Covering Planar Metrics (and Beyond): O(1) Trees Suffice

Hsien-Chih Chang | **Jonathan Conroy** | Hung Le

Lazar Milenković | Shay Solomon | Cuong Than

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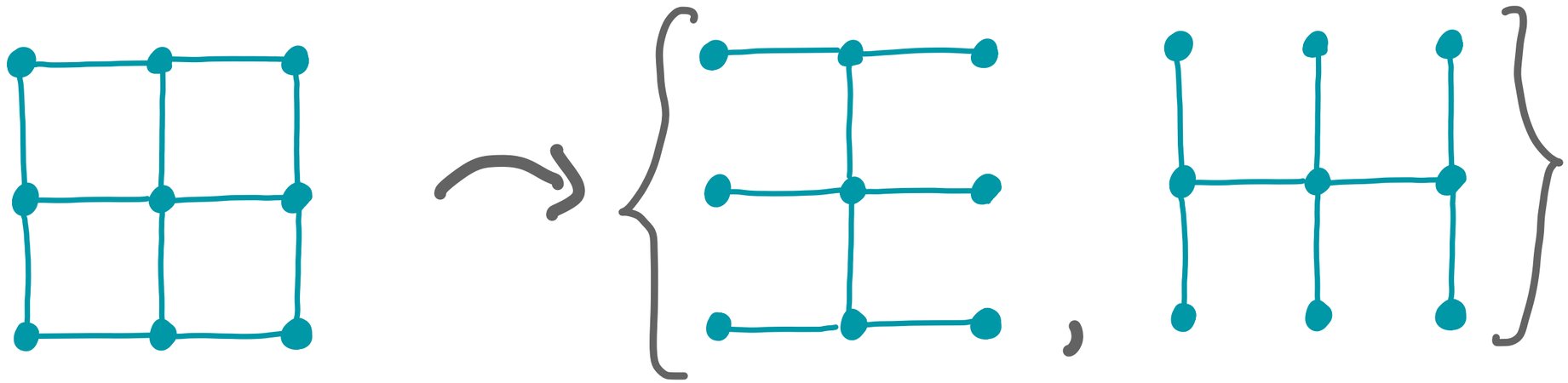


Tree Cover

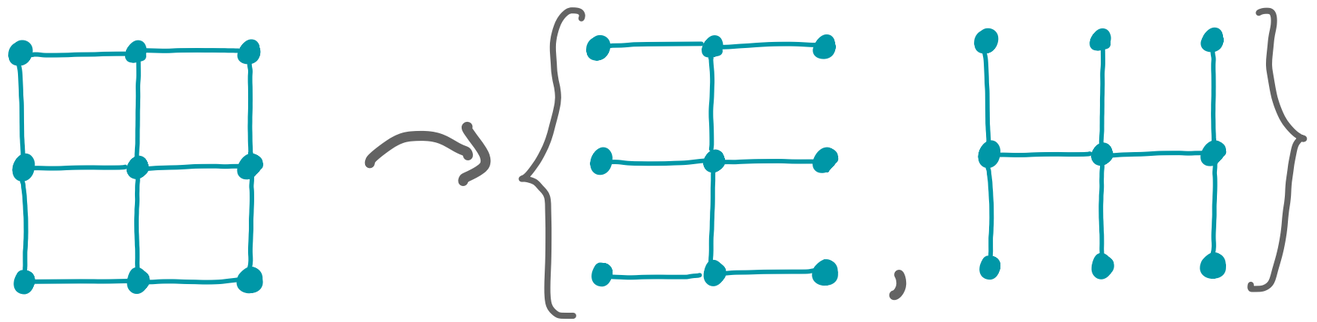
An α-*distortion tree cover* of a graph G is a set of trees $\mathcal{T}$ such that $\forall u, v \in V(G)$:

- For every tree $T \in \mathcal{T}$,

$$d_G(u, v) \leq d_T(u, v)$$

- There exists $T \in \mathcal{T}$ such that

$$d_T(u, v) \leq \alpha \cdot d_G(u, v)$$



Want to minimize *size* $|\mathcal{T}|$ and *distortion* α

Prior Work (for Planar Metrics)

Exact, and $1+\varepsilon$ Distortion

1 distortion $O(\sqrt{n})$ trees [GKR'04]	$1+\varepsilon$ distortion $O(\log^2 n)$ trees [BFN'19]

O(1) Distortion

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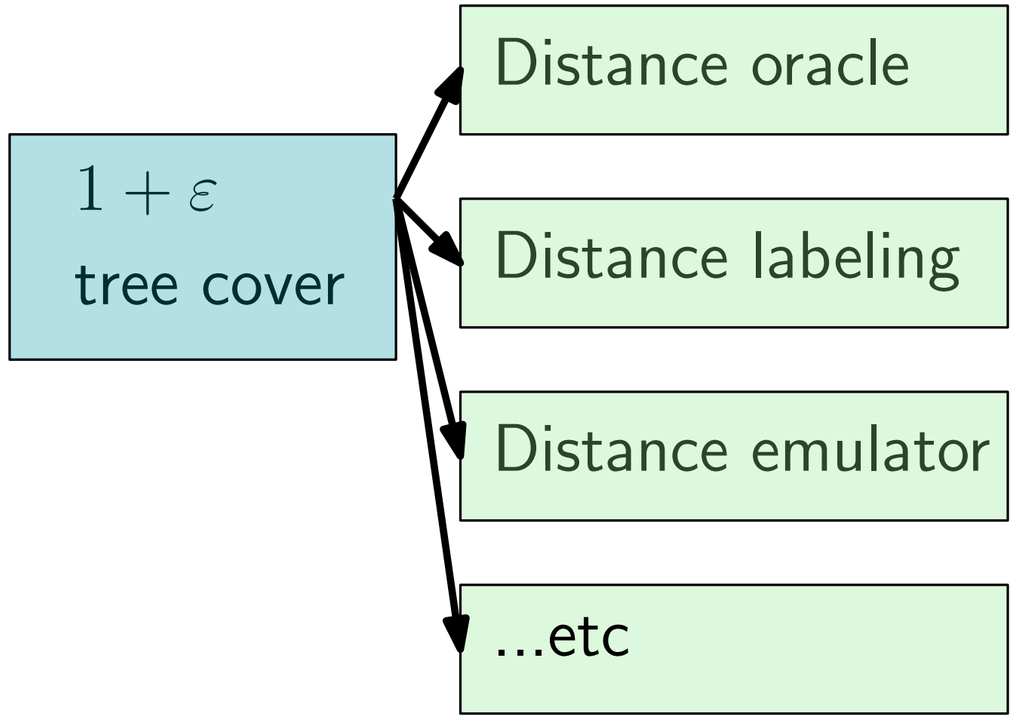
Main Result

Every planar graph has $(1+\varepsilon)$-distortion tree cover of size $O(\varepsilon^{-3} \cdot \log 1/\varepsilon)$.

Similar results for bounded-treewidth graphs and bounded-diameter minor-free graphs

Many Applications



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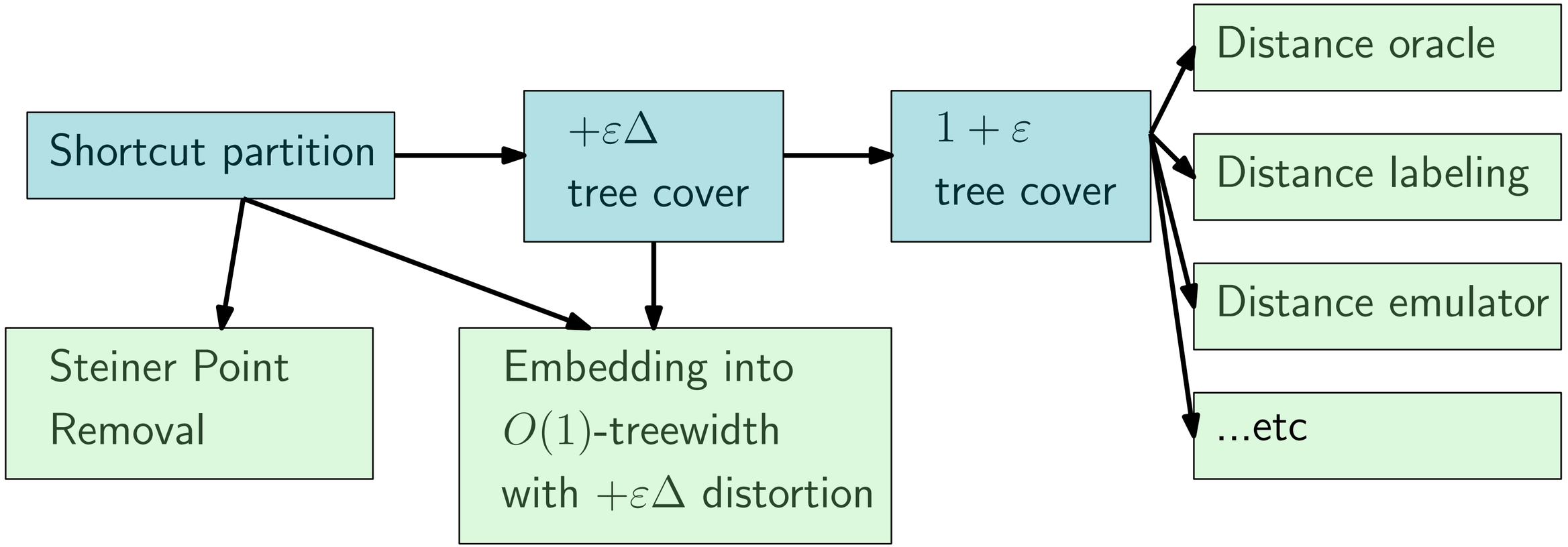

