Assignment Set 7 Lisa, Jannis, Sascha

Exercise 1

1D Euler equation : $\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$

$$f = (\underbrace{\rho u}_{= u_2}, \underbrace{\rho u^2 + p}_{\frac{u_2^2}{u_1} + p}, \underbrace{(\rho \varepsilon_{tot} + p) u}_{\frac{(u_3 + p) u_2}{u_1}})^T$$

$$\Rightarrow f = \left(u_2, \frac{u_2^2}{u_1} + p, \frac{(u_3 + p) u_2}{u_1}\right)^T$$

$$p(u_1, u_2, u_3) = (\gamma - 1) \rho \varepsilon = (\gamma - 1) \rho \left(\varepsilon_{tot} - \frac{u^2}{2}\right)$$

$$= (\gamma - 1)\left(\rho \varepsilon_{tot} - \frac{\rho u^2}{2}\right) = (\gamma - 1)\left(u_3 - \frac{u_2^2}{2u_1}\right)$$

$$\Rightarrow f = \begin{pmatrix} u_2 \\ \frac{u_2^2}{u_1} + (\gamma - 1)\left(u_3 - \frac{u_2^2}{2u_1}\right) \\ \frac{u_2}{u_1}\left(u_3 + (\gamma - 1)\left(u_3 - \frac{u_2^2}{2u_1}\right)\right) \end{pmatrix}$$

1D euler eq. :

1. $\partial_t \rho + \partial_x (\rho u) = \partial_t \rho + \partial_x (u_2) = 0$

2. $\partial_t (\rho u) + \partial_x (\rho u u) + \partial_x p$

$$= \partial_t (\rho u) + \partial_x \left(\frac{u_2^2}{u_1} + (\gamma - 1)\left(u_3 - \frac{u_2^2}{2u_1}\right)\right) = 0$$

3. $\partial_t(\rho\varepsilon) + \partial_x(\rho\varepsilon u) + p\,\partial_x u$

$$= \partial_t\left(\rho E_{tot} - \rho\frac{u^2}{2}\right) + \partial_x\left(\rho E_{tot} u - \rho\frac{u^3}{2}\right)$$

$$+ (\gamma-1)\left(U_3 - \frac{U_2^2}{2U_1}\right)\partial_x\left(\frac{U_2}{U_1}\right)$$

$$= \partial_t\left(U_3 - \frac{U_2^2}{2U_1}\right) + \partial_x\left(\frac{U_3U_2}{U_1} - \frac{U_2^3}{2U_1^2}\right)$$

$$+ (\gamma-1)\left(U_3 - \frac{U_2^2}{2U_1}\right)\partial_x\left(\frac{U_2}{U_1}\right) \;\dots$$

determine A wit $a_{ij} = \frac{f_i}{\partial u_j}$

$$A = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_2} \\ \frac{\partial f_3}{\partial u_1} & \frac{\partial f_3}{\partial u_2} & \frac{\partial f_3}{\partial u_3} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 & 0 \\ -\frac{U_2^2}{U_1^2} + \frac{(\gamma-1)U_2^2}{2U_1^2} & \frac{2U_2}{U_1} - \frac{(\gamma-1)U_2}{U_1} & (\gamma-1) \\ \frac{(\gamma-1)U_2^3}{2U_1^3} - \frac{U_2\left(U_3 + (\gamma-1)\left(U_3 - \frac{U_2^2}{2U_1}\right)\right)}{U_1^2} & \frac{(\gamma-1)U_2^2}{U_1^2} + \frac{U_3 + (\gamma-1)\left(U_3 - \frac{U_2^2}{2U_1}\right)}{U_1} & \frac{\gamma U_2}{U_1} \end{pmatrix}$$

Exercise 2

a)

1D euler eq.:

1. $\partial_t(\rho) + \partial_x(\rho u) = \partial_t(\rho) + u \cdot \partial_x(\rho) + \rho \cdot \partial_x(u)$

$\Rightarrow C_{11} = u, \quad C_{12} = \rho, \quad C_{13} = 0$

2. $\partial_t(\rho u) + \partial_x(\rho u u) + \partial_x(p) = u \cdot \partial_t \rho + \rho \cdot \partial_t u$

$+ u \cdot u \, \partial_x \rho + \rho u \, \partial_x u + \rho u \, \partial_x u + \partial_x p$

$= u \cdot \partial_t \rho + \rho \cdot \partial_t u + u^2 \partial_x \rho + 2 \rho u \, \partial_x u + \partial_x p$

$= - u \cdot \partial_x(\rho u) + \rho \cdot \partial_t u + u^2 \partial_x \rho + 2 \rho u \partial_x u + \partial_x p$

$= \cancel{- u^2 \partial_x \rho} - \cancel{\rho u} \partial_x u + \rho \cdot \partial_t u + \cancel{u^2 \partial_x \rho} + \cancel{2} \rho u \, \partial_x u + \partial_x p$

$= \partial_t u + u \partial_x u + \frac{1}{\rho} \partial_x p$

$\Rightarrow C_{21} = 0, \quad C_{22} = u, \quad C_{23} = \frac{1}{\rho}$

3. $\partial_t(\rho \varepsilon) + \partial_x(\rho \varepsilon u) + p \, \partial_x u \quad \Big| \; \varepsilon = \frac{p}{(\gamma - 1)\rho}$

