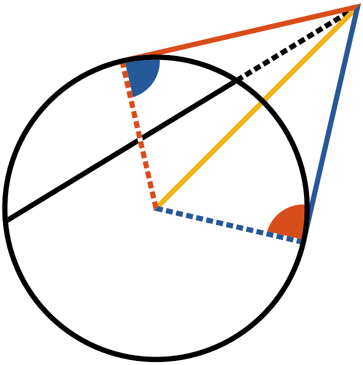
draw and .

Then $$\times \quad = {}^2 - {}^2 \text{ (??)},$$

that is, $$\times \quad = {}^2 - {}^2,$$

$$\therefore \quad \times \quad = {}^2 \text{ (??)}.$$

Q. E. D.



If from a point outside of a circle two straight lines be drawn, the one cutting the circle, the other meeting it, and if the rectangle contained by the whole cutting line and its external segment be equal to the square of the line meeting the circle, the latter is a tangent to the circle.

Draw from the given point, a tangent to the circle,

and draw from the centre , , and ,

$^2 = \times$ (??)

but $^2 = \times$ (??),

$\therefore$ $^2 = $ ^{2},

and $\therefore$ $=$.

Then in and

and $=$ and ,

and is common,

$\therefore$ $=$ (??);

but $=$ a right angle (??),

$\therefore$ $=$ a right angle,

and $\therefore$ is a tangent to the circle (??).

Q. E. D.

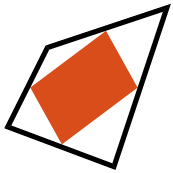
[redacted]

Book IV

Definitions

IV.1

A rectilinear figure is said to be *inscribed in* another, when all the angular points of the inscribed figure are on the sides of the figure in which it is said to be inscribed.

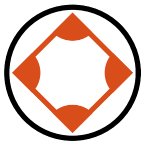


IV.2

A figure is said to be *described about* another figure, when all the sides of the circumscribed figure pass through the angular points of the other figure.

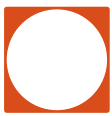
IV.3

A rectilinear figure is said to be *inscribed in* a circle, when the vertex of each angle of the figure is in the circumference of the circle.



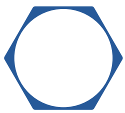
IV.4

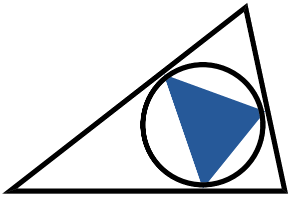
A rectilinear figure is said to be *circumscribed about* a circle, when each of its sides is a tangent to the circle.



IV.5

A circle is said to be *inscribed in* a rectilinear figure, when each side of the figure is a tangent to the circle.

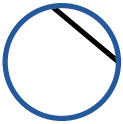




IV.6

A circle is said to be *circumscribed about* a rectilinear figure, when the circumference passes through the vertex of each angle of the figure.

 is circumscribed.



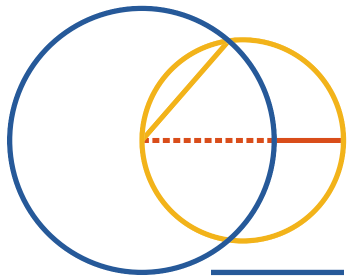
IV.7

A straight line is said to be *inscribed in* a circle, when its extremities are in the circumference.

The Fourth Book of the Elements is devoted to the solution of problems, chiefly relating to the inscription and circumscription of regular polygons and circles.

A regular polygon is one whose angles and sides are equal.

In a given circle to place a straight line, equal to a given straight line ( ), not greater than the diameter of the circle.



Draw , the diameter of ;

and if = , then the problem is solved.

But if be not equal to ,

> (??);

make = (??)

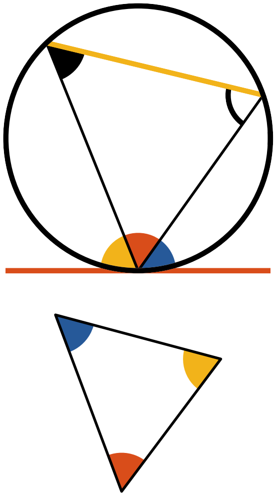
with as radius,

describe , cutting ,

and draw , which is the line required.

For = = (?? and ??).

Q. E. D.



IN *a given circle* *to inscribe a triangle equiangular to a given triangle.*

To any point of the given circle draw , a tangent (??);

and at the point of contact make = (??)

and in like manner = ,

and draw .

∵ = (??)

and = (??)

∴ = ;

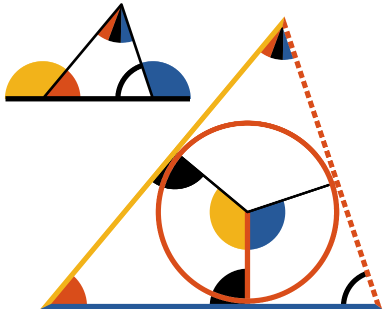
also = for the same reason.

∴ = (??),

and therefore the triangle inscribed in the circle is equiangular to the given one.

Q. E. D.

A BOUT *a given circle* *to circumscribe a triangle equiangular to a given triangle.*



Produce any side , of the given triangle both ways;
from the centre of the given circle draw , any radius.

Make = (??)

and = .

At the extremities of the radii, draw , and , tangents to the given circle. (??)

The four angles of , taken together are equal to four right angles. (??)

but and are right angles (??)

∴ + = , two right angles

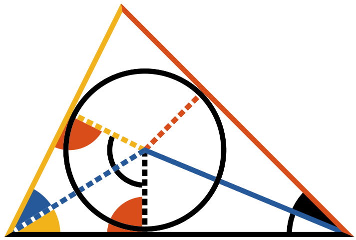
but = (??)

and = (??) and ∴ = .

In the same manner it can be demonstrated that
= ;
∴ = (??)

and therefore the triangle circumscribed about the given circle is equiangular to the given triangle.

Q. E. D.



I*n a given triangle* *to inscribe a circle.*

Bisect and (??) by and ; from the point where these lines meet draw , and respectively perpendicular to , and .

In and

= , = and common,

∴ = (?? and ??).

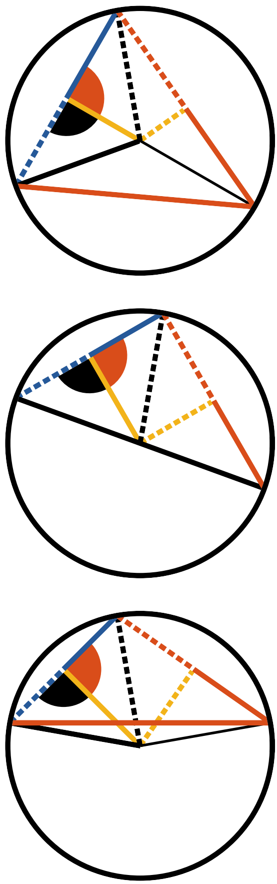
In like manner, it may be shown also that

= ,

∴ = = ;

hence with any one of these lines as radius, describe and it will pass through the extremities of the other two; and the sides of the given triangle, being perpendicular to the three radii at their extremities, touch the circle (??), which is therefore inscribed in the given triangle.

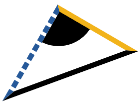
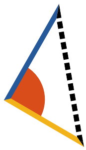
Q. E. D.



T*o describe a circle about a given triangle.*

Make = and = (??)

From the points of bisection draw and ⊥ and respectively (??), and from their point of concourse draw , and and describe a circle with any one of them, and it will be the circle required.

In  and 

= (??),

common,

= (??),

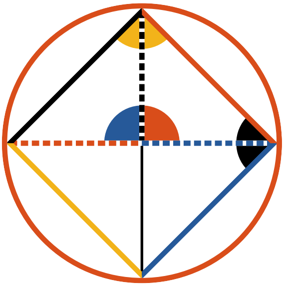
∴ = (??).

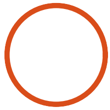
In like manner it may be shown that

= .

∴ = = ; and therefore a circle described from the concourse of these three lines with any one of them as a radius will circumscribe the given triangle.

Q. E. D.



IN *a given circle*  *to inscribe a square.*

Draw the two diameters of the circle ⊥ to each other, and draw , , and .

 is a square.

For, since  and  are, each of them, in a semicircle, they are right angles (**??**),

∴ ‖ (**??**):

and in like manner ‖ .

And ∵  =  (**??**),

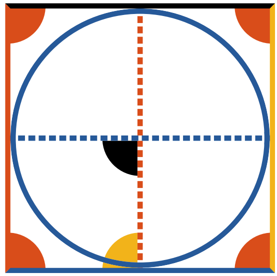
and = = (**??**).

∴ = (**??**);

and since the adjacent sides and angles of the parallelogram  are equal, they are all equal (**??**); and ∴

, inscribed in the given circle, is a square.

Q. E. D.



ABOUT *a given circle* *to circumscribe a square.*

Draw two diameters of the given circle perpendicular to each other, and through their extremities draw , , and tangents to the circle;

and is a square.

= a right angle, (??)

also = (??),

∴ || ; in the same manner it can be demonstrated that || , and also that and || ;

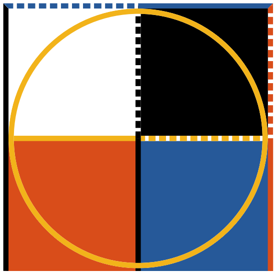
∴ is a parallelogram,

and ∵ = = = = they are all right angles (??);

it is also evident that , , and are equal.

∴ is a square.

Q. E. D.



T*o inscribe a circle in a given square.*

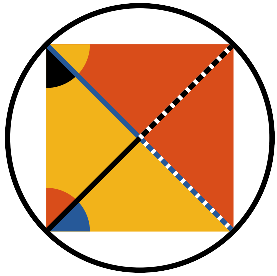
Make = ,
and = ,
draw || ,
and ||
(??)

∴ is a parallelogram;
and since = (??)
=
∴ is equilateral (??)

In like manner it can be shown that =
are equilateral parallelograms;
∴ = = = .

and therefore if a circle be described from the concourse of these lines with any one of them as radius, it will be inscribed in the given square (??).

Q. E. D.



To *describe a circle about a given square* .

Draw the diagonals and intersecting each other;

then, because and have their sides equal, and the base common to both,

= (??),

or is bisected:

in like manner it can be shown that is bisected;

but = ,

hence = their halves,

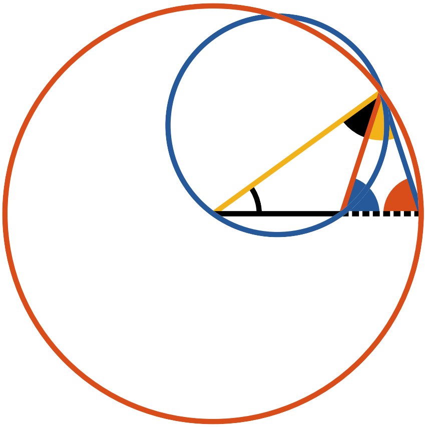
∴ = (??);

and in like manner it can be proved that

= = = .

If from the confluence of these lines with any one of them as radius, a circle be described, it will circumscribe the given square.

Q. E. D.



T*o construct an isosceles triangle, in which each of the angles at the base shall be double of the vertical angle.*

Take any straight line
and divide it so that × = ^{2} (??)

With as radius, describe and place in it from the extremity of the radius, = (??);

draw .

Then is the required triangle.

For, draw and describe about (??)

Since × = ^{2} = ^{2},

∴ is tangent to (??)

∴ = (??),

add to each, ∴ + = + ;

but + or = (??):

since = (??)

consequently = + = (??)

∴ = (??)

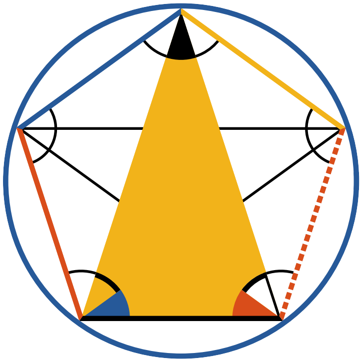
∴ = = (??)

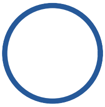
∴ = (??)

∴ = = = + = twice ;

and consequently each angle at the base is double of the vertical angle.

Q. E. D.



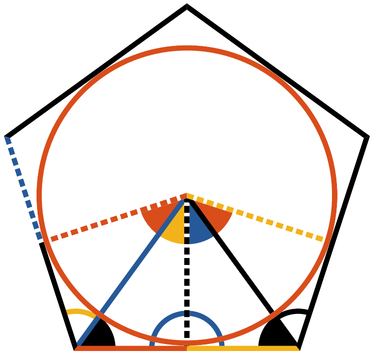
I*n a given circle*  *to inscribe an equilateral and equiangular pentagon.*

Construct an isosceles triangle, in which each of the angles at the base shall be double of the angle at the vertex, and inscribe in the given circle a triangle  equiangular to it (**??**);

Bisect  and  (**??**),
draw , , and .

Because each of the angles , , ,  and  are equal, the arcs upon which they stand are equal (**??**); and ∴ , , , and which subtend these arcs are equal (**??**) and ∴ the pentagon is equilateral, it is also equiangular, as each of its angles stand upon equal arcs (**??**).

Q. E. D.



To describe an equilateral and equiangular pentagon about a given circle .

Draw five tangents through the vertices of the angles of any regular pentagon inscribed in the given circle (??).

These five tangents will form the required pentagon.

Draw .

In and

= (??)

= , and common;

∴ = and ∴ = (??)

∴ = twice , and = twice .

In the same manner it can be demonstrated that

= twice , and = twice ;

but = (??),

∴ their halves = , also = ,

and common;

∴ = and = ,

∴ = twice .

In the same manner it can be demonstrated that

= twice ,

but =

∴ = ;

In the same manner it can be demonstrated that the other sides are equal, and therefore the pentagon is equilateral, it is also equiangular, for

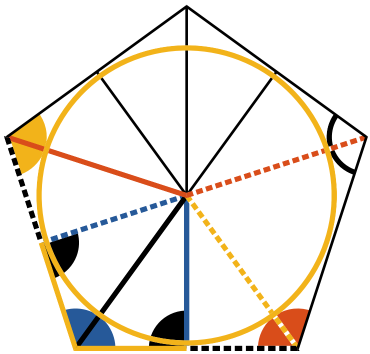
= twice and = twice ,

and therefore  = ,

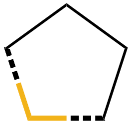
∴  = ;

in the same manner it can be demonstrated that the other angles of the described pentagon are equal.

