Density of the solution of the Skorokhod stochastic reflection problem driven by a fractional Brownian motion with Hurst parameter $h > \frac{1}{2}$

HARESS El Mehdi

05/2019

Contents

1 Introduction

Let X_t be the solution of a stochastic differential equation with diffusion coefficient σ and drift b of the form

$$\forall t \in [0,T],\ X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds,$$

where B is a fractional Brownian motion with Hurst parameter h, and the integral is in the sense of Stratonovich.

Recall that the Stratonovich integral of a process $\{u_t, t \in [0,T]\}$ is a deterministic integral defined when the quantity

$$(2\epsilon)^{-1} \int_0^T u_s(B_{\min((s+\epsilon),T)} - B_{\max((s-\epsilon),0)})ds,$$

converges in probability as $\epsilon \to 0$.

These equations can describe many phenomena such as physical systems or thermal fluctuations. Usually when studying a phenomenon, in addition to having the stochastic equation that describes it, we work in a bounded domain and we want our solution not to exceed some boundary $\{L_t, t \in [0,T]\}$. Theoretically, we want to reflect X_t each time each time it touches L_t. Otherwise, we don't want to change X_t. In probability theory and related fields, this problem is known as the Skorokhod problem and has been studied a lot in the past years

- For example, we can prove the existence of a solution when B is a Brownian motion. In fact, in the case of a locally bounded drift b with linear growth and a positive diffusion coefficient ($\sigma(x) > \delta > 0$), T. Zhang proved the existence and uniqueness of a strong solution to the stochastic differential equations with reflecting boundary [8].

- Furthermore, we can prove some properties of the solution when B is a Brownian motion. In [4], T. Nilssen and T. Zhang consider a one-dimensional stochastic differential equation with reflection. They prove the Malliavin differentiability of the solution considering a bounded and measurable drift b and a diffusion coefficient $\sigma \in C_b^1(\mathbf{R})$ bounded away from 0.

Note : The Malliavin calculus is a set of mathematical techniques that extend the field of calculus of variations from deterministic functions to stochastic processes.

- Moreover, we can still prove the existence of a solution even when the equation is not driven by a Brownian motion and the integral is not in the sense of Stratonovich. In [6], A. Richard, E. Tanré and S. Torres study one dimensional differential equations driven by a rough noise (and thus, a rough integral) with Hölder continuity $\beta \in (\frac{1}{3}, 1)$, which are reflected on some stochastic boundary process $(L_t)_{t\in[0,T]}$. They restrict their study to $\mathbf{R}$ valued processes in order to give a concise and simple proof by penalization. They consider a drift $b = 0$ and a diffusion coefficient $\sigma \in C_b^4(\mathbf{R}, (\mathbf{R^d})')$.

- Finally, we can prove even stronger properties of the solution when considering a similar problem. In fact, when studying an elliptic stochastic partial differential equation on some domain D with reflection of the type
$$\forall x \in D,\ -\Delta u(x) + f(u(x)) = \dot{W}(x) + \eta$$
with Dirichlet-type boundary conditions, f a continuous non-decreasing function and $\{\dot{W}(x), x \in D\}$a white noise on D. S. Tindel proves in [7] the absolute continuity of the law of the solution.

In this paper, we are interested in the Skorokhod problem for one dimensional differential equations (σ and b are real functions) driven by the fractional Brownian motion B with Hurst parameter h, which are reflected on the boundary $L_t = 0\ \forall t \in [0, T]$.

We will use the Stratonovich integral and consider $\mathbf{R}$ valued processes in order to give a proof via penalization and obtain the density of the solution via Malliavin Calculus.

Finally, we suppose that $b = 0$ and σ is a bounded lipschitz function in $C_b^2(\mathbf{R}, \mathbf{R})$. Later, when we want to prove the density, we assume that σ is constant $= 1$.

The proofs given in this paper will resemble the proofs given in richard2019penalisation and [7] but will become much simpler in our case because of our strong assumptions.

