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Anul II
Specializarea Fizică

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Examen - Mecanică Teoretică
Biletul nr. 5

Subiectul 1.

1. $\text{rot}\,\vec{A} = ?$

$\vec{A} = k r^2 \vec{r}$, $k = \text{const}$

$\text{rot}(\varphi \cdot \vec{A}) = \text{grad}\,\varphi \times \vec{A} + \varphi\, \text{rot}\,\vec{A}$

În cazul de față:

$\text{rot}(k r^2 \vec{r}) = k\, \text{rot}(r^2 \vec{r}) = \text{grad}\, r^2 \times \vec{r} + r^2 \text{rot}\,\vec{r}$

$\text{rot}\,\vec{r} = (\vec{u_i} \frac{\partial}{\partial x_i}) \times (x_j \vec{u_j}) = \varepsilon_{ijk} \vec{u_k} \frac{\partial x_j}{\partial x_i} = \varepsilon_{ijk} \vec{u_k} \cdot \delta_{ij} = 0$

$\text{grad}\, r^2 = \vec{u_i} \frac{\partial (r^2)}{\partial x_i} = 2 x_i \vec{u_i} = 2\vec{r}$

Dar $\text{grad}\, r^2 \times \vec{r} = 2\vec{r} \times \vec{r} = 0$

Deci $\text{rot}(k r^2 \vec{r}) = \vec{0}$

2. $\vec{M} = \nabla(r^n) = \vec{u_i} \frac{\partial (r^n)}{\partial x_i} = \vec{u_i} \cdot n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x_i}$

$\frac{\partial r}{\partial x_i} = \frac{x_i}{r} \Rightarrow \vec{M} = \vec{u_i}\, n\, r^{n-1} \cdot \frac{x_i}{r} \Rightarrow \vec{M} = n \cdot r^{n-2} \cdot \vec{r}$

$\vec{M} = n \cdot r^{n-1} \cdot \vec{u_r}$

3. $\vec{A} = (r \sin\varphi) \vec{u_r} + (r^2 \cos\varphi) \vec{u_\varphi} + 2r e^{-5z} \vec{k}$

$\text{div}(\vec{A})$ în punctul $(\frac{1}{2}, \frac{\pi}{2}, 0)$

Divergența unui câmp vectorial în coordonate cilindrice este:

$$\text{div}(\vec{A}) = \frac{1}{r} \cdot \frac{\partial}{\partial r}(r \cdot A_r) + \frac{1}{r} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

$\frac{\partial}{\partial r}(r \cdot A_r) = \frac{\partial}{\partial r}(r^2 \sin\varphi) = 2r \sin\varphi$

$\frac{\partial A_\varphi}{\partial \varphi} = \frac{\partial}{\partial \varphi}(r^2 \cos\varphi) = -r^2 \sin\varphi$

$\frac{\partial A_z}{\partial z} = \frac{\partial}{\partial z}(2r e^{-5z}) = 2r \cdot (-5) \cdot e^{-5z} = -10 r \cdot e^{-5z}$

Așadar, $\text{div}(\vec{A}) = 2\sin\varphi - r \sin\varphi - 10 r \cdot e^{-5z}$

$\text{div}(\vec{A})(\frac{1}{2}, \frac{\pi}{2}, 0) = 2 \sin\frac{\pi}{2} - \frac{1}{2} \cdot \sin\frac{\pi}{2} - 10 \cdot \frac{1}{2} \cdot e^0 = -\frac{7}{2} \Rightarrow \text{div}(\vec{A})(\frac{1}{2}, \frac{\pi}{2}, 0) = -\frac{7}{2}$

Subiectul 3

1.) $(\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}) = (a_i L_i, b_j L_j) = a_i b_j (L_i, L_j)$

$(L_i, L_j) = (\varepsilon_{ikl}\cdot x_k p_l, L_j) = \varepsilon_{ikl}[x_k(p_l, L_j) + (x_k, L_j)\cdot p_l]$

$(p_l, L_j) = (p_l, \varepsilon_{jqr} x_q p_r) = \varepsilon_{jqr}\cdot(p_l, x_q p_r) =$

$= \varepsilon_{jqr}\cdot[x_q\cdot(p_l, p_r) + (p_l, x_q)\cdot p_r]$

$(p_l, p_r) = 0$

$(p_l, x_q) = -\delta_{lq}$

$(p_l, L_j) = -\varepsilon_{jqr}\cdot\delta_{lq}\cdot p_r = -\varepsilon_{jlr}\cdot p_r = \varepsilon_{ljr} p_r$

$(x_k, L_j) = (x_k, \varepsilon_{jmn} x_m p_n) = \varepsilon_{jmn}[x_m\cdot(x_k, p_n) + (x_k, x_m)\cdot p_n]$