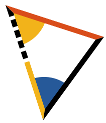
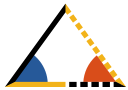
Q. E. D.



To *inscribe a circle in a given equiangular and equilateral pentagon.*

Let  be a given equiangular and equilateral pentagon; it is required to inscribe a circle in it.

Make = , and = (??)
Draw , , , , etc.
∵ = , = , and
common to the two triangles

 and  ;

∴ = and = (??)

And ∵ = = twice

∴ twice , hence is bisected by .

In like manner it may be demonstrated that  is bisected by , and that the remaining angle of the polygon is bisected in a similar manner.

Draw , , etc. perpendicular to the sides of the pentagon.

Then in the two triangles  and 

we have = (??), common,

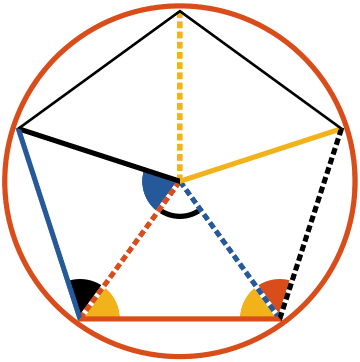
and = = a right angle ;

∴ = (??)

In the same way it may be shown that the five perpendiculars on the sides of the pentagon are equal to one another.

Describe  with any one of the perpendiculars as radius, and it will be the inscribed circle required. For if it does not touch the sides of the pentagon, but cut them, then a line drawn from the extremity at right angles to the diameter of a circle will fall within the circle, which has been shown to be absurd (**??**).

Q. E. D.



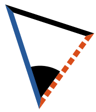
To *describe a circle about a given equilateral and equiangular pentagon.*

Bisect and by and , and from the point of section, draw , and .

=,

=,

∴ = (??);

and since in  and ,

= , and common,

also = ;

∴ = (??).

In like manner it may be proved that

= = .

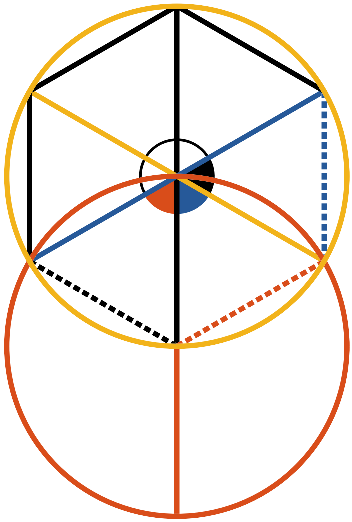
and therefore

= = = = .

Therefore if a circle be described from the point where these five lines meet, with any one of them as a radius, it will circumscribe the given pentagon.

Q. E. D.

To *inscribe an equilateral and equiangular hexagon in a given circle* .



From any point in the circumference of the given circle describe passing through its centre, and draw the diameters , and ; draw , , , etc. and the required hexagon is inscribed in the given circle.

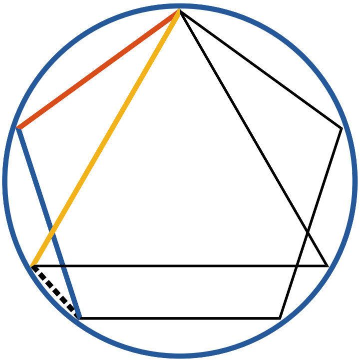
Since passes through the centres of the circles, and are equilateral triangles, hence

= = one-third of (??) but =

(??);

∴ = = = one-third of (??), and the angles vertically opposite to these are all equal to one another (??), and stand on equal arches (??), which are subtended by equal chords (??); and since each of the angles of the hexagon is double of the angle of an equilateral triangle, it is also equiangular.

Q. E. D.



T*o inscribe a circle in an equilateral and equiangular quindecagon in a given circle.*

Let and be the sides of an equilateral pentagon inscribed in the given circle, and the side of an inscribed equilateral triangle.

$$\left.\begin{array}{l}\text{The arc subtended}\\ \text{by and }\end{array}\right\} = \frac{2}{5} = \frac{6}{15} \left\{\begin{array}{l}\text{of the whole}\\ \text{circumference}\end{array}\right.$$

$$\left.\begin{array}{l}\text{The arc subtended}\\ \text{by }\end{array}\right\} = \frac{1}{3} = \frac{5}{15} \left\{\begin{array}{l}\text{of the whole}\\ \text{circumference}\end{array}\right.$$

$$\text{Their difference} = \frac{1}{15}$$

$\therefore$ the arc subtended by $= \frac{1}{15}$ difference of the whole circumference.

Hence if straight lines equal to be placed in the circle (**??**), an equilateral and equiangular quindecagon will be thus inscribed in the circle.

Q. E. D.

Book V

Definitions

V.1

A less magnitude is said to be an aliquot part or submultiple of a greater magnitude, when the less measures the greater; that is, when the less is contained a certain number of times exactly in the greater.

V.2

A greater magnitude is said to be a multiple of a less, when the greater is measured be the less; that is, when the greater contains the less a certain number of times exactly.

V.3

Ratio is the relation which one quantity bears to another of the same kind, with respect to magnitude.

V.4

Magnitudes are said to have a ratio to one another, when they are of the same kind; and the one which is not the greater can be multiplied so as to exceed the other.

The other definitions will be given throughout the book where their aid is first required.

Axioms

V.1

Equimultiples or equisubmultiples of the same, or of equal magnitudes, are equal.

$$\begin{aligned} \text{If } A &= B, \text{ then} \\ \text{twice } A &= \text{twice } B, \\ \text{that is, } 2A &= 2B; \\ 3A &= 3B; \\ 4A &= 4B; \\ &\text{etc. etc.} \\ \text{and } \frac{1}{2} \text{ of } A &= \frac{1}{2} \text{ of } B; \\ \frac{1}{3} \text{ of } A &= \frac{1}{3} \text{ of } B; \\ \frac{1}{4} \text{ of } A &= \frac{1}{4} \text{ of } B; \\ &\text{etc. etc.} \end{aligned}$$

V.2

A multiple of a greater magnitude is greater than the same multiple of a less.

$$\begin{aligned} \text{Let } A &> B, \\ \text{then } 2A &> 2B; \\ 3A &> 3B; \\ 4A &> 4B; \\ &\text{etc. etc.} \end{aligned}$$

V.3

That magnitude, of which a multiple is greater than the same multiple of another, is greater than the other.

$$\begin{aligned} \text{Let } 2A &> 2B, \\ \text{then } A &> B; \\ \text{or, let } 3A &> 3B, \\ \text{then } A &> B; \\ \text{or, let } mA &> mB, \\ \text{then } A &> B. \end{aligned}$$

If any number of magnitudes be equimultiples of as many others, each of each: what multiple soever any one of the first is of its part, the same multiple shall of the first magnitudes taken together be of all the others taken together.

Let be the same multiple of ,

that is of ,

that is of .

Then is evident that

is the same multiple of

which that is of ;

because there are as many magnitudes

in $=$

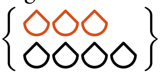
as there are in $=$.

The same demonstration holds in any number of magnitudes, which has here been applied to three.

∴ If any number of magnitudes, etc.

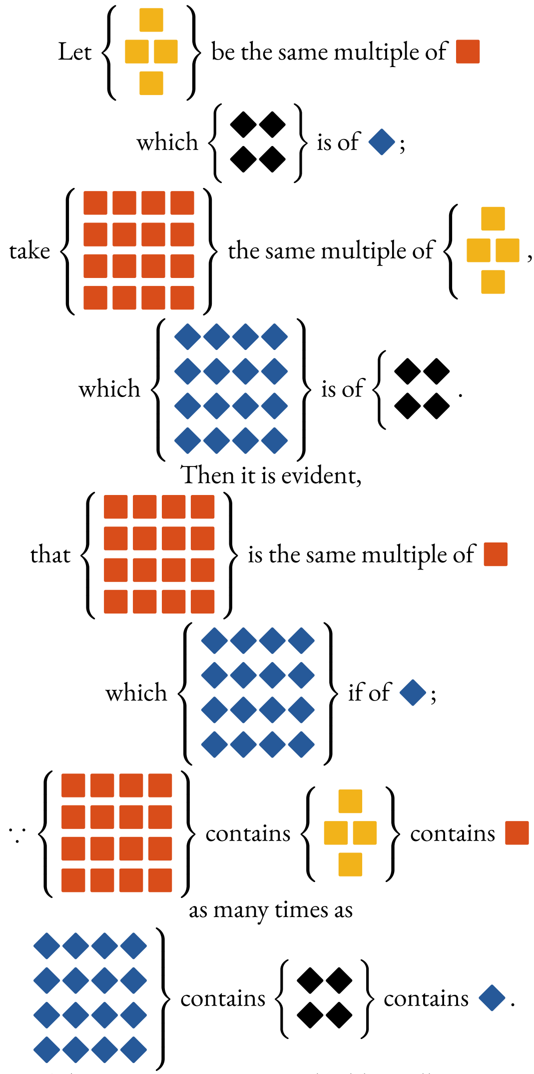
If the first magnitude be the same multiple of the second that the third of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth, then shall the first, together with the fifth, be the same multiple of the second that the third, together with the sixth, is of the fourth.

Let , the first, be the same multiple of , the second, that , the third, is of , the fourth; and let , the fifth, be the same multiple of , the second, that , the sixth, is of , the fourth.

Then it is evident, that , the first and fifth together, is the same multiple of , the second, that , the third and sixth together, is of the same multiple of , the fourth; because there are as many magnitudes in  =  as there are in  = .

∴ If the first magnitude, etc.

If the first of four magnitudes be the same multiple of the second that the third is to the fourth, and if any equimultiples whatever of the first and third be taken, those shall be equimultiples; one of the second, and the other of the fourth.

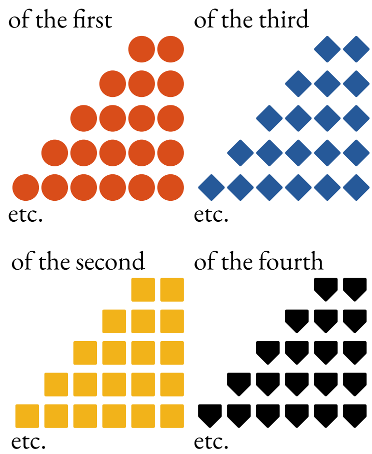


The same reasoning is applicable in all cases.

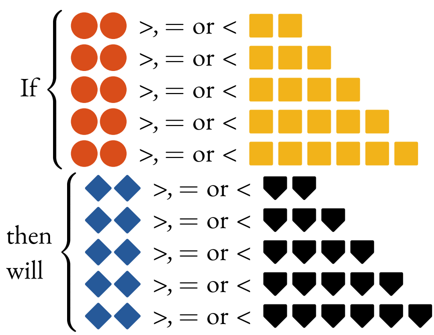
∴ If the first of four, etc.

V.5 Definition V.

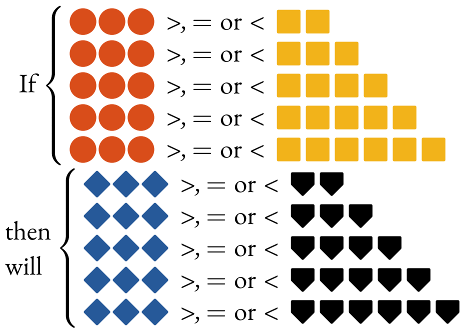
Four magnitudes, , , , , are said to be proportionals when every equimultiple of the first and third be taken, and every equimultiple of the second and fourth, as,



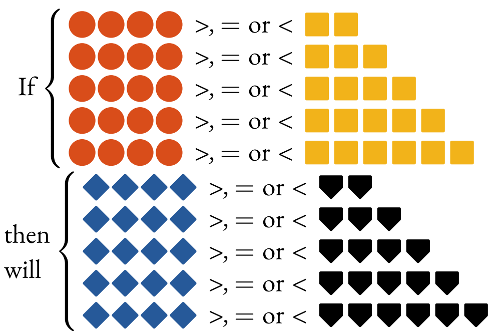
Then taking every pair of equimultiples of the first and third, and every pair of equimultiples of the second and fourth,



That is, if twice the first be greater, equal, or less than twice the second, twice the third will be greater, equal, or less than twice the fourth; or, if twice the first be greater, equal, or less than three times the second, twice the third will be greater, equal, or less than three times the fourth, and so on, as above expressed.



In other terms, if three times the first be greater, equal, or less than twice the second, three times the third will be greater, equal, or less than twice the fourth; or, if three times the first be greater, equal, or less than three times the second, then will three times the third be greater, equal, or less than three times the fourth; or if three times the first be greater, equal, or less than four times the second, then will three times the third be greater, equal, or less than four times the fourth, and so on. Again,



And so on, with any other equimultiples of the four magnitudes, taken in the same manner.

Euclid expresses this definition as follows :—

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, and the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

In future we shall express this definition generally, thus:

If M ● $>, =$ or $<$ m ■,
when M ◆ $>, =$ or $<$ m ▼,

Then we infer that ●, the first, has the same ratio to ■, the second, which ◆, the third, has to ▼ the fourth; expressed in the succeeding demonstrations thus:

● : ■ :: ◆ : ▼;
or thus, ● : ■ = ◆ : ▼;
or thus, $\dfrac{●}{■} = \dfrac{◆}{▼}$;
and is read,
"as ● is to ■, so is ◆ to ▼."
And if ● : ■ :: ◆ : ▼ we shall infer if
M ● $>, =$ or $<$ m ■, then will
M ◆ $>, =$ or $<$ m ▼.

That is, if the first be to the second, as the third is to the fourth; then if M times the first be greater than, equal to, or less than m times the second, then shall M times third be greater than, equal to, or less than m times the fourth, in which M and m are not to be considered particular multiples, but every pair of multiples whatever; nor are such marks as , , , etc. to be considered any more than representatives of geometrical magnitudes.

The student should throughly understand this definition before proceeding further.

