UNIT I - ONE DIMENSIONAL RANDOM VARIABLES

Random Variables, Probability Functions.

Discrete random variables.

1. Soln:

Given:

$X : p(x) = c(x^2+4)$, $x = 0, 1, 2 \ldots$

$\Rightarrow \sum_{x=0}^{n} p(x) = 1$

$\sum_{x=0}^{2} c(x^2+4) = 1$

$c[(0+4)+(1+4)+(4+4)] = 1$

$c[4+5+8] = 1$

$c[17] = 1$

$c = 1/17$.

2. Given:

x	0	1	2	3	4
$P(x)$	0.41	0.37	0.16	0.05	0.01

$P(X \le 0) = P(X=0) = 0.41$

$P(X \le 1) = P(X=0) + P(X=1) = 0.41+0.37 = 0.78$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2) = 0.41+0.37+0.16 = 0.94$

$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3) = 0.99$

$P(X \le 4) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$.

3. Given:

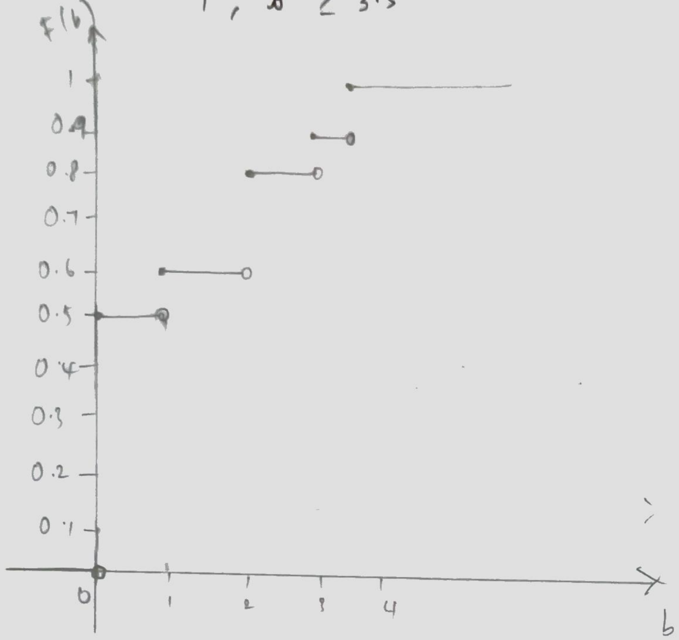
$$F(b) = \begin{cases} 0, & b < 0 \\ 1/2, & 0 \le b < 1 \\ 3/5, & 1 \le b < 2 \\ 4/5, & 2 \le b < 3 \\ 9/10, & 3 \le b < 3.5 \\ 1, & b \ge 3.5 \end{cases}$$

$1/2 = 0.5$

$3/5 = 0.6$

$4/5 = 0.8$

$9/10 = 0.9$



x	0	1	2	3	3.5
$P(x)$	$1/2$	$1/10$	$1/5$	$1/10$	$1/10$

Continuous random variables

4. Given:

$$f(x) = \begin{cases} k(1-x^2), & 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a). k:

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^4 k(1-x^2)\,dx = 1$$

$$k\left[x - \frac{x^3}{3}\right]_0^4 = 1$$

$$k\left[4 - \frac{64}{3} - 0 + 0\right] = 1$$

$$k\left[\frac{12-64}{3}\right] = 1$$

$$k\left[\frac{-52}{3}\right] = 1$$

$$\boxed{k = -3/52}$$

b). $P(|X| > 1)$

$$P(|X| > 1) = P(-1 < X < 1)$$

$$= \int_{-1}^{0} f(x)\,dx + \int_0^1 f(x)\,dx$$

$$= 0 + \int_0^1 \left(-3/52\right)(1-x^2)\,dx$$

$$= \frac{-3}{52}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{-3}{52}\left[1 - \frac{1}{3}\right]$$

$$= \frac{-3}{52}\left[\frac{3-1}{3}\right]$$

$$= -2/52$$

ii). $P(|X| > 1)$.

