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Parțial (2p) of 9,50

(2) a) $x''+16x'+64x=0$

Ec. liniară cu coef. constanți, omogenă

Ec. caracteristică: $\lambda^2+16\lambda+64=0$ (2p)

$(\lambda+8)^2=0 \rightarrow \lambda+8=0 \rightarrow \lambda_{1,2}=-8 \in \mathbb{R}$ dublă

S.f.s: $\{e^{-8t}, te^{-8t}\}$

$x(t)=C_1 e^{-8t}+C_2 t e^{-8t}, t\in\mathbb{R}, C_1, C_2 \in \mathbb{R}$ const.

b) $\begin{cases} x''+8x=0 \\ x(0)=1, x'(0)=4 \end{cases} \rightarrow \lambda^2+8=0 \rightarrow \lambda^2=-8$

$\lambda_{1,2}=\pm\sqrt{8}\, i$

S.f.s: $\{\cos\sqrt{8}t, \sin\sqrt{8}t\}$

$x(t)=C_1\cos\sqrt{8}t+C_2\sin\sqrt{8}t, t\in\mathbb{R}, C_1, C_2\in\mathbb{R}$ const.

$x(0)=C_1\cos 0+C_2\sin 0=1 \Rightarrow C_1=1$

$x'(t)=-\sqrt{8}C_1\sin\sqrt{8}t+\sqrt{8}C_2\cos\sqrt{8}t$ (2,5p)

$x'(0)=-\sqrt{8}C_1\sin 0+\sqrt{8}C_2\cos 0 \Rightarrow \sqrt{8}C_2=4$

$\Rightarrow C_2=\frac{{}^{\sqrt{8})}4}{\sqrt{8}}=\frac{4\sqrt{8}}{8}=\frac{\sqrt{8}}{2}=\frac{2\sqrt{2}}{2}=\sqrt{2}$

$x(t)=\cos\sqrt{8}t+\sqrt{2}\sin\sqrt{8}t, \forall t\in\mathbb{R}$

c) $x''+2x'-3x=12e^{3t}$

$x(t)=x_0(t)+x_p(t)$

Ec. caracteristică: $\lambda^2+2\lambda-3=0 \rightarrow \Delta=4+4\cdot3=4^2$

$\lambda_1=\frac{-2+4}{2}=1\in\mathbb{R}$, $\lambda_2=\frac{-2-4}{2}=-3\in\mathbb{R}$

S.f.s: $\{e^t, e^{-3t}\}$

$x_0(t) = \mathcal{C}_1 e^{t} + \mathcal{C}_2 e^{-3t}$, $t \in \mathbb{R}$, $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}$ const. ✓

$f(t) = 12e^{3t} \to \gamma = 3 \to \ell = 0$, $P(t) = 12$

$x_p(t) = t^{\ell} e^{\gamma t} Q(t) = t^0 e^{3t} \cdot a = ae^{3t}$

$x_p'(t) = 3ae^{3t}$; $x_p''(t) = 9ae^{3t}$

3p

$x'' + 2x' - 3x = 12e^{3t}$

$9ae^{3t} + 2 \cdot 3ae^{3t} - 3 \cdot ae^{3t} = 12e^{3t} \to 9ae^{3t} + 6ae^{3t} - 3ae^{3t} = 12e^{3t}$

$12ae^{3t} = 12e^{3t} \to a = 1 \to x_p(t) = e^{3t}$, $t \in \mathbb{R}$

$x(t) = \mathcal{C}_1 e^{t} + \mathcal{C}_2 e^{-3t} + e^{3t}$, $t \in \mathbb{R}$, $\mathcal{C}_1, \mathcal{C}_2 \in \mathbb{R}$ const. ✓

sau

$x(t) = \mathcal{C}_1(t) e^{t} + \mathcal{C}_2(t) e^{-3t}$

$$\begin{cases} \mathcal{C}_1'(t) e^{t} + \mathcal{C}_2'(t) e^{-3t} = 0 \\ \mathcal{C}_1'(t) e^{t} - 3\mathcal{C}_2'(t) e^{-3t} = 12e^{3t} \end{cases} \overset{"-"}{\Longrightarrow} 4\mathcal{C}_2'(t) e^{-3t} = -12e^{3t}$$

$$\begin{cases} -\mathcal{C}_1'(t) e^{t} - \mathcal{C}_2'(t) e^{-3t} = 0 \\ \mathcal{C}_1'(t) e^{t} - 3\mathcal{C}_2'(t) e^{-3t} = 12e^{3t} \end{cases} (+)$$

