UNIT I - ONE DIMENSIONAL RANDOM VARIABLES
Problem set

Random Variables, Probability functions

Discrete random variables

1. Determine the value of c so that the following function can serve as a probability mass function of a discrete random variable X: $p(x) = c(x^2 + 4),\ x = 0, 1, 2$

2. The probability distribution of X, the no. of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
$p(x)$	0.41	0.37	0.16	0.05	0.01

   Construct the cumulative distribution of X.

3. If the distribution function of X is given by

$$F(b) = \begin{cases} 0,\ b < 0 \\ 1/2,\ 0 \le b < 1 \\ 3/5,\ 1 \le b < 2 \\ 4/5,\ 2 \le b < 3 \\ 9/10,\ 3 \le b < 3.5 \\ 1,\quad b \ge 3.5 \end{cases}.$$

   calculate the probability mass function of X.

Continuous random variables

4. If the probability density of a random variable X is given by $f(x)) = \begin{cases} k(1-x^2),\ 0 < x < 4 \\ 0,\ \ otherwise \end{cases}$,

   find a)k b) $P(|X| > 1)$ c) $P(2X + 3 > 5)$

5. A system consisting of one original unit plus a spare can function for a random amount of time X. If the density of$X$ is given (in units of months) by

$$f(x) = \begin{cases} Cxe^{-x/2}, x > 0 \\ 0,\ \ otherwise \end{cases}$$

   what is the probability that the system functions for at least 5 months?

6. The probability density function of a random variable is

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2 - x, & 1 < x \le 2 \\ 0 & otherwise \end{cases}$$

   Compute the cumulative distribution function $F(x)$.

7. A continuous random variable X has the following probability distribution given by

$$f(x) = \begin{cases} a(1+x^2), & 2 \le x \le 5 \\ 0 & otherwise \end{cases}$$

Find $P(X < 4)$.

Moments, moment generating functions and their properties

8. A continuous random variable has a p.d.f $f(x) = kx^2e^{-x},\ x > 0$. Find k, mean and variance.

9. There are three envelopes containing \$100, \$200 and \$6,000 respectively. A player selects an envelope and keeps what is in it. Find the expected winnings E of the player.

10. A player tosses two fair dice. If the sum is 7 or 11, the player wins \$7, otherwise the player loses \$2. Determine the expected value of the game.

11. Obtain the moment generating function of the random variable X having probability density function

$$f(x) = \begin{cases} x, & 0 < x \le 1 \\ 2-x, & 1 < x \le 2 \\ 0 & otherwise \end{cases}$$

12. Suppose the moment generating function of a random variable X is (i) $e^t(5-4e^t)^{-1}$, find $P(X = 3)$ (ii) $\left(\frac{1}{2}\right)^{10}(e^t+1)^{10}$, find $P(X \le 2)$ (iii) $e^{3(e^t-1)}$, find $P(X = 0)$.

Binomial Random Variable

13. A radar system has a probability of 0.1 of detecting a certain target during a single scan. Find the probability that the target will be detected (i) atleast 2 times in four consecutive scan (ii) atleast once in twenty scans.

14. It is known that screws produced by a certain company will be defective with probability 0.01 independently of each other. The company sells the screws in packages of 10 and offers a money back guarantee that atmost 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

15. A communication system consists of n components each of which will function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

16. If the probability of success is 0.09, how many trials are needed to have a probability of atleast one success as 1/2 or more?

17. If the probability is 0.20 that a downtime of an automated production process will exceed 2 minutes, find the probability that 3 out of 8 down times of the process will exceed 2 minutes.

Poisson Random Variable

18. Suppose that the number of typographical errors on a single page of the book has a Poisson distribution with parameter $\lambda = 1/2$. Calculate the probability that there is atleast one error on the given page.

19. If X is a Poisson random variable such that $P(X = 2) = \frac{2}{3}P(X = 1)$, find $P(X = 0)$.

20. If the probability that an individual will suffer an infection is 0.001, determine the probability that out of 2000 individual (i) exactly 3 (ii) more than 2 individual will suffer an infection.

21. Ten percent of the tools produced in a certain manufacturing company turn out to be defective. Find the probability that in a sample of 10 tools chosen at random, exactly 2 will be defective by using (i) binomial distribution (ii) the Poisson approximation to the binomial distribution.

22. In a company, arrival of telephone calls follow a Poisson distribution. The board receives 4 calls per minute. If the board is capable of handling at the most 6 calls per minute, what is the probability that the board will saturate in one minute.

23. At a busy traffic intersection the probability p of an individual car having an accident is very small, say, $p = 0.0001$. However, during a certain peak hours of the day, say between 4.00p.m and 6 p.m., a large number of cars, say 1000 pass through the intersection. Under these conditions, what is the probability of 2 or more accidents occurring during that period.

24. In a certain factory manufacturing blades, there is a small chance of $\frac{1}{500}$ for any blade to be defective. The blades are in packets of 10. Use Poisson distribution to calculate the approximate number of packets containing (i) no defective (ii) one defective (iii) 2 defective blades respectively in a consignment of 1000 packets.

Geometric Random Variable

25. If one copy of the magazine out of 10 copies bears a special prize following geometric distribution, determine the mean and variance.

