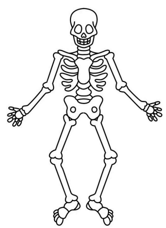
How to Protect Yourself from Threatening Skeletons: Optimal Padded Decomposition in Minor-free Graphs



Jonathan Conroy

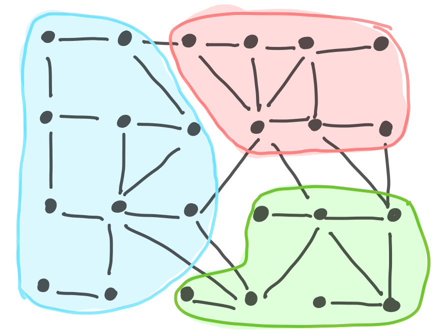
Dartmouth College

Arnold Filtser

Bar-Ilan University

Stochastic Clustering

Undirected graph $G = (V, E)$



Want to partition V:

- each cluster has small diameter
- nearby vertices are in same cluster with good probability

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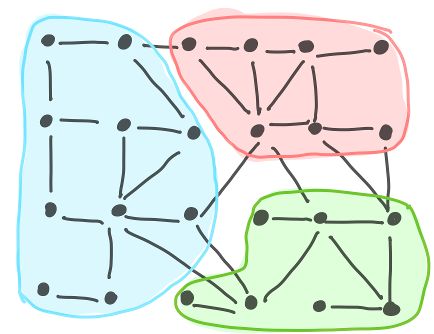

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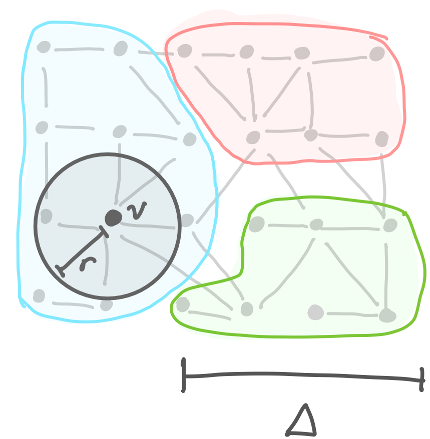
One motivation: divide & conquer

Padded Decomposition

Graph $G=(V,E)$ has (β, Δ)-padded decomposition if there is stochastic partition of V such that

- $\forall$ cluster C, $\mathrm{diam}(C) \le \Delta$
- $\forall\, v \in V$ and $\forall\, r > 0$, ball $B(v,r)$ is cut w.p. $\le \beta \cdot \frac{r}{\Delta}$

padding parameter



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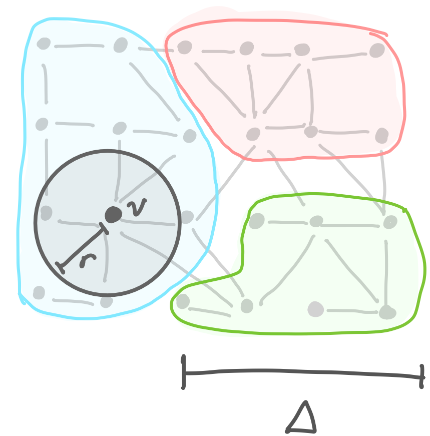

Example:

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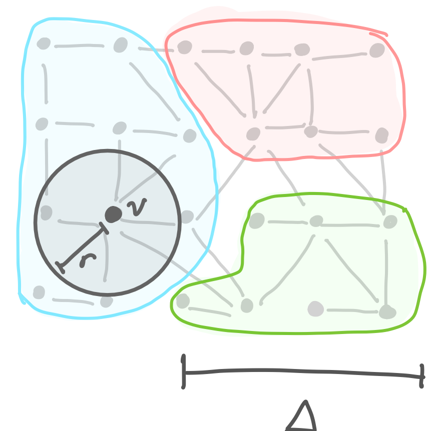

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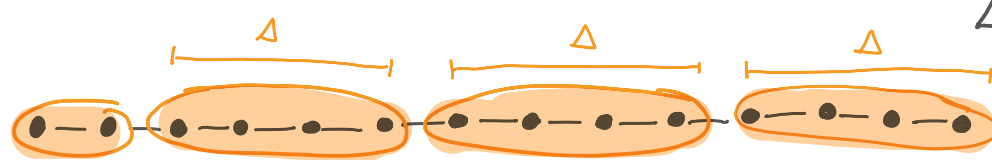

Padded Decomposition

Thm. [Bartal '96] Every n-vertex graph has padding parameter $\beta = O(\log n)$. This is tight (for expanders).

