Stochastic differential equations (SDEs) -The Skorokhod problem-

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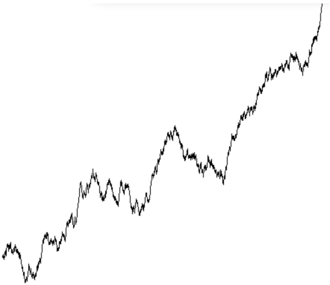
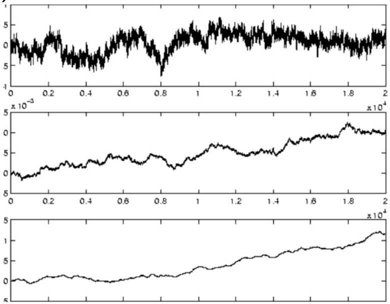
Plan

Brownian motion

$\{W_t, t \in [0, T]\}$: the only centered guassian process with continuous trajectories that has a covariance : $\mathbb{E}(W_t W_s) = min(s, t)$

Trajectory = for a fixed ω, $t \to W_t(\omega)$





Hurst parameter $h \in (0, 1)$

Stochastic differential equations

Stochastic differential equation = Differential equaition + noise.

$$\frac{dx(t)}{dt} = f(x(t), t) + \epsilon$$

SDEs often take the form :

$$\forall t \in [0, T],\ X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s.$$

- $\{B_t, t \in [0, T]\}$ is a fractional brownian motion (fbm)
- σ is the diffusion coefficient
- b is the drift

Stochastic differential equations

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$$\forall t \in [0, T],\ X_t = \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s.$$

- B has continuous trajectories, yet they are not differentiable
- Stochastic integrals : Probabilistic and deterministic integrals

Stochastic differential equations

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- Rough integral : we define the integral for every ω
- Stratonovich integral : define as a limit in probability :

$$\lim_{\epsilon \to 0}(2\epsilon)^{-1} \int_0^T u_s(B_{\min((s+\epsilon),T)} - B_{\max((s-\epsilon),0)})ds.$$

Definition of the Skorokhod problem

A couple (Y, K) is a solution of the Skorokhod problem if

$$\forall t \in [0, T],\ \ Y_t = \int_0^t \sigma(Y_s) dB_s + K_t$$

and

- $\forall t \in [0, T], Y_t \geq 0$
- K is non decreasing
- $\forall t \in [0, T], \int_0^t (Y_s) dK_s = 0$, or equivalently, $\int_0^t \mathbf{1}_{Y_s \neq L_s} dK_s = 0$

Interpretation

Let's consider the SDE

$$\forall t \in [0, T], \; Y_t = \int_0^t \sigma(Y_s) dB_s,$$

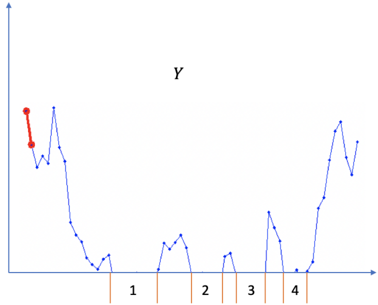
- $b = 0$
- $\sigma \in C_b^2(\mathbf{R})$
- $Y_t(\omega) \in \mathbf{R}$

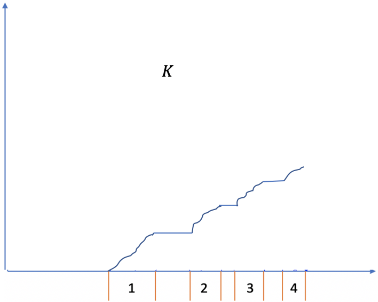
For every ω, we want $t \to Y_t$ to be non-negative

Interpretation

The idea is to introduce a process $\{K_t, t \in [0, T]\}$ such that

$$\forall t \in [0, T], \;\; Y_t = \int_0^t \sigma(Y_s)dB_s + K_t$$





Context

Solution of the Skorokhod problem $= (Y, K)$

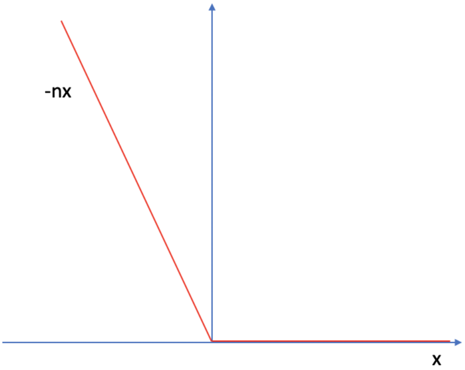
- A solution exists when B is a brownian motion [4]
- That solution has a Malliavin derivative. [1]
- We can still prove that a solution exists when considering rough paths [2]
- We can prove that the solution has a density with stronger restrictions on the parameters [3]

Penalization method

Penalization : Approximate K_t by a sequence,

$$\forall t \in [0, T], \;\; Y_t^n = y_0 + \int_0^t \sigma(Y_s^n) dB_s + \int_0^t \psi_n(Y_s^n) ds.$$

$$\psi_n(x) = (nx)_-$$



Results

- For every t in $[0, T]$, Y_t^n converge a.s to a variable Y_t
- $\int_0^t \psi_n(Y_s^n)ds$ converge a.s to a variable K_t
- (Y, K) is a solution of the Skorokhod problem
- Y_t has a density (Malliavin Calculus) when $\sigma = 1$ and B has a Hurst parameter $h > \frac{1}{2}$

The drift estimate problem

- We will consider the following stochastic differential equation :
$\forall t \in [0, T],\ Y_t = y_0 + \int_0^t b_{\vartheta_0}(Y_s)ds + \sigma B_t$
- The unkwon parameter ϑ_0 lies in a certain set Θ which will be specified. $\{b_\vartheta, \vartheta \in \Theta\}$ is a known family
- Our aim is to get an accurate estimation of ϑ_0 according to some discrete observations of Y

Estimators for the drift estimate problem

- Least squares estimator
- Maximum likelihood estimator
- Strong hypotheses

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