$= \varepsilon_{jmn}\cdot[x_m\cdot\delta_{kn} + 0] = \varepsilon_{jmk}\cdot x_m = \varepsilon_{kjm}\cdot x_m$

Deci: $(L_i, L_j) = \varepsilon_{ikl}\cdot[x_k\cdot\varepsilon_{ljr}\cdot p_r + p_l\cdot\varepsilon_{kjm} x_m]$

$= x_k p_r\cdot\varepsilon_{lik}\cdot\varepsilon_{ljr} + p_l x_m\cdot\varepsilon_{kli}\cdot\varepsilon_{kjm}$

$= x_k p_r\cdot(\delta_{ij}\delta_{kr} - \delta_{ir}\delta_{kj}) + p_l x_m\cdot(\delta_{lj}\delta_{im} - \delta_{lm}\delta_{ij})$

$= x_k p_k\cdot\delta_{ij} - p_i x_j + p_j x_i - p_l x_l\cdot\delta_{ij}$

$= p_j x_i - p_i x_j = \varepsilon_{ijm}\cdot L_m$

Așadar, $(\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}) = a_i b_j \varepsilon_{ijm} L_m = \varepsilon_{ijm} a_i b_j L_m =$

$\varepsilon_{ijm} = (\vec{u_i}\times\vec{u_j})\cdot\vec{u_m} \Rightarrow$

$\Rightarrow (\vec{a}\cdot\vec{L}, \vec{b}\cdot\vec{L}) = (\vec{a}\times\vec{b})\cdot\vec{L}$

2.) $(\vec{a}\cdot\vec{r}, L_j) = a_i\cdot(x_i, L_j)$

Calculat mai sus, $(x_i, L_j) = \varepsilon_{ijm} x_m$ $\Big\}\Rightarrow$

$\Rightarrow (\vec{a}\cdot\vec{r}, L_j) = a_i\cdot x_m\cdot\varepsilon_{ijm} = \varepsilon_{jmi}\cdot a_i\cdot x_m$

$= \varepsilon_{jmi}\cdot x_m\cdot a_i$

$\varepsilon_{jmi} = (\vec{u_m}\times\vec{u_i})\cdot\vec{u_j} \Rightarrow (\vec{a}\cdot\vec{r}, L_j) = (\vec{r}\times\vec{a})_j$

Subiectul 2

1. 2 grade de libertate $\Rightarrow n = 2$

$V = V(r)$, câmp conservativ

Coordonatele generalizate sunt r și θ

a) $L = T - V$

$$L = \frac{m}{2} |\vec{v}|^2 - V(r)$$

$$\vec{v} = \dot{r}\,\vec{u_r} + r\dot{\theta}\,\vec{u_\theta}$$

$$|\vec{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Impulsurile generalizate sunt definite prin relațiile:

$$\dot{p_j} = \frac{\partial L}{\partial q^j}, \quad p_j = \frac{\partial L}{\partial \dot{q}^j}$$

Integralele prime ce apar în acest caz sunt impulsurile generalizate ale coordonatelor ciclice (cele care nu apar în expresia lagrangeanului)

$$\left.\begin{aligned} p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \end{aligned}\right\} \Rightarrow$$

θ nu apare în expresia lagrangeanului deci p_θ este o integrală primă $\left(\dot{p_\theta} = \frac{\partial L}{\partial \theta} = 0\right)$

$$H = T + V \Rightarrow H = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

Știm că: $\dot{r} = \frac{p_r}{m}$, $\dot{\theta} = \frac{p_\theta}{m r^2}$;

$$H = \frac{m}{2}\cdot\frac{p_r^2}{m^2} + \frac{m r^2}{2}\cdot\frac{p_\theta^2}{m^2 r^4} + V(r) \Rightarrow$$

$$\Rightarrow H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{2 m r^2} + V(r)$$

b) $$\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p_j} = -\frac{\partial H}{\partial q^j}$$

$$\begin{cases} \dot{r} = \frac{\partial H}{\partial p_r} \Rightarrow \dot{r} = \frac{p_r}{m} \\ \dot{p_r} = -\frac{\partial H}{\partial r} \Rightarrow \dot{p_r} = -V(r)' + \frac{p_\theta^2}{m r^3} \\ \dot{\theta} = \frac{\partial H}{\partial p_\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{m r^2} \\ \dot{p_\theta} = -\frac{\partial H}{\partial \theta} = 0 \end{cases}$$

$$\begin{cases} \dot{r} = \frac{p_r}{m} \\ \dot{p}_r = -V(r)' + \frac{p_\theta^2}{m r^3} \\ \dot{\theta} = \frac{p_\theta}{m r^2} \\ p_\theta = m r^2 \dot{\theta} = const \end{cases}$$