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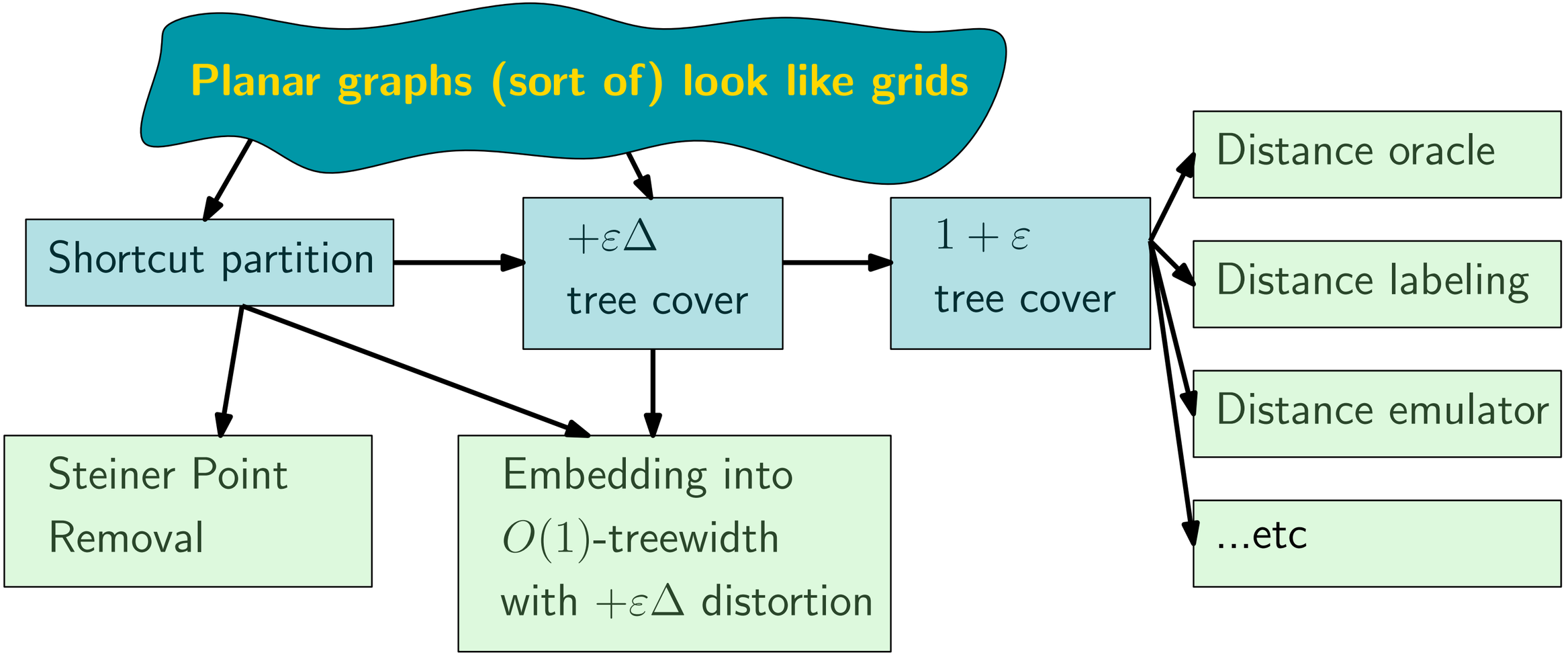
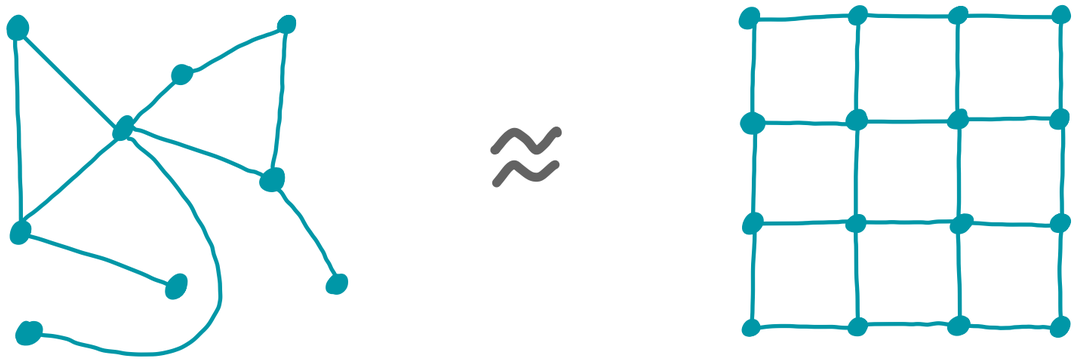


Technical Overview

Planar graphs (sort of) look like grids

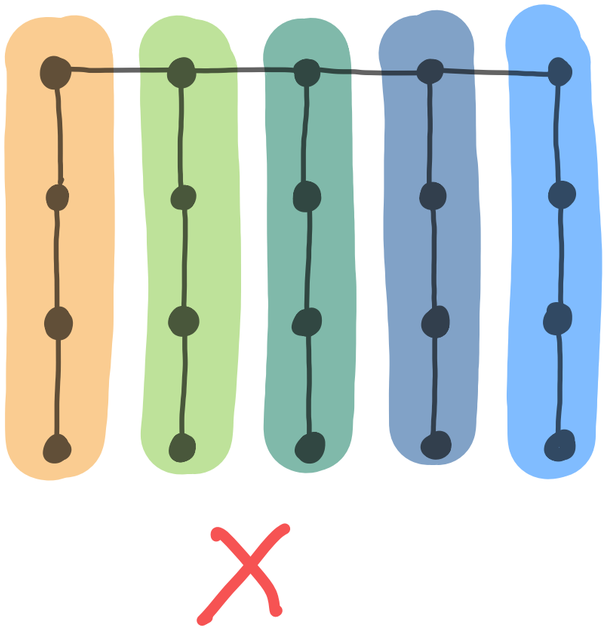
Shortcut partition

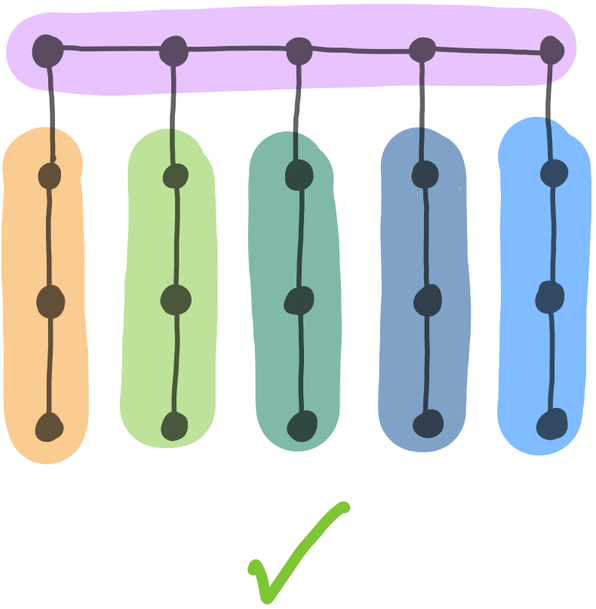
- Shortcut partition...
- ...is easy for grids
- ... and planar graphs $\approx$ grids