If the first of four magnitudes have the same ratio to the second, which the third has to the fourth, then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth; viz., the equimultiple of the first shall have the same ratio to that of the second, which the equimultiple of the third has to that of the fourth.

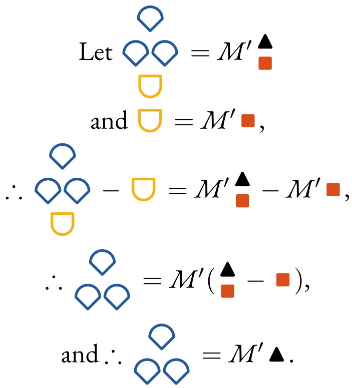
Let ● : ■ :: ◆ : ⬟, then 3● : 2■ :: 3◆ : 2⬟, every equimultiple of 3● and 3◆ are equimultiples of ● and ◆, and every equimultiple of 2■ and 2⬟, are equimultiples of ■ and ⬟ (??)

That is, M times 3● and M times 3◆ are equimultiples of ● and ◆, and m times 2■ and m2⬟ are equimultiples of 2■ and 2⬟; but ● : ■ :: ◆ : ⬟ (??); ∴ if M3● $<, =,$ or $>$ m2■, then M3◆ $<, =$, or $>$ m2⬟ (??) and therefore 3● : 2■ :: 3◆ : 2⬟ (??)

The same reasoning holds good if any other equimultiple of the first and third be taken, any other equimultiple of the second and fourth.

∴ If the first four magnitudes, etc.

I F *one magnitude be the same multiple of another, which a magnitude taken from the first is of a magnitude taken from the other, the remainder shall be the same multiple of the remainder, that the whole is of the whole.*



∴ If one magnitude, etc.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two, the remainders are either equal to these others, or equimultiples of them.

Let $\diamond\diamond\diamond\diamond = M' \blacksquare$; and $\cup\cup = M' \blacktriangle$

then $\diamond\diamond\diamond\diamond - m' \blacksquare = M' \blacksquare - m' \blacksquare = (M' - m') \blacksquare$

and $\cup\cup - m' \blacktriangle = M' \blacktriangle - m' \blacktriangle = (M' - m') \blacktriangle$.

Hence, $(M' - m') \blacksquare$ and $(M' - m') \blacktriangle$ are equimultiples of $\blacksquare$ and $\blacktriangle$, and equal to $\blacksquare$ and $\blacktriangle$, when $M' - m' = 1$.

$\therefore$ If two magnitudes be equimultiples, etc.

If the first of the four magnitudes has the same ratio to the second which the third has to the fourth, then if the first be greater than the second, the third is also greater than the fourth; and if equal, equal; if less, less.

Let ● : ■ :: ⛊ : ◆; therefore, by the fifth definition, if ●● > ■■, then will ⛊⛊ > ◆◆;
but if ● > ■, then ●● > ■■ and
⛊⛊ > ◆◆,
and ∴ ⛊ > ◆.
Similarly, if ● =, or < ■, then will ⛊ =, or < ◆.
∴ If the first of four, etc.

V.14 Definition XIV

Geometricians make use of the technical term "Invertendo," by inversion, when there are four proportionals, and it is inferred, that the second is to the first as the fourth to the third.

Let $A : B :: C : D$, then, by "invertendo" it is inferred $B : A :: D : C$

IF *four magnitudes are proportionals, they are proportionals also when taken inversely.*

Let  :  ::  : ,

then, inversely,  :  ::  : .

If M  $<$ m , then M  $<$ m 

by the fifth definition.

Let M  $<$ m , that is, m  $>$ M ,

$\therefore$ M  $<$ m , or, m  $>$ M ;

$\therefore$ if m  $>$ M , then will m  $>$ M .

In the same manner it may be shown,

that if m  $=$ or $<$ M ,

then will m  $=$, or $<$ M ;

and therefore, by the fifth definition, we infer

that  :  ::  : .

$\therefore$ If four magnitudes, etc.

If *the first be the same multiple of the second, or the same part of it, that the third is of the fourth; the first is to the second, as the third is to the fourth.*

,

Let , the first, be the same multiple of , the second,

that , the third, is of , the fourth.

Then : :: :

take M , m , M , m ;

because is the same multiple of

that is of (according to the hypothesis);

and M if taken the same multiple of

that M is of ,

∴ (according to the third proposition),

M is the same multiple of

that M is of .

Therefore, if M be of a greater multiple than m is,

then M is a greater multiple of than m is;

that is, if M be greater than m , then M will be greater than m ;

in the same manner it can be shown, if M be equal

m , then M will be equal m .

And, generally, if M $>, =$ or $<$ m

than M will be $>, =$ or $<$ m ;

$\therefore$ by the fifth definition,

: :: :

Next, let be the same part of

that is of .

In this case also : :: : .

For, because is the same part of that is of

,

therefore is the same multiple of

that is of .

Therefore, by the preceding case,

: :: : ;

and $\therefore$: :: : ,

by proposition B.

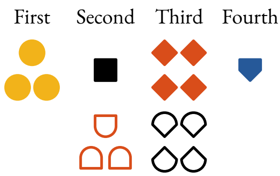
$\therefore$ If the first be the same multiple, etc.

If the first be to the second as the third to the fourth, and if the first be a multiple, or a part of the second; the third is the same multiple, or the same part of the fourth.

Let : :: : ;

and first, let be a multiple ;

shall be the same multiple of .



Take = .

Whatever miltiple is of

take the same multiple of ,

then, ∵ : :: :

and of the second and fourth, we have taken equimultiples,

and therefore (??)

: :: : , but (??),

= ∴ (??) =

and is the same multiple of

that is of .

Next, let  :  ::  : ,

and also  a part of ;

then  shall be the same part of .

Inversely (**??**),  :  ::  : ,

but  is a part of ;

that is,  is a multiple of ;

∴ by the preceding case,  is the same multiple of 

that is,  is the same part of 

that  is of .

∴ if the first be to the second, etc.

EQUAL *magnitudes have the same ratio to the same magnitude, and the same has the same ratio to equal magnitudes.*

Let ● $=$ ◆ and ■ any other magnitude;

then ● : ■ $=$ ◆ : ■ and ■ : ● $=$ ■ : ◆.

$\because$ ● $=$ ◆,

$\therefore M$● $= M$◆;

$\therefore$ if M● $>, =$ or $< m$■, then

M◆ $>, =$ or $< m$■,

and $\therefore$ ● : ■ $=$ ◆ : ■ (??).

From the foregoing reasoning it is evident that,

if m■ $>, =$ or $< M$●, then

m■ $>, =$ or $< M$◆

$\therefore$ ■ : ● $=$ ■ : ◆ (??).

$\therefore$ Equal magnitudes, etc.

V.15 Definition VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition), the multiple of the first is greater than that of the second, but multiple of the third is not greater than the multiple of the fourth; then the first is said to have to the second a greater ratio than the third magnitude has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

If, among the equimultiples of four magnitudes, compared in the fifth definition, we should find ●●●●● $>$ ■■■■, but ◆◆◆◆◆ $=$ or $>$ ⬟⬟⬟⬟, or if we should find any particular multiple M' of the first and third, and a particular multiple m' of the second and fourth, such, that M' times the first is $> m'$ times the second, but M' times the third is not $> m'$ times the fourth, i. e. $=$ or $< m'$ times the fourth; then the first is said to have to the second a greater ratio than the third has to the fourth; or the third has to the fourth, under such circumstances, a less ratio than the first has to the second: although several other equimultiples may tend to show that the four magnitudes are proportionals.

This definition will in future be expressed thus :—

If M' ⬟ $> m'$ ⬠, but M' ■ $=$ or $< m'$ ◆,
then ⬟ : ⬠ $>$ ■ : ◆.

In the above general expression, M' and m' are to be considered particular multiples, not like the multiples M and m introduced in the fifth definition, which are in that definition considered to be every pair of multiples that can be taken. It must also be here observed, that ⬟, ⬠, ■, and the like symbols are to be considered merely the representatives of geometrical magnitudes.

In a partial arithmetical way, this may be set forth as follows:

Let us take the four numbers 8, 7, 10, and 9.

First	Second	Third	Fourth
8	7	10	9
16	14	20	18
24	21	30	27
32	28	40	36
40	35	50	45
48	42	60	54
56	49	70	63
64	56	80	72
72	63	90	81
80	70	100	90
88	77	110	99
96	84	120	108
104	91	130	117
112	98	140	126
etc.	etc.	etc.	etc.

Among the above multiples we find $16 > 14$ and $20 > 18$; that is, twice the first is greater than twice the second, and twice the third is greater than twice the fourth; and $16 < 21$ and $20 < 27$; that is, twice the first is less than three times the second, and twice the third is less than three times the fourth; and among the same multiples we can find $72 > 56$ and $90 > 72$; that is, 9 times the first is greater than 8 times the second, and 9 times the third is greater than 8 times the fourth. Many other equimultiples might be selected, which would tend to show that the numbers 8, 7, 10, and 9 were proportionals, but they are not, for we can find multiple of the first > a multiple of the second, but the same multiple of the third that has been taken of the first not > the same multiple of the fourth which has been taken to the second; for instance, 9 times the first is > 10 times the second, but 9 times the third is not > 10 times the fourth, that is $72 > 70$, but $90 \not> 90$, or 8 times the first we find > 9 times the second, but 8 times the third is not greater than 9 times the fourth, that is, $64 > 63$, but

80 not > 81. When any such multiples as these can be found, the first (8) is said to have the second (7) a greater ratio than the third (10) has to the fourth (9), and on the contrary the third (10) is said to have to the fourth (9) a less ratio than the first (8) has to the second (7).

OF *unequal magnitudes the greater has a greater ratio to the same than the less has: and the same magnitude has a greater ratio to the less than it has to the greater.*

Let ▲🟥 and 🟨 be two unequal magnitudes,
and 🔵 any other.

We shall first prove that ▲🟥 which is the greater of the two unequal magnitudes, has a greater ratio to 🔵 than 🟨, the less, has to 🔵; that is, ▲🟥 : 🔵 > 🟨 : 🔵;

take M' ▲🟥, m' 🔵, M' 🟨, m' 🔵;
such, that M' ▲ and M' 🟥 shall be each > 🔵;
also take m' 🔵 the least multiple of 🔵,
which will make m' 🔵 > M' 🟨 $=$ M' 🟥;
$\therefore$ M' 🟨 is $\not>$ m' 🔵,

but M' ▲🟥 is $>$ m' 🔵, for,
as m' 🔵 is the first multiple which first becomes
$>$ M' 🟥, than $(m' - 1)$ 🔵 or m' 🔵 $-$ 🔵 is $\not>$ M' ▲,
and 🔵 $\not>$ M' ▲
$\therefore$ m' 🔵 $-$ 🔵 $+$ 🔵 must be $<$ M' 🟥 $+$ M' ▲;

that is, m' 🔵 must be $<$ M' ▲🟥;

$\therefore$ M' ▲🟥 is $>$ m' 🔵; but it has been shown above that M' 🟨 is $\not>$ m' 🔵, therefore, by the seventh definition,

▲🟥 has to 🔵 a greater ratio than 🟨 : 🔵.
Next we shall prove that 🔵 has a greater ratio to 🟨, the less, than it has to ▲🟥, the greater;

or, 🔵 : 🟨 > 🔵 : ▲🟥.

Take m' 🔵, M' 🟨, m' 🔵 and M' ▲🟥,

the same as in the first case, such that
M' ▲ and M' ■ will be each > ●, and m' ● the least
multiple of ●, which first becomes greater than
M' ■ $= M'$ ■.
$\therefore m'$ ● $-$ ● is $\not>$ M' ■,
and ● is $\not>$ M' ▲; consequently
m' ● $-$ ● $+$ ● is $<$ M' ■ $+$ M' ▲;
$\therefore m'$ ● is $<$ M' $\overset{▲}{■}$, and $\therefore$ by the seventh definition,
● has to ■ a greater ratio than ● has to $\overset{▲}{■}$.
$\therefore$ Of unequal magnitudes, etc.

The contrivance employed in this proposition for finding among the multiples taken, as in the fifth definition, a multiple of the first greater than the multiple of the second, but the same multiple of the third which has been taken of the first, not greater than the same multiple of the fourth which has been taken of the second, may be illustrated numerically as follows :—

The number 9 has a greater ratio to 7 than 8 has to 7: that is, $9 : 7 > 8 : 7$; or $8 + 1 : 7 > 8 : 7$.

The multiple of 1, which first becomes greater than 7, is 8 times, therefore, we may multiply the first and third by 8, 9, 10, or any other greater number; in this case, let us multiply the first and third by 8, and we have $64 + 8$ and 64: again, the first multiple of 7 which becomes greater than 64 is 10 times; then, by multiplying the second and fourth by 10, we shall have 70 and 70; then, arranging these multiples, we have—

8 times the first	10 times the second	8 times the third	10 times the fourth
$64 + 8$	70	64	70

Consequently $64 + 8$, or 72, is greater than 70, but 64 is not greater than 70, $\therefore$ by the seventh definition 9 has a greater ratio to 7 than 8 has to 7.

The above is merely illustrative of the foregoing demonstration, for this property could be shown of these or other numbers very readily in the following manner; because, if an antecedent contains its consequent a greater number of times than another antecedent contains its consequent, or when a fraction is formed of an antecedent for the numerator, and its consequent for the denominator be greater than another fraction which is formed of another antecedent for the numerator and its consequent for the denominator, the ratio of the first antecedent to its consequent is greater than the ratio of the last antecedent to its consequent.

Thus, the number 9 has a greater ratio to 7, than 8 has to 7, for $\frac{9}{7}$ is greater than $\frac{8}{7}$.

Again, $17 : 19$ is a greater ratio than $13 : 15$, because $\frac{17}{19} = \frac{17\times15}{19\times15} = \frac{255}{185}$, and $\frac{13}{15} = \frac{13\times19}{15\times19} = \frac{247}{185}$, hence it is evident that $\frac{255}{185}$ is greater than $\frac{247}{185}$, $\therefore$ $\frac{17}{19}$ is greater than $\frac{13}{15}$, and, according to what has been above shown, 17 has to 19 a greater ratio than 13 has to 15.

