A Steiner Tree VC Set System in Minor-free (Di)Graphs

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Abstract

We propose a set system of maximum-covering minimum-density partial Steiner trees for planar and minor-free graphs. We show that this system has VC dimension at most $h-1$ for edge-weighted K_h-minor-free graphs, both directed and undirected. We also consider its geometric interpretation as a range space, proving it to be piercing.

In addition, we demonstrate how one can form a junction tree set system of bounded VC dimension from such Steiner trees. This is motivated by refining the junction tree set cover approach used in Chekuri and Jain's polylogarithmic approximation algorithm for Directed Steiner Forest in planar graphs [CJ25].

1 Introduction

Network design is a fundamental question in the fields of algorithms and combinatorial optimization, motivated by applications in energy, telecommunications, transportation, and logistics. In the *Directed Steiner Tree (DST)* problem, we have an edge-weighted directed graph $G = (V, E)$, a root $r \in V$, and a set of k terminals $S \subseteq V$. The goal is to find a minimum cost subgraph $H \subseteq G$ that contains an r-to-s path for every terminal $s \in S$, which will naturally form an out-tree from r. DST is NP-hard but admits an $O(k^{\varepsilon})$-approximation in polynomial-time [CCC+99]. If one allows for quasi-polynomial time, an $O(\log^2 k/\log\log k)$-approximation is known, with a tight matching lower bound assuming both the Projection Games Conjecture and $\mathsf{NP} \subseteq \mathsf{ZPTIME}(2^{n^{\delta}})$ for some $0 < \delta < 1$ [GLL19]. Additionally, DST has been shown to have an integrality gap of $\Omega(\sqrt{k})$ [ZK02] and $\Omega(n^{\delta})$ for some $\delta > 0$ [LL22]. However, it is not known whether DST admits a polynomial-time polylogarithimic approximation in general graphs.

Directed Steiner Tree is a special case of the *Directed Steiner Forest (DSF)* problem, which generalizes the root-to-terminals requirement to an arbitrary set of k terminal pairs $\{(s_i, t_i)\}_{i\in[k]}$; the subgraph H must contain every s_i-to-t_i path. DSF is known *not* to admit a polylogarithimic approximation in general graphs unless $\mathsf{P} = \mathsf{NP}$ due to a reduction from the Label Cover problem [DK99] thanks to the PCP Theorem. The best known algorithms are able to achieve $O(k^{1/2+\varepsilon})$ [CEGS11] and $O(n^{2/3+\varepsilon})$ [BBM+13] approximation ratios.

Recently, there has been an exciting line of work investigating these problems in the context of *planar digraphs*, with various results showing separation of the planar case from the general case for both DST and DSF. Planar graphs are a well-studied class of sparse graphs that have far-reaching applications across computer science, including but not limited to circuit design, terrain modeling, meshes, and visualization. Along with their generalization as minor-free graphs, planar graphs are at the center of graph theory and algorithm research. A key characteristic of planar graphs is that they have *separators*, which are well-structured vertex cuts that divide a graph into balanced chunks. Friggstad and Mousavi [FM23] make use of Thorup's shortest-path planar separator [Tho04] to show a polynomial-time $O(\log k)$-approximation for DST on planar graphs. Building on these techniques, Chekuri, Jain, Kulkarni, Zheng, and Zhu [CJK+24] obtain poly-time polylogarithmic approximations for the planar case of various generalizations of DST, as well as proving an integrality gap of $O(\log^2 k)$ for the cut-based LP relaxation of planar DST.

Chekuri and Jain [CJ25] make use of this integrality gap result to give a poly-time $O(\log^6 k)$-approximation for DSF on planar graphs. Their algorithm relies on the key notion of a *junction tree*, which is a subgraph with root r that can be decomposed into an in-tree and out-tree both rooted at r. As simple and well-structured objects on digraphs, junction trees have seen use in algorithms research for network design problems such as Buy-at-Bulk [CHKS10, Ant11, CEKP18, GKL24, CJ24] in addition to DSF [CEGS11, FKN12, GLQ21]. Chekuri and Jain [CJ25] first show that there exists a junction tree spanning some subset of terminal pairs with density at most $O(\log^2 k)$ times the optimum solution's density, where we define *density* to be the ratio of cost per terminal covered. Next, they give an algorithmic method to produce a junction tree of density at most $O(\log^5 k)$ times the optimum density, by finding a suitable set of terminals to cover and then 'gluing' together an in-tree and out-tree, both computed by rounding the DST linear program. With the ability to generate an (approximate) minimum density partial solution, their algorithm performs *greedy set cover* algorithm to build a DSF solution that covers all terminal pairs, iteratively adding to their solution a near-optimal partial tree over the remaining uncovered pairs until all pairs are covered.

Research Question. In using the greedy set cover approach, the algorithm of Chekuri and Jain loses an additional $O(\log k)$-factor in its approximation ratio; our primary research question is whether we can shave off this factor by more a refined technique, taking advantage of the simple structure of junction trees and the geometric nature of planar graphs.

Greedy set cover is a classic approximation technique that achieves $\Theta(\log n)$-approximation ratio[1], which is optimal for arbitrary set systems. However, $o(\log n)$-approximations have been found for certain cases of set cover, and in particular, there is significant amount of work on the geometric set cover problem. In geometric set cover, the set system consists of well-structured geometric objects on a ground set of points embedded into low-dimensional Euclidean space. This framework lends itself very well to studying set systems on planar graphs.

