Excercise 1

$$\frac{1}{2}\dot{\vec{r}}^{\,2} - \frac{\mu}{r} = \text{Const.}$$

energy is conserved, therefore the energy at the periapsis and apoapsis are the same

$$\frac{1}{2}\dot{\vec{r}}_p^{\,2} - \frac{\mu}{r_p} = \frac{1}{2}\dot{\vec{r}}_a^{\,2} - \frac{\mu}{r_a}$$

$$\Leftrightarrow \frac{1}{2}\left(\dot{\vec{r}}_p^{\,2} - \dot{\vec{r}}_a^{\,2}\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

angular moment $\vec{h} = \vec{r} \times \vec{v} = \vec{r}_a \times \dot{\vec{r}}_a = \vec{r}_p \times \dot{\vec{r}}_p = \text{const.}$

let's look at absolute values: $h = r_a v_a = r_p v_p$

$$\Rightarrow v_p = \frac{r_a v_a}{r_p}$$

$$\Rightarrow \frac{1}{2}\left(v_p^2 - v_a^2\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$\Leftrightarrow \frac{v_a^2}{2}\left(\frac{r_a^2 - r_p^2}{r_p^2}\right) = \mu\left(\frac{1}{r_p} - \frac{1}{r_a}\right)$$

$$\Leftrightarrow \frac{v_a^2}{2} = \mu\left(\frac{r_p - r_a}{r_p r_a}\right)\cdot\left(\frac{r_p^2}{r_p^2 - r_a^2}\right) = \mu\,\frac{r_p^{\cancel{2}}\,\cancel{(r_p - r_a)}}{r_a \cancel{r_p}\,\cancel{(r_p - r_a)}(r_p + r_a)}$$

$$= \mu\,\frac{r_p}{r_a(r_p + r_a)}$$

With $2a = r_a + r_p$

$$\Rightarrow \frac{v_a^2}{2} = \mu\,\frac{2a - r_a}{2a\, r_a} \Leftrightarrow \frac{v_a^2}{2} - \frac{\mu}{r_a} = \boxed{-\frac{\mu}{2a}}$$

const. we are looking for

$$\Rightarrow \frac{1}{2}\dot{\vec{r}}^{\,2} - \frac{\mu}{r} = -\frac{\mu}{2a} \Leftrightarrow \dot{\vec{r}}^{\,2} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

Exercise 2

$$\left.\begin{aligned} & G \equiv 1 \\ & m_1 + m_2 = M \equiv 1 \\ & r = r_1 - r_2 \equiv 1 \\ & \Rightarrow a = 1 \end{aligned}\right\} \quad \begin{aligned} & T^2 = \frac{4\pi^2}{\mu} a^3 \\ & \mu = G(m_1 + m_2) \equiv 1 \end{aligned}$$

$$\Rightarrow T^2 = 4\pi^2 a^3 \Leftrightarrow T = 2\pi a^{2/3} = \underline{\underline{2\pi}}$$