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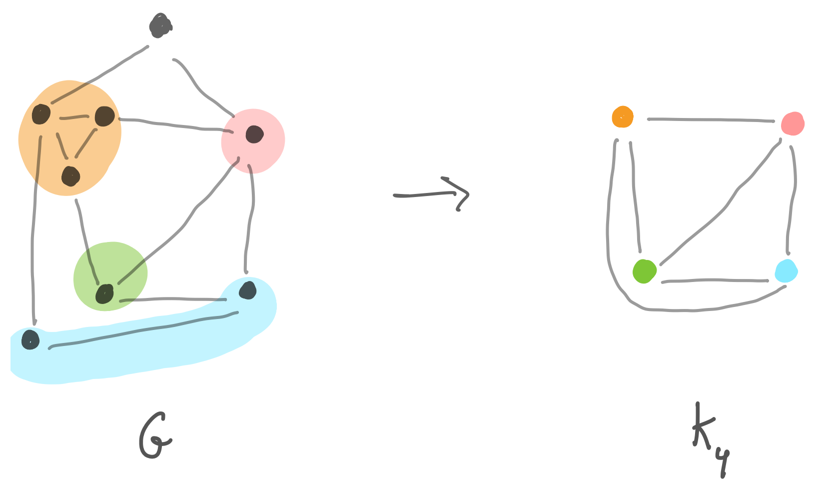
Applications

- Multicommodity max flow/min cut gap
- Flow sparsifiers
- Steiner point removal
- Spanners
- Metric embeddings into trees / ℓ_p / ℓ_∞
- Average distortion, vertex cuts, and treewidth approximation
- Compact routing schemes
- Distributed deadlock prevention
- Ultrametric covers
- Universal TSP and Steiner tree
- Oblivious buy-at-bulk

K_r - Minor-free Graphs

Can we get $\beta = o(\log n)$ for restricted graph classes?

Graph minor: obtained by deleting edges/vertices & contracting edges



K_r - Minor-free Graphs

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$O(r^3)$ [Klein, Plotkin, Rao 93]

$\Omega(\log r)$ [Bartal 96]

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Special cases

Genus g: $O(\log g)$
[Lee, Sidiropoulos 10]

Pathwidth pw: $O(\log pw)$
[Abraham, Gavoille, Gupta, Neiman, Talwar 14]

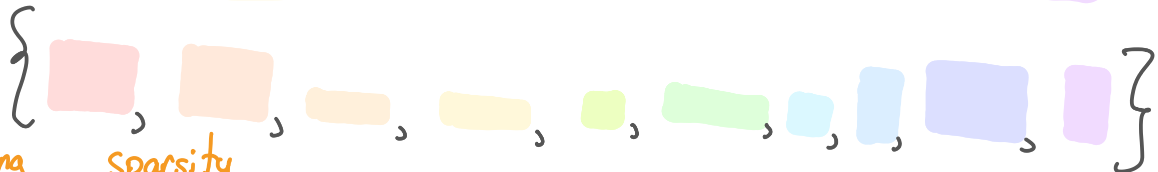
Treewidth tw: $O(\log tw)$
[Filtser, Friedrich, Issac, Kumar, Le, Mallek, Zeif 24]

Our main result

Thm. Every K_r-minor-free graph has padding parameter

$$\beta = O(\log r)$$

Sparse Cover



A $(\overset{\text{padding}}{\rho}, \overset{\text{sparsity}}{\tau}, \Delta)$-sparse cover is a set of (not disjoint) clusters

- each cluster has diam $\leq \Delta$
- $\forall$ vertex v, ball $B(v, \Delta/\rho)$ is contained in some cluster
- $\forall$ vertex v, v is contained in $\leq \tau$ clusters

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Observation. By taking the union of $O(\log n)$ samples of padded decomposition:

(β, Δ)-padded decomposition $\Rightarrow$ $(\frac{1}{\beta}, O(\log n), \Delta)$-sparse cover

Our results

Thm. (ρ, τ, Δ)-sparse cover $\Rightarrow$

$(O(\rho \log \tau), \Delta)$-padded decomposition

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Thm. (ρ, τ, Δ)-sparse cover $\Rightarrow$