2 Notations

Let $W = \{W(h), h \in H\}$ denote an isonormal Gaussian process associated with the Hilbert space H. We assume that W is defined on a complete probability space (Ω,F,P). Let S be the set of random variables F, such that F has the form

$$F = f(W(h_1), ..., W(h_n)),$$

where f belongs to the set of infinitely continuously differentiable function from $\mathbf{R^n}$ to $\mathbf{R}$, such that f and all of its derivatives have polynomial growth. Then we can define the derivative operator,

$$DF = \sum_1^n \partial_i f(W(h_1), ..., W(h_n))h_i.$$

We will also call L^p the set $L^p(\Omega, \mathscr{F}, \mathbf{P})$ and $\mathbf{D^{1,p}}$ the closure of S with respect to the norm

$$\|F\| = [E(|F|^p) + E(\|DF\|_H^p)]^{\frac{1}{p}}.$$

Usually and in this paper, H is a set of functions (recall that in the white noise case, $H = L^2(T, B, \mu)$ where μ is a σ-finite atomless measure on a measurable space (T,B)). Hence, we can use the notations,

$$D_t F = DF(t);$$

$$D^h F =< DF, h >_H;$$

$$\|DF\| =< DF, DF >_H .$$

In this paper, we will note D the derivative operator associated with a fractional Brownian motion B ($W = B$ and $D_r B_t = \mathbf{1}_{[0,t]}(r)$) and D^W to the derivative operator associated with a Brownian motion W.

Let T be a positive real number. Let f be a function of one variable, we define $\partial f_{s,t} = f_t - f_s$. For β in (0,1), $C^{\beta-Hol}$ is the space of Hölder continuous functions on $[0, T]$, with the semi norm defined by

$$\|f\|_{\beta,[0,T]} = \sup_{0\leq s\leq t\leq T} \frac{| \partial f_s, t |}{| t - s |^\beta}.$$

We also remind the p variation of f

$$\|f\|_{p-var}^p = \sup_\pi \sum_{i=0}^{n-1} \mid \partial f_{t_i,t_{i+1}} \mid^p,$$

where the supremum is taken over all finite subdivisions $\pi = t_0 < ... < t_n$ of $[0,T]$.

3 Preliminaries

3.1 Malliavin calculus

We will use the following theorems to prove the existence of the density. For the proof, see [5] and [[7], **Theorem 2.1 and 2.2**].

Theorem 3.1. *Let X_t be the solution of*

$$\forall t \in [0,T], X_t = x_0 + B_t + \int_0^t b(X_s)ds,$$

where b is supposed to be globally Lipschitz with linear growth. Then X_t belongs to $\mathbf{D}^{\mathbf{1},\infty}$ and the derivative $D_r X_t$ satisfies the following equation

$$\forall t \in [0,T],\ \forall r \in [0,T],\ D_r X_t = \int_0^t b'(X_s) D_r X_s ds + \mathbf{1}_{[0,t]}(r).$$

Theorem 3.2. *Let $(F_n, n \in \mathbf{N})$ a family of elements in $\mathbf{D}^{\mathbf{1},\mathbf{2}}$ converging to F in L^p for $p > 1$. Suppose that $(DF_n, n \in \mathbf{N})$ is a bounded family in L^p. Then $F \in \mathbf{D}^{\mathbf{1},\mathbf{p}}$, $F_n \in \mathbf{D}^{\mathbf{1},\mathbf{p}}$ for every n, and there exists a sub-sequence of $(F_n, n \in \mathbf{N})$ converging to DF in the weak topology of L^p.*

Theorem 3.3. *Let F be a random variable belonging to the space $\mathbf{D}^{1,2}$. If $\left\|D^W F\right\| > 0$ on Ω_0, then the measure $(\mathbf{1}_{\mathbf{\Omega_0}} P) \circ F^{-1}$ is absolutely continuous with the respect to the Lebesgue measure.*

Remark: $(\mathbf{1}_{\mathbf{\Omega_0}} P) \circ F^{-1}(a) = \mathbf{P}(\{\omega \in \Omega_0, F(\omega) = a\})$.

3.2 The Skorokhod problem

We suppose throughout the paper that σ is a lipschitz function in $C_b^2(\mathbf{R}, \mathbf{R})$ (bounded by M and with a lipschitz constant k).

Definition 3.1. We say that (Y,K) is a solution to the Skorokhod problem with diffusion coefficient σ started from $y_0 \geq 0$ and reflected on $L = 0$ if
(i) (Y,K) satisfies the RDE

$$\forall t \in [0,T], Y_t = y_0 + \int_0^t \sigma(Y_s)dB_s + K_t; \tag{3.1}$$

(ii) $\forall t \in [0,T], Y_t \geq 0$;
(iii) K is non decreasing;
(iv) $\forall t \in [0,T], \int_0^t (Y_s)dK_s = 0$, or equivalently, $\int_0^t \mathbf{1}_{Y_s \neq L_s} dK_s = 0$.