So that the general terms upon which a greater, equal, or less ratio exists are as follows :—

If $\frac{A}{B}$ be greater than $\frac{C}{D}$, A is said to have to B a greater ratio than C has to D; if $\frac{A}{B}$ be equal to $\frac{C}{D}$, then A has to B the same ratio which C has to D; and if $\frac{A}{B}$ be less to $\frac{C}{D}$, A is said to have to B a less ratio than C has to D.

The student should understand all up to this proposition perfectly before proceeding further, in order fully to comprehend the following propositions of this book. We therefore strongly recommend the learner to commence again, and read up to this slowly, and carefully reason at each step, as he proceeds, particularly guarding against the mischievous system of depending wholly on the memory. By following these instructions, he will find that the parts which usually present considerable difficulties will present no difficulties whatever, in prosecuting the study of this important book.

MAGNITUDES *which have the same ratio to the same magnitude are equal to one another; and those to which the same magnitude has the same ratio are equal to one another.*

Let ◆ : ■ :: ● : ■, then ◆ = ●.

For, if not, let ◆ > ●, then will

◆ : ■ > ● : ■ (**??**), which is absurd according to the hypothesis.

∴ ◆ is $\not>$ ●.

In the same manner it may be shown, that

● is $\not>$ ◆,

∴ ◆ = ●.

Again, let ■ : ◆ :: ■ : ●,

then will ◆ = ●.

For (invert.) ◆ : ■ :: ● : ■,

therefore, by the first case, ◆ = ●.

∴ Magnitudes which have the same ratio, etc.

This may be shown otherwise, as follows :—

Let $A : B = A : C$, then $B = C$, for, as the fraction $\frac{A}{B}$ = the fraction $\frac{A}{C}$, and the numerator of one equal to the numerator of the other, therefore, denominators of these fractions are equal, that is $B = C$.

Again, if $B : A = C : A$, $B = C$. For, as $\frac{B}{A} = \frac{C}{A}$, B must $= C$.

THAT *magnitude which has a greater ratio than another has unto the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has unto another magnitude, is the less of the two.*

Let ⬟ : ■ > ● : ■, then ⬟ > ●.

For if not, let ⬟ = or < ●;

then, ⬟ : ■ = ● : ■ (??) or

⬟ : ■ < ● : ■ (??) and (invert.), which is absurd according to the hypothesis.

∴ ⬟ is ≠ or < ●, and

∴ ⬟ must be > ●.

Again, let ■ : ● > ■ : ⬟,

then, ● < ⬟.

For if not, ● must be > or = ⬟,

then ■ : ● < ■ : ⬟ (??) and (invert.);

or ■ : ● = ■ : ⬟ (??), which is absurd (??);

∴ ● is ≯ or = ⬟,

and ∴ ● must be < ⬟.

∴ That magnitude which has, etc.

RATIOS *that are the same to the same ratio, are the same to each other.*

Let ${\color{blue}\blacklozenge} : {\color{blue}\blacksquare} = {\color{red}\bullet} : {\color{orange}\blacktriangledown}$ and ${\color{red}\bullet} : {\color{orange}\blacktriangledown} = \blacktriangle : \bullet$,

then will ${\color{blue}\blacklozenge} : {\color{blue}\blacksquare} = \blacktriangle : \bullet$.

For if $M{\color{blue}\blacklozenge} >, =$ or $< m{\color{blue}\blacksquare}$,

then $M{\color{red}\bullet} >, =$ or $< m{\color{orange}\blacktriangledown}$,

and if $M{\color{red}\bullet} >, =$ or $< m{\color{orange}\blacktriangledown}$,

then $M\blacktriangle >, =$ or $< m\bullet$ (??);

$\therefore$ if $M{\color{blue}\blacklozenge} >, =$ or $< m{\color{blue}\blacksquare}$, $M\blacktriangle >, =$ or $< m\bullet$

and $\therefore$ (??) ${\color{blue}\blacklozenge} : {\color{blue}\blacksquare} = \blacktriangle : \bullet$.

$\therefore$ Ratios that are the same etc.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

Let

■ : ⬤ = ◡ : ⬡ = ◆ : ⛊ = • : ▼ = ▲ : ●;

then will ■ : ⬤ = ■ + ◡ + ◆ + • + ▲ :

⬤ + ⬡ + ⛊ + ▼ + ●.

For if M■ > m⬤, then M◡ > m⬡,

and M◆ > m⛊, M• > m▼,

also M▲ > m● (??)

Therefore, if M■ > m⬤, then will

M■ + M◡ + M◆ + M• + M▲, or

M(■ + ◡ + ◆ + • + ▲) be greater than

m⬤ + m⬡ + m⛊ + m▼ + m●, or

m(⬤ + ⬡ + ⛊ + ▼ + ●).

In the same way it may be shown, if M times one of the antecedents be equal to or less than m times one of the consequents, M times all the antecedents taken together, will be equal to or less than m times all the consequents taken together. Therefore, by the fifth definition, as one of the antecedents is to its consequent, so are all the antecedents taken together to all the consequents taken together.

∴ If any number of magnitudes, etc.

I*F the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth has to the sixth; the first shall also have to the second a greater ratio than the fifth to the sixth.*

Let ⬟ : ⊔ = ■ : ◆, but ■ : ◆ > ⬡ : ●,
then ⬟ : ⊔ > ⬡ : ●
For, ∵ ■ : ◆ > ⬡ : ●, there are some multiples
(M' and m') of ■ and ⬡, and of ◆ and ●,
such that M' ■ > m' ◆,
but M' ⬡ $\ngtr$ m' ●, by the seventh definition.
Let these multiples be taken, and take the same multiples
of ⬟ and ⊔.
∴ (??) if M' ⬟ >, =, or < m' ⊔;
then will M' ■ >, =, or < m' ◆,
but M' ■ > m' ◆ (construction);
∴ M' ⬟ > m' ⊔,
but M' ⬡ is $\ngtr$ m' ● (construction);
and therefore by the seventh definition,
⬟ : ⊔ > ⬡ : ●
∴ If the first has to the second, etc.

If the first has the same ratio to the second which the third has to the fourth; then, if the first be greater than the third, the second shall be greater than the fourth; and if equal, equal; and if less, less.

Let ⬟ : ⊔ :: ■ : ◆, and first suppose
⬟ > ■, then will ⊔ > ◆.
For ⬟ : ⊔ > ■ : ⊔ (??), and by the hypothesis
⬟ : ⊔ = ■ : ◆;
∴ ■ : ◆ > ■ : ⊔ (??),
∴ ◆ < ⊔ (??), or ⊔ > ◆.
Secondly, let ⬟ = ■, then will ⊔ = ◆.
For ⬟ : ⊔ = ■ : ⊔ (??),
and ⬟ : ⊔ = ■ : ◆ (??);
∴ ■ : ⊔ = ■ : ◆ (??),
and ∴ ⊔ = ◆ (??).
Thirdly, if ⬟ < ■, then will ⊔ < ◆;
∵ ■ > ⬟ and ■ : ◆ = ⬟ : ⊔;
∴ ◆ > ⊔, by the first case,
that is, ⊔ < ◆.
∴ If the first has the same ratio, etc.

MAGNITUDES *have the same ratio to one another which their equimultiples have.*

Let ● and ■ be two magnitudes;

then, ● : ■ :: M'● : M'■.

For ● : ■ = ● : ■

= ● : ■

= ● : ■

∴ ● : ■ :: 4● : 4■. (**??**).

And as the same reasoning is generally applicable, we have

● : ■ :: M'● : M'■.

∴ Magnitudes have the same ratio, etc.

V.8 Definition XIII

The technical term permutando, or alternando, by permutation or alternately, is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shown in the following proposition :—

Let  :  ::  : ,
by "permutando" or "alternando" it is inferred
 :  ::  : .

It may be necessary here to remark that the magnitudes , , , , must be homogeneous, that is, of the same nature or similitude of kind; we must therefore, in such cases, compare lines with lines, surfaces with surfaces, solids with solids, etc. Hence the student will readily perceive that a line and a surface, a surface and a solid, or other heterogenous magnitudes, can never stand in the relation of antecedent and consequent.

IF *four magnitudes of the same kind be proportionals, they are also proportionals when taken alternately.*

Let $\blacktriangledown : \sqcup :: \blacksquare : \blacklozenge$, then $\blacktriangledown : \blacksquare :: \sqcup : \blacklozenge$.
For $M\blacktriangledown : M\sqcup :: \blacktriangledown : \sqcup$ (??),
and $M\blacktriangledown : M\sqcup :: \blacksquare : \blacklozenge$ (??) and (??);
also $m\blacksquare : m\blacklozenge :: \blacksquare : \blacklozenge$ (??);
$\therefore M\blacktriangledown : M\sqcup :: m\blacksquare : m\blacklozenge$ (??),
and $\therefore$ if $M\blacktriangledown >, =$ or $< m\blacksquare$,
then will $M\sqcup >, =,$ or $< m\blacklozenge$ (??);
therefore, by the fifth definition,
$\blacktriangledown : \blacksquare :: \sqcup : \blacklozenge$
$\therefore$ If four magnitudes of the same kind, etc.

V.16 Definition XVI

Dividendo, by division, when there are four proportionals, and it is inferred, that the excess of the first above the second is to the second, as the excess of the third above the fourth, is to the fourth.

Let $A : B :: C : D$;
by "dividendo" it is inferred
$A - B : B :: C - D : D$.

According to the above, A is supposed to be greater than B, and C greater than D; if this be not the case, but to have B greater than A, and D greater than C, B and D can be made to stand as antecedents, and A and C as consequents, by "invertion"

$B : A :: D : C$;
then, by "dividendo," we infer
$B - A : A :: D - C : C$

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately: that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

Let $\blacktriangledown + \cup : \cup :: \blacksquare + \blacklozenge : \blacklozenge$,
then will $\blacktriangledown : \cup :: \blacksquare : \blacklozenge$.
Take $\blacktriangledown > m\cup$ to each add $M\cup$,
then we have $M\blacktriangledown + M\cup > m\cup + M\cup$,
or $M(\blacktriangledown + \cup) > (m+M)\cup$:
but $\because \blacktriangledown + \cup : \cup :: \blacksquare + \blacklozenge : \blacklozenge$ (??),
and $M(\blacktriangledown + \cup) > (m+M)\cup$;
$\therefore M(\blacksquare + \blacklozenge) > (m+M)\blacklozenge$ (??);
$\therefore M\blacksquare + M\blacklozenge > m\blacklozenge + M\blacklozenge$;
$\therefore M\blacksquare > m\blacklozenge$, by taking $M\blacklozenge$ from both sides:
that is, when $M\blacktriangledown > m\cup$, then $M\blacksquare > m\blacklozenge$.
In the same manner it may be proved, that if
$M\blacktriangledown =$ or $< m\cup$, then will $M\blacksquare =$ or $< m\blacklozenge$;
and $\therefore \blacktriangledown : \cup :: \blacksquare : \blacklozenge$ (??)
$\therefore$ If magnitudes taken jointly, etc.

V.15 Definition XV

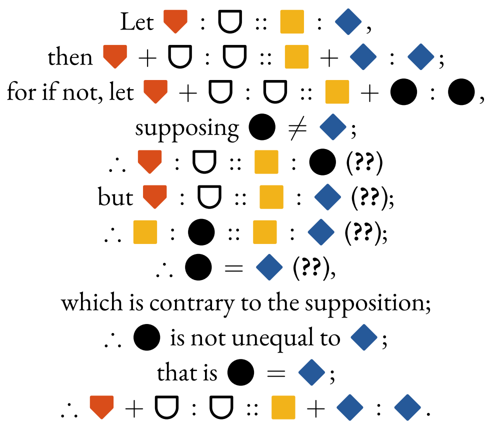
The term componendo, by composition, is used when there are four proportionals; and it is inferred that the first together with the second is to the second as the third together with the fourth is to fourth.

Let $A : B :: C : D$;
then, by term "componendo," it is inferred that
$A + B : B :: C + D : D$

By "invertion" B and D may become the first and the third, and A and C the second and fourth, as

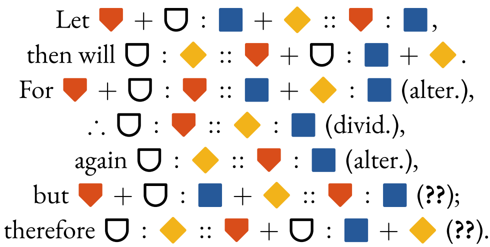
$B : A :: D : C$,
then, by "componendo," we infer that
$B + A : A :: D + C : C$.

I*F magnitudes, taken separately, be proportionals, they shall also be proportionals taken jointly: that is, if the first be to the second as the third is to the fourth, the first and second together shall be to the second as the third and fourth together is to the fourth.*



∴ If magnitudes, taken separately, etc.

If a whole magnitude be to a whole, as a magnitude taken from the first, is to a magnitude taken from the other; the remainder shall be to the reminder, as the whole to the whole.



∴ If a whole magnitude be to a whole, etc.

V.17 Definition XVII

The term "convertendo," by conversion, is made use of by geometricians, when there are four proportionals, and it is inferred, that the first is to its excess above the second, as the third is to its excess above the fourth. See the following proposition :—

IF *four magnitudes be proportionals, they are also proportionals by conversion: that is, the first is to its excess above the second, as the third is to its excess above the fourth.*

Let ●○ : ○ :: ■◆ : ◆,
then shall ●○ : ● :: ■◆ : ■.
∵ ●○ : ○ :: ■◆ : ◆;
∴ ● : ○ :: ■ : ◆ (divid.),
∴ ○ : ● :: ◆ : ■ (inver.),
∴ ●○ : ● :: ■◆ : ■ (compo.).
∴ If four magnitudes, etc.

V.18 Definition XVIII

"Ex æquali" (sc. distantiâ), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: "of this there are two following kinds, which arise from the different order in which the magnitudes are taken, two and two."

V.19 Definition XIX

"Ex æquali," from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second to the third of the first rank, so is the second to the third of the other; and so in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proposition. It is demonstrated in (**??**).