$$P(X>1) = \int_{1}^{4} f(x)dx + \int_{4}^{\infty} f(x)dx.$$

$$= \int_{1}^{4} (-3/52)(1-x^2)dx$$

$$= -3/52 \left[x - \frac{x^3}{3}\right]_0^4$$

$$= \frac{-3}{52}\left[4 - \frac{64}{3}\right].$$

$$= \frac{-3}{52}\left[\frac{12-64}{3}\right]$$

$$= \frac{-3}{52} \times \frac{-52}{3}$$

$$= 1$$

iii). $P(2X+3>5)$

$$P(2X+3>5) = P(2X > 5-3)$$

$$= P(2X > 2)$$

$$= P(X>1)$$

$$= 1.$$

5) Given :

$$f(x) = \begin{cases} cxe^{-x/2}, & x > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$dv = e^{-x/2}dx$

$u = x \qquad v = \dfrac{e^{-x/2}}{-1/2}$

$u' = 1 \qquad v_1 = \dfrac{e^{-x/2}}{1/4}$

$u'' = 0$

$\int u\,dv = uv - u'v_1 + \ldots$

$$c\int_0^{\infty} x e^{-x/2}\,dx = 1$$

$$c\left[\frac{xe^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4}\right]_0^{\infty} = 1$$

$$c\left[-2xe^{-x/2} - 4e^{-x/2}\right]_0^{\infty} = 1$$

$$c\left[-4e^{-\infty} + 4e^0\right] = 1$$

$$c[0 + 4] = 1$$

$$c = 1/4$$

The probability that the system functions for atleast 5 months.

$$P(X \geq 5) = \frac{1}{4}\int_5^{\infty} xe^{-x/2}\,dx$$

$$= \frac{1}{4}\left[-2xe^{-x/2} - 4e^{-x/2}\right]_5^{\infty}$$

$$= \frac{1}{4}\left[0 - 0 + 10e^{-5/2} - 4e^{-5/2}\right]$$

$$= \frac{1}{4}\left[10e^{-5/2} - 4e^{-5/2}\right]$$

$$= \frac{1}{4}\left[6e^{-5/2}\right]$$

$$= \frac{3}{2}e^{-5/2} = 3/2(0.082) = 0.123.$$

6. Given:

$$f(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2-x, & 1 < x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

for $x < 0$,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = 0.$$

for $0 < x \leq 1$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{x} x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^x = \frac{x^2}{2}.$$

for $1 < x \leq 2$,

$$F(x) = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= 2x - \frac{x^2}{2} - 1$$

for $x > 2$,

$$F(x) = 1.$$

1. Given:

$$f(a) = \begin{cases} a(1+x^2), & 2 \le x \le 5 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$a \int_{2}^{5} (1+x^2)\,dx = 1$$

$$a\left[x + \frac{x^3}{3}\right]_2^5 = 1$$

$$a\left[5 + \frac{125}{3} - 2 - \frac{8}{3}\right] = 1$$

$$a\left[3 + \frac{117}{3}\right] = 1$$

$$a\left[\frac{9+117}{3}\right] = 1$$

$$a\left[\frac{126}{3}\right] = 1$$

$$a = 1/42.$$

$$P(X<4) = \int_2^4 \frac{1}{42}(1+x^2)\,dx$$

$$= \frac{1}{42}\left[x + \frac{x^3}{3}\right]_2^4$$

$$= \frac{1}{42}\left[4 + \frac{64}{3} - 2 - \frac{8}{3}\right]$$

$$= \frac{1}{4}\left[2 + \frac{56}{3}\right]$$

$$= \frac{1}{42}\left[\frac{6+56}{3}\right] = \left(\frac{62}{3}\right)\left(\frac{1}{42}\right)$$

$$= 0.492.$$

Moment, Moment Generating functions and their properties:

8. Given:

$$f(x) = kx^2 e^{-x}, \quad x > 0.$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$k\int_0^{\infty} x^2 e^{-x}\,dx = 1$$

$$k\left[-x^2 e^{-x} - 2x e^{-x} - 2e^{-x}\right]_0^{\infty} = 1$$

$$k[2] = 1$$

$dv = e^{-x}dx$

$u = x^2 \quad v = -e^{-x}$

$u' = 2x \quad v_1 = e^{-x}$

$u'' = 2 \quad v_2 = -e^{-x}$

$u''' = 0 \quad v_3 = e^{-x}$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