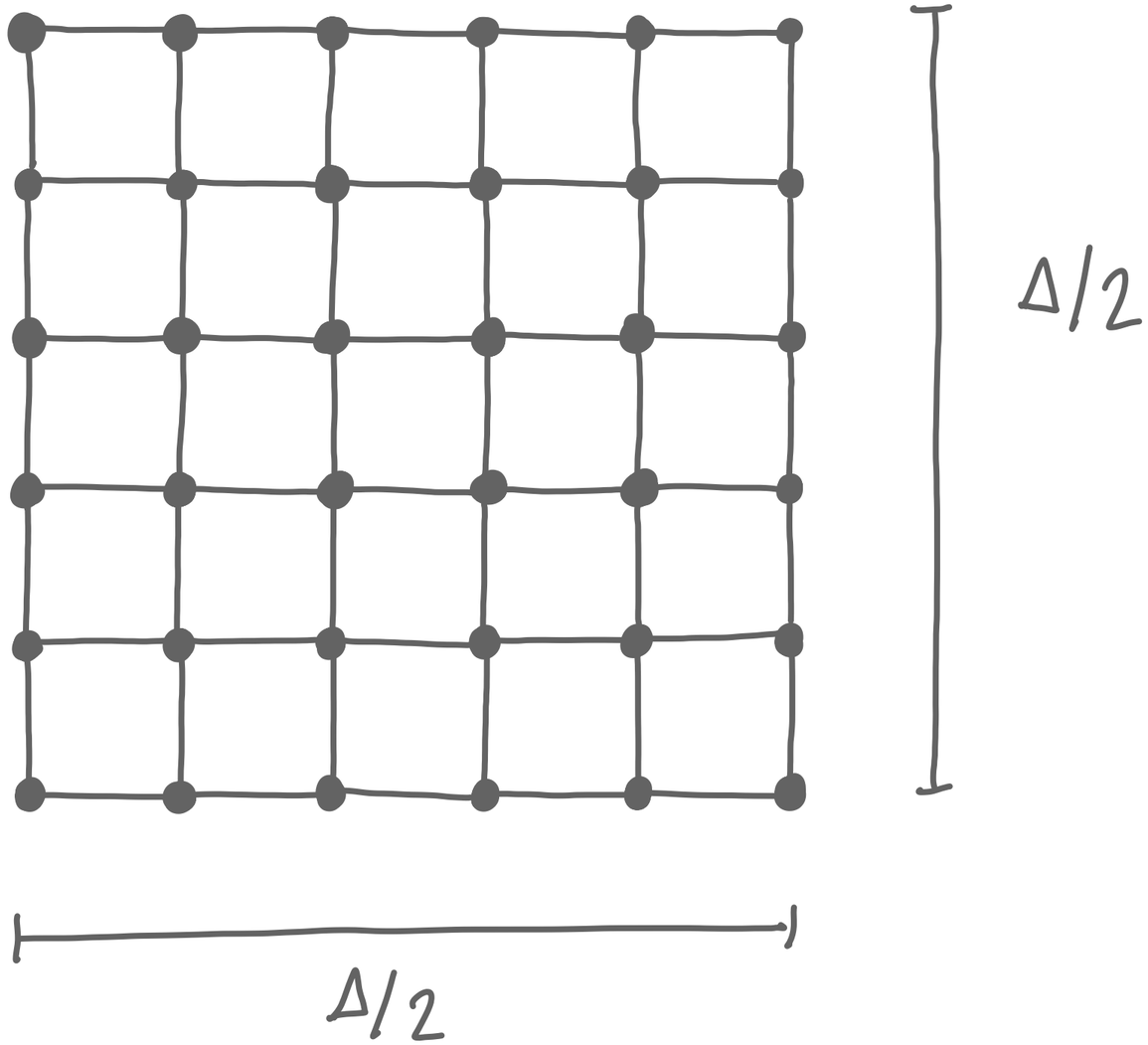
Shortcut partition

- Similar to [Filtser'20] *scattering partition*
- Rough idea: Given graph G with diameter Δ. Want to partition vertices into clusters of diameter $\varepsilon\Delta$, such that any shortest-path intersects only $O(1/\varepsilon)$ clusters





Shortcut partition for grid graph



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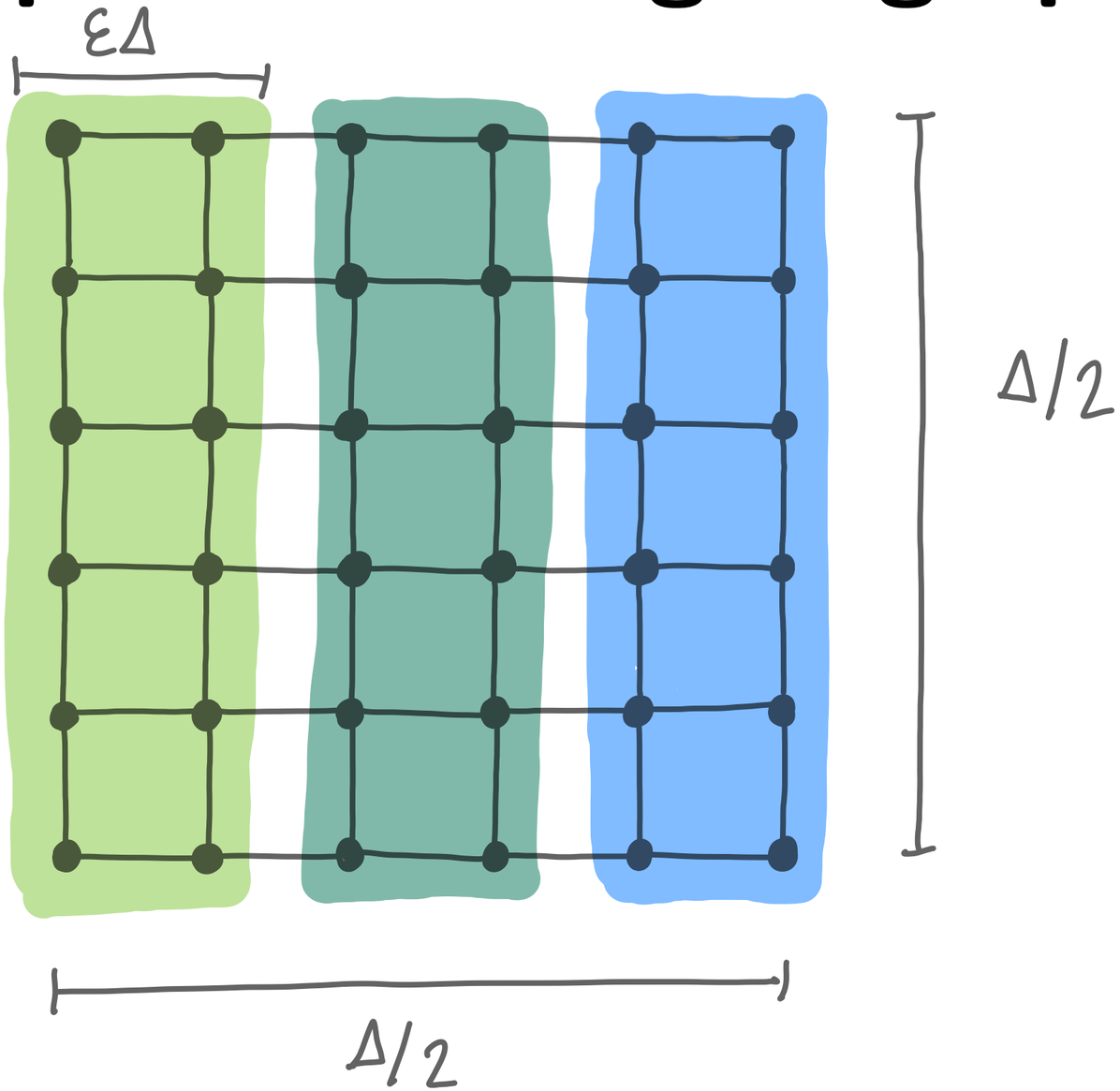