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[Filtser 25]

another proof of

$(O(1), 2^{O(r)}, \Delta)$-sparse cover $\Rightarrow$ $(O(r), \Delta)$-padded decomposition

Our results

Thm. (ρ, τ, Δ)-sparse cover $\Rightarrow$

$(O(\rho \log \tau), \Delta)$-padded decomposition

Thm. K_r-minor-free graphs have

$(O(1), O(r^4), \Delta)$ - sparse cover, $\forall \Delta > 0$

Overview

1. Bounded threateners program

2. Cop decomposition for K_r-minor-free graphs

3. $O(\log r)$ padding

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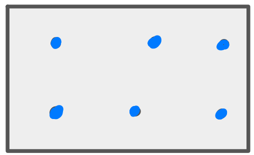

"for apart" points $\Rightarrow$ good padding

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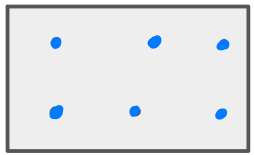

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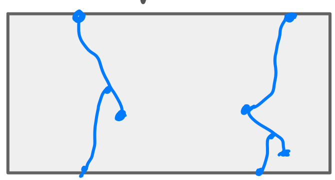

"far apart shortest-path separators"

3. $O(\log r)$ padding

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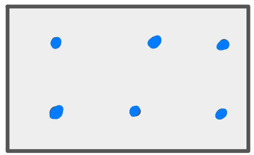

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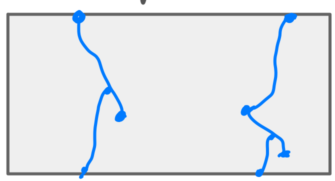

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divide & conquer on cop decomposition (NEW)

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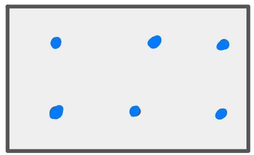

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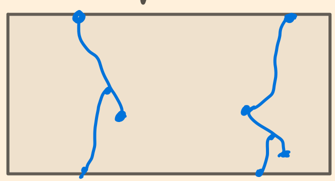

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[Bar96, AGGNT14]

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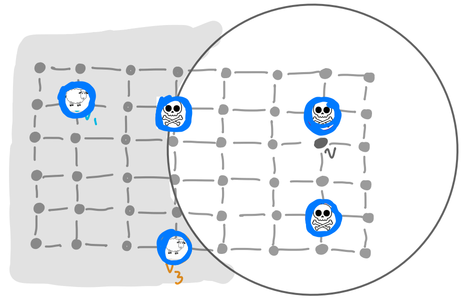


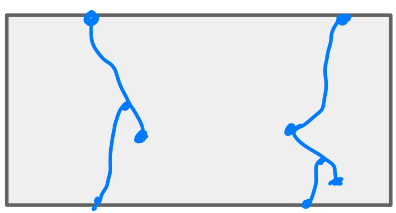
Goal: Find net points s.t. every vertex is close to

$\leq \tau$ net points ("threateners")

$\Rightarrow O(\log \tau)$ padded decomposition

[And86, AGGNT14, CCLMST24]