3.3 Penalization method

We define ψ_n for each n such that

$$\forall y \in \mathbf{R}, \psi_n(y) = \begin{cases} 0 & \text{if } y > 0, \\ \text{smooth convex interpolation} & \text{if } -\frac{1}{n} < y < 0, \\ \frac{-1}{2} - ny & \text{if } y \leq -\frac{1}{n}, \end{cases}$$

and $\forall n \in \mathbf{N}, \psi_n \in \mathbf{C}^\infty, \psi_n' \in \mathbf{C_b}^\infty, \psi_n \leq \psi_{n+1}$ and $-\frac{1}{2} - ny \leq \psi_n(y) \leq ny_-$.
We consider the equation

$$\forall t \in [0,T],\ Y_t^n = y_0 + \int_0^t \sigma(Y_s^n)dB_s + \int_0^t \psi_n(Y_s^n)ds. \tag{3.2}$$

We'll see that for each n, there exists a unique solution to (3.2) and that the sequence of pairs $(Y_.^n, \int_0^. \psi_n(Y_s)ds)_{n\in\mathbf{N}}$ converges uniformly in $C[0,T]$ to a solution of the Skorokhod problem (Y,K).

3.4 Penalization estimates

Theorem 3.4. *Let $p \in [1,3)$, and assume that $(g^n)_n \in \mathbf{C}^{p-var}[0,T])$ are continuous functions such that $g_0^n = 0$. Set $C > 0$ and for each n, let f^n be the solution to:*

$$\begin{cases} f_t^n = f_0^n + g_t^n + C\int_0^t \psi_n(f_u^n), \quad \forall t \in [0,T], \\ f_0^n = f_0. \end{cases}$$

Then, $\forall t \in [0,T], \forall n \in \mathbf{N}, \mid f_t^n \mid \leq \sqrt{26}\, \|g_.^n + f_0\|_{\infty,[0,T]}$
Moreover, if there exists a modulus of continuity which is uniform in n for

$(g_n)_n$ *then* $\sup_{t\in[0,T]}(f^n_t)_- \xrightarrow{n->\infty} 0$
Now if $(g^n)_n \in \mathbf{C}^{\beta-Hol}[0,T])$ *and* $f_0 > 0$ *then*

$$\forall t \in [0,T], \forall n \in \mathbf{N}, \psi_n(f^n_t) \leq C_n(C^{-\beta} + C^{1-\beta})n^{1-\beta}$$

where $C_n = A(\|g^n_.\|_{\beta-Hol,[0,T]} + \frac{1}{2}CT^{1-\beta})$ *and* A *is a constant.*

See [[6], **Lemma 3.3**] for the proof.

3.5 Comparison theorem for ODEs

Theorem 3.5. *Let* f *and* g *be real continuous functions such that* $\forall x \in [a,b],\ f'(x) \leq \phi(f(x))$ *and* $f(a) = \alpha$ *where* ϕ *is a Lipschitz function. Suppose that* $\forall x \in [a,b],\ g'(x) = \phi(f(x))$ *and* $g(a) \geq \alpha$. *Then* $\forall x \in [a,b],\ f(x) \leq g(x)$.

Proof. Denote $h = f - g$ and suppose there exist $b > a$ such that $h(b) > 0$. Since h is continuous and $h(a) < 0$, there exists c in $[a,b)$ such that $\forall x \in (c,b],\ h(x) > 0$ and $h(c) = 0$ ($c = \inf\{d \in [a,b],\ \forall x \in [d,b],\ h(x) > 0\}$). Thus, we get that $\forall x \in [c,b]$,

$$0 \leq h'(x) = f'(x) - g'(x) \leq \phi(g(x)) - \phi(g(x)) \leq k \mid f(x) - g(x) \mid = kh(x).$$

From this inequality, we have that $\forall x \in [c,b],\ (h(x)\exp(-Lx))' \leq 0$, hence $h(b) \leq h(c)\exp(L(b-c)) = 0$, which is a contradiction.

4 Penalization

4.1 Doss-Sussman representation

Theorem 4.1. *Consider the equation*

$$Y_t = y + \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds,$$

where $\sigma \in C^2_b(\mathbf{R},\mathbf{R})$ *and* $b \in C^1_b(\mathbf{R},\mathbf{R})$. *Then, the unique solution of the equation is given by*

$$Y_t = \alpha(B_t, Z_t),$$

where Z_t is the solution of

$$Z_t = y_0 + \int_0^t (\frac{\partial \alpha}{\partial y}(B_s, Z_s))^{-1} b(\alpha(B_s, Z_s) ds, \tag{4.1}$$

and $\alpha(x, y)$ is the solution of $\begin{cases} \frac{\partial \alpha}{\partial x}(x, y) = \sigma(\alpha(x, y)), \\ \alpha(0, y) = y \end{cases}$.