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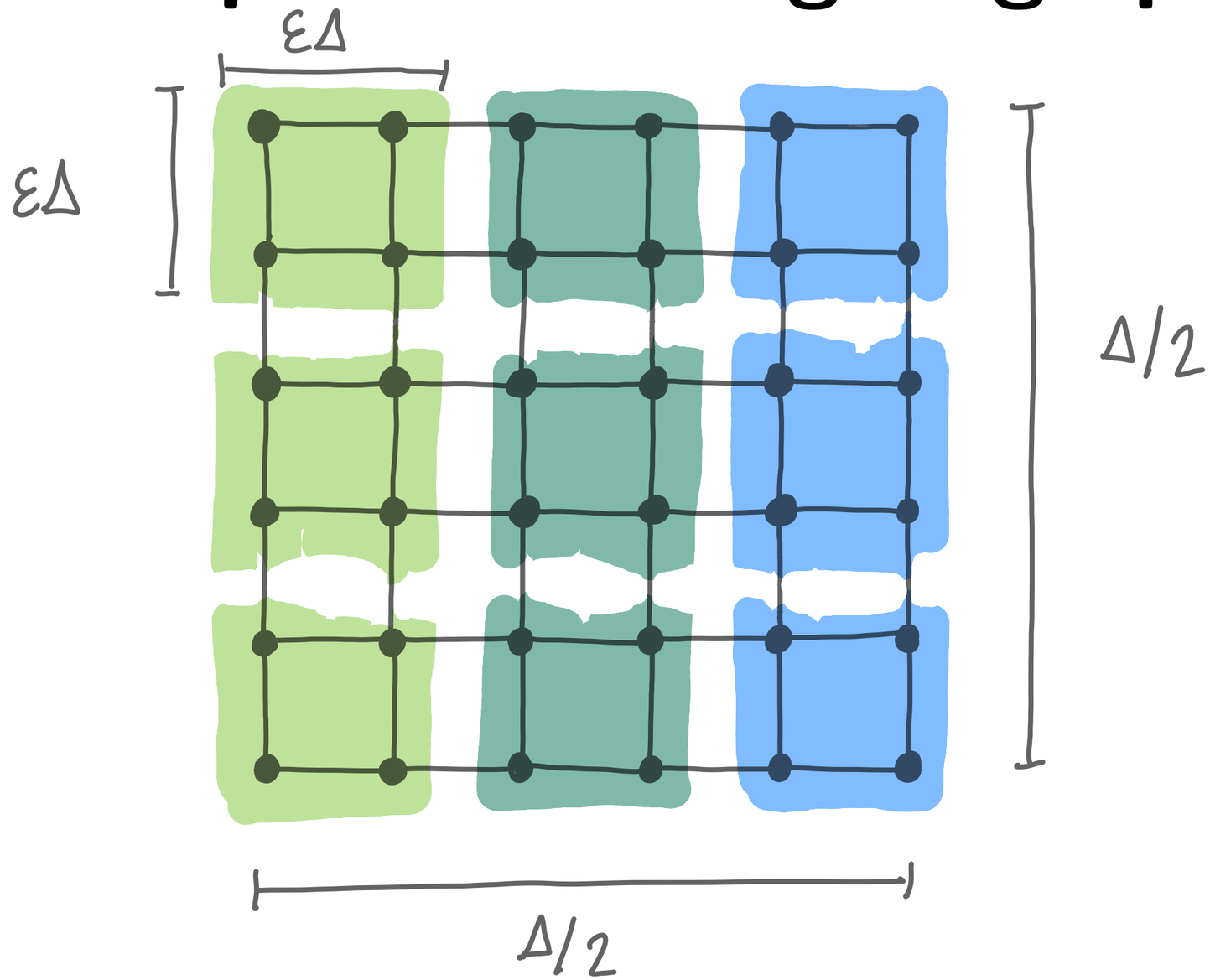
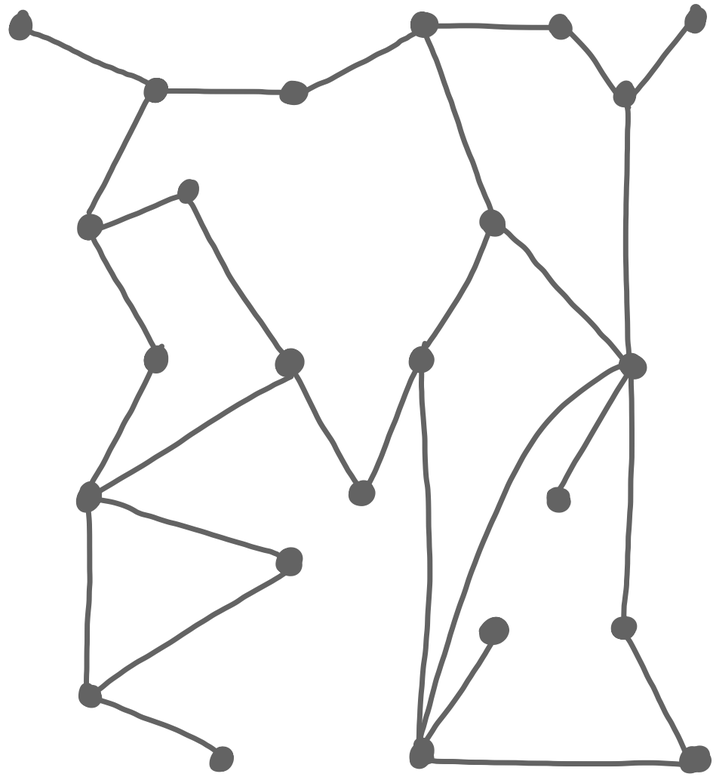


Shortcut partition for planar graph: 1st try

Partition graph into ordered sequence of "columns".

Each column is the $\varepsilon\Delta$-neighborhood around a shortest path



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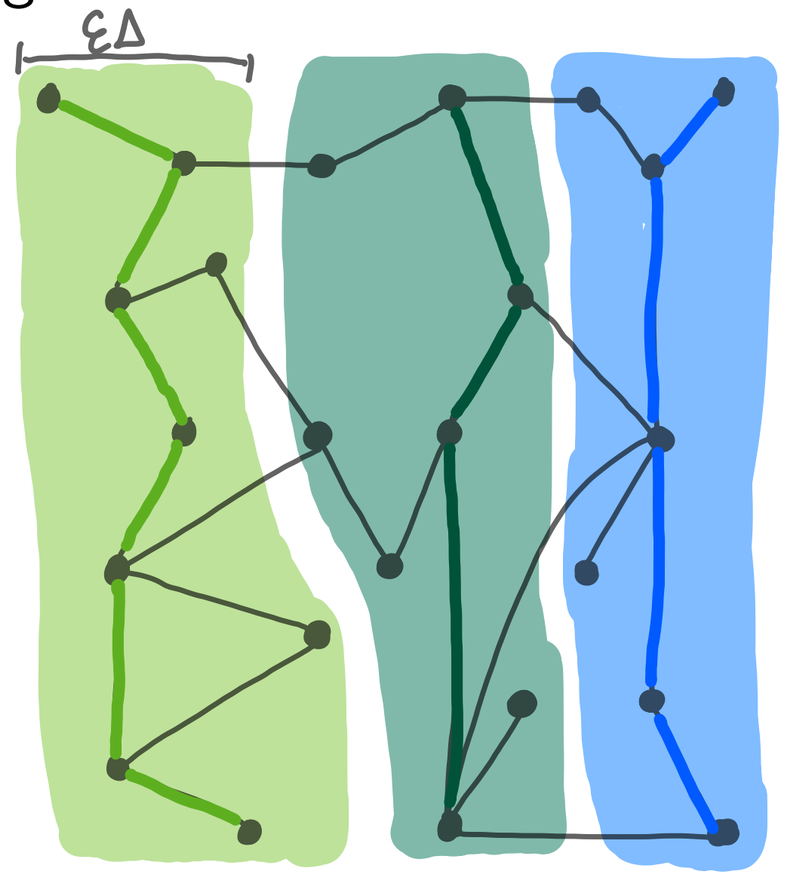

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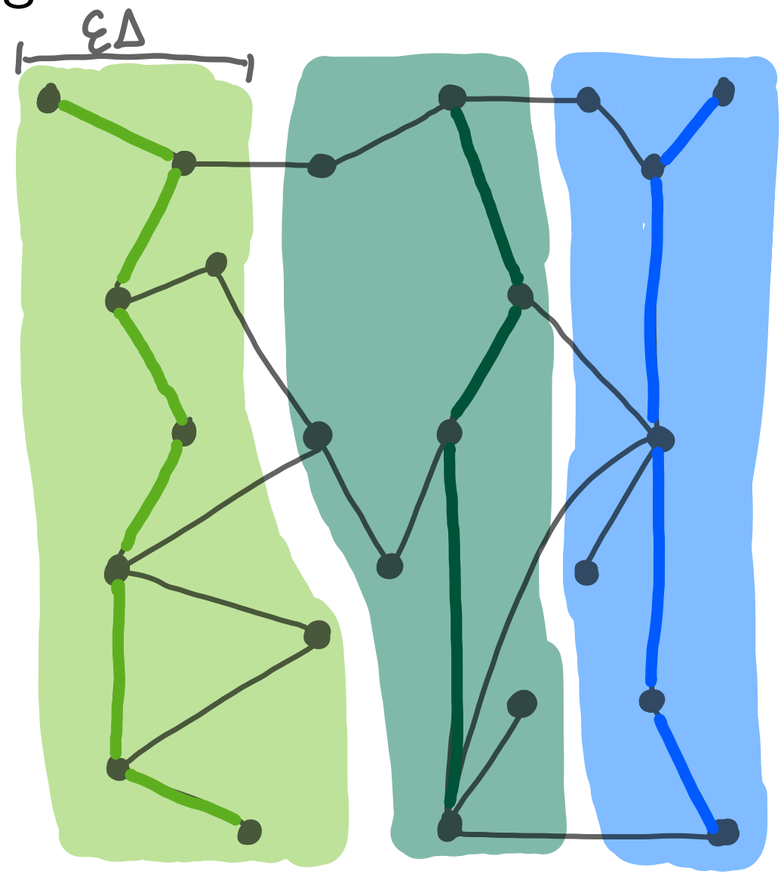
Edges only between columns that are adjacent in sequence