2. Cop decomposition of minor-free graphs



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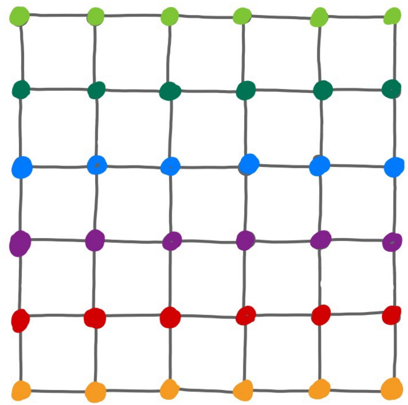
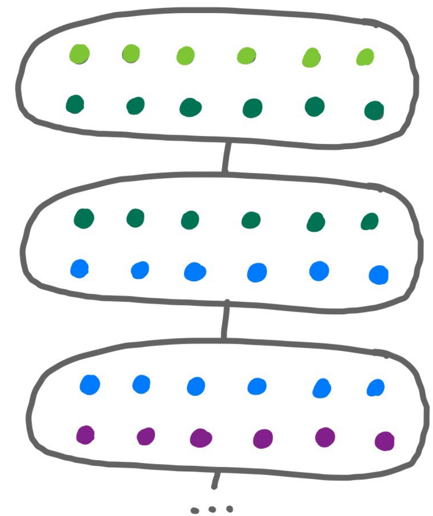
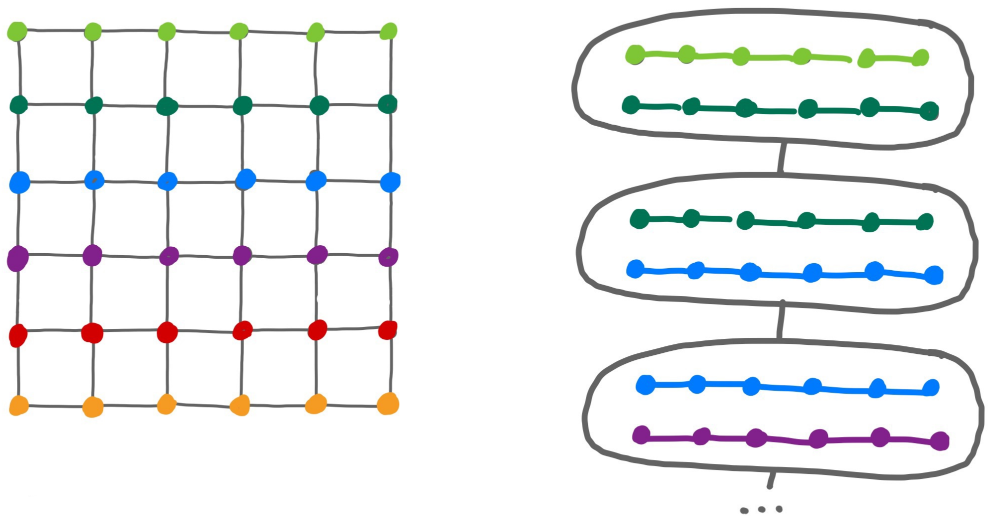
K_r-Minor-free graphs have treewidth $O_r(\sqrt{n})$





Tree decomposition with bags of $\leq O_r(\sqrt{n})$ vertices

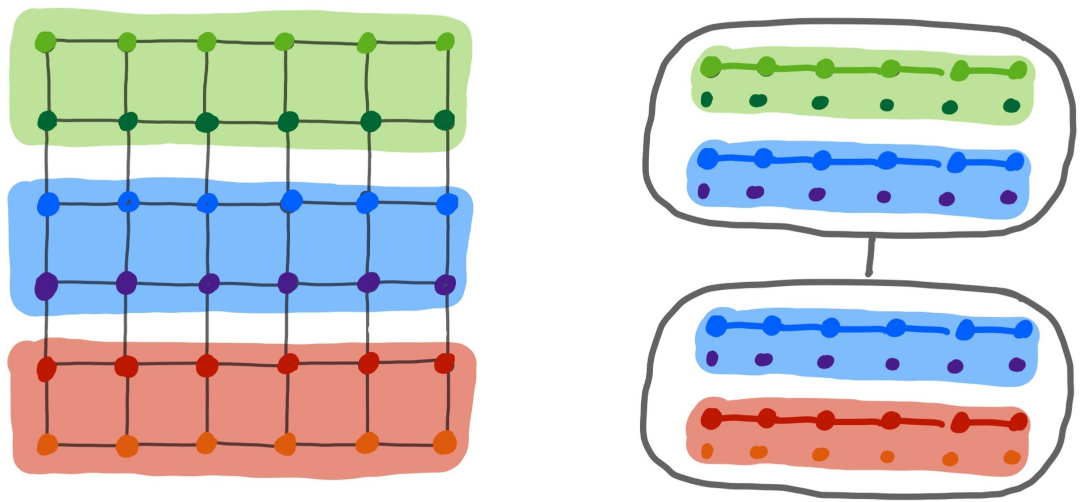
[Andreae 86] cop decomposition



Vertices of G are partitioned into paths.