$-4\mathcal{C}_2'(t) e^{-3t} = 12e^{3t} \to \mathcal{C}_2'(t) = -\frac{12e^{3t+3t}}{4} = -3e^{6t}$

$\mathcal{C}_2(t) = -3\int e^{6t}\,dt = -3 \cdot \frac{e^{6t}}{6} = -\frac{e^{6t}}{2} + K_2$

$\mathcal{C}_1'(t) e^{t} = -\mathcal{C}_2'(t) e^{-3t} \Rightarrow \mathcal{C}_1'(t) = 3e^{6t} \cdot e^{-3t-t} = 3e^{2t}$

~~$\mathcal{C}_1(t) = \frac{1}{2}\int e^{3t}\,dt = \frac{1}{2} \cdot \frac{e^{3t}}{3} + K_2 = \frac{e^{3t}}{6} + K$~~

~~$x(t) = \left(\frac{e^{6t}}{\cdot} + K_1\right) e^{t} + \left(\frac{e^{3t}}{6} + K_2\right) e^{-3t} \Rightarrow$~~

~~$x(t) = K_1 e^{t} + K_2 e^{-3t} + e^{+6t}$~~

$\mathcal{C}_1(t) = 3\int e^{2t}\,dt = 3 \cdot \frac{e^{2t}}{2} + K_1 = \cancel{e^{3t} + K_1}$

$$x(t) = \left(e^{\cancel{3}t} \cdot \frac{3}{2} + K_1\right)e^{t} + \left(-\frac{e^{6t}}{2} + K_2\right)e^{-3t}$$

$$x(t) = K_1 e^{t} + K_2 e^{-3t} + \frac{3}{2}e^{3t} - \frac{1}{2}e^{3t}$$

$$x(t) = K_1 e^{t} + K_2 e^{-3t} + e^{3t}, \quad K_1, K_2 \in \mathbb{R} \text{ const}$$
$$t \in \mathbb{R}$$

(3) $\begin{cases} x' = \dfrac{\cos(x^2)}{t+1} + 2e^{-t} \\ x(0) = 1 \end{cases}$ $\quad t+1 \neq 0 \to t \neq -1$

$t_0 = 0, \ x_0 = 1; \quad f: \mathbb{R}\setminus\{-1\} \times \mathbb{R} \to \mathbb{R}, \ f(t,x) = \dfrac{\cos x^2}{t+1} + 2e^{-t}$

Fie $a = \frac{1}{2}, \ b = 1$

$$\Delta = [t_0 - a, t_0 + a] \times [x_0 - b, x_0 + b] = \underbrace{\left[-\tfrac{1}{2}, \tfrac{1}{2}\right]}_{t} \times \underbrace{[0, 2]}_{x}$$

i) f continuă pe Δ (funcții elementare)

ii) f Lipschitz în raport cu x (coordonata spațială) pe Δ

$$\frac{\partial f}{\partial x}(t,x) = \frac{-\sin(x^2)}{t+1} \cdot 2x \quad \text{continuă pe } \Delta$$

$\Rightarrow$ Th. Picard $\Rightarrow$ Pr. Cauchy are o soluție unică:

$x : [t_0 - \delta, t_0 + \delta] \to [x_0 - b, x_0 + b]$

$x : [-\delta, \delta] \to [0, 2]$

1,75p

$$\delta = \min\left\{a, \frac{b}{M}\right\} = \min\left\{\frac{1}{2}, \frac{1}{M}\right\} = \min\left\{\frac{1}{2}, \frac{3}{14}\right\}$$

$$|f(t,x)| = \left|\frac{\cos x^2}{t+1} + 2e^{-t}\right| \leq \frac{|\cos x^2|}{\underbrace{|t+1|}_{=t+1}} + 2e^{-t} = \frac{1}{-\frac{1}{2}+1} + 2e^{1/2}$$

$$= \frac{2}{\cancel{3}} + 2e^{1/2} \simeq \frac{2}{3} + \frac{3}{4} \simeq \frac{14}{3}$$

$$\to \delta = \frac{3}{14} \to x : \left[-\frac{3}{14}, \frac{3}{14}\right] \to [0, 2]$$