$\Rightarrow \boxed{k = 1/2.}$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^{\infty} x \frac{1}{2} x^2 e^{-x}\,dx$$

$$= \frac{1}{2}\int_0^{\infty} x^3 e^{-x}\,dx$$

$$= \frac{1}{2}\left[-x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x}\right]_0^{\infty}$$

$$= \frac{1}{2}(6)$$

$$= 3.$$

$dv = e^{-x}dx$

$u = x^3 \quad v = -e^{-x}.$

$u' = 3x^2 \quad v_1 = e^{-x}$

$u'' = 6x \quad v_2 = -e^{-x}$

$u''' = 6 \quad v_3 = e^{-x}$

$u^{iv} = 0$

$\int u\,dv = uv - u'v_1 + u''v_2 - \ldots$

$\boxed{\text{Mean, } E(X) = 3.}$

$$E(X^2) = \frac{1}{2}\int_0^{\infty} x^4 e^{-x}\,dx.$$

$$= \frac{1}{2}\left[-x^4 e^{-x} - 4x^3 e^{-x} - 12x^2 e^{-x} - 24x e^{-x} - 24e^{-x}\right]_0^{\infty}$$

$dv = e^{-x}dx$

$u = x^4 \quad v = -e^{-x}$

$u' = 4x^3 \quad v_1 = e^{-x}$

$u'' = 12x^2 \quad v_2 = -e^{-x}$

$u''' = 24x \quad v_3 = e^{-x}$

$u^{iv} = 24 \quad v_4 = -e^{-x}$

$u^{v} = 0 \quad v_5 = e^{-x}$

$\int u\,dv = uv - u'v_1 + u''v_2 - \ldots$

$= \frac{1}{2}(24)$

$E(X^2) = 12$

Variance $= E(x^2) - (E(x))^2$

$= 12 - 9$

$= 3.$

9. Given: $S = \{100, 200, 6000\}$.

soln:

$P(X = 100) = 1/3.$

$P(X = 200) = 1/3$

$P(X = 6000) = 1/3.$

$E(X) = \sum x p(x) = 100 \cdot \frac{1}{3} + 200 \cdot \frac{1}{3} + 6000 \cdot \frac{1}{3}$

$= 33.33 + 66.67 + 2000$

$= 2100.$

10. Given:

~~P(X=7 or 11)~~ =

$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}.$

$P(X=7) = 6/36 = 1/6.$

$P(X=11) = 2/36 = 1/18.$

$$P(X = 7 \text{ or } 11) = \frac{6}{36} + \frac{2}{36} = \frac{6+2}{36} = \frac{8}{36} = \frac{2}{9}.$$

The probability for the otherwise is given by $\frac{28}{36}$.

$$\Rightarrow E(X) = \cancel{\frac{2}{9}} \cancel{\times} \frac{1}{6} \times 7 + \frac{1}{18} \times 11 + \frac{28}{36} \times (-2).$$

$$= 1.16 + 0.611 - 1.55$$

$$= 0.221.$$

11. <u>Given:</u>

$$f(x) = \begin{cases} x & , \quad 0 < x \leq 1 \\ 2-x & , \quad 1 < x \leq 2 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx.$$

$$= \int_0^1 e^{tx} \cdot x\, dx + \int_1^2 e^{tx} \cdot (2-x)\, dx.$$

$\Rightarrow$ $u = x$, $dv = e^{tx} dx$, $v = e^{tx}/t$

$u' = 1$, $v_1 = e^{tx}/t^2$

$u'' = 0$

$u = 2-x$, $dv = e^{tx} dx$, $v = e^{tx}/t$

$u' = -1$, $v_1 = e^{tx}/t^2$

$u'' = 0$

$$\int u\, dv = uv - u'v_1 + u''v_2 - \ldots$$

ii). $p(X \geq 1) = 1 - p(X < 1)$.

$= 1 - p(X = 0)$

$= 1 - \left({}^{20}C_0 (0 \cdot 1)^0 (0 \cdot 9)^{20}\right)$

$= 1 - \left(1 \times 1 \times (0 \cdot 9)^{20}\right)$.