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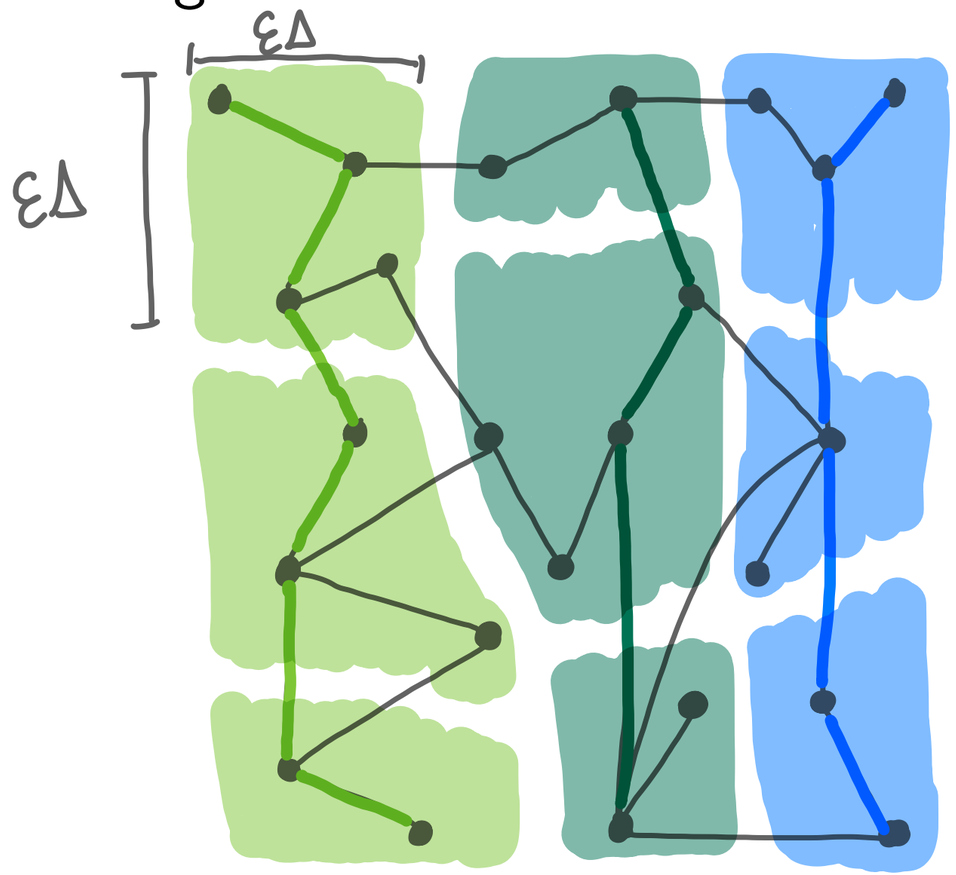
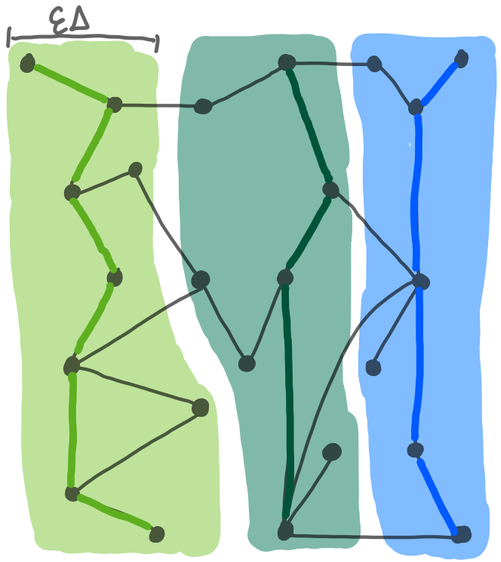


Obstacles to Gridlike Structure

Want: Partition graph into ordered sequence of columns with width $\varepsilon\Delta$

"Gridlike structure"



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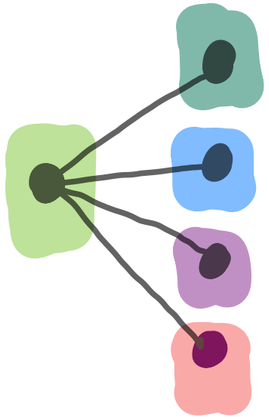
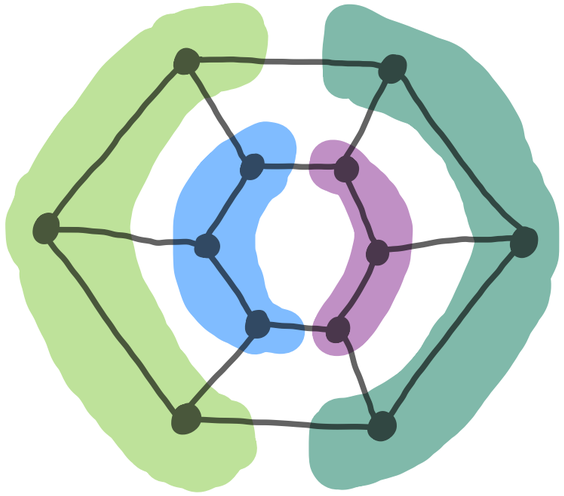


Obstacle 2: Wheels



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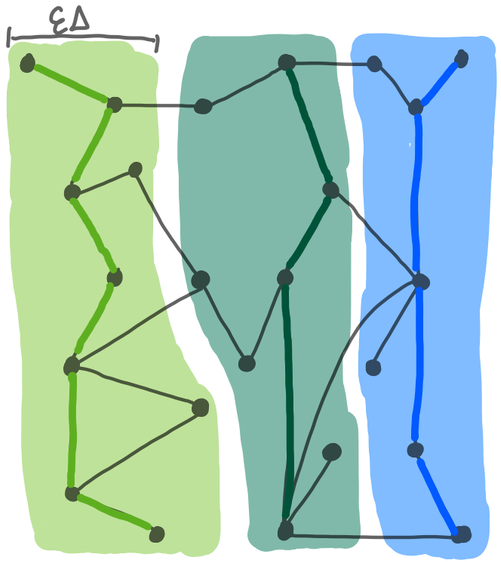

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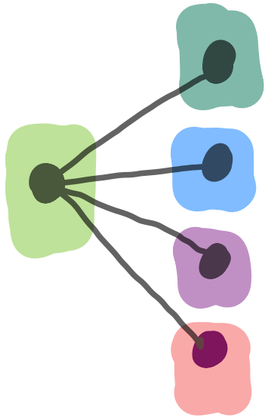
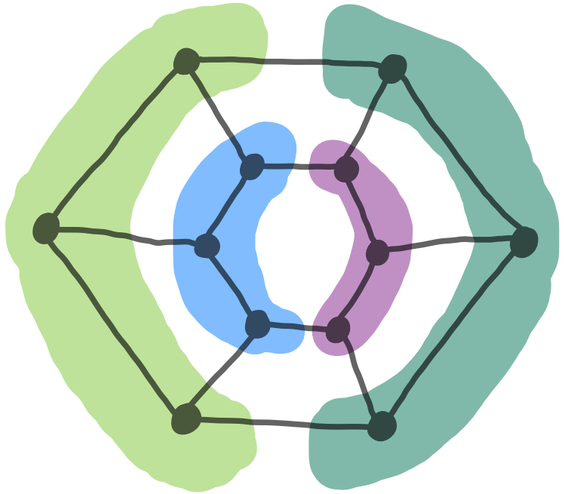


Fix: Allow tree of columns

Obstacle 2: Wheels



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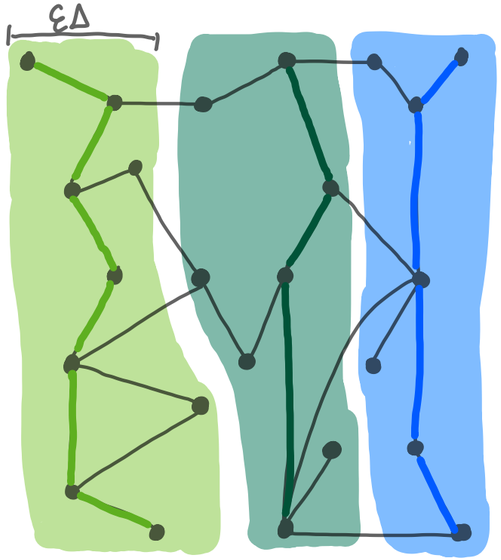

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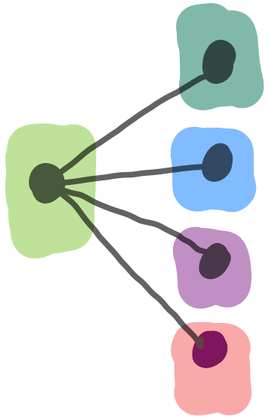