One way to measure the complexity of geometric objects is *VC dimension*, introduced by Vapnik and Chervonenkis [VC71] for statistics, but since then, finding application across a wide range of fields in theoretical computer science such as machine learning and geometry. As a tool for planar graph algorithms, Chepoi, Estellon, and Vaxés [CEV07] proved that the set system of undirected distance balls on K_h-minor-free graphs has VC dimension $h-1$. Their minor-building approach has since been used to bound the VC dimension of sets systems for other geometric shapes used for computing planar and minor-free graph diameter [LP19, DHV20, CGL24, KZ24], on both undirected and directed graphs.

Returning to DSF and covering the terminal pairs with junction trees, VC dimension provides us a pathway to circumvent its $\Theta(\log n)$-approximation bound. For set systems of bounded VC dimension, the algorithm of Brönnimann and Goodrich [BG94] gives an $O(\log \mathsf{OPT})$-approximation for set cover. Their algorithm makes use of the multiplicative weights update (MWU) framework, an iterative 'evolutionary' process applicable for a wide host of problems (see the survey [AHK12]). Brönnimann and Goodrich use the core idea of *ε-nets*: given a set system $(X, \mathcal{S})$, an *ε-net* is a representative subset of points in X requiring nonempty intersections with every set in $\mathcal{S}$ of at least ε fraction of total size of X. A pair of classic results [HW86, BEHW86] showed that set systems of constant VC dimension have ε-nets of size $O(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$, and it is the extra logarithmic factor $\log\frac{1}{\varepsilon}$ that leads to the $O(\log \mathsf{OPT})$-approximation ratio for set cover. Crucially, for set systems which admit *linear-size ε-nets*, i.e. a net of size $O(\frac{1}{\varepsilon})$, the results of Brönnimann and Goodrich instead provide an $O(1)$-approximation.

While the result of $O(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$-size ε-nets is tight for general VC set systems [KPW92], various geometric set systems have been shown to admit linear-size nets. Some such results include half-spaces in $\mathbb{R}^3$ [Mat92], unit-cubes in $\mathbb{R}^3$ [CV05], and pseudo-disks in $\mathbb{R}^2$ [PR08]. Notably, in order to show this latter result on pseudo-disks, Pyrga and Ray give a general framework for proving the existence of (near-)linear-size ε-nets [PR08]. Broadly speaking, their framework requires (a) an ε-net of size dependent only on ε, and (b) a connectivity-sketching subgraph called a *sparse support*. For the former, the previously mentioned $O(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon})$-size nets due to [HW86, BEHW86]

[1] Here, n is the number of elements in the ground set of the set cover instance; in this case we have n equals to k, the number of terminal pairs to be covered.

are sufficient for any set system of bounded VC dimension. Sparse supports are more difficult to come by, but Raman and Ray [RR20] show the existence of planar (and thus sparse) supports for *non-piercing* regions.[2] Of note, the PTAS for set cover on non-piercing regions [GRRR16] is not applicable here, as the algorithm of [CJ25] is limited in access to the set system and only able to sample individual junction trees.

Main Result. Our main result is to propose a set system of Steiner trees with VC dimension $h-1$ on K_h-minor-free graphs. Explicitly, our set system is made up of the *maximum-covering minimum-density* partial Steiner tree rooted at each vertex in the graph. We find that the minimum-density property alone is not sufficient to define the set system, and maximum-covering is necessary as a tie-breaking scheme. Using this set system of Steiner trees, we are able to construct a set system of junction trees with VC dimension $O(h)$. We also show that this set system is piercing, ruling out the improved approximation techniques previously discussed.

Our result of gluing Steiner trees to form junction trees is a promising application of VC theory to a combinatorial graph algorithm. It may be of independent interest to consider solution instances to other design problems as composites of well-structured partial solutions.

We lay out the key definitions for our work in section 2 and then define our proposed set system $\overrightarrow{\mathcal{T}}_G(S)$ in the beginning of section 3. Section 3.1 contains the proof of our main result, Theorem 3.1, relying on the key idea of *tree extensions* from sections 3.2 and 3.3. We show $\overrightarrow{\mathcal{T}}_G(S)$ to be piercing in section 3.4, consider its undirected counterpart in section 3.5, and explore alternative set systems in section 3.6. Lastly, we show how to glue together junction trees in section 4 and discuss further directions in section 5.

2 Preliminaries

Partial Steiner Trees. Let $G = (V, E, w)$ be a directed K_h-minor-free graph with non-negative valued edge weights $w : E \to \mathbb{R}_{\geq 0}$, and let $S \subseteq V$ be a set of terminals. A *partial Steiner tree* $T \subseteq G$ is a directed out-tree with some root $r \in V$, also known as a partial arborescence. We say T covers the terminals $S \cap T$ and define the *density* of a partial Steiner tree, or indeed any subgraph of G, as the ratio of its cost to number of terminals covered:

$$\rho(H) \coloneqq \frac{w(H)}{|S \cap H|}$$

Thus, a subgraph that covers no terminals has infinite density.