26. A basketball player makes 90% of her free throws. What is the probability she will miss for the first time on the seventh shot?

27. Suppose that a trainee soldier shoots a target in an independent fashion. If the probability that the target is shot on any one shot is 0.7,

    (a) What is the probability that the target would be hit on tenth attempt?

    (b) What is the probability that it takes him less than 4 shots?

    (c) What is the probability that it takes him an even number of shots?

28. If the probability is 0.05 that a certain kind of measuring devices will show executive drift, what is the probability that the sixth of those measuring devices tested will be the first to show executive drift.

29. Let X be a discrete random variable having geometric distribution with parameter p. Show that for any two positive integer s and t, $P(X > s+t/X > s) = P(X > t)$.

30. In a company, a group of particular item is accepted, if m or more items are tested before the first defective is found. If $m = 5$ and the group consists of 15 percent defective item, what is the probability that it will be accepted?

Uniform Random Variable

31. Find the cumulative distribution function of a uniformly distributed random variable over the range a to b.

32. If X is uniformly distributed over $(-a, a), a > 0$, find a so as to satisfy $P(X \geq 1) = \frac{1}{3}$.

33. Trains arrive at a station at 15 minutes intervals starting at 4 a.m. If a passenger arrive at a station at a time that is uniformly distributed between 9.00 and 9.30, find the probability that he has to wait for the train for (i) less than 6 minutes (ii) more than 10 minutes.

Exponential Random Variable

34. The mileage which car owners get with a certain kind of radial tyre is a random variable having an exponential distribution with mean 40,000km. Find the probability that one of these tyres will last (i) atleast 20,000 km and (ii) atmost 30,000 km.

35. Prove that a continuous random variable X following an exponential distribution satisfies the memoryless property (i.e) $P(X > s + t/X > s) = P(X > t)$.

36. The time required to repair a machine is exponentially distributed with parameter 1/2
    (a) What is the probability that the repair time exceeds 2 hours?
    (b) What is the conditional probability that a repair takes at least 10 hours given that its duration exceeds 9 hours?

37. In a construction site, 3 lorries unload material per hour, on an average. What is the probability that the time between arrival of successive lorries will be (i) atleast 30 minutes (ii) less than 10 minutes assuming it follows exponential distribution.

Gamma Random Variable

38. In a certain city, the daily consumption of electric power in millions of kilowatt-hours can be treated as a random variable following Gamma distribution with parameters $\lambda = 1/2$ and $n = 3$. If the power plant of this city has a daily capacity of 12 million kilowatt-hours, what is the probability that this power supply will be inadequate on any given day.

39. Suppose that an average of 30 customers per hour arrive at a shop in accordance with a Poisson process. That is, if a minute is our unit, then $\lambda = 1/2$. What is the probability that the shopkeeper will wait more than 5 minutes before both of the first two customers arrive.

Normal Random Variable

40. The mean weight of 500 students at a certain college is 151 lb and the standard deviation is 15 lb. Assuming that the weights are normally distributed, find how many students weigh (i)between 120 lb to 155 lb (ii) more than 185 lb.

41. In a class of 50, the average mark of students in a subject is 48 and standard deviation is 24. Find the number of students who get (i) above 50 (ii) between 35 to 50.

42. In a class of students, the heights of the students is normally distributed. Six percent have height below 60 inches and 39 percent are between 60 and 70 inches. Find the mean and standard deviation of height.

43. In a normal distribution, 7 percent of items are below 35 and 11 percent of the items are above 63. Find the mean and standard deviation of the distribution.

44. In an engineering examination, a student is considered to have failed, secured second class, first class and distinction accordingly as he scores less than 45 percent, between 45 percent an 60 percent, between 60 percent and 75 percent, and above 75 percent respectively. In a particular year, 10 percent of the students failed in the examination and 5 percent of the students get distinction. Find the percentage of students who have got first class and second class assuming normal distribution of marks.

45. Find the probability of getting between 3 and 6 heads inclusive in 10 tosses of a fair coin by using (i) the binomial distribution (ii) the normal approximation to the binomial distribution.

46. A fair coin is tossed 500 times. Find the probability that the number of heads will not differ from 250 by (i) more than 10 and (ii) more than 30.

Functions of a Random Variable

47. Let X be a random variable with c.d.f. $F_X(x)$ and p.d.f. $f_X(x)$. Let $Y = ax + b$, where a and b are real constants and $a \neq 0$.
a) Find the cdf of Y in terms of $F_X(x)$
b) Find the pdf of Y in terms of $f_X(x)$.

48. Let $Y = ax + b$. Determine the p.d.f of Y, if
(a) X is a uniform r.v. over $(0, 1)$.
(b) $N(0, 1)$

49. Let X be a random variable with p.d.f. $f_X(x)$. Let $Y = X^2$. Find the pdf of Y.

50. Let $Y = X^2$. Find and sketch the pdf of Y if X is a uniform r.v. over $(-1, 2)$.

51. Let $Y = e^x$. Find the pdf of Y if X s a
(a) uniform r.v. over $(0, 1)$
(b) normal r.v. with parameters μ and σ^2.

52. Let $Y = tanX$. Find the pdf of Y if X is a uniform r.v. over $(-\pi/2, \pi/2)$.