① a) $\underbrace{(e^{2t}-2tx+\sin x)}_{P(t,x)}dx - \underbrace{(x^2-2xe^{2t})}_{Q(t,x)}dt = 0$

$$\begin{cases} \frac{\partial P}{\partial t}(t,x) = 2e^{2t} - 2x \\ \frac{\partial Q}{\partial x}(t,x) = -2x + 2e^{2t} \end{cases} \Big| \to \frac{\partial P}{\partial t}(t,x) = \frac{\partial Q}{\partial x}(t,x) \text{ (EE)}$$

$\Rightarrow \exists$ o funcție $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ de clasă C^2 astfel încât:

$$\begin{cases} \frac{\partial F}{\partial x}(t,x) = P = e^{2t} - 2tx + \sin x \\ \frac{\partial F}{\partial t}(t,x) = Q = 2xe^{2t} - x^2 \end{cases}$$

Ep

$\frac{\partial F}{\partial x}(t,x) = e^{2t} - 2tx + \sin x \Big| \int dx$

$F(t,x) = \int e^{2t}dx - 2t\int x\,dx + \int \sin x\,dx = e^{2t}x - 2t\frac{x^2}{2} - \cos x + \mathcal{C}(t)$

$F(t,x) = e^{2t}x - x^2t - \cos x + \mathcal{C}(t)$

$\left(\frac{\partial F}{\partial t}(t,x) = \right) \frac{\partial}{\partial t}\left(e^{2t}x - x^2t - \cos x + \mathcal{C}(t)\right) = 2xe^{2t} - x^2$

$\cancel{F(x) =}\ 2xe^{2t} - x^2 + \mathcal{C}'(t) = 2xe^{2t} - x^2 \to \mathcal{C}'(t) = 0$

$\Rightarrow \mathcal{C}(t) = \text{const.} = k$

$F(t,x) = e^{2t}x - x^2t - \cos x + k$

Forma implicită a soluției: $e^{2t}x - x^2t - \cos x = \text{const.}$

① c) $t^2x' = 3tx - x^2$

~~Caz I~~: $t^2 \neq 0 \Rightarrow x' = 3\frac{x}{t} - \frac{x^2}{t^2} = 3\cdot\frac{x}{t} - \left(\frac{x}{t}\right)^2$ (EO)

$u = \frac{x}{t} \to x = t\cdot u \to x' = u + tu'$

$u + tu' = 3u - u^2 \Rightarrow tu' = 2u - u^2 \mid : t, t \neq 0 \Rightarrow u' = \frac{2u - u^2}{t}$

$u' = \underbrace{\frac{1}{t}}_{f(t)} \cdot \underbrace{u(2-u)}_{g(u)}$ (EVS)

① c) continuare : $u' = \frac{1}{t} \cdot u(2-u)$

$\frac{du}{dt} = \frac{1}{t} \cdot u(2-u) \Rightarrow \frac{du}{u(2-u)} = \frac{1}{t} dt \quad \Big| \int$

În acest caz: $u \neq 0$, $2-u \neq 0 \Rightarrow u \neq 2$

$\Rightarrow \int \frac{du}{u(2-u)} = \int \frac{1}{t} dt$

1,75p

$\int \frac{du}{u(2-u)} = -\int \frac{du}{u(u-2)} = -\frac{1}{2} \ln |u^2-2u|$??

$\int \frac{1}{t} dt = \ln |t|$

$-\frac{1}{2} \ln |u^2-2u| = \ln |t| + C$, $C = \ln K$

$(u^2-2u)^{-1/2} = \pm Kt = at \Big|^{(-2)}$ $a \in \mathbb{R}$ const.

$u^2 - 2u = \frac{1}{a^2t^2} \quad \Rightarrow u = \frac{x}{t} \Rightarrow \left(\frac{x}{t}\right)^2 - 2\frac{x}{t} = \frac{1}{a^2t^2}$

Caz II : $t^2 = 0 \overset{?}{\Rightarrow} x(t) = 0$

$\frac{x^2}{t^2} - 2\frac{x}{t} = \frac{1}{a^2t^2}$

$\hookrightarrow u = 0 \Rightarrow x = 0$; $u = 2 \Rightarrow x = 2t$ soluții

① b) $\begin{cases} x' = -3t^2x + x^2e^{t^3}\cos t \\ x(0) = 1 \end{cases}$

$x' = -3t^2x + x^2e^{t^3}\cos t = -3t^2x + e^{t^3}\cos t \, x^2$ (EB)

