Error of Simpson's rule:

$$E_2[f] = I_2 - Q_2$$

$$Q_2[f] = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \quad (I)$$

$$x_0 = a; \qquad x_1 = \frac{a+b}{2} = a+h; \qquad x_2 = b = a+2h$$

$$I[f] = \int_a^{a+2h} f(x)\,dx \overset{\text{Taylor}}{=} 2h f(a) + 2h^2 f'(a) + \frac{4h^3}{3} f''(a) = \frac{2h^4}{3} f'''(a) + \frac{4h^5}{15} f^{(4)}(a) + \dots \qquad (II)$$

Taylor expansion of f:

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \frac{h^3}{6} f'''(a) + \frac{h^4}{24} f^{(4)}(a) + \dots$$

$$f(a+2h) = f(a) + 2h f'(a) + \underset{2}{\cancel{\tfrac{4}{2}}} h^2 f''(a) + \underset{4/3}{\cancel{\tfrac{8}{6}}} h^3 f'''(a) + \underset{2/3}{\cancel{\tfrac{16}{24}}} h^4 f^{(4)}(a) + \dots$$

Insert into (I)

$$\Longrightarrow Q_2[f] = \frac{h}{3}\left[f(a) + 4f(a+h) + f(a+2h)\right]$$

$$= \frac{h}{3}\left\{f(a) + 4\left[f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \frac{h^3}{6}f'''(a) + \frac{h^4}{24}f^{(4)}(a) + \dots\right] + \left[f(a) + 2hf'(a) + 2h^2 f''(a) + \frac{4}{3}h^3 f'''(a) + \frac{2}{3}h^4 f^{(4)}(a) + \dots\right]\right\}$$

$$\overset{(II)}{=} \int_a^{a+2h} f(x)\,dx - \underbrace{\frac{h^5}{90} f^{(4)}(a)}$$

All terms except for this one cancel each other out.
This one can be found as:

$$\frac{h}{3}\left(4\frac{h^4}{24}f^{(4)}(a) + \frac{2}{3}h^4 f^{(4)}(a)\right) - \underbrace{\frac{4h^5}{15}f^{(4)}(a)}_{\text{from (II)}} = \frac{7}{90}h^5 f^{(4)}(a)$$

$$\Rightarrow E_2[f] = I[f] - Q_2[f] = \frac{h^5}{90} f^{(4)}(a) = \frac{f^{(4)}(a)}{90}\left(\frac{b-a}{2}\right)^5$$