Tree decomposition with bags of $\leq O(r^2)$ shortest* paths

[Abraham, Gavoille, Gupta, Neiman, Talwar 14] Cop decomposition

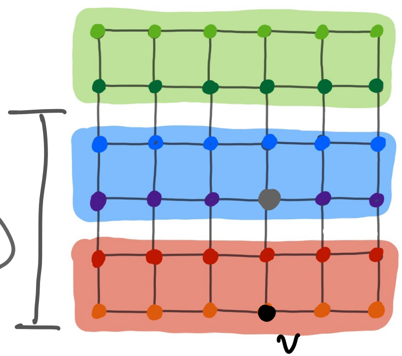


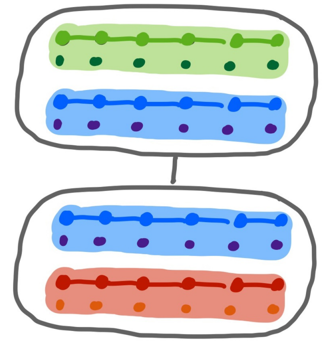
Vertices of G are partitioned into supernodes := shortest* path + neighborhood

Tree decomposition with bags of $\leq O(r^2)$ supernodes

[Abraham, Gavoille, Gupta, Neiman, Talwar 14] Cop decomposition

$2^{O(r)}$ supernodes within distance $O(\Delta)$





For any vertex v, in expectation there are

$2^{O(r)}$ "nearby" paths above v in the cop decomposition

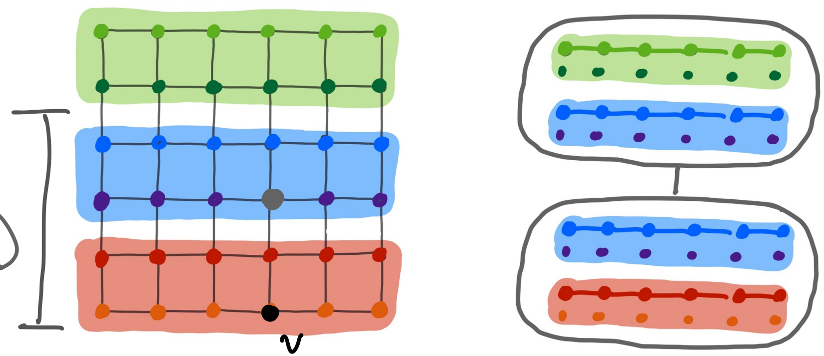
↳ threatening skeletons

$\Rightarrow O(r)$ padding

[Chang, C., Le, Milenkovic, Solomon, Than 24] Buffered Cop Decomposition

$2^{O(r)}$ supernodes

within distance $O(\Delta)$



For any vertex v, ~~in expectation~~ deterministically there are

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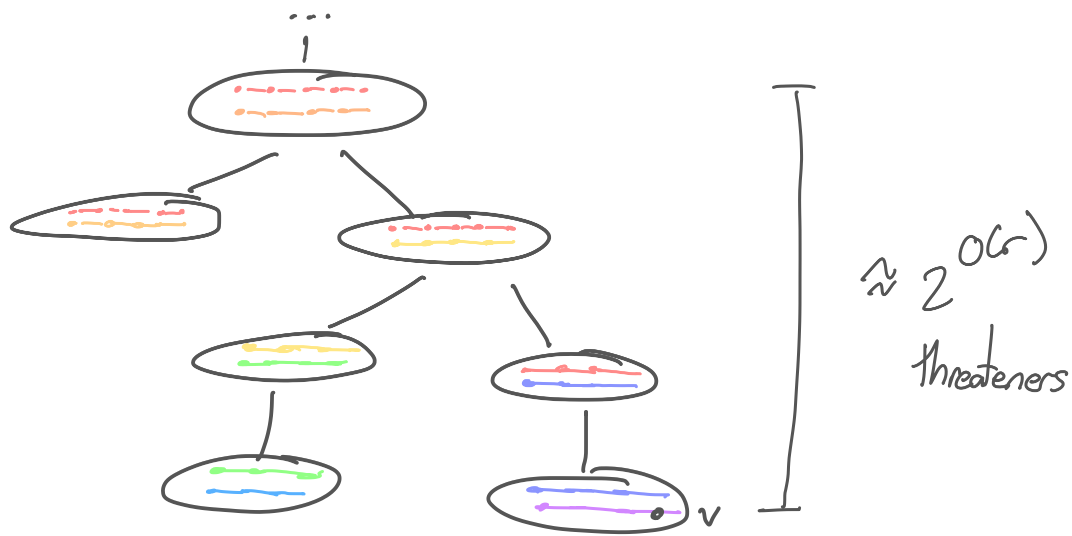
[Fil 25] ↳ threatening skeletons $\Rightarrow O(r)$ padding

3. $O(\log r)$ padding

divide & conquer on cop decomposition

Current Status



Can we improve the buffered cop decomposition so every vertex has poly(r) threatening skeletons?