$$\ddot{r} = \frac{\dot{p}_r}{m} \Rightarrow \ddot{r} = \frac{1}{m}\left(-V(r)' + \frac{m^2 r^4 \dot{\theta}^2}{m r^3}\right)$$

$$\left.\begin{aligned} \ddot{r} &= -\frac{V(r)'}{m} + r\dot{\theta}^2 \\ \dot{\theta} &= \frac{p_\theta}{m r^2} \end{aligned}\right\} \Rightarrow$$

$$\ddot{r} = -\frac{V(r)'}{m} + r \cdot \frac{p_\theta^2}{m^2 r^4}$$

$$\ddot{r} = \frac{V(r)'}{m} - \frac{p_\theta^2}{m^2 r^3}$$

Astfel, avem:

$$\begin{cases} \ddot{r} = \frac{V'(r)}{m} + \frac{p_\theta^2}{m^2 r^3} \\ \dot{\theta} = \frac{p_\theta}{m r^2} \\ p_\theta = const \end{cases} \Leftrightarrow \begin{cases} \ddot{r} = \frac{p_\theta^2}{m^2 r^3} - \frac{V'(r)}{m} \\ \dot{\theta} = \frac{p_\theta}{m r^2} \\ p_\theta = const \end{cases}$$

2. Oscilatorul bidimensional

m, planul xOy

T, Q_j

a.) Ecuația Lagrange de speța a II-a:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$n = 2$ grade de libertate $\Rightarrow$ x, y coordonate generalizate

$$T = \frac{m}{2} \cdot v^2 = \frac{m}{2} \cdot (\dot{x}^2 + \dot{y}^2)$$

$$Q_x = \vec{F}_{ex} \cdot \frac{\partial(x\vec{i})}{\partial x} + \vec{F}_{ey} \cdot \frac{\partial(y\vec{j})}{\partial x} = \vec{F}_{ex} \cdot \vec{i} = F_{ex}$$

$$Q_y = \vec{F}_{ex} \cdot \vec{i}\,\frac{\partial x}{\partial y} + \vec{F}_{ey} \cdot \vec{j} \cdot \frac{\partial y}{\partial y} = F_{ey}$$

b.) Deci:

$$\frac{\partial T}{\partial \dot{x}} = m\dot{x} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial T}{\partial \dot{y}} = m\dot{y} \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{y}}\right) = m\ddot{y}$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial y} = 0$$

$$\Rightarrow \begin{cases} m\ddot{x} = F_{ex} \\ m\ddot{y} = F_{ey} \end{cases}$$

$$\Leftrightarrow \begin{cases} m\ddot{x} = -k_1 x \\ m\ddot{y} = -k_2 y \end{cases} \Leftrightarrow \begin{cases} m\ddot{x} + k_1 x = 0 \\ m\ddot{y} + k_2 y = 0 \end{cases} \Rightarrow \begin{cases} \ddot{x} + \frac{k_1}{m}x = 0 \\ \ddot{y} + \frac{k_2}{m}y = 0 \end{cases}$$

$$\Rightarrow x = A_1 \cos \omega_1 t + A_2 \sin \omega_1 t \ , \quad \omega_1 = \sqrt{\frac{k_1}{m}}$$

$$y = B_1 \cos \omega_2 t + B_2 \sin \omega_2 t \ , \quad \omega_2 = \sqrt{\frac{k_2}{m}}$$

$$x(0) = x_0 \Rightarrow A_1 = x_0$$

$$\left.\begin{array}{l} \dot{x}(0) = \dot{x}_0 \\ \dot{x}(t) = -\omega_1 A_1 \sin \omega_1 t + A_2 \omega_1 \cos \omega_1 t \end{array}\right\} \Rightarrow A_2 \omega_1 = \dot{x}_0$$

$$A_2 = \frac{\dot{x}_0}{\omega_1}$$

$$y(0) = y_0 \Rightarrow B_1 = y_0$$

$$\dot{y}(t) = -\omega_2 B_1 \sin \omega_2 t + B_2 \omega_2 \cos \omega_2 t$$

$$\dot{y}(0) = \dot{y}_0 \Rightarrow \dot{y}(0) = B_2 \omega_2 = \dot{y}_0 \Rightarrow B_2 = \frac{\dot{y}_0}{\omega_2}$$

Deci ecuațiile de mișcare sunt:

$$\begin{cases} x = x_0 \cos \omega_1 t + \frac{\dot{x}_0}{\omega_1} \sin \omega_1 t , \quad \omega_1 = \sqrt{\frac{k_1}{m}} \\ y = y_0 \cos \omega_2 t + \frac{\dot{y}_0}{\omega_2} \sin \omega_2 t , \quad \omega_2 = \sqrt{\frac{k_2}{m}} \end{cases}$$