For the proof, see [[1],**Proposition** 6]

In (3.2), σ is in $C_b^2(\mathbf{R}, \mathbf{R})$, but ψ_n is not bounded, therefore we can't exactly apply the theorem above. Nevertheless, throughout the paper, we assume that there exists a Doss-Sussman representation for (3.2) for each n in whichever interval the solution exists. We write $Y_t^n = \alpha(B_t, Z_t^n)$.

4.2 Existence of a global solution

In general, it is a difficult task to obtain a global existence in an RDE when the drift is unbounded. We first derive a local existence [See...] and a Doss-Sussman representation on the small interval where the solution exists. Global existence is then achieved by stability.
In fact, in view of [[3], **Lemma 1.**], we know that either Y^n is a global solution on $[0, T]$, or that there is some time θ such that for any $t \in [0, \theta)$, $(Y_s^n)_{s \in [0,t]}$ is a solution to (3.2) and $\lim_{t \to \theta} \mid Y_t^n \mid = \infty$. In the remaining of the article, we will obtain an upper bound on Y_t^n for any t, and since Y^n is continuous, this will be incompatible with a global solution.
That's why, for the rest of the article, we will write T instead of θ or θ_n.

5 Convergence of the penalized processes

Let $\beta = h - \epsilon$ where h is the Hurst parameter of B.

5.1 Growth of the penalized processes

One can notice that

$$\frac{\partial \alpha}{\partial y}(x, y) = \exp(-\int_0^x \sigma'(\alpha(s, y)) ds > 0,$$

Since $\forall t \in [0, T], Z_t^n$ satisfies (4.1), Z_t^{n+1} satisfies

$$Z_t^{n+1} = y_0 + \int_0^t \frac{\partial \alpha}{\partial y}(B_s, Z_s^{n+1})^{-1} \psi_{n+1}(\alpha(B_s, Z_s^{n+1}) ds,$$

Hence, since $\psi_n \leq \psi_{n+1}$ and using the comparison theorem for ODEs, we deduce that $\forall t \in [0,T], Z_t^n \leq Z_t^{n+1}$.
Again, since $\frac{\partial \alpha}{\partial y}(x,y) > 0$, we have that $\forall t \in [0,T], \alpha(B_s, Z_s^n) \leq \alpha(B_s, Z_s^{n+1})$, or $\forall t \in [0,T], Y_t^n \leq Y_t^{n+1}$.

5.2 Boundness of the penalized processes

Since σ' is bounded (by M), we have that

$$\forall s,t \in [0,T], \int_0^{B_t} \sigma'(\alpha(z,y_0))dz - \int_0^{B_s} \sigma'(\alpha(z,y_0))dz = \int_{B_s}^{B_t} \sigma'(\alpha(z,y_0))dz \leq M(B_t - B_s).$$

The fBm B has continuous trajectories, which gives us that both $(t \to \exp(-\int_0^{B_t} \sigma'(\alpha(z,y_0))dz))$ and $(t \to \exp(\int_0^{B_t} \sigma'(\alpha(z,y_0))dz))$ are continuous function on $[0,T]$. For a fixed ω, denote $t_{1,\omega}$ and $t_{2,\omega}$ the points where they reach their maximum respectively. It means that

$$\max(\left\| \frac{\partial \alpha}{\partial y}(B_., y_0) \right\|_{\infty,[0,T]}, \left\| (\frac{\partial \alpha}{\partial y}(B_., y_0))^{-1} \right\|_{\infty,[0,T]}) = \max(\frac{\partial \alpha}{\partial y}(B_{t_{1,\omega}}, y_0), (\frac{\partial \alpha}{\partial y}(B_{t_{2,\omega}}, y_0))^{-1}).$$

Again, since σ' is bounded, $(y \in \mathbf{R} \to \sigma'(\alpha(s,y)))$ is bounded independently on s, therefore, there exist a random constant C such that,

$$\sup_{y_0 \in \mathbf{R}} \max(\left\| \frac{\partial \alpha}{\partial y}(B_., y_0) \right\|_{\infty,[0,T]}, \left\| (\frac{\partial \alpha}{\partial y}(B_., y_0))^{-1} \right\|_{\infty,[0,T]}) \leq C < \infty.$$

Note that this constant depends on B and α, we will call it C_B^α.