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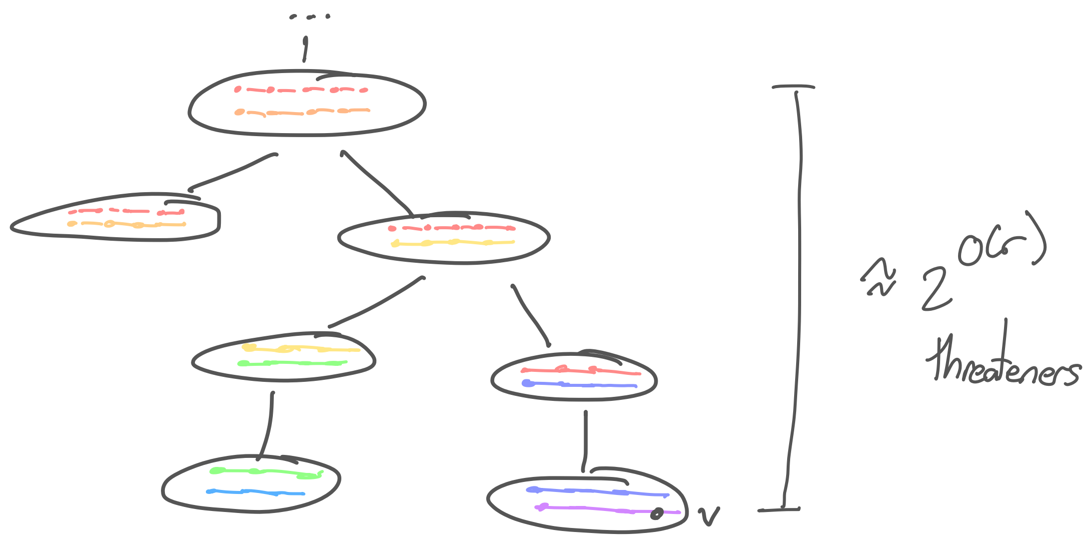

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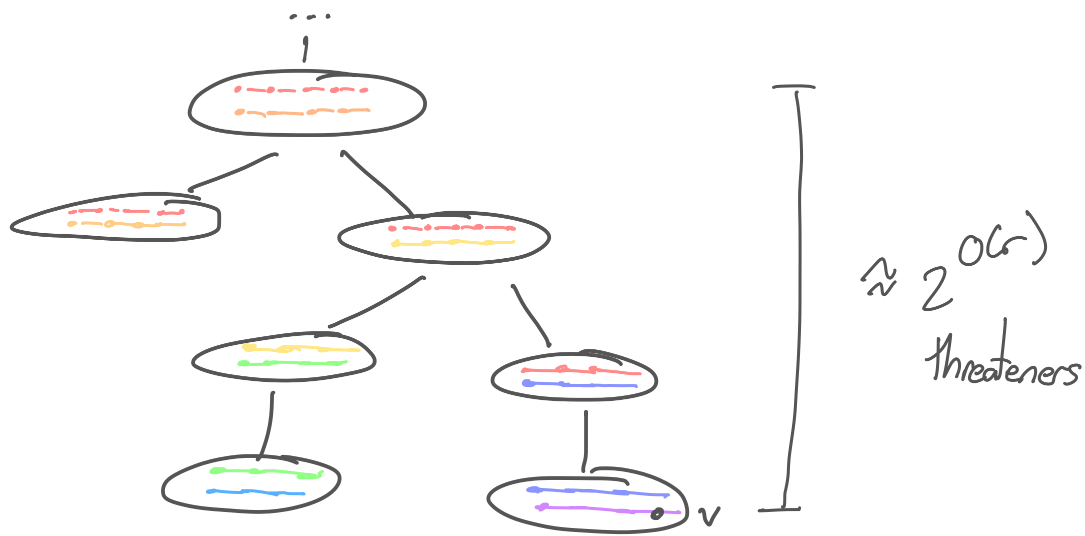