VC dimension. Let $\mathcal{S} = (X, \mathcal{F})$ be a set system, i.e. $\mathcal{F} \subseteq 2^X$. For any $Y \subseteq X$, we define the *projection* of $\mathcal{F}$ onto Y to be:

$$\mathcal{F}|_Y \coloneqq \{S \cap Y : S \in \mathcal{F}\}$$

We say that $\mathcal{F}$ *shatters* $A \subseteq X$ if $\mathcal{F}|_A = 2^A$. Define the VC dimension of a set system, *VC*($\mathcal{S}$), as the maximum cardinality of finite sets that $\mathcal{F}$ shatters, or ∞ if $\mathcal{F}$ can shatter sets of arbitrarily large cardinality.

[2] We say two regions A and B pierce if either $A \setminus B$ or $B \setminus A$ are disconnected, and a family of regions is *non-piercing* if they are pairwise non-piercing.

3 VC dimension of Steiner Tree Set Systems

We propose a set system of partial Steiner trees built on the following objects, which use minimum density as the primary criterion and then maximum terminals covered as a tie-breaker.

Definition 3.1. A tree T is a *maximum-covering minimum-density partial Steiner tree rooted at* r, or simply an *r-MMPS-tree*, if it (a) has minimum density among all partial Steiner trees rooted at r and (b) covers the most terminals among all minimum density trees rooted at r.

For a given root r, let T_r denote an r-MMPS-tree; note, there may be multiple such r-MMPS-trees. However, the maximum-covering and minimum-density properties force that that every r-MMPS-tree covers a consistent set of terminals.

Lemma 3.1. *Fix $r \in V$. Every r-MMPS-tree covers the same terminals, i.e. $T_r \cap S = T'_r \cap S$.*

Therefore, a root r uniquely defines the terminals covered by an r-MMPS-tree, regardless of choice of representative tree T_r. We then choose a set system on S of *all* r-MMPS-trees, mirroring the systems of balls from [CEV07, LWN23]:

$$\overrightarrow{\mathcal{T}}_G(S) \coloneqq (S, \{S \cap T_v : v \in V\})$$

This brings us to our main result, which is to bound the VC dimension of this set system.

Theorem 3.1. *The VC dimension of $\overrightarrow{\mathcal{T}}_G(S)$ is at most $h-1$.*

3.1 Proof of Theorem 3.1

Our proof of Theorem 3.1 follows the minor-building approach of Proposition 1 in [CEV07], which shows a VC dimension bound on the system of undirected distance balls on planar graphs. Let G be a K_h-minor-free directed graph with terminal set S. Assume to the contrary there exists a set of h terminals $A = \{a_i\}_{i\in[h]} \subseteq S$ that is shattered by $\overrightarrow{\mathcal{T}}_G(S)$. For each pair of distinct vertices a_i, a_j, let T_{ij} be an r_{ij}-MMPS-tree such that $T_{ij} \cap A = \{a_i, a_j\}$ for some root $r_{ij} \in V$. Now, define P_{ij} to be the path in T_{ij} from a_i to a_j. Lastly, for each terminal $a_i \in A$, we define the subgraph S_i to be the union of all sections of these paths that are shared by more than one P_{ij}:

$$S_i \coloneqq \bigcup \left\{P_{ij} \cap P_{ij'} : j, j' \in [h], j \neq j'\right\}.$$

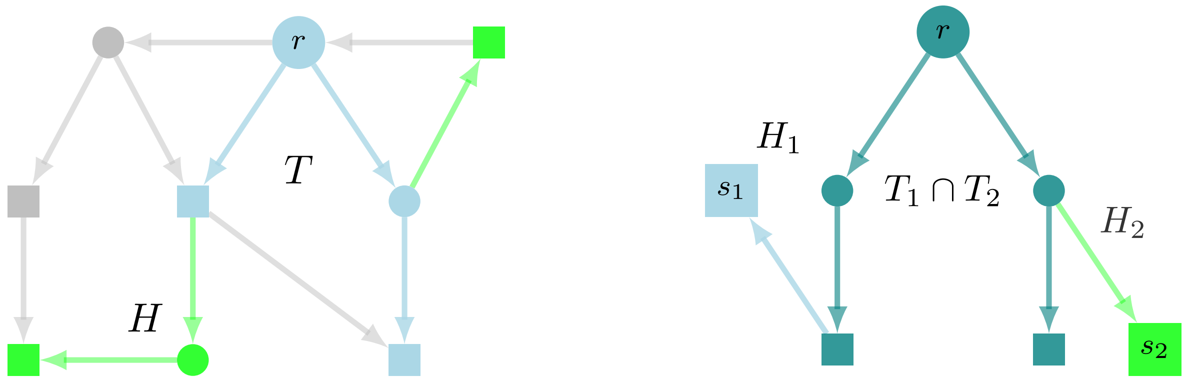
By a pair of disjointness results on P_{ij}'s, Chepoi et al. crucially show that these S_i's are also disjoint.

Claim 1 ([CEV07], Claim 3)**.** *The sets $S_1, \ldots, S_h$ are pairwise disjoint.*

As the sets of $S_1, \ldots, S_h$ induce disjoint connected subgraphs of G by Claim 1, we can contract subgraph S_i each into a single vertex v_i. We similarly replace the non-contracted part of each path P_{ij} with an edge (v_i, v_j), giving the complete graph K_h on h vertices. Therefore, G contains a K_h-minor and we have a contradiction, concluding the proof of Theorem 3.1.