Let's first prove that $\forall t \in [0,T]$ $(Z_t^n)_{n \in \mathbf{N}}$ is bounded. Let's define $\tilde{Z}_t^n$ as the solution of the random ODE

$$\tilde{Z}_t^n = y_0 + C_B^\alpha \int_0^t \psi_n(\alpha(B_s, \tilde{Z}_s^n))ds,$$

such that $\tilde{Z}_t^n \geq Z_t^n$ for all $t \in [0,T]$ (using the comparison theorem (**Theorem 3.5**) and the fact that $C_B^\alpha \geq \frac{\partial \alpha}{\partial y}(B_s, Z_s)$ $\forall s \in [0,T]$).

Note that

$$\alpha(B_t, Z_t^n) = \alpha(B_t, y_0) + \int_{y_0}^{Z_t^n} \frac{\partial \alpha}{\partial y}(B_t, s)ds \geq \alpha(B_t, y_0) + (C_B^\alpha)^{-1}(Z_t^n - y_0),$$

And

$$| \alpha(B_t, y_0) - \alpha(B_s, y_0) | \leq \left\| \frac{\partial \alpha(., y_0)}{\partial x} \right\|_{\infty,[0,T]} | B_t - B_s | \leq C_\beta \left\| \frac{\partial \alpha(., y_0)}{\partial x} \right\|_{\infty,[0,T]} | t-s |^\beta,$$

where C_β is a constant that depends on β. Recall that $\frac{\partial \alpha}{\partial x} = \sigma(\alpha)$, and since σ is bounded, there exist a constant $C_{\beta,\sigma}$ such that

$$| \alpha(B_t, y_0) - \alpha(B_s, y_0) | \leq C_{\beta,\sigma} | t - s |^\beta .$$

Since ψ_n is a decreasing function, we get that

$$\psi_n(\alpha(B_t, Z_t^n)) \leq \psi_n(\alpha(B_t, y_0) + (C_B^\alpha)^{-1}(Z_t^n - y_0)) \leq \psi_n(\alpha(B_0, y_0) - C_{\beta,B} t^\beta + (C_B^\alpha)^{-1}(Z_t^n - y_0)).$$

Since $\alpha(B_0, y_0) = y_0 \geq 0$, we have

$$\forall t \in [0,T], \tilde{Z_t^n} \leq y_0 + C_B^\alpha \int_0^t \psi_n(-C_{\beta,B} s^\beta + (C_B^\alpha)^{-1}(\tilde{Z_s^n} - y_0)) ds.$$

Now let's define $\forall t \in [0,T], \overline{Z_t^n}$ as the solution of the equation

$$\overline{Z_t^n} = -C_{\beta,B} t^\beta + \int_0^t \psi_n(\overline{Z_s^n}) ds.$$

Notice that

$$X_t^n := -C_{\beta,B} t^\beta + (C_B^\alpha)^{-1}(\tilde{Z_t^n} - y_0) = -C_{\beta,B} t^\beta + (C_B^\alpha)^{-1} C_B^\alpha \int_0^t \psi_n(\alpha(B_s, \tilde{Z_s^n}) ds$$

$$\leq -C_{\beta,B} t^\beta + \int_0^t \psi_n(-C_{\beta,B} t^\beta + (C_B^\alpha)^{-1}(\tilde{Z_t^n} - y_0)) = -C_{\beta,B} t^\beta + \int_0^t \psi_n(X_t^n).$$

Hence, by the comparison theorem for ODEs (**Theorem 3.5**, we have that $\forall t \in [0,T], \ X_t^n \leq \overline{Z_t^n}$, which means that

$$-C_{\beta,B} t^\beta + (C_B^\alpha)^{-1}(\tilde{Z_t^n} - y_0) \leq \overline{Z_t^n}.$$

Using the penalization estimates (**Theorem 3.4**), $\overline{Z^n}$ satisfies :

$$| \overline{Z_t^n} | \leq \sqrt{26}(C_{\beta,B}t^\beta), \tag{5.1}$$

which leads to the following bound, there exists $A > 0$ which depends on σ, β, T, B such that

$$\forall t \in [0,T], Z_t^n \leq y_0 + A.$$

This ensures the bound of $(Z^n)_{n\in\mathbf{N}}$ and since α is continuous on y, it also ensures the bound of $(Y^n)_{n\in\mathbf{N}}$.
Therefore, we have proved that the penalized processes converge a.s.

6 Uniform convergence of the penalized processes

6.1 Continuity of Z and Y

Let's fix $s \in [0,T)$ and define $Z_{s,t}^n$ as the solution of

$$\forall t \in [s,T], Z_{s,t}^n = Z_s^n + C_B^\alpha \int_s^t \psi_n(\alpha(B_u, \tilde{Z}_u^n)du.$$

Notice that $\tilde{Z}_t^n - Z_s^n \leq Z_{s,t}^n - Z_s$.