$= 1 - 0 \cdot 1242$

$= 0 \cdot 8758$

14. <u>soln:</u>

15.

$$M_x(t) = \left[\frac{x e^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^1 + \left[\frac{(2-x)e^{tx}}{t} + \frac{e^{tx}}{t^2}\right]_1^2$$

$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} - 0 + \frac{e^0}{t^2} + 0 + \frac{e^{2t}}{t^2} - \frac{e^t}{t} - \frac{e^t}{t^2}\right]$$

$$= \frac{e^{2t}}{t^2} - \frac{2e^t}{t^2} + \frac{1}{t^2}$$

$$= \frac{1}{t^2}(e^{2t} - 2e^t + 1)$$

$$= \frac{1}{t^2}(e^t - 1)^2.$$

Binomial Random Variable.

13. soln:

Given:

$p = 0.1$

$\Rightarrow q = 0.9$

$$P(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0, 1, 2 \ldots n.$$

(i). $p(X \geq 2) = P(X=2) + P(X=3) + P(X=4).$

$$= 4C_2 (0.1)^2 (0.9)^2 + 4C_3 (0.1)^3 (0.9) + 4C_4 (0.1)^4 (0.9)^0$$

$$= 6(0.01)(0.81) + 4(0.001)(0.9) + 0.0004$$

$$= 0.0486 + 0.00036 + 0.0004$$

$$= 0.04936.$$

15. soln:

$\cancel{P(X \geq 3)} =$

P(Components of a system function efficiently for a 5 - component system) $= P(x=3) + P(x=4) + P(x=5)$

$= {}^5C_3\, p^3(1-p)^2 + {}^5C_4\, p^4(1-p)^1 + {}^5C_5\, p^5(1-p)^0$

$= 10\left[p^3(1-p)^2\right] + 5\left[p^4(1-p)\right] + p^5$

$= 10\left[p^3(1+p^2-2p)\right] + 5\left[p^4 - p^5\right] + p^5$

$= 10\left[p^3 + p^5 - 2p^4\right] + 5p^4 - 5p^5 + p^5$

$= 10p^3 + 10p^5 - 20p^4 + 5p^4 - 5p^5 + p^5$

$= 6p^5 - 15p^4 + 10p^3$

P(Components of a system function efficiently for a 3 - component system) $= P(x=2) + P(x=3)$

$= {}^3C_2\, p^2(1-p) + {}^3C_3\, p^3(1-p)^0$

$= 3(p^2 - p^3) + p^3$

$= 3p^2 - 3p^3 + p^3$

$= 3p^2 - 2p^3$.

$\Rightarrow \quad p^2(3-2p) = p^3(6p^2 - 15p + 10)$

$3 - 2p = p(6p^2 - 15p + 10)$

$6p^3 - 15p^2 + 10p - 3 + 2p = 0$

$6p^3 - 15p^2 + 12p - 3 = 0$

$3(2p^3 - 5p^2 + 4p - 1) = 0$

$(p-1)(2p^2-3p+1)=0.$

$(p-1)(p-1)(p-\tfrac{1}{2})=0$

$\Rightarrow p=1, \tfrac{1}{2}, 1.$

$$\begin{array}{r|rrrr} 1 & 2 & -5 & 4 & -1 \\ & 0 & 2 & -3 & 1 \\ \hline & 2 & -3 & 1 & \boxed{0} \end{array}$$

$a=2,\ b=-3,\ c=1$

$\dfrac{-b\pm\sqrt{b^2-4ac}}{2a} = \dfrac{3\pm\sqrt{9-8}}{4}$

$= \dfrac{3\pm1}{4} = \tfrac{4}{4}, \tfrac{2}{4}.$

17

16. <u>Given:</u>

$p=0.09 \Rightarrow q=0.91$

$P(X\geq 1) \geq \tfrac{1}{2}$

$P(X=0) \leq \tfrac{1}{2}.$

${}^nC_0(0.09)^1(0.91)^n \leq \tfrac{1}{2}$

$(0.91)^n \leq \tfrac{1}{2}.$

$\log(0.91)^n \leq \log(\tfrac{1}{2})$

$n\log(0.91) \leq \log(\tfrac{1}{2})$

$n \geq \dfrac{\log(\tfrac{1}{2})}{\log(0.91)}$

$n \geq \dfrac{0.3010}{-0.094}$

$n \geq 7.36$

$\boxed{n=8}$

17. Soln:

$p = 0.020$, $q = 0.8$

$n = 8$.