3.2 Tree Extensions and Proof of Claim 1

In order to show that each S_i are pairwise disjoint in Claim 1, Chepoi et al. prove a pair of claims regarding disjointness of different P_{ij}'s that rely on the triangle inequality. This property comes because distance balls include vertices by shortest distances, which form a metric on undirected graphs. For our r-MMPS-trees, we instead go through the key idea of *tree extensions*.



(a) (b)

Figure 1: (a) Subgraph H (green) is a tree extension of T (blue) rooted at r; (b) T_1 (blue and teal) and T_2 (green and teal) are two r-MMPS-trees covering different terminal sets. Observe, H_1 is an extension of T_2 and H_2 is an extension of T_1.

Definition 3.2. Let T be a tree rooted at r. We call a subgraph H of G where $H \cap T = \varnothing$ an *extension* of T if the disjoint union $T \sqcup H$ is also a tree rooted at r.

Note, H need not be connected; see Figure 1a. Unconventionally, we require the connective edge(s) between T and H to be in $E(H)$, despite the endpoints in T not being vertices in H. This crucially makes it so that T and H partition both the vertices and edges of $T \sqcup H$.

With tree extensions defined, we show the property of our set system that allows us to prove our theorem in the manner of [CEV07].

Lemma 3.2. *Fix $r \in V$ and let T be an r-MMPS tree.*

(a) For any subgraph $H \subset T$ such that $T \setminus H$ is tree rooted at r and H is its tree extension, we have $\rho(H) \leq \rho(T) \leq \rho(T \setminus H)$.

(b) For every subgraph $H \subset G$ that is a tree extension of T, we have $\rho(H) > \rho(T)$.

Fundamentally, Lemma 3.2 states that r-MMPS-trees must include tree extensions with lower density, and only include such tree extensions. This lemma comes as a result of our particular choice of set system, requiring our Steiner trees to be both minimum-density and maximum-covering.

As a warmup, we prove Lemma 3.1, which states that an r-MMPS-tree covers a consistent set of terminals.

Lemma 3.1. *Fix $r \in V$. Every r-MMPS-tree covers the same terminals, i.e. $T_r \cap S = T'_r \cap S$.*

Proof. For some r, assume to the contrary T_1, T_2 are r-MMPS-trees covering a different set of terminals, i.e. $S \cap T_1 \neq S \cap T_2$. By definition, $\rho(T_1) = \rho(T_2)$ and $|S \cap T_1| = |S \cap T_2|$. This latter equality in combination with our assumption also gives that both $S \cap (T_1 \setminus T_2)$ and $S \cap (T_2 \setminus T_1)$ must be nonempty; we choose some terminals $s_1 \in S \cap (T_1 \setminus T_2)$ and $s_2 \in S \cap (T_2 \setminus T_1)$.

Let H_1 be the component of $T_1 \setminus T_2$ containing s_1. As $r \in T_2$, we have $r \notin H_1$, so that H_1 is an extension of both $T_1 \setminus H_1$ and T_2. We similarly define H_2 to be an extension of $T_2 \setminus H_2$ and T_1; see Figure 1b.

Without loss of generality, let $\rho(H_1) \leq \rho(H_2)$. Since T_2 is an r-MMPS-tree, $\rho(H_2) \leq \rho(T_2)$ by Lemma 3.2(a). However, we now have that H_1 is an extension of T_2 with $\rho(H_1) \leq \rho(T_2)$, which is a contradiction to Lemma 3.2(b). $\square$

We now proceed to proving Claim 1, proving two results on the disjointness of P_{ij}'s for our r-MMPS-trees.

Lemma 3.3. *For any four distinct vertices $a_i, a_j, a_{i'}, a_{j'} \in A$, the paths P_{ij} and $P_{i'j'}$ are disjoint.*

Proof. Suppose that P_{ij} and $P_{i'j'}$ share a common vertex. Since P_{ij} contains a vertex of $T_{i'j'}$, we have that a_i and a_j are in different components of $T_{ij} \setminus T_{i'j'}$; label these components $H_i \ni a_i, H_j \ni a_j$ respectively. If r_{ij} is in one of these components, we will assume it is in H_j without loss of generality, such that we have $r_{ij} \notin H_i$. We include in H_i the connective edge between H_i and $T_{ij} \setminus H_i$ so that H_i is a tree extension of $T_{ij} \setminus H_i$ as well as $T_{i'j'}$. We similarly define $H_{i'}, H_{j'}$ for $T_{i'j'}$ and assume $r_{i'j'} \notin H_{i'}$, giving that $H_{i'}$ is an extension of $T_{i'j'} \setminus H_{i'}$ and T_{ij}; see Figure 2.