Since

$$\forall t \in [0,T],\ Z_{s,t}^n \leq Z_s^n + C_B^\alpha \int_s^t \psi_n(Y_s^n - C_{\beta,B}(u-s)^\beta + (C_B^\alpha)^{-1}(\tilde{Z}_u^n - \tilde{Z}_s^n))du,$$

we have

$$\tilde{Z}_t^n \leq \tilde{Z}_s^n + C_B^\alpha \int_s^t \psi_n(-(Y_s^n)_- - C_{\beta,B}(u-s)^\beta + (C_B^\alpha)^{-1}(\tilde{Z}_u^n - \tilde{Z}_s^n)du.$$

Using the same logic we used to get (5.1), we define as the solution of

$$\dot{Z_{s,t}^n} = -(Y_s^n)_- - C_{\beta,B}(t-s)^\beta + \int_s^t \psi_n(\dot{Z_{s,u}^n} - (Y_s^n)_-)du.$$

Then, we define $\dot{X^n_{s,t}} = -(Y^n_s)_- - C_{\beta,B}(t-s)^\beta + C^\alpha_B(\tilde{Z}_t - \tilde{Z}_s)$, such that

$$\dot{X^n_{s,t}} \leq -(Y^n_s)_- - C_{\beta,B}(t-s)^\beta + \int_s^t \psi_n(-(Y^n_s)_- - C(u-s)^\beta + C^\alpha_B(\tilde{Z}_u - \tilde{Z}_s)du$$

$$= -(Y^n_s)_- - C_{\beta,B}(t-s)^\beta + \int_s^t \psi_n(\dot{X^n_{s,u}})$$

$$\leq -(Y^n_s)_- - C_{\beta,B}(t-s)^\beta + \int_s^t \psi_n(\dot{X^n_{s,u}} - (Y^n_s)_-)du.$$

Therefore, by the comparison theorem, we get that

$$\forall t \in [s,T], -(Y^n_s)_- - C_{\beta,B}(t-s)^\beta + C^\alpha_B(\tilde{Z}_t - \tilde{Z}_s) \leq \dot{Z^n_{s,t}}.$$

Using the penalization estimates (**Theorem 3.4**), we get that

$$\dot{Z^n_{s,t}} - (Y^n_s)_- \leq \sqrt{26} C_{\beta,B}(t-s)^\beta.$$

Therefore there exists two constants $K_{\sigma,\beta}$ and $L_{\sigma,\beta}$ depending on σ and β such that

$$\tilde{Z}^n_t - \tilde{Z}^n_s \leq K_{\sigma,\beta}(t-s)^\beta + L_{\sigma,\beta}(Y^n_s)_-.$$

Since $Z^n_t - \tilde{Z}^n_s \geq Z^n_t - Z^n_s$, we get that

$$Z^n_t - Z^n_s \leq K_{\sigma,\beta}(t-s)^\beta + L_{\sigma,\beta}(Y^n_s)_-. \tag{6.1}$$

$$\begin{aligned}
&\mid Y^n_t - Y^n_s \mid = \mid \alpha(B_t, Z^n_t) - \alpha(B_s, Z^n_s) \mid \\
&= \mid \alpha(B_t, Z^n_t) - \alpha(B_s, Z^n_t) + \alpha(B_s, Z^n_t) - \alpha(B_s, Z^n_s) \mid \\
&\leq \mid \alpha(B_t, Z^n_t) - \alpha(B_s, Z^n_t) \mid + \mid \alpha(B_s, Z^n_t) - \alpha(B_s, Z^n_s) \mid
\end{aligned}$$

$$\leq C_\beta \left\| \frac{\partial \alpha(.,y_0)}{\partial x} \right\|_{\infty,[0,T]} \mid t-s \mid^\beta + C^\alpha_B \mid Z^n_t - Z^n_s \mid . \tag{6.2}$$

Proposition 6.1. $\lim_{n\to\infty} \sup_{s\in[0,T]} (Y_s^n)_- = 0$

Proof. The idea is to use the penalization estimates by finding a modulus of continuity which is uniform in n for $\{\int_0^\cdot \sigma(Y_u^n)du\}_{n\in\mathbf{N}}$. See [4], Proposition 4.7]

Therefore, by taking the limit as n $\to \infty$ in the inequalities (6.1) and (6.2), we get that Z and Y are continuous.