Thus, if there be two ranks of magnitudes, A, B, C, D, E, F, the first rank, and L, M, N, O, P, Q, the second, such that $A : B :: L : M$, $B : C :: M : B$, $C : D :: N : O$, $D : E :: O : P$, $E : F :: P : Q$; we infer by the term "ex æquali" that $A : F :: L : Q$

V.20 Definition XX

"Ex æquali in proportione perturbatâ seu inordinatâ," from equality in perturbate, or disorderly proportion. This term is used when the first magnitude is to the second of the first rank as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is in the 18th definition. It is demonstrated in (**??**).

Thus, if there be two ranks of magnitudes, A, B, C, D, E, F, the first rank, and L, M, N, O, P, Q, the second, such that $A : B :: P : Q$, $B : C :: O : P$, $C : D :: N : O$, $D : E :: M : N$, $E : F :: L : M$; the term "ex æquali in proportione perturbatâ seu inordinatâ" infers that $A : F :: L : Q$

If there be three magnitudes, and other three, which taken two and two, have the same ratio; then, if the first be greater than the third, the fourth shall be greater than the sixth; and if equal, equal; and if less, less.

Let , , , the first three magnitudes,
and , , , be the other three,
such that  :  ::  : , and
 :  ::  : .
Then, if  >, =, or < ,
then will  >, =, or < .
From the hypothesis, by alternando, we have
 :  ::  : ,
and  :  ::  : 
∴  :  ::  :  (??);
∴ if  >, =, or < ,
then will  >, =, or <  (??).
∴ If there be three magnitudes, etc.

If there be three magnitudes, and other three which have the same ratio, taken two and two, but in a cross order; then if the first magnitude be greater than the third, the fourth shall be greater than the sixth; and if equal, equal, and if less, less.

Let ▼, ⬟, ■, be the first three magnitudes,
and ◆, ⬠, ●, the other three,
such that ▼ : ⬟ :: ⬠ : ●,
and ⬟ : ■ :: ◆ : ⬠.
Then if ▼ >, =, or < ■,
then will ◆ >, =, or < ●.
First, let ▼ be > ■:
then, because ⬟ is any other magnitude,
▼ : ⬟ > ■ : ⬟ (??);
but ⬠ : ● :: ▼ : ⬟ (??);
∴ ⬠ : ● > ■ : ⬟ (??);
and ∵ ⬟ : ■ :: ◆ : ⬠ (??);
∴ ■ : ⬟ :: ⬠ : ◆ (inv.),
and it was shown that ⬠ : ● > ■ : ⬟,
∴ ⬠ : ● > ⬠ : ◆ (??);
∴ ● < ◆,
that is ◆ > ●.
Secondly, let ▼ = ■; then shall ◆ = ●.
For ∵ ▼ = ■,
▼ : ⬟ = ■ : ⬟ (??);
but ▼ : ⬟ = ⬠ : ● (??),
and ■ : ⬟ = ⬠ : ◆ (hyp. and inv.),
∴ ⬠ : ● = ⬠ : ◆ (??),
∴ ◆ = ● (??).
Next, let ▼ be < ■, then ◆ shall be < ●;
for ■ > ▼,
and it has been shown that ■ : ⬟ = ⬠ : ◆,
and ⬟ : ▼ = ● : ⬠;
∴ by the first case ● is > ◆,
that is, ◆ < ●.
∴ If there be three, etc.

IF *there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.*

N.B.— This is usually cited by the words "ex æquali," or "ex æquo."

First, let there be magnitudes ${\color{red}\blacktriangledown}$, ${\color{blue}\blacklozenge}$, ${\color{goldenrod}\blacksquare}$,

and as many others ${\color{red}\blacklozenge}$, ${\color{blue}\lozenge}$, ${\color{goldenrod}\bullet}$,

such that

${\color{red}\blacktriangledown} : {\color{blue}\blacklozenge} :: {\color{red}\blacklozenge} : {\color{blue}\lozenge}$,

and ${\color{blue}\blacklozenge} : {\color{goldenrod}\blacksquare} :: {\color{blue}\lozenge} : {\color{goldenrod}\bullet}$,

then shall ${\color{red}\blacktriangledown} : {\color{goldenrod}\blacksquare} :: {\color{red}\blacklozenge} : {\color{goldenrod}\bullet}$.

Let these magnitudes, as well as any equimultiples whatever of the antecedents and consequents of the ratios, stand as follows :—

${\color{red}\blacktriangledown}$, ${\color{blue}\blacklozenge}$, ${\color{goldenrod}\blacksquare}$, ${\color{red}\blacklozenge}$, ${\color{blue}\lozenge}$, ${\color{goldenrod}\bullet}$,

and

$M{\color{red}\blacktriangledown}$, $m{\color{blue}\blacklozenge}$, $N{\color{goldenrod}\blacksquare}$, $M{\color{red}\blacklozenge}$, $m{\color{blue}\lozenge}$, $N{\color{goldenrod}\bullet}$,

$\because {\color{red}\blacktriangledown} : {\color{blue}\blacklozenge} :: {\color{red}\blacklozenge} : {\color{blue}\lozenge}$,

$\therefore M{\color{red}\blacktriangledown} : m{\color{blue}\blacklozenge} :: M{\color{red}\blacklozenge} : m{\color{blue}\lozenge}$ (??).

For the same reason

$m{\color{blue}\blacklozenge} : N{\color{goldenrod}\blacksquare} :: m{\color{blue}\lozenge} : N{\color{goldenrod}\bullet}$;

and because there are three magnitudes

$M{\color{red}\blacktriangledown}$, $m{\color{blue}\blacklozenge}$, $N{\color{goldenrod}\blacksquare}$,

and other three, $M{\color{red}\blacklozenge}$, $m{\color{blue}\lozenge}$, $N{\color{goldenrod}\bullet}$,

which, taken two and two, have the same ratio;

$\therefore$ if $M{\color{red}\blacktriangledown} >, =$ or $< N{\color{goldenrod}\blacksquare}$

then will $M{\color{red}\blacklozenge} >, =$ or $< N{\color{goldenrod}\bullet}$, by (??),

and $\therefore {\color{red}\blacktriangledown} : {\color{goldenrod}\blacksquare} :: {\color{red}\blacklozenge} : {\color{goldenrod}\bullet}$ (??).

Next, let there be four magnitudes, ${\color{blue}\blacktriangledown}$, $\blacklozenge$, ${\color{goldenrod}\blacksquare}$, ${\color{red}\blacklozenge}$,

and other four, ${\color{blue}\lozenge}$, $\bullet$, ${\color{goldenrod}\rule{1em}{0.5em}}$, ${\color{red}\blacktriangle}$,

which, taken two and two, have the same ratio,

that is to say, ${\color{blue}\blacktriangledown} : \blacklozenge :: {\color{blue}\lozenge} : \bullet$,

$\blacklozenge : {\color{goldenrod}\blacksquare} :: \bullet : {\color{goldenrod}\rule{1em}{0.5em}}$,

and : :: : ,
then shall : :: : ;
for, ∵ , , , are three magnitudes,
and , , , other three,
which, taken two and two, have the same ratio;
therefore, by the foregoing case : :: : ,
but : :: : ;
therefore again, by the first case, : :: : ;
and so on, whatever the number of magnitudes be.
∴ If there be any number, etc.

IF *there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first of the others has to the last of the same.*
N.B.— This is usually cited by the words "ex æquali in proportione perturbatâ;" or "ex æquo perturbato."

First, let there be three magnitudes ▼, ∪, ■,
and other three, ◆, ⬠, ●,
which, taken two and two in a cross order, have the same ratio;
that is, ▼ : ∪ :: ⬠ : ●,
and ∪ : ■ :: ◆ : ⬠,
then shall ▼ : ■ :: ◆ : ●.

Let these magnitudes and their respective equimultiples be arranged as follows :—

▼, ∪, ■, ◆, ⬠, ●,
M▼, *M*∪, *m*■, *M*◆, *m*⬠, *m*●,
then ▼ : ∪ :: *M*▼ : *M*∪ (??);
and for the same reason
⬠ : ● :: *m*⬠ : *m*●;
but ▼ : ∪ :: ⬠ : ● (??),
∴ *M*▼ : *M*∪ :: ⬠ : ● (??);
and ∵ ∪ : ■ :: ◆ : ⬠ (??),
∴ *M*∪ : *m*■ :: *M*◆ : *m*⬠ (??);
then, because there are three magnitudes,
M▼, *M*∪, *m*∪,
and other three *M*◆, *m*⬠, *m*●,
which, taken two and two in a cross order, have the same ratio;
therefore, if *M*▼ >, =, or < *m*■,
then will *M*◆ >, =, or < *m*● (??),
and ∴ ▼ : ■ :: ◆ : ◆ (??).
Next, let there be four magnitudes,
▼, ∪, ■, ◆,

and other four,

, , , ,

which, when taken two and two in a cross order, have the same ratio;

namely, : :: : ,

: :: : ,

and : :: : ,

then shall : :: : .

For, ∵ , , are three magnitudes,

and , , , other three,

which, taken two and two in a cross order have the same ratio,

therefore, by the first case, : :: : ,

but : :: : ,

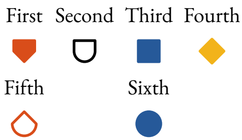
therefore again, by the first case,

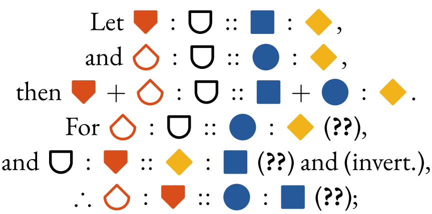
: :: : ;

and so on, whatever be the number of such magnitudes.

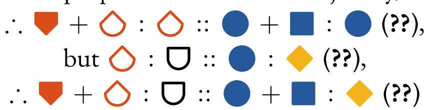
∴ If there be any number, etc.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same which the sixth has to the fourth, the first and fifth together shall have to the second the same ratio which the third and sixth together have to the fourth.





and, because these magnitudes are proportionals, they are proportionals when taken jointly,



∴ If the first, etc.

I*F four magnitudes of the same kind are proportionals, the greatest and least of them together are greater than the other two together.*

Let four magnitudes, + , + , , and ,
of the same kind, be proportionals, that is to say,
+ : + :: : ,
and let + be the greatest of the four, and
consequently by (?? and ??), is the least;
then will + + be > + + ;
∵ + : + :: : ,
∴ : :: + : + (??),
but + > + (??),
∴ > (??);
to each of these add + ,
∴ + + > + + .
∴ If four magnitudes, etc.

V.10 Definition X

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

For example, if A, B, C, be continued proportionals, that is, $A : B :: B : C$, A is said to have to C the duplicate ratio of $A : B$;

$$\text{or } \frac{A}{C} = \text{ the square of } \frac{A}{B}.$$

This property will be more readily seen of the quantities

$$ar^2, ar, a, \text{ for } ar^2 : ar :: ar : a;$$

$$\text{and } \frac{ar^2}{a} = r^2 = \text{ the square of } \frac{ar^2}{ar} = r,$$

$$\text{or of } a, ar, ar^2;$$

$$\text{for } \frac{a}{ar^2} = \frac{1}{r^2} = \text{ the square of } \frac{a}{ar} = \frac{1}{r}.$$

V.11 Definition XI

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which is has to the second; and so on, quadruplicate, etc. increasing the denomination still by unity, in any number of proportionals.

For example, let A, B, C, D, be four continued proportionals, that is, $A : B :: B : C :: C : D$; A is said to have to D, the triplicate ratio of A to B;

$$\text{or } \frac{A}{D} = \text{ the cube of } \frac{A}{B}.$$

This definition will be better understood, and applied to a greater number of magnitudes than four that are continued proportionals, as follows :—

Let ar^3, ar^2, ar, a, be four magnitudes in continued proportion, that is, $ar^3 : ar^2 :: ar^2 : ar :: ar : a$,

$$\text{then } \frac{ar^3}{a} = r^3 = \text{ the cube of } \frac{ar^3}{ar^2} = r.$$

Or, let ar^5, ar^4, ar^3, ar^2, ar, a, be six magnitudes in proportion, that is

$$ar^5 : ar^4 :: ar^4 : ar^3 :: ar^3 : ar^2 :: ar^2 : ar :: ar : a,$$

$$\text{then the ratio } \frac{ar^5}{a} = r^5 = \text{ the fifth power of } \frac{ar^5}{ar^4} = r$$

Or, let a, ar, ar^2, ar^3, ar^4, be five magnitudes in continued proportion; then $\frac{a}{ar^4} = \frac{1}{r^4} =$ the fourth power of $\frac{a}{ar} = \frac{1}{r}$.

V.A Definition A

To know a compound ratio :—

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth; and so on, unto the last magnitude.

A B C D
E F G H K L
M N

For example, if A, B, C, D, be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F, and B to C the same ratio that G has to H, and C to D the same that K has to L; then by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed; if M has to N the same ratio which A has to D, then for shortness sake, M is said to have to N the ratio compounded of the ratios E to F, G to H, and K to L.

This definition may be better understood from an arithmetical or algebraical illustration; for, in fact, a ratio compounded of several other ratios, is nothing more than a ratio which has for its antecedent the continued product of all antecedents of the ratios compounded, and for its consequent the continued product of all the consequents of the ratios compounded.

Thus, the ratio compounded of the ratios of

$$2 : 3, 4 : 7, 6 : 11, 2 : 5,$$

$$\text{is the ratio of } 2 \times 4 \times 6 \times 2 : 3 \times 7 \times 11 \times 5,$$
$$\text{or the ratio of } 96 : 1155, \text{ or } 32 : 385.$$

And of the magnitudes A, B, C, D, E, F, of the same kind, $A : F$ is the ratio compounded of the ratios of

$$A : B, B : C, C : D, D : E, E : F;$$
$$\text{for } A \times B \times C \times D \times E : B \times C \times D \times E \times F,$$
$$\text{or } \frac{A \times B \times C \times D \times E}{B \times C \times D \times E \times F} = \frac{A}{F}, \text{ or the ratio of } A : F.$$

RATIOS *which are compounded of the same ratios are same to one another.*

$$\begin{aligned} \text{Let } A : B &:: F : G, \\ B : C &:: G : H, \\ C : D &:: H : K, \\ \text{and } D : E &:: K : L, \end{aligned}$$

A B C D E
F G H K L

Then the ratio which is compounded by the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is the same as the ratio compounded of the ratios $F : G$, $G : H$, $H : K$, $K : L$, or the ratio of $F : L$.

$$\begin{gathered} \text{For } \frac{A}{B} = \frac{F}{G}, \\ \frac{B}{C} = \frac{G}{H}, \\ \frac{C}{D} = \frac{H}{K}, \\ \text{and } \frac{D}{E} = \frac{K}{L}; \\ \therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{F \times G \times H \times K}{G \times H \times K \times L}, \\ \text{and } \therefore \frac{A}{E} = \frac{F}{L}, \end{gathered}$$

or the ratio of $A : E$ is the same as the ratio $F : L$.