Without loss of generality, let $\rho(H_i) \leq \rho(H_{i'})$. By Lemma 3.2(a), we have $\rho(H_{i'}) \leq \rho(T_{i'j'})$, so therefore $\rho(H_i) \leq \rho(T_{i'j'})$. However, H_i is an extension of $T_{i'j'}$ with lesser density, giving a contradiction to Lemma 3.2(b). $\square$

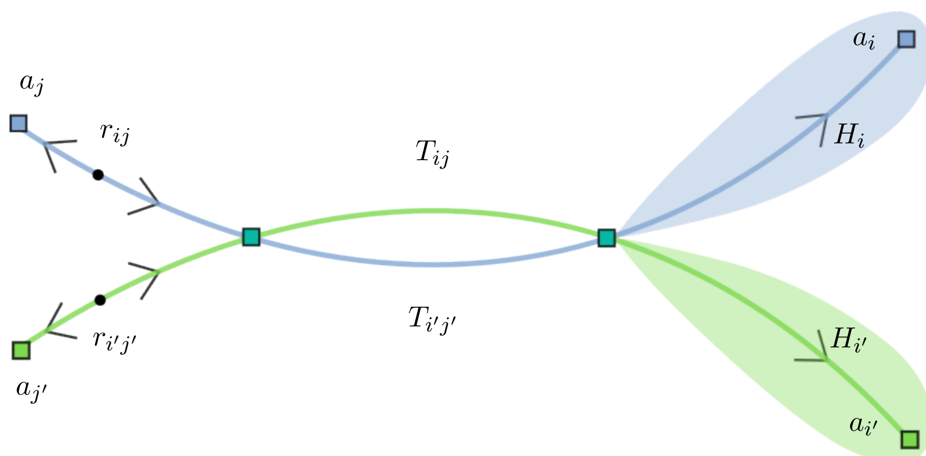


Figure 2: T_{ij} (green and teal) is an r_{ij}-MMPS-tree and $T_{i'j'}$ (blue and teal) is an $r_{i'j'}$-MMPS-tree. Observe, H_i is an extension of $T_{i'j'}$ and $H_{i'}$ is an extension of T_{ij}.

Lemma 3.4. *For any three distinct vertices $a_i, a_j, a_k \in A$, the intersection $P_{ij} \cap P_{ik} \cap P_{jk}$ is empty.*

Proof. Assume to the contrary there exists $x \in P_{ij} \cap P_{ir} \cap P_{jr}$. Thus, a_i and a_j are in different components of $T_{ij} \setminus \{x\}$; label these components of $T^i_{ij} \ni a_i, T^j_{ij} \ni a_j$. If r_{ij} is in one of these components, we will assume it is in T^j_{ij} without loss of generality so that $r_{ij} \notin T^i_{ij}$. If we similarly define $T^j_{jk}, T^k_{jk}, T^i_{ik}, T^k_{ik}$, there are two (distinct under permutation) different possibilities for the placements of r_{ik} and r_{jk}:

- Symmetric/triangular: $r_{jk} \notin T^j_{jk}$ and $r_{ik} \notin T^k_{ik}$. Let H_i be the component of $T_{ij} \setminus T_{jk}$ containing a_i. Since $H_i \subseteq T^i_{ij}$, then $r_{ij} \notin H_i$, and since $r_{jk} \in T_{jk}$, then $r_{jk} \notin H_i$. Therefore, H_i is a tree extension of both $T_{ij} \setminus H_i$ and T_{jk}. We similarly define H_j to be the component of $T_{jk} \setminus T_{ik}$ containing a_j such that it is a tree extension of $T_{jk} \setminus H_j$ and T_{ik}, and H_k to be the component of $T_{ik} \setminus T_{ij}$ containing a_k such that it is a tree extension of $T_{ik} \setminus H_k$ and T_{ij}.

  Without loss of generality, let $\rho(H_i) \leq \rho(H_j) \leq \rho(H_k)$. Since H_j is a tree extension of $T_{jk} \setminus H_j$, Lemma 3.2(a) gives $\rho(H_j) \leq \rho(T_{jk})$. However, we now have $\rho(H_i) \leq \rho(T_{jk})$, and as H_i is an extension of T_{jk}, Lemma 3.2(b) gives a contradiction.

- Asymmetric: $r_{jk} \notin T_{jk}^k$ and $r_{ik} \notin T_{ik}^k$. We define H_i as in the symmetric case, and then define H_k to be the component of $T_{jk} \setminus T_{ij}$ containing a_k. Here, H_k is an extension of both $T_{jk} \setminus H_k^{jk}$ and T_{ij}. At this point, we once more split into cases:
  - If $\rho(H_i) \leq \rho(H_k)$, then we make use of the fact that H_i extends T_{jk}. As H_k is an extension of $T_{jk} \setminus H_k$, Lemma 3.2(a) gives $\rho(H_k) \leq \rho(T_{jk})$. Thus, $\rho(H_i) \leq \rho(T_{jk})$, a contradiction by Lemma 3.2(b).
  - Otherwise, $\rho(H_k) \leq \rho(H_i)$. Since H_i is an extension of $T_{ij} \setminus H_i$, we have $\rho(H_i) \leq \rho(T_{ij})$ by Lemma 3.2(a), combining to get $\rho(H_k) \leq \rho(T_{ij})$. Since H_k is an extension to T_{ij}, Lemma 3.2(b) gives a contradiction. ☐

With Lemmas 3.3 and 3.4 established, we can finally show Claim 1. This proof follows exactly as in [CEV07], but we reproduce it for completeness.