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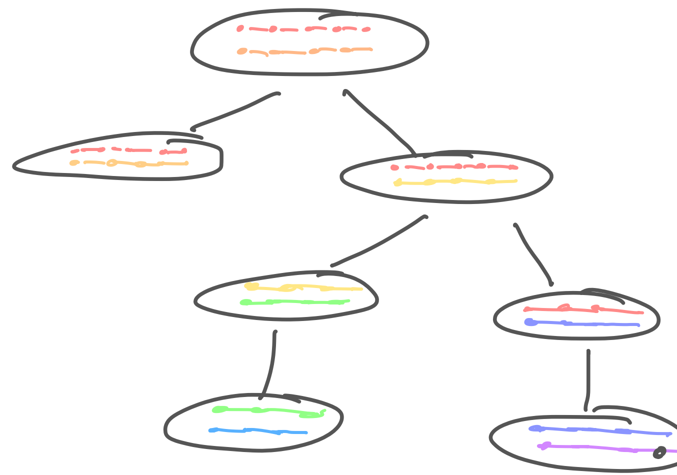
Our idea: Divide & conquer on cop decomposition!

Divide & Conquer: $O(\log r + \log\log n)$ padding

First (incorrect) idea:

1. Pick centroid bag X
2. Randomly grow balls around all paths in X
3. Recurse



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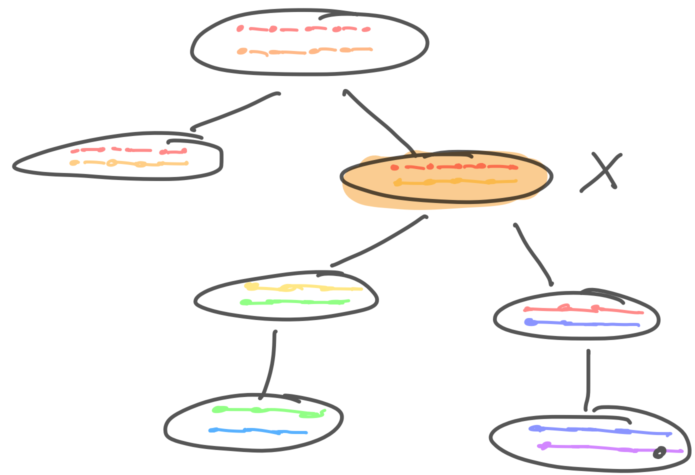

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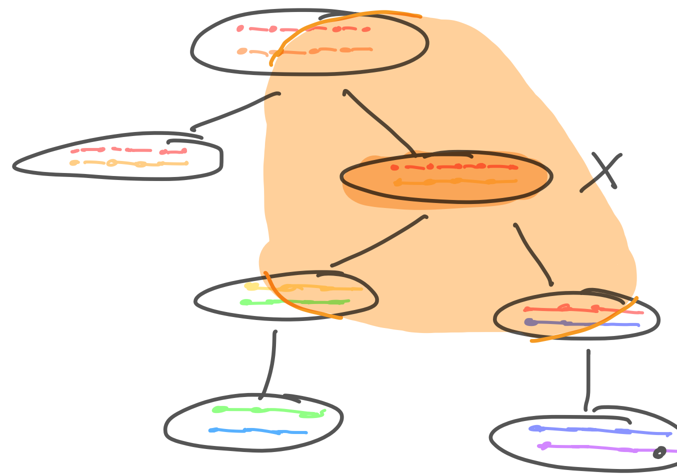

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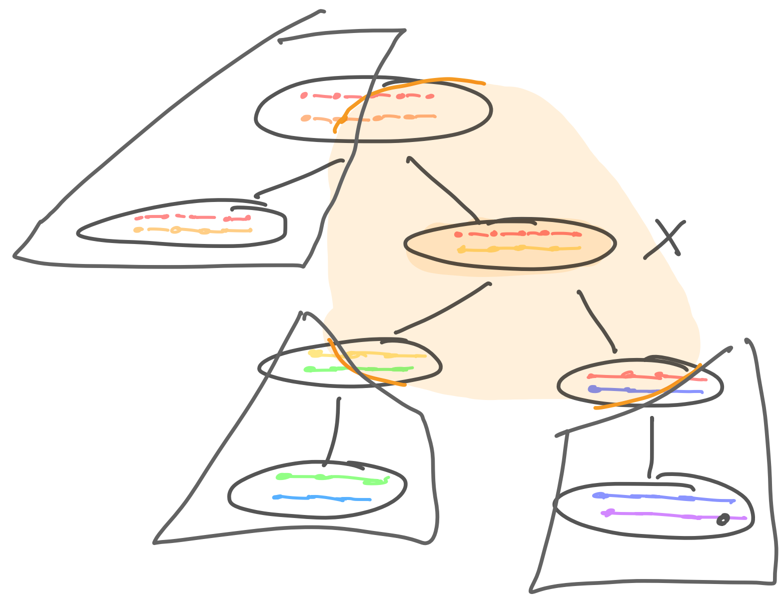