$= \frac{1}{(\gamma-1)} \partial_t p + \frac{1}{(\gamma - 1)} \partial_x(p u) + p \, \partial_x u$

$= \partial_t p + u \partial_x p + p \partial_x u + (\gamma - 1) p \partial_x u$

$= \partial_t p + u \partial_x p \; \cancel{+ p \partial_x u} + \gamma p \partial_x u \; \cancel{- p \partial_x u}$

$= \partial_t p + u \partial_x p + \gamma p \, \partial_x u \Rightarrow C_{31} = 0, \; C_{32} = \gamma p,$

$C_{33} = u$

$$\Rightarrow \quad C = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \gamma P & u \end{pmatrix}$$

b)

is C quadratic? $\rightarrow$ yes it's a 3×3 Matrix ✓

has the characteristic polynomial 3 roots?

$$\chi_c = \det(C - \lambda \underline{I}) = \begin{vmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & \frac{1}{\rho} \\ 0 & \gamma P & u-\lambda \end{vmatrix}$$

$$= (u-\lambda)^3 - \frac{\gamma P}{\rho}(u-\lambda) = -\lambda^3 + 3u\lambda^2 - 3u^2\lambda + u^3 - \frac{\gamma P}{\rho}u + \frac{\gamma P}{\rho}\lambda$$

$$= -\lambda^3 + 3u\lambda^2 + \left(\frac{\gamma P}{\rho} - 3u^2\right)\lambda + u\left(u^2 - \frac{\gamma P}{\rho}\right)$$

test $\lambda = u$

$$\Leftrightarrow \cancel{-u^3} + \cancel{3}u^3 + \cancel{\frac{\gamma P}{\rho}}u - \cancel{3}u^3 + \cancel{u^3} - \cancel{\frac{\gamma P}{\rho}}u = 0 \quad ✓$$

rest:

$$-\lambda^3 + 3u\lambda^2 + \left(\frac{\gamma P}{\rho} - 3u^2\right)\lambda + u\left(u^2 - \frac{\gamma P}{\rho}\right) : (\lambda - u) = -\lambda^2 + 2u\lambda + \left(\frac{\gamma P}{\rho} - u^2\right)$$

$$-\ (-\lambda^3 + u\lambda^2)$$

$$2u\lambda^2 + \left(\frac{\gamma P}{\rho} - 3u^2\right)\lambda$$

$$-\ (2u\lambda^2 - 2u^2\lambda)$$

$$\left(\frac{\gamma P}{\rho} - u^2\right)\lambda + u\left(u^2 - \frac{\gamma P}{\rho}\right)$$

$$-\ \left(\left(\frac{\gamma P}{\rho} - u^2\right) + u\left(u^2 - \frac{\gamma P}{\rho}\right)\right)$$

$$0$$

$$\Rightarrow \quad \lambda^2 - 2u\lambda - \left(\frac{\gamma P}{\rho} - u^2\right)$$

$$\Rightarrow \lambda_{2/3} = u \pm \sqrt{u^2 + \frac{\gamma P}{\rho} - u^2} = u \pm \sqrt{\frac{\gamma P}{\rho}}$$

$$\Rightarrow \lambda_1 = u, \quad \lambda_2 = u + \sqrt{\frac{\gamma P}{\rho}}, \quad \lambda_3 = u - \sqrt{\frac{\gamma P}{\rho}}$$

$\Rightarrow$ 3 roots ✓

$$\Rightarrow \chi_C(\lambda) = -(\lambda - u)\left(\lambda - \left[u + \sqrt{\frac{\gamma P}{\rho}}\right]\right)\left(\lambda - \left[u - \sqrt{\frac{\gamma P}{\rho}}\right]\right)$$

3. are the geom. and alge. Multiplicity the same?

alg. multiplicity for all 3 λ is $\underline{1}$

Calculate eigenspaces

$$E_i = \text{Kernel}(C - \lambda_i I)$$

$$\Rightarrow \quad E_1 = \text{Kernel}(C - \lambda_1 I) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$E_2 = \text{Kernel}(C - \lambda_2 I) = \left\langle \left(\frac{\rho}{\gamma P}, \frac{1}{\sqrt{\rho \gamma P}}, 1 \right)^T \right\rangle$$

$$E_3 = \text{Kernel}(C - \lambda_2 I) = \left\langle \left(\frac{\rho}{\gamma P}, -\frac{1}{\sqrt{\rho \gamma P}}, 1 \right) \right\rangle$$

$\Rightarrow$ eigenvectors

$\dim(E_1) = \dim(E_2) = \dim(E_3) \Rightarrow$ geom. multipl.

$=$ alg. multipl. ✓

$\Rightarrow$ C can be diagonalized

$$C = T D T^{-1} \quad \text{with} \quad T = \begin{pmatrix} 1 & \frac{\rho}{\gamma p} & \frac{\rho}{\gamma p} \\ 0 & \frac{1}{\sqrt{\rho \gamma p}} & -\frac{1}{\sqrt{\rho \gamma p}} \\ 0 & 1 & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} u & 0 & 0 \\ 0 & u + \sqrt{\frac{\gamma p}{\rho}} & 0 \\ 0 & 0 & u - \sqrt{\frac{\gamma p}{\rho}} \end{pmatrix}$$