The same may be demonstrated of any number of ratios so circumstanced.

$$\begin{aligned} \text{Next, let } A : B &:: K : L, \\ B : C &:: H : K, \\ C : D &:: G : H, \\ D : E &:: F : G, \end{aligned}$$

Then the ratio which is compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, or the ratio of $A : E$, is the same as the ratio compounded of the ratios of $K : L$, $H : K$, $G : H$, $F : G$, or the ratio of $F : L$

$$\text{For } \frac{A}{B} = \frac{K}{L},$$
$$\frac{B}{C} = \frac{H}{K},$$
$$\frac{C}{D} = \frac{G}{H},$$
$$\text{and } \frac{D}{E} = \frac{F}{G};$$
$$\therefore \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{K \times H \times G \times F}{L \times K \times H \times G},$$
$$\text{and} \therefore \frac{A}{E} = \frac{F}{L},$$

or the ratio of $A : E$ is the same as the ratio $F : L$.

$\therefore$ Ratios which are compounded, etc.

IF *several ratios be the same to several ratios, each to each, the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.*

A	B	C	D	E	F	G	H	P	Q	R	S	T
a	b	c	d	e	f	g	h	V	W	X	Y	Z

$$\begin{array}{llll} \text{If} & A : B :: a : b & \text{and } A : B :: P : Q & \text{and } a : b :: V : W \\ & C : D :: c : d & C : D :: Q : R & c : d :: W : X \\ & E : F :: e : f & E : F :: R : S & e : f :: X : Y \\ \text{and} & G : H :: g : h & G : H :: S : T & f : h :: Y : Z \end{array}$$

then $P : T = V : Z$.

For $\frac{P}{Q} = \frac{A}{B} = \frac{a}{b} = \frac{V}{W}$

$$\frac{Q}{R} = \frac{C}{D} = \frac{c}{d} = \frac{W}{X}$$

$$\frac{R}{S} = \frac{E}{F} = \frac{e}{f} = \frac{X}{Y}$$

$$\frac{S}{T} = \frac{G}{H} = \frac{g}{h} = \frac{Y}{Z}$$

and $\therefore \frac{P \times Q \times R \times S}{Q \times R \times S \times T} = \frac{V \times W \times X \times Y}{W \times X \times Y \times Z}$,

and $\therefore \frac{P}{T} = \frac{V}{Z}$,

or $P : T = V : Z$.

$\therefore$ If several ratios, etc.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio, compounded of these remaining ratios.

$A\ B\ C\ D\ E\ F\ G\ H$
$P\ Q\ R\ S\ T\ X$

Let $A : B$, $B : C$, $C : D$, $D : E$, $E : F$, $F : G$, $G : H$, be the first ratios, and $P : Q$, $Q : R$, $R : S$, $S : T$, $T : X$, the other ratios; also, let $A : H$, which is compounded of the first ratios, be the same as the ratio of $P : X$, which is the ratio compounded of the other ratios; and, let the ratio of $A : E$, which is compounded of the ratios of $A : B$, $B : C$, $C : D$, $D : E$, be the same as the ratio of $P : R$, which is compounded of the ratios $P : Q$, $Q : R$.

Then the ratio which is compounded of the remaining first ratios, that is, the ratio compounded of the ratios $E : F$, $F : G$, $G : H$, that is, the ratio of $E : H$, shall be the same as the ratio of $R : X$, which is compounded of the ratios of $R : S$, $S : T$, $T : X$, the remaining other ratios.

$$\text{Because } \frac{A \times B \times C \times D \times E \times F \times G}{B \times C \times D \times E \times F \times G \times H} = \frac{P \times Q \times R \times S \times T}{Q \times R \times S \times T \times X},$$

$$\text{or } \frac{A \times B \times C \times D}{B \times C \times D \times E} \times \frac{E \times F \times G}{F \times G \times H} = \frac{P \times Q}{Q \times R} \times \frac{R \times S \times T}{S \times T \times X},$$

$$\text{and } \frac{A \times B \times C \times D}{B \times C \times D \times E} = \frac{P \times Q}{Q \times R}$$

$$\therefore \frac{E \times F \times G}{F \times G \times H} = \frac{R \times S \times T}{S \times T \times X},$$

$$\therefore \frac{E}{H} = \frac{R}{X},$$
$$\therefore E : H = R : X.$$

$\therefore$ If a ratio which, etc.

If there be any number of ratios, and any number of other ratios, such that the ratio which is compounded of ratios, which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios, which are the same, each to each, to the last ratios—and if one of the first ratios, or the ratio which is compounded of ratios, which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios, which are the same, each to each, to several of the last ratios—then the remaining ratio of the first; or, if there be more than one, the ratio which is compounded of ratios, which are the same, each to each, to the remaining ratios of the first, shall be the same to the remaining ratio of the last; or, if there be more than one, to the ratio which is compounded of ratios, which are the same, each to each, to these remaining ratios.

$h \quad k \quad m \quad n \quad s$

$A\ B,\ C\ D,\ E\ F,\ G\ H,\ K\ L,\ M\ N \quad a\,b\,c\,d\,e\,f\,g$

$O\ P,\ Q\ R,\ S\ T,\ V\ W,\ X\ Y \quad h\,k\,l\,m\,n\,p$

$a \quad b \quad k \quad m \quad e \quad f \quad g$

Let $A : B$, $C : D$, $E : F$, $G : H$, $K : L$, $M : N$, be the first ratios, and $O : P$, $Q : R$, $S : T$, $V : W$, $X : Y$, the other ratios;

$$\text{and let } A : B = a : b,$$
$$C : D = b : c,$$
$$E : F = c : f,$$
$$G : H = d : e,$$
$$K : L = e : f,$$
$$M : N = f : g.$$

Then, by the definition of a compound ratio, the ratio of $a : g$ is compounded of the ratios of $a : b$, $b : c$, $c : d$, $d : e$, $e : f$, $f : g$, which are the same as the ratio of $A : B$, $C : D$, $E : F$, $G : H$, $K : L$, $M : N$, each to each.

$$\text{Also, } O : P = h : k,$$
$$Q : R = k : l,$$
$$S : T = l : m,$$
$$V : W = m : n,$$
$$X : Y = n : p,$$

Then will the ratio of $h : p$ be the ratio compounded of the ratios of $h : k$, $k : l$, $l : m$, $m : n$, $n : p$, which are the same as the ratios of $O : P$, $Q : R$, $S : T$, $V : W$, $X : Y$, each to each.

$$\therefore \text{by the hypothesis } a : g = h : p.$$

Also, let the ratio which is compounded of the ratios of $A : B$, $C : D$, two of the first ratios (or the ratios of $a : c$ for $A : B = a : b$, and $C : D = b : c$), be the same as the ratio of $a : d$, which is compounded of the ratios of $a : b$, $b : c$, $c : d$, which are the same as the ratios of $O : P$, $Q : R$, $S : T$, three of the other ratios.

And let the ratios of $h : s$, which is compounded of the ratios of $h : k$, $k : m$, $m : n$, $n : s$, which are the same as the remaining first ratios, namely, $E : F$, $G : H$, $K : L$, $M : N$; also, let the ratio of $e : g$, be that which is compounded of the ratios $e : f$, $f : g$, which are the same, each to each, to the remaining other ratios, namely $V : W$, $X : Y$. Then the ratios of $h : s$ shall be the same as the ratio of $e : g$; or $h : s = e : g$.

For

$$\frac{A \times C \times E \times G \times K \times M}{B \times D \times F \times H \times L \times N} = \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g},$$

$$\text{and } \frac{O \times Q \times S \times V \times X}{P \times R \times T \times W \times Y} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p},$$

by the composition of the ratios;

$$\therefore \frac{a \times b \times c \times d \times e \times f}{b \times c \times d \times e \times f \times g} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (??)},$$

$$\text{or } \frac{a \times b}{b \times c} \times \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times l}{k \times l \times m} \times \frac{m \times n}{n \times p},$$

$$\text{but } \frac{a \times b}{b \times c} = \frac{A \times C}{B \times D} = \frac{O \times Q \times S}{P \times R \times T} = \frac{a \times b \times c}{b \times c \times d} = \frac{h \times k \times l}{k \times l \times m};$$

$$\therefore \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{m \times n}{n \times p}.$$

$$\text{And } \frac{c \times d \times e \times f}{d \times e \times f \times g} = \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p} \text{ (??)},$$

$$\text{and } \frac{m \times n}{n \times p} = \frac{e \times f}{f \times g} \text{ (??)},$$

$$\therefore \frac{h \times k \times l \times m \times n}{k \times l \times m \times n \times p} = \frac{ef}{fg},$$

$$\therefore \frac{h}{s} = \frac{e}{g},$$

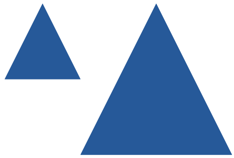
$$\therefore h : s = e : g.$$

∴ If there be any number, etc.

Book VI

Definitions

VI.1



Rectilinear figures are said to be similar, when they have their several angles equal, each to each, and the sides about the equal angles proportional.

VI.2

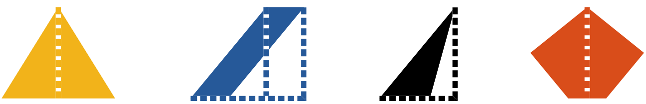
Two sides of one figure are said to be reciprocally proportional to two sides of another figure when one of the sides of the first is to the second, as the remaining side of the second is to the remaining side of the first.

VI.3

A straight line is said to be cut in extreme and mean ratio, when the whole is to the greater segment, as the greater segment is to the less.

VI.4

The altitude of any figure is the straight line drawn from its vertex perpendicular to its base, or the base produced.



TRIANGLES *and parallelograms having the same altitude are to one another as their bases.*

Let the triangles  and  have a common vertex, and their bases and in the same straight line.

Produce both ways, take successively on produced lines equal to it; and on produced lines successively equal to it; and draw lines from the common vertex to their extremities.

The triangles  thus formed are all equal to one another, since their bases are equal. (**??**)

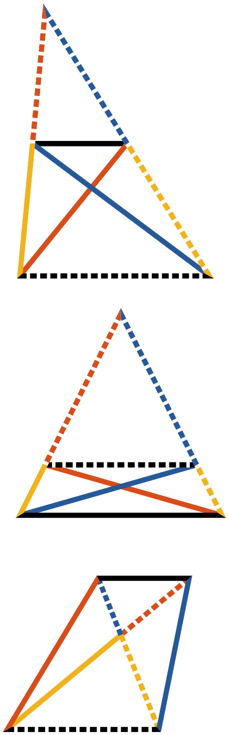
$\therefore$  and its base are respectively equimultiples of  and the base .

In like manner  and its base are respectively equimultiples of  and the base .

$\therefore$ if m or 6 times  $>, =$ or $<$ n or 5 times  then m or 6 times $>, =$ or $<$ n or 5 times , m and n stand for every multiple taken as in the fifth definition of the Fifth Book. Although we have only shown that this property exists when m equal 6, and n equal 5, yet it is evident that the property holds good for every multiple value that may be given to m, and to n.

Parallelograms having the same altitude are the doubles of the triangles, on their bases, and are proportional to them (Part I), and hence their doubles, the parallelograms, are as their bases (**??**).

Q. E. D.



IF *a straight line* *be drawn parallel to any side* *of a triangle, it shall cut the other sides, or those sides produced, into proportional segments*

And if any straight line *divide the sides of a triangle, or those sides produced, into proportional segments, it is parallel to the remaining side* .

Part I.

Let ‖ , then shall

: :: : .

Draw and ,

and = (??);

∴ : :: : (??);

but : :: : (??),

∴ : :: : (??).

Part II.

Let : :: : .

then || .

Let the same construction remain,

∵ : :: : (??) ;

and : :: :

but : :: : (??)

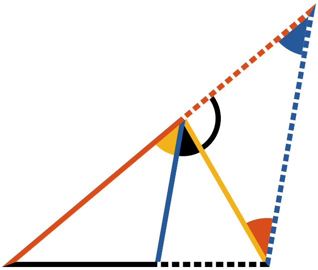
∴ : :: : (??)

∴ = (??);

but they are on the same base , and at the same side of it, and

∴ || (??).

Q. E. D.



A RIGHT *line ( ) bisecting the angle of a triangle, divides the opposite side into segments (,) proportional to the conterminous sides (,).*

And if a straight line ( ) drawn from any angle of a triangle divide the opposite side ( ) into segments (,) proportional to the conterminous sides (,), it bisects the angle.

Part I.

Draw ∥ , to meet ;

then, = (??).

∴ = ; but = , ∴ = ,

∴ = (??);

and ∵ ∥ ,

: :: : (??);

but = ;

∴ : :: : (??).

Part II.

Let the same construction remain,

and : :: : (??);

but : :: : (??)

∴ : :: : (??).

and ∴ = (??),

and ∴ = (??);

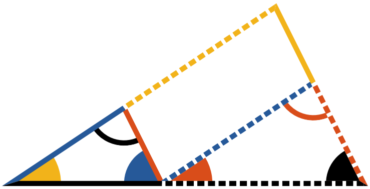
but since ∥ ; = ,

and = (??);

∴ = , and = ,

and ∴ bisects .

Q. E. D.



IN *equiangular triangles (and) the sides about the equal angles are proportional, and the sides which are opposite to the equal angles are homologous.*

Let the equiangular triangles be so placed that two sides , opposite to equal angles and may be conterminous, and in the same straight line; and that the triangles lying at the same side of that straight line; and that the triangles lying at the same side of that straight line, may have the equal angles not conterminous, i. e. opposite to , and to .