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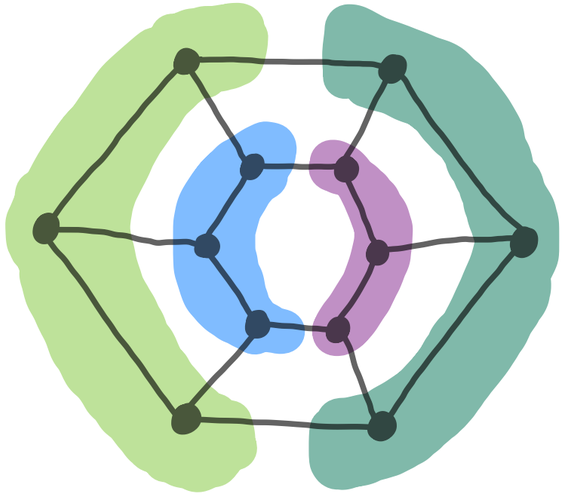
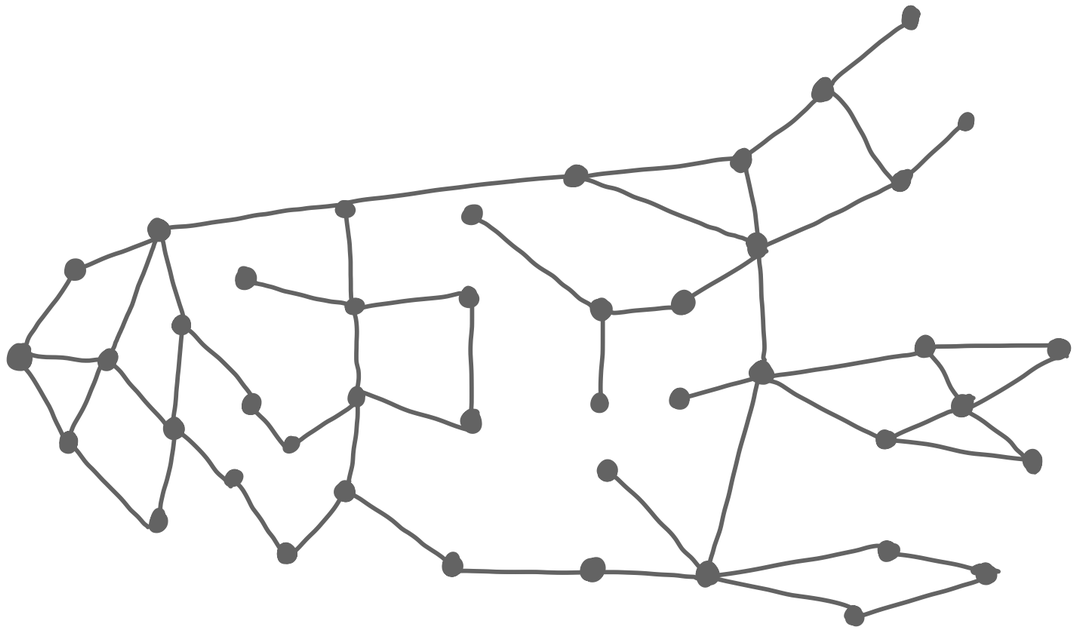


Fix: Nesting only occurs $\frac{1}{\varepsilon}$ times.

Gridtree Hierarchy

Thm (Informal). Every planar graph can be partitioned into a tree of columns of width $\varepsilon\Delta$, with $O(1/\varepsilon)$ layers of nesting.



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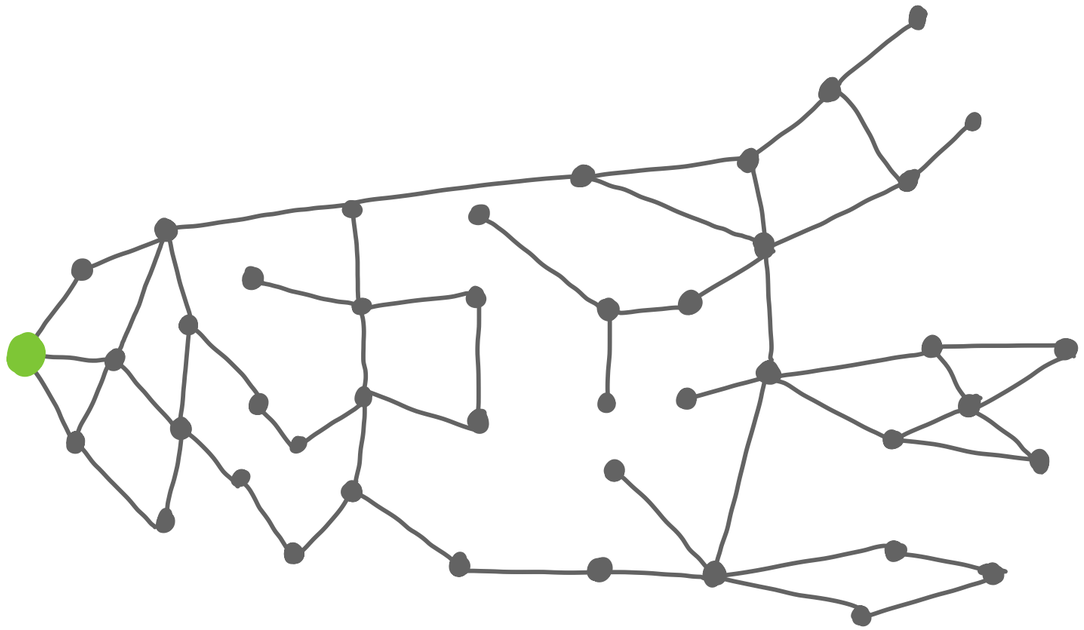

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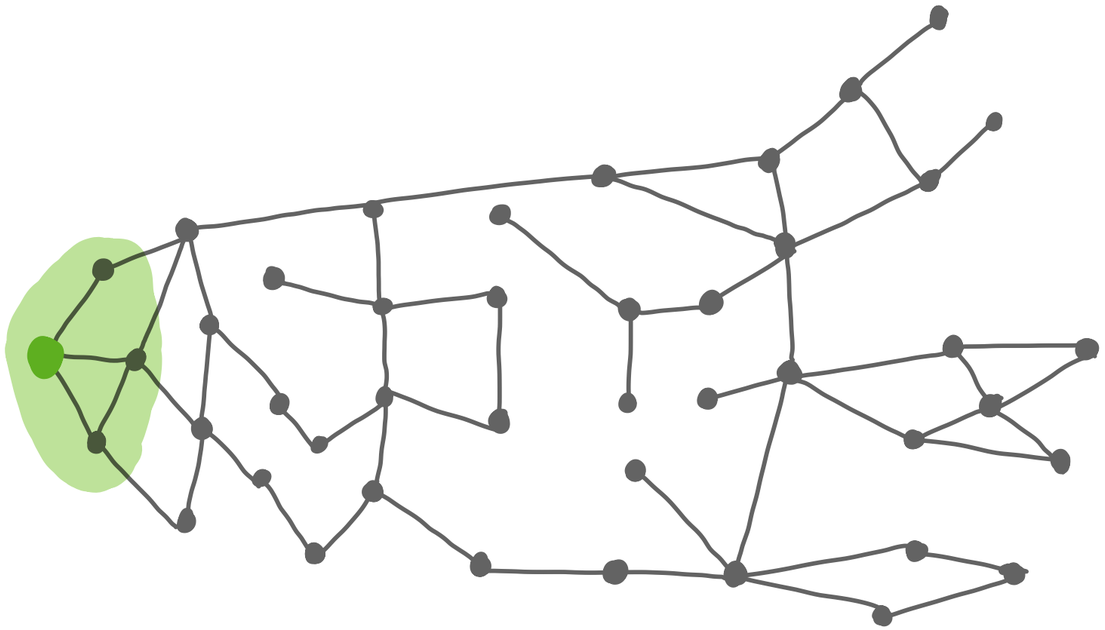

