Note Title 01-10-2020

MA 323

Lecture #07

Topic: Generation of geometric Brownian motion and jump-diffusion sample paths

We begin our discussion on methods for simulating paths for a variety of stochastic processes relevant to financial engineering. The emphasis is on methods for exact simulation of continuous time processes at a discrete set of dates.

The word "exact" here implies that the distribution of the simulated values coincides with the distribution of the continuous time process on the simulation time grid.

§ Brownian Motion

One Dimension

By a "standard" one dimensional Brownian motion on $[0, T]$, we mean a "stochastic process" $\{ W(t), 0 \le t \le T \}$

with the following properties:

(i) $W(0) = 0$

(ii) The mapping $t \longmapsto W(t)$ is, with probability 1, a continuous function on $[0, T]$

(iii) The increments $\{W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_k) - W(t_{k-1})\}$ are independent for any k and any $0 \leq t_0 < t_1 < \cdots < t_k \leq T$.

(iv) $W(t) - W(s) \sim N(0, t-s)$, for any $0 \leq s < t \leq T$.

Note: $W(t) \sim N(0, t)$ for $0 < t \leq T$.

For constants μ and $\sigma > 0$, we call a process $X(t)$ a Brownian motion with drift μ and diffusion coefficient σ^2, represented as $X \sim BM(\mu, \sigma^2)$ if

$$\frac{X(t) - \mu t}{\sigma}$$

is a standard Brownian motion.

Accordingly, a Brownian motion may be constructed from a standard Brownian motion by setting

$$X(t) = \mu t + \sigma W(t).$$

It follows that $X(t) \sim N(\mu t, \sigma^2 t)$.

Furthermore, $X(t)$ solves the SDE:

$$dX(t) = \mu\, dt + \sigma\, dW(t)$$

The assumption of $X(0) = 0$ is a normalization.

However, one may construct a Brownian motion with parameters μ and σ^2 with the initial value x by just adding x to each $X(t)$.

For time dependent (but deterministic) $\mu(t)$ and $\sigma(t) > 0$, a Brownian motion, with drift μ and diffusion coefficient $\sigma^2 > 0$, may be defined via the SDE:

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t).$$

Equivalently,

$$X(t) = X(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s)$$

where $X(0)$ is a constant.

The process $X(t)$ has continuous sample paths and independent increments. Each increment $X(t) - X(s)$ is normally distributed with mean

$$E\left[X(t)-X(s)\right] = \int_s^t \mu(u)\,du$$

and variance

$$\mathrm{Var}\left[X(t)-X(s)\right] = \mathrm{Var}\left[\int_s^t \sigma(u)\,dW(u)\right]$$

$$= \int_s^t \sigma^2(u)\,du.$$

Random Walk Construction:

In the simulation of Brownian motion, we focus on the simulation of $(W(t_1), W(t_2), \ldots, W(t_n))$ or $(X(t_1), X(t_2), \ldots, X(t_n))$ at a fixed set of points $0 < t_1 < t_2 < \cdots < t_n$

Let $Z_1, Z_2, \ldots, Z_n$ be independent $N(0,1)$ variables, generated using methods already discussed in the course.

For a standard Brownian motion, we set $t_0 = 0$ and $W(t_0) = 0$, and the subsequent values are generated as follows:

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i}\; Z_{i+1}, \quad i = 0, 1, 2, \ldots, n-1$$

For $X \sim BM(\mu, \sigma^2)$ with constant μ and σ, we set $t_0 = 0$ and $W(t_0) = 0$, and the subsequent values are generated as follows:

$$X(t_{i+1}) = X(t_i) + \mu(t_{i+1} - t_i) + \sigma\sqrt{t_{i+1} - t_i}\; Z_{i+1}$$

$i = 0, 1, 2, \ldots, n-1.$

When the coefficients are time dependent, the recursion becomes:

$$X(t_{i+1}) = X(t_i) + \int_{t_i}^{t_{i+1}} \mu(s)\, ds + \sqrt{\int_{t_i}^{t_{i+1}} \sigma^2(u)\, du}\; Z_{i+1},$$

$i = 0, 1, 2, \ldots, n-1.$

Using the Euler approximation for the integrals, we obtain

$$X(t_{i+1}) = X(t_i) + \mu(t_i)(t_{i+1} - t_i) + \sigma(t_i)\sqrt{t_{i+1} - t_i}\; Z_{i+1}$$

$i = 0, 1, 2, \ldots, n-1.$

∮ Geometric Brownian Motion

One Dimension

A stochastic process $S(t)$ is a geometric Brownian motion if $\log S(t)$ is a Brownian motion with initial value $\log S(0)$.

In other words, a geometric Brownian motion is simply an exponential Brownian motion

Whereas ordinary Brownian motion can take negative values, which is undesirable in a stock price modelling, geometric Brownian motion is always positive because the exponential function only takes positive values.

In the context of asset pricing the percentage changes

$$\frac{S(t_2)-S(t_1)}{S(t_1)}, \quad \frac{S(t_3)-S(t_2)}{S(t_2)}, \; \ldots\ldots\ldots, \; \frac{S(t_n)-S(t_{n-1})}{S(t_{n-1})}$$

are independent for $t_1 < t_2 < t_3 < \cdots < t_n$ (rather than absolute changes $S(t_{i+1}) - S(t_i)$).

A geometric Brownian motion process is specified through an SDE of the form:

$$\frac{dS(t)}{S(t)} = \mu\, dt + \sigma\, dW(t)$$

Using Itô's Lemma, we obtain,

$$d \log S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma\, dW(t)$$

Here μ : Drift parameter and σ : Volatility parameter

Solving, with the initial value of $S(0)$, we obtain

$$S(t) = S(0)\ \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)t + \sigma W(t)\right)$$

More generally, if $u < t$, then

$$S(t) = S(u)\ \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(t-u) + \sigma\left(W(t) - W(u)\right)\right).$$

Since the increments of W are independent and normally distributed, we therefore have the following recursive procedure for simulating values of S at $0 = t_0 < t_1 < t_2 < \cdots < t_n$:

$$S(t_{i+1}) = S(t_i) \exp\left(\left(\mu - \tfrac{1}{2}\sigma^2\right)(t_{i+1} - t_i) + \sigma \sqrt{t_{i+1} - t_i}\, Z_{i+1} \right),$$

$$i = 0, 1, 2, \ldots, n-1$$

With $Z_1, Z_2, \ldots, Z_n$ being independent $N(0,1)$ variables.