Draw and .

Then, ∵ = , || (??);

and for a like reason || ,

∴ is a parallelogram.

But : :: : (??);

and since = (??),

: :: : ;

and by alternation

: :: : (??).

In like manner it may be shown, that

: :: : ;

and by alternation, that

: :: : ;

but it has been already proved that

: :: :

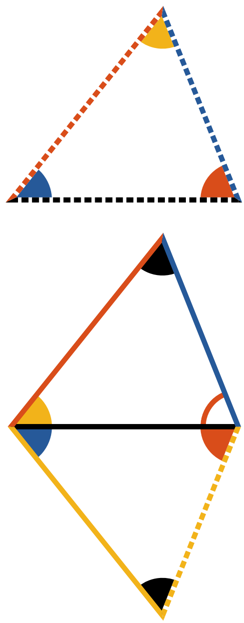
and therefore, ex æquali,

: :: : (??),

therefore the sides about the equal angles are proportional, and those which are opposite to the equal angles are homologous.

Q. E. D.

If two triangles have their sides proportional (: :: :) and (: :: :) they are equiangular, and the equal angles are subtended by the homologous sides.



From the extremities of , draw and , making

= , = (??);

and consequently = (??),
and since the triangles are equiangular,

: :: : (??);

but : :: : (??);

∴ : :: : , and consequently = (??). In like manner it may be shown that

= .

Therefore, the two triangles having a common base , and their sides equal, have also equal angles opposite to equal sides, i. e.

= and = (??).

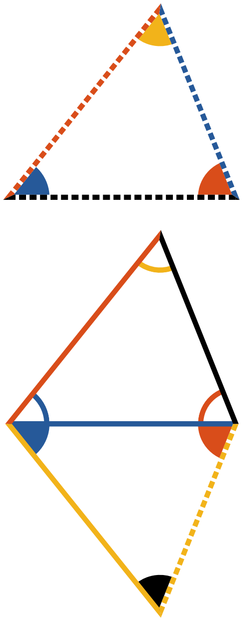
but = (??) and ∴ = ;

for the same reason = ,

and consequently = (??);

and therefore the triangles are equiangular, and it is evident that the homologous sides subtended by the equal angles.

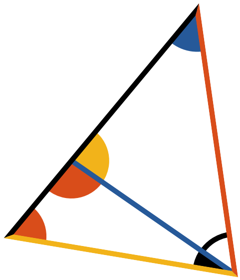
Q. E. D.

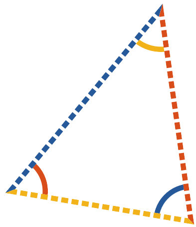


If two triangles (and) have one angle ( ) of the one, equal to one angle ( ) of the other, and the sides about the equal angles proportional, the triangles shall be equiangular, and have those angles equal which the homologous sides subtend.

From the extremities of , one of the sides of
, about ,
draw and ,
making = ,
and = ;
then = (??),
and two triangles being equiangular,
: :: : (??);
but : :: : (??)
∴ : :: : (??),
and consequently = (??);
∴ = in every respect (??).
But = (??),
and ∴ = ;
and since also = ,
= (??);
and ∴ and are equiangular, with their equal angles opposite to homologous sides.

Q. E. D.





If two triangles (and) have one angle in each equal (equal to), the sides about two other angles (: :: :), and each of the remaining angles (and) either less or not less than a right angle, the triangles are equiangular, and those angles are equal about which the sides are proportional.

First let it be assumed that the angles and are each less than a right angle: then if it be supposed that and contained by the proportional sides, are not equal, let be greater, and make = .

∵ = (??) and = (??)

∴ = (??);

∴ : :: : (??),

but : :: : (??)

∴ : :: : ;

∴ = (??),

and ∴ = (??).

But is less than a right angle (??)

∴ is less than a right angle;

and ∴ must be greater than a right angle (??), but it has been proven = and therefore less than a right angle, which is absurd. ∴ and are not unequal;

∴ they are equal, and since = (??)

∴ = (??), and therefore the triangles are equiangular.

But if and be assumed to be each not less than a right angle, it may be proved as before, that the triangles are equiangular, and have the sides about equal the angles proportional (**??**).

Q. E. D.

In a right angled tiangle (), if a perpendicular () be drawn from the right angle to the opposite side, the triangles (, ) on each side of it are similar to the whole triangle and to each other.

∵  =  (??),

and  common to  and ;

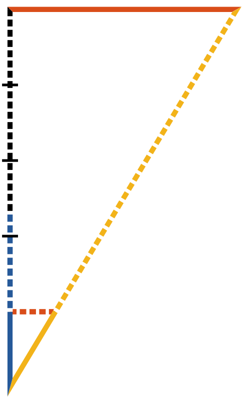
 =  (??);

∴  and  are equiangular; and consequently have their sides about the equal angles proportional (??), and are therefore similar (??).

In like manner it may be proved that  is similar to ; but  has been shown to be similar to ;

∴  and  are similar to the whole and to each other.

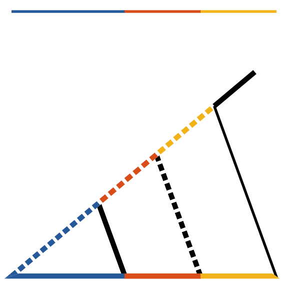
Q. E. D.



FROM *a given straight line ( ) to cut off any required part.*

From either extremity of the given line draw
making any angle with ;
and produce till the whole produced line
contains as often as
contains the required part.
Draw ,
and draw || .
is the required part of .
For since ||
: :: : (??),
and by composition (??);
: :: : ;
but contains as often as
contains the required part (??);
∴ is the required part.

Q. E. D.



T*o divide a given straight line ( ) similarly to a given divided line ( ).*

From either extremity of the given line draw making any angle;

take , and equal to , and respectively (??);

draw , and draw and || to it.

Since are ||,

: :: : (??),

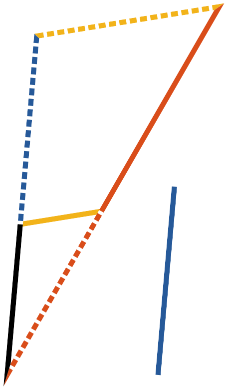
or : :: : (??),

and : :: : (??),

: :: : (??),

and ∴ the given line is divided similarly to .

Q. E. D.



To find a third proportional to two given straight lines (and).

At either extremity of the given line draw

making an angle;

take = , and draw ;

make = ,

and draw || ; (??)

is the third proportional to and .

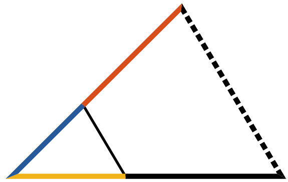
For since || ,

∴ : :: : (??);

but = = (??);

∴ : :: : (??).

Q. E. D.



T*o find a fourth proportional to three given lines* { }.

Draw and making any angle;

take = ,

and = ,

also = ,

draw ,

and ∥ (??);

is the fourth proportional.

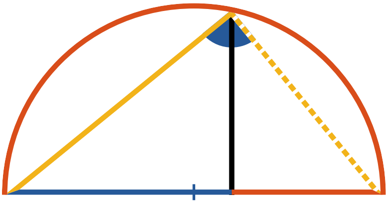
On account of the parallels,

: :: : (??);

but { } = { } (??);

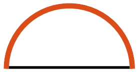
∴ : :: : (??).

Q. E. D.



T*o find a mean proportional between two given straight lines* { }.

Draw any straight line ,
make = and = ;
bisect ; and from the point of bisection as a centre, and half line as a radius, describe a semicircle



,
draw ⊥ :
is the mean proportional required.
Draw and .
Since is a right angle (**??**),
and is ⊥ from it upon the opposite side,
∴ is a mean proportional between and (**??**),
and ∴ between and (**??**).

Q. E. D.

EQUAL *parallelograms and , which have one angle in each equal, have the sides about the equal angles reciprocally proportional*
(: :: :)
And parallelograms which have one angle in each equal, and the sides about them reciprocally proportional, are equal.

Let and ; and and , be so placed that and may be continued right lines. It is evident that they may assume this position. (**??**, **??** and **??**)

Complete .

Since = ;

∴ : :: : (**??**)

∴ : :: : (**??**)

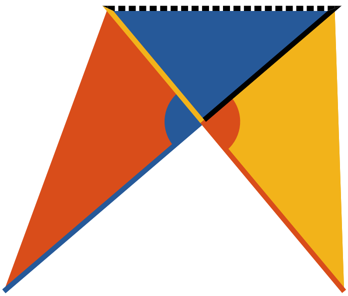
The same construction remaining:

: :: : (**??**)
: (**??**)
: (**??**)

∴ : :: : (**??**)

and ∴ = (**??**).

Q. E. D.



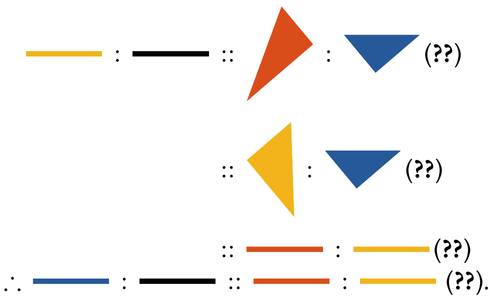
EQUAL *triangles, which have one angle in each equal (=), have the sides about the equal angles reciprocally proportional* (: :: :)

And two triangles which have an angle of the one equal to an angle of the other, and the sides about the equal angles reciprocally proportional, are equal.

I.

Let the triangles be so placed that the equal angles and may be vertically opposite, that is to say, so that and may be in the same straight line. Whence also and must be in the same straight line (??).

Draw , then



II.

Let the same construction remain, and

 :  :: : (??)

and : ::  :  (??)

but : :: : , (??)

$\therefore$  :  ::  :  (??);

$\therefore$  =  (??).

Q. E. D.

If four straight lines be proportional (: :: :) the rectangle (×) contained by the extremes, is equal to the rectangle (×) contained by the means.

And if the rectangle contained by the extremes be equal to the rectangle contained by the means, the four straight lines are proportional.

Part I.

From the extremities of and draw and ⊥ to them and = and respectively:

complete the parallelograms and .

and since,

: :: : (??)

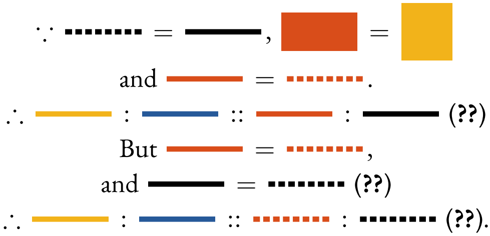
∴ : :: : (??)

∴ = (??),

that is, the rectangle contained be the extremes, equal to the rectangle contained be the means.

Part II.

Let the same construction remain;



Q. E. D.

If three striaght lines be proportional (: :: :) the rectangle under the extremes is equal to the square of the mean.

And if the rectangle under the extremes be equal to the square of the mean, the three straight lines are proportional.

Part I.

Assume = ,

and since : :: : ,

then : :: : ,

$\therefore$ $\times$ = $\times$

(??).

But = ,

$\therefore$ $\times$ = $\times$ or =

^{2};

therefore, if the three straight lines are proportional, the rectangle contained be the extremes is equal to the square of the mean.

Part II.

Assume = ,

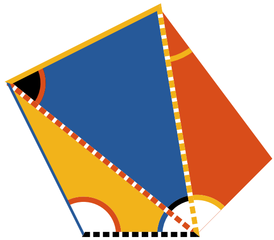
then $\times$ = $\times$,

$\therefore$: :: : (??),

and : :: : .

Q. E. D.

O*N a given straight line ( ) to construct a rectilinear figure similar to a given one ( ) and similarly placed.*



Resolve the given figure into triangles by drawing the lines and .

At the extremities of make

= and = ;

again at the extremities of make

= and = ;

in like manner make

= and = .

Then = .

It is evident from the construction and (??) that the figures are equiangular; and since the triangles and are equiangular;

then by (??), : :: :

and : :: :

Again, because and are equiangular,

: :: :

∴ ex æquali,

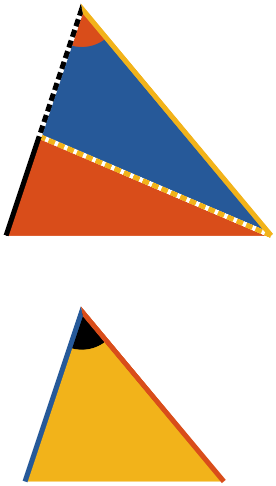
: :: : (??)

In like manner it may be shown that the remaining sides of the two figures are proportional.

∴ by (??)

is similar to and similarly situated; and on the given line .

Q. E. D.



SIMILAR *triangles (and) are to one another in the duplicate ratio of their homologous sides.*

Let and be equal angles, and and homologous sides of the similar triangles and and on the greater of these lines take a third proportional, so that

: :: : ;

draw .

: :: : (??);

∴ : :: : (??),

but : :: : (??),

∴ : :: :

consequently = for they have the sides about the equal angles and reciprocally proportional (??);

∴ : :: : (??);

but : :: : (??),

∴ : :: : ,

that is to say, the triangles are to one another in the duplicate ratio of their homologous sides and (??).

Q. E. D.

SIMILAR *polygons may be divided into the same number of similar triangles, each similar pair of which are proportional to the polygons; and the polygons are to each other in the duplicate ratio of their homologous sides.*

Draw and , and and , resolving the polygons into triangles. Then because the polygons are similar, = , and : :: :

∴ and are similar,

and = (??);

but = because they are angles of similar polygons; therefore the remainders and are equal;

hence : :: : ,

on account of the similar triangles,

and : :: : ,

on account of the similar polygons,

∴ : :: : ,

ex æquali (??), and as these proportional sides contain equal angles, the triangles and are similar (??).

In like manner it may be shown that the triangles and are similar.

But is to in the duplicate ratio of to (??),

and is to in like manner, in the duplicate ratio of to ;

∴ : :: : (??).

Again  is to  in the duplicate ratio of to , and  is to  in the duplicate ratio of to .

 :  ::  :  ;

::  : 

and as one of the antecedents is to one of the consequents, so is the sum of all the antecedents to the sum of all the consequents; that is to say, the similar triangles have to one another the same ratio as the polygons (??).