Claim 1 ([CEV07], Claim 3)**.** *The sets $S_1, \ldots, S_h$ are pairwise disjoint.*

Proof. Assume to the contrary that two sets S_i and S_j with $i \neq j$ are not disjoint, i.e. some vertex x lies in their intersection. More specifically, let $x \in (P_{ik} \cap P_{ik'}) \cap (P_{jl} \cap P_{jl'})$. We proceed by cases:

- If $i \in \{l, l'\}$ and $j \in \{k, k'\}$, say $i = l'$ and $j = k'$. In the case $k = l$, then $x \in P_{ik} \cap P_{ik'} \cap P_{kk'}$ gives a contradiction to Lemma 3.4, and if $k \neq l$, then $x \in P_{ik} \cap P_{k'l}$ gives a contradiction by Lemma 3.3.
- If $i \notin \{l, l'\}$ but $j \in \{k, k'\}$, say $j = k'$. If $k \in \{l, l'\}$, assume $k = l'$, so that $x \in P_{ik} \cap P_{jl}$ gives a contradiction by Lemma 3.4.
- Otherwise, $i \notin \{l, l'\}$ and $j \notin \{k, k'\}$. Since $k \neq k'$ and $l \neq l'$, we can pick a vertex from each pair, say k, l, such that $k \neq l$. Thus, $x \in P_{ik} \cap P_{jl}$, gives a contradiction by Lemma 3.3.

Thus, the subgraphs S_i and S_j are disjoint. ☐

3.3 Proof of Lemma 3.2

To begin, we state a simple arithmetic property known as the *mediant inequality*. If we have fractions $\frac{a}{c} \leq \frac{b}{d}$, then their mediant $\frac{a+b}{c+d}$ lies between them:

$$\frac{a}{c} \leq \frac{a+b}{c+d} \leq \frac{b}{d}$$

Recall that for a partial Steiner tree T, we define its extensions H such that T and H partition both the vertices and edges of $T \sqcup H$. Thus, the density of $T \sqcup H$ is the mediant of densities $\rho(T)$ and $\rho(H)$:

$$\rho(T \sqcup H) = \frac{w(T \sqcup H)}{|S \cap (T \sqcup H)|} = \frac{w(T) + w(H)}{|S \cap T| + |S \cap H|}$$

We now restate Lemma 3.2 and give its proof:

Lemma 3.2. *Fix $r \in V$ and let T be an r-MMPS tree.*

(a) For any subgraph $H \subset T$ such that $T \setminus H$ is tree rooted at r and H is its tree extension, we have $\rho(H) \leq \rho(T) \leq \rho(T \setminus H)$.

(b) For every subgraph $H \subset G$ that is a tree extension of T, we have $\rho(H) > \rho(T)$.

Proof of Lemma 3.2(a). Let $H \subset T$ be a subgraph such that $T \setminus H$ is a tree rooted at r and H is its tree extension. Since T is an r-MMPS-tree, it has minimum density of all partial Steiner trees rooted at r, giving $\rho(T) \leq \rho(T \setminus H)$. As H is a tree extension of $T \setminus H$, the density $\rho(T)$ is the mediant of $\rho(T \setminus H)$ and $\rho(H)$. Therefore, $\rho(H) \leq \rho(T) \leq \rho(T \setminus H)$. □

Proof of Lemma 3.2(b). Let H be a tree extension of T, and assume to the contrary $\rho(H) \leq \rho(T)$. Observe that $T \sqcup H$ is a partial Steiner tree containing r; its density is the mediant of the densities of T and H, so we have $\rho(H) \leq \rho(T \sqcup H) \leq \rho(T)$. Thus, $T \sqcup H$ has density no greater than T, and since $\rho(H) \leq \rho(T)$ is finite, H must cover at least one terminal. This gives $|S \cap (T \sqcup H)| > |S \cap T|$. In particular, $T \sqcup H$ is a partial Steiner tree containing r with density less than or equal to T and covering more terminals than T, which is a contradiction to T being an r-MMPS-tree. □

3.4 $\overrightarrow{\mathcal{T}}_G(S)$ is Piercing

In previous sections, we have worked with $\overrightarrow{\mathcal{T}}_G(S)$ as a set system on the set of terminals S. However, if G is planar graph, we can also interpret our r-MMPS-trees as a geometric set system over the vertices of G, which can be embedded into the plane. We call $\overrightarrow{\mathcal{T}}_G(S)$ a *range space* in $\mathbb{R}^2$, where each T_r is a *range* on $V \subseteq \mathbb{R}^2$. This interpretation of $\overrightarrow{\mathcal{T}}_G(S)$ allow us to discuss geometric properties, which can be used to prove the existence of linear-size ε-nets.

Before looking at these geometric properties, we first must deal with a loose thread, being that a tree T_r is not uniquely defined on our graph. Lemma 3.1 uniquely defines $S \cap T_r$, but there could be many such trees that cover the same set of terminals with the same cost. To account for this, we refer to the Isolation Lemma [MVV87], which gives a tie-breaking scheme for r-MMPS-trees. More precisely, the result states that we can augment weights on edges in G to obtain a modified graph $\hat{G}$ satisfying (a) every tree $\hat{T}_r$ is unique, where $\hat{T}_r$ denotes T_r in $\hat{G}$, and (b) every tree $\hat{T}_r$ is an r-MMPS-tree in G.