$\begin{cases} a(t) = -3t^2 \\ b(t) = e^{t^3}\cos t \\ \alpha = 2 \end{cases}$

$\alpha = 2 > 0 \rightarrow x(t) = 0$ (soluție)

$y = x^{1-\alpha} = x^{1-2} = x^{-1} = \frac{1}{x}$

$\Rightarrow x = \frac{1}{y} \Rightarrow x' = -\frac{y'}{y^2}$

$\frac{-y'}{y^2} = -3\frac{t^2}{y} + \frac{1}{y^2} \cdot e^{t^3}\cos t \Big| \cdot (-y^2)$

$y' = 3t^2y - e^{t^3}\cos t$ (EL) cu funcția necunoscută $y = y(t)$

$y' = 3t^2 y - e^{t^3}\cos t$ (EL)

$a(t) = 3t^2$

$b(t) = -e^{t^3}\cos t$

(FVC): $y = y_0 e^{\int_{t_0}^{t} a(s)ds} + \int_{t_0}^{t} e^{\int_s^t a(p)dp} \cdot b(s)\,ds$

$y_0 = y(0) = \text{const.}$

2p

$\int_{t_0}^{t} a(s)\,ds = \int_0^t 3s^2 ds = 3\frac{s^3}{3}\Big|_0^t = s^3\Big|_0^t = t^3$

$x(0) = 1 \Rightarrow t_0 = 0$ ✓

$y = y_0 \cdot e^{t^3} + \int_0^t e^{\frac{t^3}{s^3}} \cdot (-e^{s^3}\cos s)\,ds = y_0 e^{t^3} + \int_0^t (\cancel{e^{t^3} \div e^{s^3}})(-e^{s^3}\cos s)ds$ $e^{t^3 - s^3}$

$y = y_0 e^{t^3} + e^{t^3}\int_0^t -\frac{e^{s^3}}{e^{s^3}}\cos s\,ds = y_0 e^{t^3} - e^{t^3}\int_0^t \cos s\,ds$

$y = y_0 e^{t^3} - e^{t^3}\cdot \sin s\Big|_0^t = y_0 e^{t^3} - (e^{t^3}\sin t - e^{t^3}\cdot \sin 0)$

$y = y_0 e^{t^3} - e^{t^3}\sin t \Rightarrow y = \underbrace{(y_0 - \sin t)}_{\cancel{\mathcal{C}}} e^{t^3} = \cancel{\mathcal{C} e^{t^3}}$ $\sin t$ nu e constant, depinde de t!

$x = \frac{1}{y} \rightarrow x(t) = \frac{1}{\mathcal{C}e^{t^3}}$, $t \in \mathbb{R}$

Condiția $x(0)=1$?

d) $x = tx' + \ln\{(1+(x')^2\}$ (EC) $\rightarrow$ caz I: $x''=0$ (soluție generală); caz II: soluții parametrice

(EC): $x = tx' + \varphi(x') \rightarrow \varphi(x') = \ln[(1+(x')^2]$

Caut soluții x ~~nenule~~, de clasă C^2 și derivez:

$\cancel{x'} = \cancel{x'} + tx'' + 2x'x'' \cdot \frac{1}{1+(x')^2}$

$\Rightarrow x''\left(t + 2x' \cdot \frac{1}{1+(x')^2}\right) = 0 \rightarrow x''\left(t + \frac{2x'}{1+(x')^2}\right) = 0$

Caz I: $x''=0 \,||\int dt \rightarrow x'=C$, $C\in\mathbb{R}$ const $||\int dt \Rightarrow$

$\Rightarrow x(t) = Ct + D$, $C, D \in \mathbb{R}$ const.

$Ct + D = tC + \ln(1+C^2) \Rightarrow D = \ln(1+C^2)$

Soluția generală: $x(t) = Ct + \ln(1+C^2)$ ✓

Caz II: $t + \frac{2x'}{1+(x')^2} = 0$

(2p)

Metoda parametrică $\rightarrow$ notez $x' = p$

$$\begin{cases} t(p) = -\frac{2p}{1+p^2}, \ p\in\mathbb{R} \text{ parametru} \\ x(p) = -\frac{2p}{1+p^2}\cdot p + \ln(1+p^2) = -\frac{2p^2}{1+p^2} + \ln(1+p^2) \end{cases}$$

Soluția parametrică

singulară