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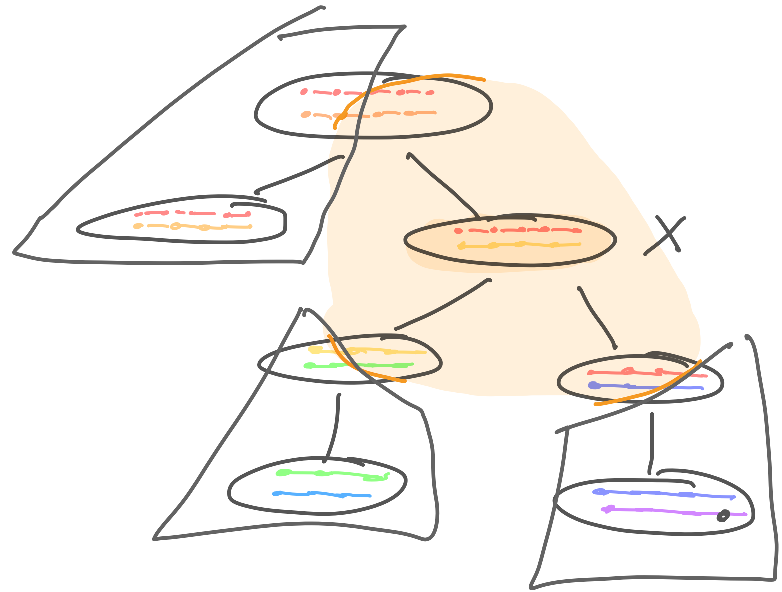
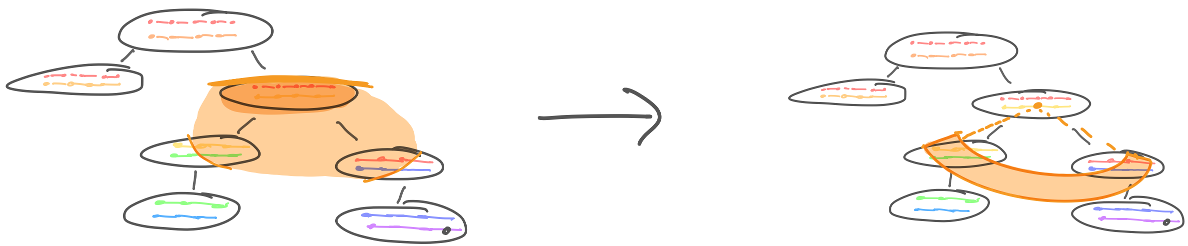


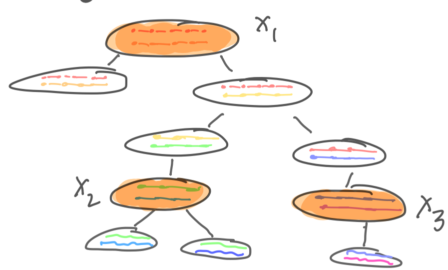
$O(\text{poly}(r) \cdot \log n)$ threatening events

Technical Challenges

1) In cop decomposition, we can only grow balls "downward"



2) Remove many bags at once to avoid $O(\log \log n)$



Conclusion

- We show K_r-minor-free graphs have padding $\beta = O(\log r)$

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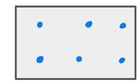

"far apart" points $\Rightarrow$ good padding

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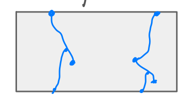

"far apart shortest-path separators"

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<u>divide & conquer</u> on cop decomposition (NEW)

Open directions

- Our guarantees are for <u>weak</u> diameter.

  Can we get <u>strong</u> diameter padded decomposition / sparse cover?

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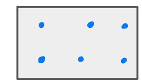

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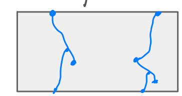

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