6.2 Uniform convergence

Dini's theorem states that if $\{f_n\}$ is an increasing sequence of continuous real-valued functions on a compact space, which converges point-wise to a continuous function f, then the convergence is uniform. Therefore, we are now able to conclude that the convergences are uniform (by taking $f_n = Y_n$ in Dini's theorem).

7 Identification of the limit process

7.1 Convergence of the stochastic integral

Proposition 7.1. *The following convergence holds a.s. uniformly in* $[0, T]$*,*

$$\int_0^t \sigma(Y_s^n)dB_s \to \int_0^t \sigma(Y_s)dB_s, as\ n \to \infty.$$

Proof. See [[6], Proposition 4.9].

7.2 Points of increase of K

Let's define $K_t^n = \int_0^t \psi_n(Y_s^n)ds, t \in [0, T]$.
$\forall t \in [0, T]$, $K_t^n = Y_t^n - y_0 - \int_0^t \sigma(Y_s^n)dB_s$, therefore K^n also converges uniformly to some process K.
For each n, K_n is non decreasing and so is K, it follows that dK^n weakly

converges towards dK and since Y^n also converges uniformly to Y,

$$0 \geq \int_0^t (Y_s^n)\psi_n(Y_s^n)ds = \int_0^t (Y_s^n)dK_s^n \to \int_0^t (Y_s)dK_s.$$

Since $Y_s \geq 0$ and K is non decreasing, it follows that $\forall t \in [0,T]$, $\int_0^t Y_s dK_s = 0$.

Notice that since $\lim_n \sup_{t\in[0,T]}(Y_t^n)_- = 0$, we have that $Y \geq 0$. Therefore, we can affirm that (Y,K) is a solution to the Skorokhod problem.

8 Density of the limit process

In this section, we assume that $\sigma = 1$.

8.1 Y_t belongs to $\mathbf{D}^{1,2}$

Lemma 8.1. *For every $t \in [0,T]$, $Y_t \in \mathbf{D}^{\mathbf{1,2}}$, and there exist a sub-sequence of $(DY_t^n, n \in \mathbf{N})$ converging to DY in L^2.*

Proof. In order to apply **Theorem 3.2**, we have to verify that $Y_t^n \in \mathbf{D}^{\mathbf{1,p}}$ and $(DY_t^n, n \in \mathbf{N})$ is bounded in L^p for some p (In this proof, p is going to be equal to ∞).

Recall that by (Theorem 3.1.) $Y_t^n \in \mathbf{D}^{\mathbf{1,2}}$ and

$$D_r Y_t^n = \int_0^t \psi_n'(Y_s^n) D_r Y_s^n ds + \mathbf{1}_{[0,t]}(r).$$

Let's first show that $D_r Y_t^n \geq 0$.
We have for $s < r$

$$\begin{cases} \frac{dD_rY_s^n}{ds} = \psi_n'(Y_s^n)D_rY_s^n \\ D_rY_0^n = 0. \end{cases}$$

Which gives us : $D_r Y_s^n = 0$ for $s < r$.
For $t \geq r$, we have

$$\begin{cases} \frac{dD_rY_t^n}{dt} = \psi_n'(Y_t^n)D_rY_t^n \\ D_rY_r^n = \int_0^r \psi_n'(Y_s^n)D_rY_s^n ds + 1 = 1. \end{cases}$$

Hence,

$$D_r Y_t^n = \begin{cases} \exp\left(\int_r^t \psi_n'(Y_s^n)ds\right) & \text{if } r \leq t \\ 0 & \text{else.} \end{cases}$$

Therefore, $D_r Y_t^n \geq 0$. Now let's show that $D_r Y_t^n \leq 1$.
Since $\psi_n' \leq 0$, we have that $\exp\left(\int_0^t \psi_n'(Y_s^n)ds\right) \leq 1$ and therefore $D_r Y_t^n \leq 1$.
Hence, DY_t^n belongs to L^∞, which proves that $Y_t \in \mathbf{D}^{\mathbf{1},\infty}$.