We now assume without loss of generality that every tree T_r in G is unique and fix some embedding of G, and hence our range space $\mathcal{S} = \big(V \subset \mathbb{R}^2, \{V \cap T_v : v \in V\}\big)$ is well defined. We define a range space to be *non-piercing* if for every pair of regions A and B, both $A \setminus B$ and $B \setminus A$ are connected, i.e. neither B pierces A nor vice-versa. Raman and Ray [RR20] prove that every non-piercing family of regions has a planar support, which, along with Theorem 3.1, would be sufficient for the linear size ε-nets of [PR08]. However, despite the ball-like nature of our r-MMPS-trees, the set system $\overrightarrow{\mathcal{T}}_G(S)$ is indeed piercing, as seen by Figure 3.

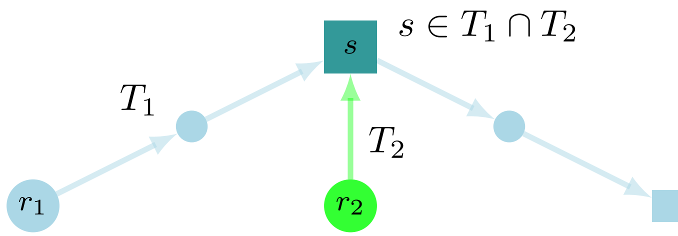


Figure 3: T_1 (blue and teal) is an r_1-MMPS-tree and T_2 (green and teal) is an r_2-MMPS-tree. Observe, $T_1 \setminus T_2$ is disconnected, so T_2 pierces T_1.

3.5 Undirected Graphs

We can also consider a set system analogous to $\overrightarrow{\mathcal{T}}_G(S)$ for undirected graphs. Since undirected trees have no explicit root, we instead define an r-MMPS-tree T_r to be a maximum-covering minimum-density partial Steiner tree *including* r. Now, we can define a set system of all r-MMPS-trees, just as in the directed case:

$$\mathcal{T}_G(S) \coloneqq (S, \{S \cap T_v : v \in V\})$$

As a corollary of Theorem 3.1, we prove that this set system has a matching bound on VC dimension.

Corollary 3.2. *The VC dimension of $\mathcal{T}_G(S)$ is at most $h-1$.*

Proof. Consider the set system $\mathcal{T}_G(S)$ on an undirected graph G, and let $\overrightarrow{G}$ be the directed graph containing both edges (u,v) and (v,u) for every undirected edge $\{u,v\}$ in G. We claim $\mathcal{T}_G(S) = \overrightarrow{\mathcal{T}}_{\overrightarrow{G}}(S)$. By definition, for any undirected tree T including a fixed node r, there is an identical directed tree of same cost rooted at r in $\overrightarrow{G}$. Similarly, for every directed tree rooted at r, there is an identical undirected tree of same cost including r. Thus, the trees in each set system are identical and the set systems are isomorphic, giving $\mathcal{T}_G(S) = \overrightarrow{\mathcal{T}}_{\overrightarrow{G}}(S)$. As a result, our VC dimension bound of $h-1$ from Theorem 3.1 applies to the undirected set system $\mathcal{T}_G(S)$. □

3.6 Other Candidate Set Systems

Since our goal is to use a set cover approach to cover all terminals at minimum cost, minimum-density was a natural criterion for forming a set system of partial solutions. We designed a set system with minimum-density Steiner trees for every root r, mirroring a set system of balls rooted from each vertex as in [CEV07, LWN23]. However, this minimum-density property alone was insufficient; without the maximum-covering tie-breaking scheme, the set system has unbounded VC dimension. To see this, consider an arbitrarily large star graph with leaves as terminals S and center r: any non-empty tree rooted at r has density exactly 1 and so every one of the 2^S possible r-rooted trees are minimum-density. The set system includes all such trees and therefore shatters S, proving that its VC dimension is ∞.

As an alternative to minimum-density among trees rooted at r, we also considered minimum-density among all trees covering a subset of terminals $A \subseteq S$. For this case, our set family would be the collection of these trees (intersected with S) for each $A \subseteq S$. First, the same star graph counterexample necessitates that this set system is also maximum-covering. However, even with this tie-breaking scheme, we can still construct a graph with arbitrarily many terminals that this set system shatters. Consider a graph with a rich core of terminals S^* and then an arbitrarily large set of terminal leaves S, each of which is (equally) costly to reach. Let T_A denote a maximum-covering minimum-density tree among all trees covering terminals $A \subseteq S$. Observe that T_A will cover exactly the terminals $S^* \cup A$, as including S^* lowers the density but adding any terminals in $S \setminus A$ raises the density. Therefore, every tree T_A projected onto S will cover exactly terminals A, and so the set system shatters S and thus has unbounded VC dimension.

4 VC dimension of Glued Junction Trees

In this section, we show how to construct a set system of junction trees with bounded VC dimension from a set system of directed Steiner trees with bounded VC dimension. This most naturally applies to $\overrightarrow{\mathcal{T}}_G(S)$, but could extend to other Steiner tree set systems to due its black box approach.