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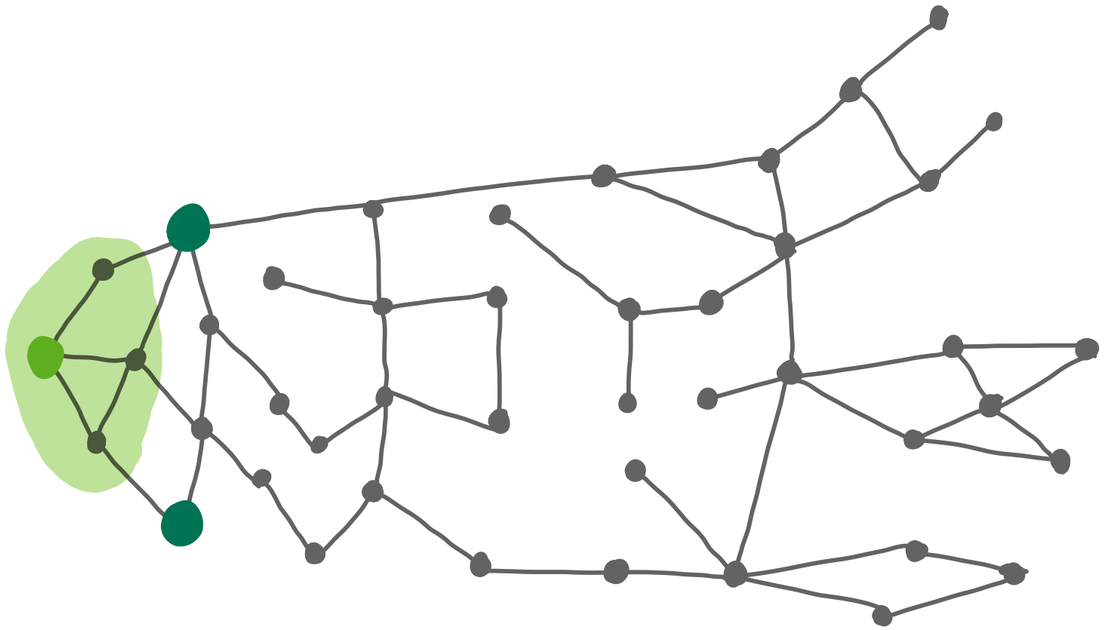

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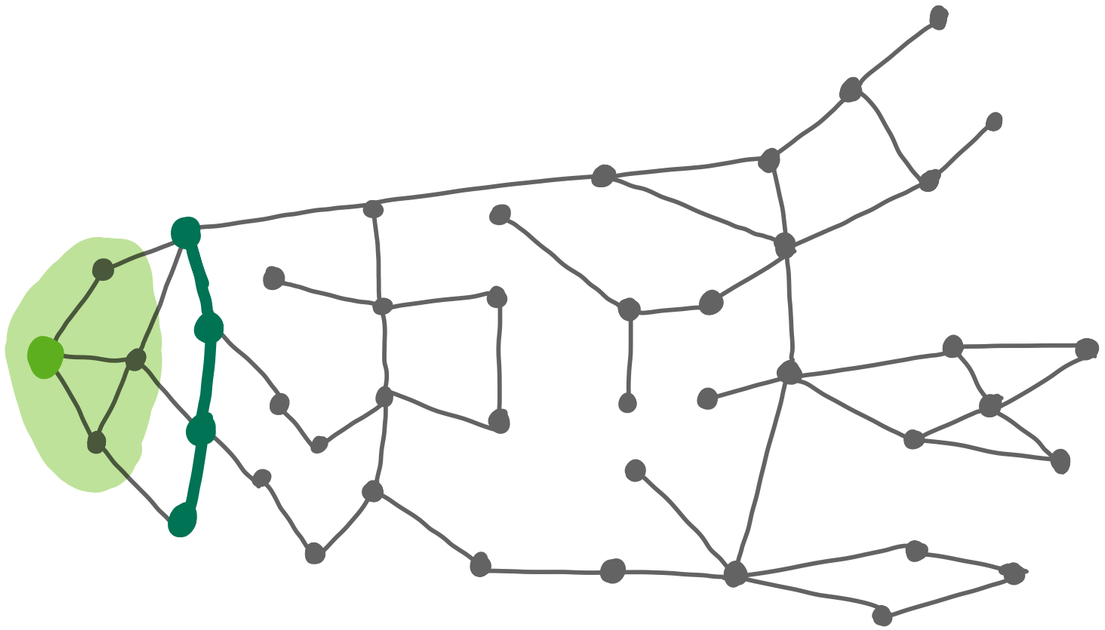

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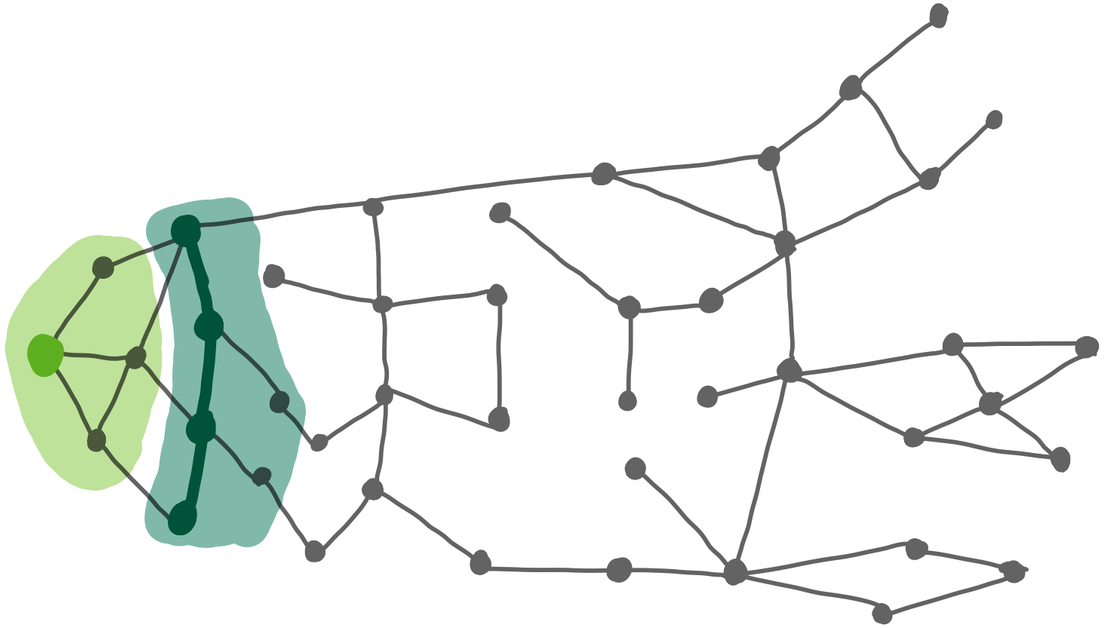

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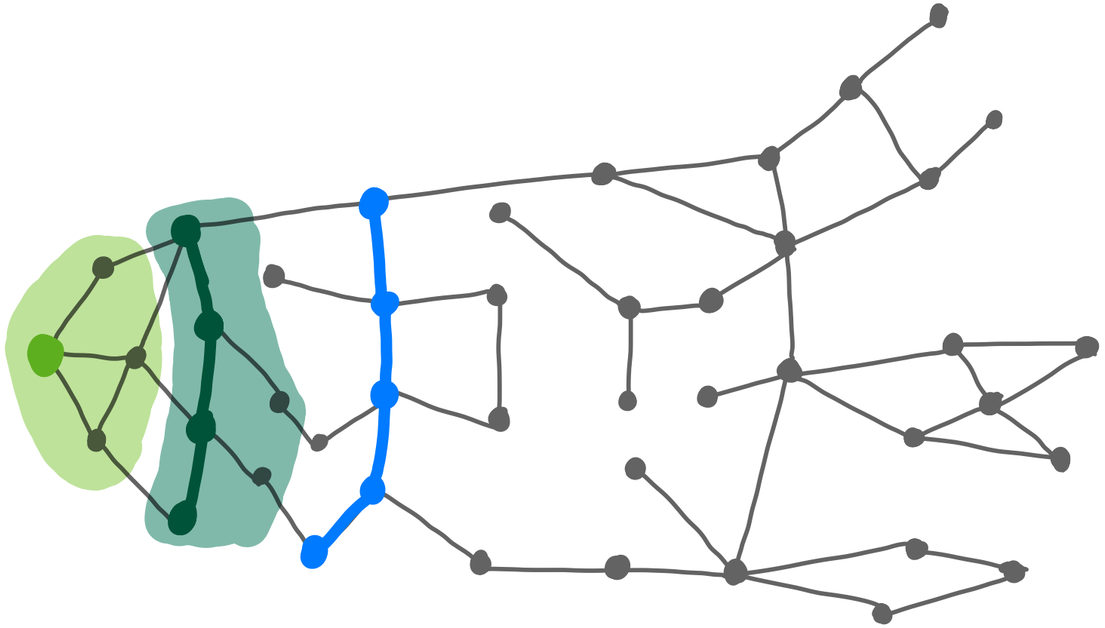