8.2 Density of Y_t

Theorem 8.2. *The law of Y_t on $\mathbf{R}_+^*$ is absolutely continuous with respect to the Lebesgue measure.*

Proof. Let's fix $t > 0$ and let $a > 0$, $k \geq 1$ and $j \geq 1$. We define $\Omega_a = \{\omega, Y_t(w) \geq a\}$, $\Omega_{a,k} = \{\omega \in \Omega_a, Y_t^n(\omega) \geq 2a \ n \geq k\}$ and $\Omega_{a,k,j} = \{\omega \in \Omega_{a,k}, Y_s(\omega) \geq a \text{ and } Y_s^k(\omega) \geq a \text{ for every } s \in B_j\}$ where $B_j = \{s, \mid s-t \mid \leq \frac{1}{j}\}$.
One can see that $\Omega = \cup_{a\geq 0}\Omega_a = \cup_{k\geq 1}\Omega_{a,k} = \cup_{k\geq 1}\cup_{j\geq 1}\Omega_{a,k,j}$. Hence, it is sufficient to show that $\|DY_t\| > 0$ on $\Omega_{a,k,j}$ for a fixed a, k and j.
On $\Omega_{a,k,j}$, we have that $Y_s^n \geq a$ for every $k \geq n$ and hence $\psi_n'(Y_s^n) = 0$ for every $s \in B_j$. Therefore for every $s \in B_j$,

$$\forall r \in [0,T], \frac{dD_r Y_s^n}{ds} = 0,$$

(both when $r \leq s$ and $r > s$).

And so on $\Omega_{a,k,j}$, for every $r \in B_j$ and for every $s \in B_j$

$$D_r Y_s^n = 1 \text{ (since } D_r Y_r^n = 1).$$

Thus, when n goes to ∞, we have on $\Omega_{a,k,j}$,

$$\forall r \in B_j, \ D_r Y_t = 1 \text{ (because } t \in B_j).$$

Since $D_r Y_t \geq 0$ (see Lemma 8.1.), we have finally proven that $\|DY_t\| > 0$ on $\Omega_{a,k,j}$.

Now, if B has a Hurst parameter $h > \frac{1}{2}$, we know that

$$D_r^W F = K_h^*(D_. F)(r),$$

where $K_h^*(\phi)(s) = (h-\frac{1}{2})c_h \int_s^T (\frac{\theta}{s})^{h-\frac{1}{2}}(\theta-s)^{h-\frac{3}{2}}\phi(\theta)d\theta$ where $c_h = \left(\frac{2h\Gamma(\frac{3}{2}-h)}{\Gamma(h+\frac{1}{2})\Gamma(2-2h)}\right)^{\frac{1}{2}}$.
Therefore $D_r^W Y_t = (h-\frac{1}{2})c_h \int_r^T (\frac{\theta}{r})^{h-\frac{1}{2}}(\theta-r)^{h-\frac{3}{2}} D_\theta Y_t d\theta$.

Now, on $\Omega_{a,k,j}$ and for $r < t$, we have that

$$D_r^W Y_t = (h - \frac{1}{2}) c_h \int_r^t (\frac{\theta}{r})^{h-\frac{1}{2}} (\theta - r)^{h-\frac{3}{2}} d\theta > 0.$$

This proves that $\left\|D^W Y_t\right\| > 0$. Using **Theorem 3.3**, we conclude that $\forall t \in [0, T]$, the law of Y_t on $\mathbf{R}_+^*$ is absolutely continuous with respect to the Lebesgue measure.

9 Uniqueness of the solution

It seems we can't prove the uniqueness of the solution (Y, K) directly via penalization. However in the case of $H > \frac{1}{2}$ (which we will consider later), the uniqueness comes from [2].

References

[1] E. Alos, J. A. León, D. Nualart, et al. Stochastic stratonovich calculus fbm for fractional brownian motion with hurst parameter less than 1/2. *Taiwanese Journal of Mathematics*, 5(3):609–632, 2001.

[2] A. Falkowski and L. Slominski. Stochastic differential equations with constraints driven by processes with bounded p-variation. *Probab. Math. Statist*, 35:343–365, 2015.

[3] A. Lejay et al. On rough differential equations. *Electronic Journal of Probability*, 14:341–364, 2009.

[4] T. Nilssen and T. Zhang. A note on the malliavin differentiability of one-dimensional reflected stochastic differential equations with discontinuous drift. *arXiv preprint arXiv:1410.0520*, 2014.

[5] D. Nualart. *The Malliavin calculus and related topics*, volume 1995. Springer, 2006.

[6] A. Richard, E. Tanré, and S. Torres. Penalisation techniques for one-dimensional reflected rough differential equations. 2019.

[7] S. Tindel et al. Quasilinear stochastic elliptic equations with reflection: the existence of a density. *Bernoulli*, 4(4):445–459, 1998.

[8] T.-S. Zhang. On the strong solutions of one-dimensional stochastic differential equations with reflecting boundary. *Stochastic processes and their applications*, 50(1):135–147, 1994.