But  is to  in the duplicate ratio of to ;

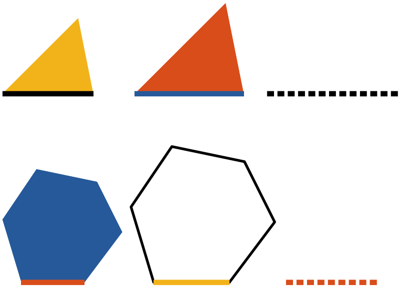
∴  is to  in the duplicate ratio of to .

Q. E. D.

RECTILINEAR *figures (*  *and*  *) which are similar to the same figure (*  *) are similar also to each other.*

Since  and  are similar, they are equiangular, and have the sides about the equal angles proportional (**??**); and since the figures  and  are also similar, they are equiangular, and have the sides about the equal angles proportional; therefore  and  are also equiangular, and have the sides about the equal angles proportional (**??**), and are therefore similar.

Q. E. D.



I*f four straight lines be proportional (: :: :), the similar rectilinear figures similarly described on them are also proportional.*

And if four similar rectilinear figures, similarly described on four straight lines, be proportional, the straight lines are also proportional.

Part I.

Take a third proportional to and , and a third proportional to and (??);

since : :: : (??)

: :: : (??)

∴ ex æquali,

: :: : ;

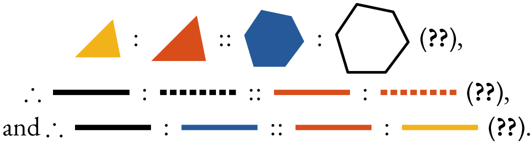
but : :: : (??),

and : :: : ;

∴ : :: : (??).

Part II.

Let the same construction remain;



Q. E. D.

EQUIANGULAR *parallelograms (and) are to one another in a ratio compounded of the ratios of their sides.*

Let two of the sides and about the equal angles be placed so that they may form one straight line.

Since + = ,

and = (??),

+ = ,

and ∴ and form one straight line (??);

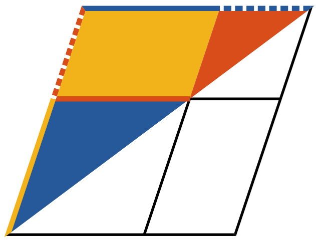
complete .

Since : :: : (??),

and : :: : (??),

has to a ratio compounded of the ratios of to , and of to .

Q. E. D.



IN *any parallelogram ( ) the parallelograms (and) which are about the diagonal are similar to the whole, and to each other.*

As and have a common angle they are equiangular;

but ∵ ∥

and are similar (??),

∴ : :: : ;

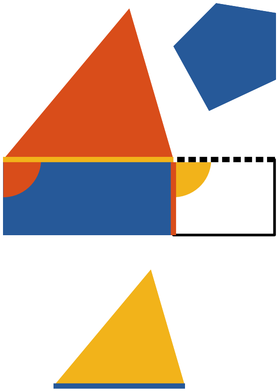
and the remaining opposite sides are equal to those,

∴ and have the sides about the equal angles proportional, and are therefore similar.

In the same manner it can be demonstrated that the parallelograms and are similar.

Since, therefore, each of the parallelograms and is similar to , they are similar to each other.

Q. E. D.



T*o describe a rectilinear figure, which shall be similar to a given rectilinear figure* (*), and equal to another* (*).*

Upon describe = ,

and upon describe = ,

and having = (??),

and then and will lie in the same straight line (?? and ??).

Between and find a mean proportional (??),

and upon describe , similar to , and similarly situated.

Then = .

For since and are similar, and

: :: : (??),

: :: : (??);

but : :: : (??);

∴ : :: : (??);

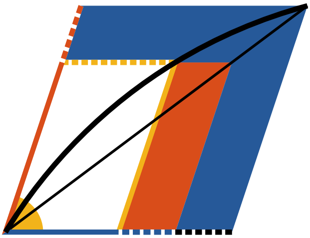
but = (??),

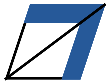
and ∴ = (??);

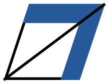
and  =  (??);

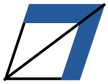
consequently,  which is similar to  is also = .

Q. E. D.



I*F similar and similarly posited parallelograms (*  *and*  *) have a common angle, they are about the same diagonal.*

For, if possible, let  be the diagonal of  and draw ‖ (??).

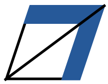
Since  and  are about the same diagonal , and have  common, they are similar (??);

∴ : :: : ;

but : :: : (??),

∴ : :: : ,

and ∴ = (??), which is absurd.

∴  is not the diagonal of  in the same manner it can be demonstrated that no other line is except .

Q. E. D.

OF *all the rectangles contained by the segments of a given straight line, the greatest is the square which is described on half the line.*

Let be the given line,

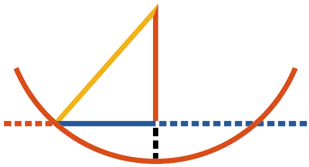
 and unequal segments,

and and equal segments;

then  > .

For it has been demonstrated already (??), that the square of half the line is equal to the rectangle contained by any unequal segments together with the square of the part intermediate between the middle point and the point of unequal section. The square described on half the line exceeds therefore the rectangle contained by any unequal segments of the line.

Q. E. D.



T*o divide a given straight line ( ) so that the rectangle contained by its segments may be equal to a given area, not exceeding the square of half the line.*

Let the given area be = ^{2}.

Bisect , or make $=$;

and if $^2 =$ ^{2},

problem is solved.

But if $^2 \neq$ ^{2},

then must $>$ (??).

Draw $\perp$ $=$;

make $=$ or ;

with as radius describe a circle cutting the given line; draw .

Then $\times$ $+$ $^2 =$ ^{2} (??)

$=$ ^{2}.

But $^2 =$ $^2 +$ ^{2} (??);

$\therefore$ $\times$ $+$ $^2 =$ $^2 +$ ^{2},

from both, take ^{2},

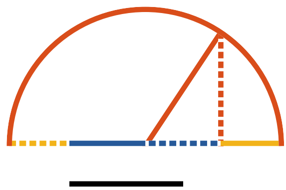
and $\times$ $=$ ^{2}.

But $=$ (??),

and $\therefore$ is so divided that

$\times$ $=$ ^{2}.

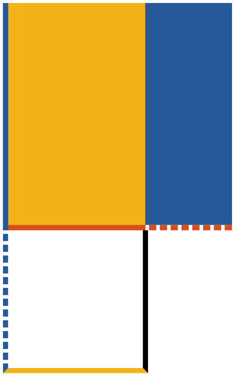
Q. E. D.



To *produce a given straight line (), so that the rectangle contained by the segments between the extremities of the given line and the point to which it is produced, may be equal to a given area, i. e. equal to the square on* .

Make = ,
and draw $\perp$ = ;
draw ;
and with the radius , describe a circle meeting
produced.
Then
$\times$ + ^{2} = ^{2} (??)
= ^{2}.
But ^{2} = ^{2} + ^{2} (??)
$\therefore$ $\times$ + ^{2} =
^{2} + ^{2},
from both take ^{2},
and $\therefore$ $\times$ = ^{2}
but = ,
$\therefore$ ^{2} = the given area.

Q. E. D.



To cut a given finite straight line ( ) in extreme and mean ratio.

On describe the square (??);

and produce , so that

$\times$ $=$ ^{2} (??);

take $=$,

and draw $\parallel$,

meeting $\parallel$ (??).

Then $=$ $\times$, and is

$\therefore =$;

and if from both these equals be taken the common part

,

, which is the square of ,

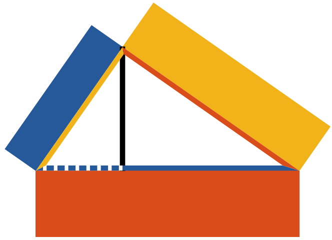
will be $=$, which is $=$ $\times$;

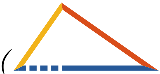
that is ^{2} $=$ $\times$;

$\therefore$: :: : ,

and is divided in extreme and mean ratio (??).

Q. E. D.



IF *any similar rectilinear figures be similarly described on the sides of a right angled triangle* ()*, the figure described on the side* ( ) *subtending the right angle is equal to the sum of the figures on the other sides.*

From the right angle draw perpendicular to ;

then

: :: :

(??).

∴ : :: :

(??).

but : :: :

(??).

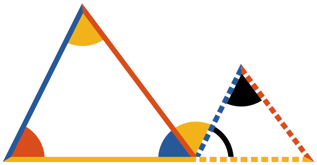
Hence + : ::

+ : ;

but + = ;

and ∴ + = .

Q. E. D.



If two triangles (and), have two sides proportional (: :: :), and be so placed at an angle that the homologous sides are parallel, the remaining sides (and) form one right line.

Since ∥ ,

= (??);

and also since ∥ ,

= (??);

∴ = ;

and since : :: : (??),

the triangles are equiangular (??);

∴ = ;

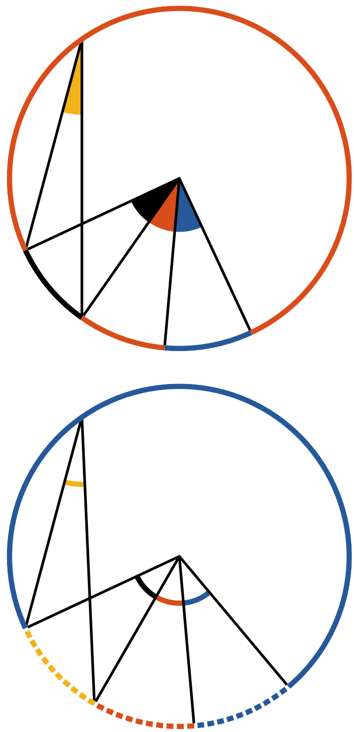
but = ;

∴ + + = + + = (??),

and ∴ and lie in the same straight line (??).

Q. E. D.

In equal circles (,), angles, whether at the centre or circumference, are in the same ratio to one another as the arcs on which they stand (: :: :); so also are sectors.



Take in the circumference of any number of arcs , , etc. each = , and also in the circumference of take any number of arcs , , etc. each = , draw the radii to the extremities of the equal arcs.

The since the arcs , , , etc. are all equal, the angles , , , etc. are also equal (??); ∴ is the same multiple of which the arc is of ; and in the same manner is the same multiple of , which the arc is of the arc .

Then it is evident (??),

if (or if m times) >, =, < (or n times)

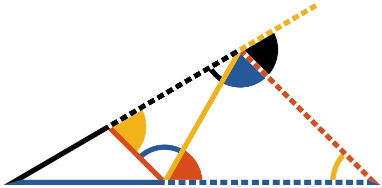
then (or m times)

>, =, < (or n times);

∴ : :: : (??), or the angles at the centre are as the arcs on which they stand; but the angles at the circumference being halves of the angles at the centre (??) are in the same ratio (??), and therefore are as the arcs on which they stand.

It is evident, that sectors in equal circles, and on equal arcs are equal (??, ??, ?? and ??). Hence, if the sectors be substituted for the angles in the above demonstration, the second part of the proportion will be established, that is, in equal circles the sectors have the same ratio to one another as the arcs on which they stand.

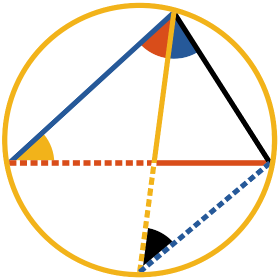
Q. E. D.



If the right line ( ) bisecting an external angle of the triangle meet the opposite side ( ) produced, that whole produced side ( ), and its external segment ( ) will be proportional to the sides (and), which contain the angle adjacent to the external bisected angle.

For if be drawn ∥ ,
then = , (??);
= , (??),
= , (??);
and ∴ = , (??),
and : :: : (??);
But also, : :: : (??);
and therefore : :: :
(??).

Q. E. D.



If an angle of a triangle be bisected by a straight line, which likewise cuts the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square of the straight line which bisects the angle.

Let be drawn, making = ;

then shall × = × + ^{2}.

About describe (??),

produce to meet the circle, and draw .

Since = (??),

and = (??),

∴ and are equiangular (??);

∴ : :: : (??);

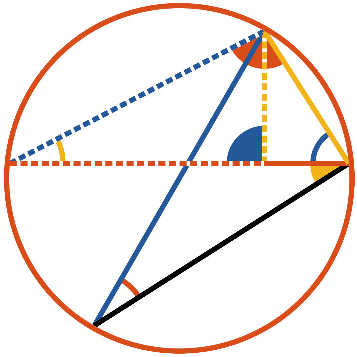
∴ × = × (??)

= × + ^{2} (??);

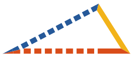
but × = × (??);

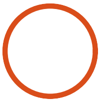
∴ × = × + ^{2}

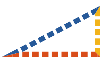
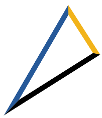
Q. E. D.



I F *from any angle of a triangle a straight line be drawn perpendicular to the base; the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circle described about the triangle.*

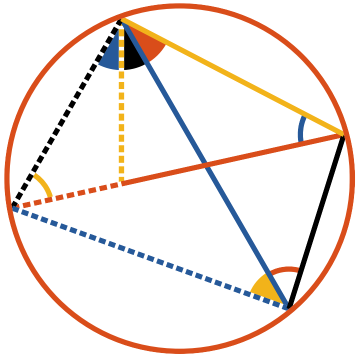
From  of 
draw ⊥ ;
then shall × = × the diameter of the described circle.

Describe  (??), draw its diameter , and
draw ;
then ∵  =  (?? and ??);
and  =  (??);

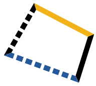
∴  is equiangular to  (??);

∴ : :: : ;
and ∴ × = × (??).

Q. E. D.



THE *rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle, is equal to both the rectangles contained by its opposite sides.*

Let  be any quadrilateral figure inscribed in

;

and draw and ;

then × =

× + × .

Make = (??),

∴ = ; and = (??);

∴ : :: : (??);

and ∴ × = × (??);

again, ∵ = (??),

and = (??);

∴ : :: : (??);

and ∴ × = × (??);

but, from above,

× = × ;

∴ × =

× + × (??).

Q. E. D.

Contents