Let $\mathcal{S}_G(X) = (X, \mathcal{F})$ be a set system of partial Steiner trees on a directed graph G with terminals X and $\mathrm{VC}(\mathcal{S}_G(X)) = d$. Our set family $\mathcal{F}$ is defined as a collection of subsets of X, where each set $\mathbf{f}^r \in \mathcal{F}$ is associated with a root r (multiple trees can share the same root).

For a DSF instance with terminal pairs $D = \{(s_i, t_i)\}_{i\in[k]}$, let $S = \{s_i\}_{i\in[k]}$ and $T = \{t_i\}_{i\in[k]}$ be the sets of sources and sinks respectively. We will define $\mathcal{S}_{\text{out}} = \mathcal{S}_G(T) = (T, \mathcal{F}_{\text{out}})$ to be a set system of out-trees flowing from r to the sinks, and similarly define $\mathcal{S}_{\text{in}} = \mathcal{S}_{\overleftarrow{G}}(S) = (S, \mathcal{F}_{\text{in}})$ to be a set system of in-trees flowing from the sources to r, where $\overleftarrow{G}$ is G with all edges reversed.

We then relabel the ground sets of each set system, letting $\mathcal{S}_{\text{out}} = ([k], \mathcal{F}_{\text{out}})$ and $\mathcal{S}_{\text{in}} = ([k], \mathcal{F}_{\text{in}})$. From these Steiner tree set systems, we form junction trees by gluing pairs of in-trees and out-trees at root r, obtaining the set family:

$$\mathcal{F}_J \coloneqq \{\mathbf{f}^r_{\text{in}} \cap \mathbf{f}^r_{\text{out}} : r \in V, \mathbf{f}^r_{\text{in}} \in \mathcal{F}_{\text{in}}, \mathbf{f}^r_{\text{out}} \in \mathcal{F}_{\text{out}}\}$$

Define the set system $\mathcal{J} = ([k], \mathcal{F}_J)$. Here, the intersection is used because junction trees only cover terminal pairs that are covered by *both* in-tree and out-tree. We now refer to a technical result on mixing range spaces:

Lemma 4.1 ([HP11], Theorem 6.2.15)**.** *Let $\mathcal{S}_1 = \left(X, \mathcal{R}^1\right), \ldots, \mathcal{S}_k = \left(X, \mathcal{R}^k\right)$ be range spaces with VC dimension $\delta_1, \ldots, \delta_k$, respectively. Let $f(\mathbf{r}_1, \ldots, \mathbf{r}_k)$ be a function that maps any k-tuples of sets $\mathbf{r} \in \mathcal{R}^1, \ldots, \mathbf{r} \in \mathcal{R}^k$ into a subset of X. Consider the range set:*

$$\mathcal{R}' \coloneqq \left\{ f(\mathbf{r}_1, \ldots, \mathbf{r}_k) : \mathbf{r}_1 \in \mathcal{R}^1, \ldots, \mathbf{r}_k \in \mathcal{R}^k \right\}$$

and the associated range space $\mathcal{S}' = (X, \mathcal{R}')$. Then, the VC dimension of $\mathcal{S}'$ is bounded by $O(k\delta \ln k)$, where $\delta \coloneqq \max_i \delta_i$.

Corollary 4.1. *$VC(\mathcal{J}) = O(d)$.*

Proof. Observe that $\mathcal{F}_J$ is a subset of the following set family, which does not require in-trees and out-trees to share a root r:

$$\mathcal{G} \coloneqq \left\{ \mathbf{f}^r_{\text{in}} \cap \mathbf{f}^{r'}_{\text{out}} : \mathbf{f}^r_{\text{in}} \in \mathcal{F}_{\text{in}}, \mathbf{f}^{r'}_{\text{out}} \in \mathcal{F}_{\text{out}} \right\}$$

Lemma 4.1 bounds the VC dimension of $([k], \mathcal{G})$ by $O(d)$, with parameters $\delta = \text{VC}(\mathcal{S}_{\text{in}}) = \text{VC}(\mathcal{S}_{\text{out}}) = d$ and $k = 2$. Since $\mathcal{F}_j \subseteq \mathcal{G}$, then we can also bound $\text{VC}(\mathcal{J}) = O(d)$. $\square$

5 Conclusion and Future Directions

In this work, we propose a set system of Steiner trees with bounded VC dimension and demonstrate a method to gluing together a VC system of junction trees. Our hope is to find a set system of Steiner trees akin to those in this work that (a) admit an $o(\log k)$-approximation for set cover and (b) contain the trees produced algorithmically in [CJ25].

For beating the $O(\log k)$-approximation ratio, any bounded VC dimension set system admitting an $O(\frac{1}{\varepsilon})$-size ε-net would be sufficient by the construction the MWU set cover algorithm of [BG94]. A promising avenue in this directions comes in non-piercing set systems, whose planar supports [RR20] produce linear-size nets by [PR08].

The other challenge of this task is tying the algorithmic Steiner trees computed by LP rounding to a VC set system. While VC theory can provide a powerful tool for analysis, it often requires extremely simple structures to be of use. On the other hand, instances of network design problems can be highly intricate, and particularly for directed graphs. It remains to be seen whether these two frameworks can be married together for algorithmic improvements on problems such as Directed Steiner Forest.

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