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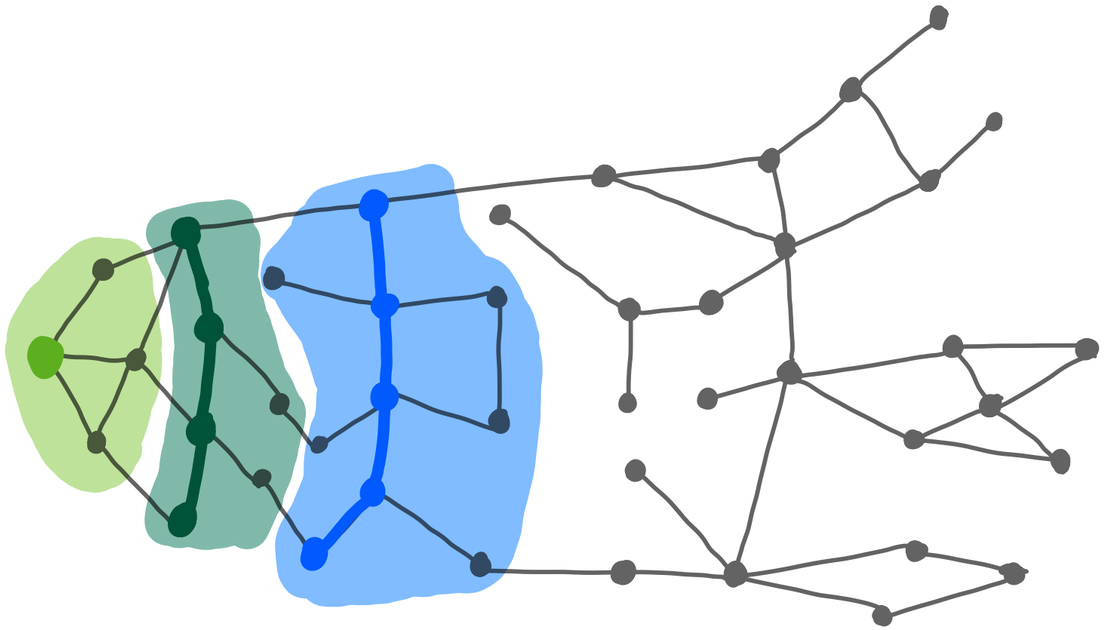

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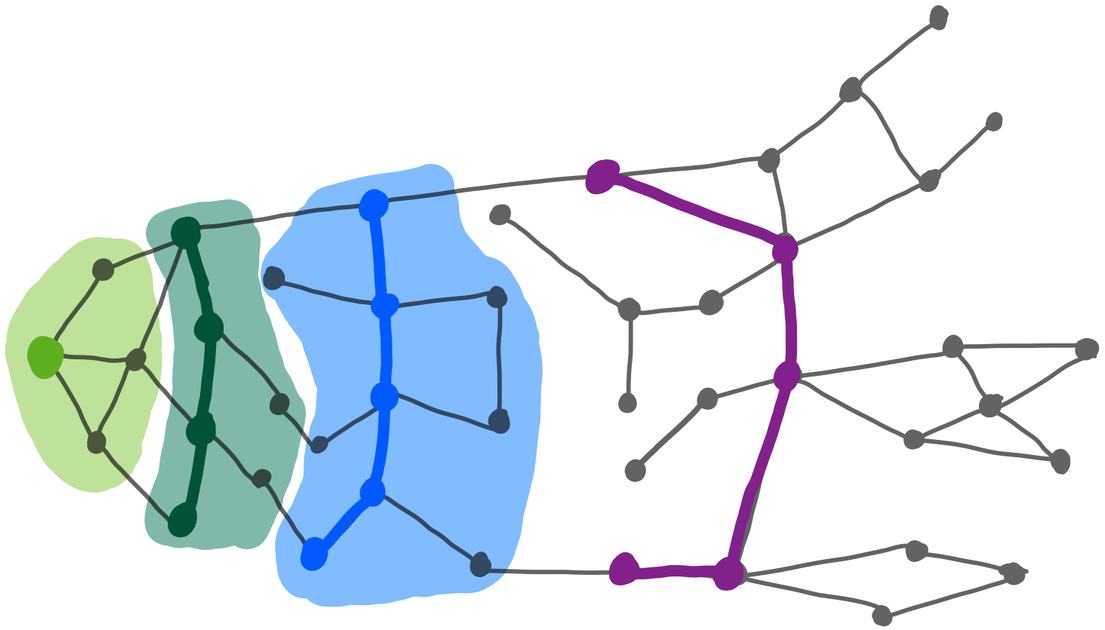

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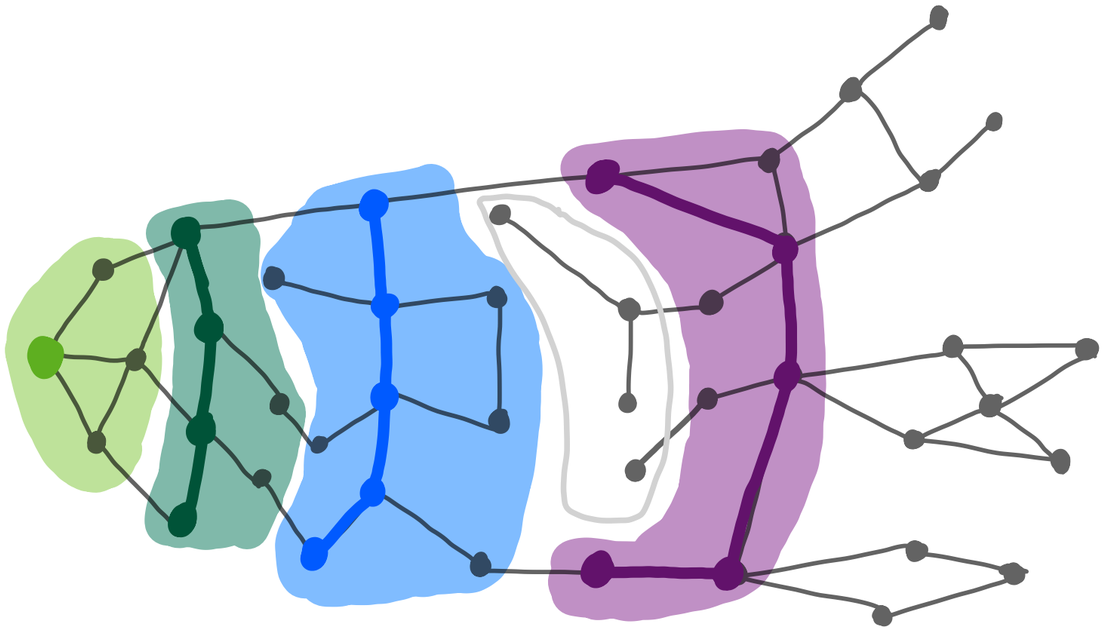

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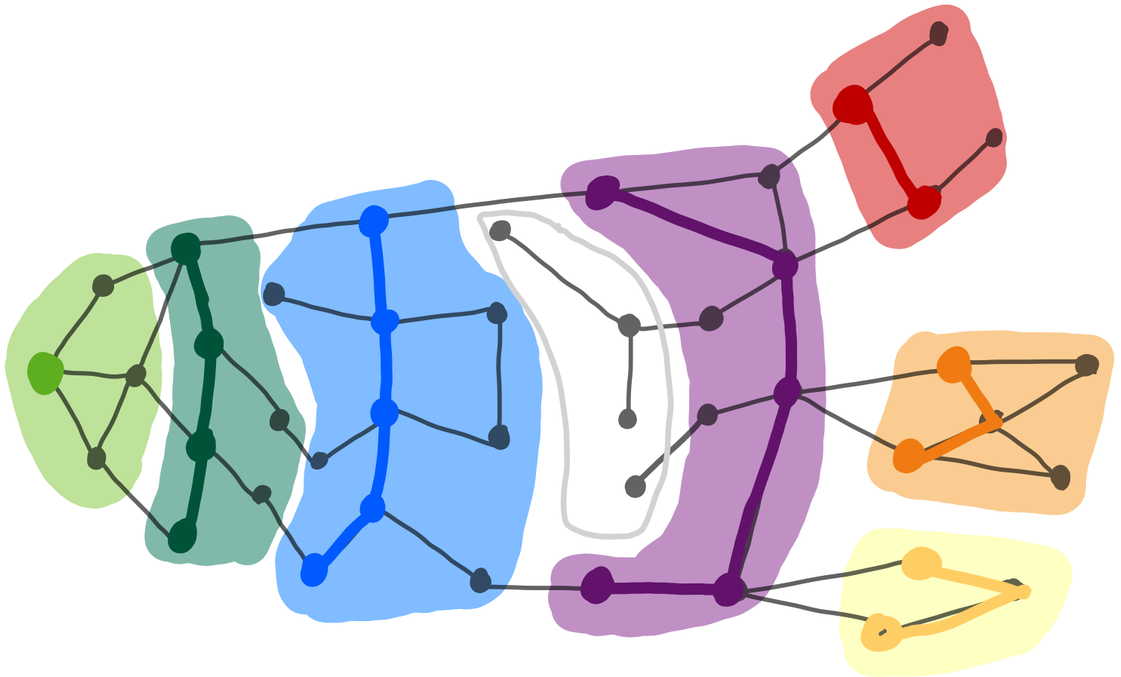

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Summary

- Many problems (e.g., tree cover) are easy on grids
- We introduce *gridtree* structure: planar graph look like grids

Open Problems

- $O(1)$-size $(1+\varepsilon)$-distortion *spanning* tree cover?
- Efficient (i.e., $o(n^2)$) construction of gridtree?
- Other applications of gridtree?

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