$P(X=3) = {}^{8}C_{3}(0.2)^{3}(0.8)^{5}$

$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1} \times 0.008 \times 0.3276$

$= 56 \times 0.0026208$

$= 0.1467$.

Poisson Random Variable.

18. Given:

$\lambda = 1/2$.

$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$

$P(X \geq 1) = 1 - P(X<1)$

$= \cancel{1 - \frac{e^{-1/2} \cdot (1/2)^{0}}{0!}} = 1 - \frac{e^{-1/2}(1/2)^{0}}{0!}$

$= 1 - \frac{0.6065(1)}{1}$

$= 1 - 0.6065$

$= 0.3935$.

19. Given:

$$P(x=2) = \frac{2}{3} P(x=1)$$

$$\frac{e^{-\lambda}\lambda^2}{2!} = \frac{2}{3}\frac{e^{-\lambda}\lambda}{1!}$$

$$\frac{\lambda}{2!} = \frac{2}{3}$$

$$\lambda = 4/3.$$

$$P(x=0) = \frac{e^{-4/3}(4/3)^0}{0!}$$

$$= e^{-4/3}$$

$$= 0.2644.$$

20). Given:

$$\lambda = 0.001.$$

(i). $$P(x=3) = \frac{e^{-0.001}(0.001)^3}{3!}$$

$$= \frac{e^{-0.001}(0.00000001)}{6}$$

$$= \frac{(0.999)(0.00000001)}{6}$$

$$= 0.000000000165$$

ii). $P(X > 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$$= 1 - \left[\frac{e^{-0.001}(0.001)^0}{0!} + \frac{e^{-0.001}(0.001)}{1!} \right]$$

$= 1 - [0.999 + 0.999(0.001)]$

$= 1 - 0.9999$

$= 0.0001$

21. <u>soln:</u>

$\cancel{\lambda} = \cancel{\frac{10}{100}}$ $\cancel{= 10.1}$

$P = 0.1,$

$q = 0.9$

$n = 10$

(i). $P(X = 2) = 10C_2 (0.1)^2 (0.9)^8$

$= 45 \times 0.01 \times 0.43$

$= 0.1935.$

ii).

$np = \lambda$

$\lambda = 10 \times 0.1$

$\lambda = 1$

23. Soln

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!}$$

$$P(X=2) = \frac{e^{-1}(1)^2}{2!}$$

$$= \frac{e^{-1}}{2}$$

$$= \frac{0.367}{2}$$

$$= 0.1839.$$

22 Soln:

$\lambda = 4$

$$P(X>6) = 1 - P(X \le 6)$$

$$= 1 - [P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)]$$

$$= 1 - \left[\frac{e^{-4}(4)^0}{0!} + \frac{e^{-4}(4)^1}{1!} + \frac{e^{-4}(4)^2}{2!} + \frac{e^{-4}(4)^3}{3!} + \frac{e^{-4}(4)^4}{4!} + \frac{e^{-4}(4)^5}{5!} + \frac{e^{-4}(4)^6}{6!}\right]$$

$$= 1 - \left[e^{-4}\left[1+4+8+\frac{64}{6}+\frac{256}{24}+\frac{1024}{120}+\frac{4096}{720}\right]\right]$$

$$= 1 - [e^{-4}(13+10.67+10.67+8.53+5.69)]$$

$$= 1 - [e^{-4}(48.56)]$$

$$= 1 - [0.183(48.56)]$$

$$= 1 - 0.8732$$

$$= 0.1268$$

23. Soln:

$p = 0.0001$, $n = 1000$.

$\lambda = np = 1000 \times 0.0001$

$\lambda = 0.1$

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [e^{-0.1} + 0.1e^{-0.1}]$

$= 1 - [e^{-0.1}(1+0.1)]$

$= 1 - 1.1(e^{-0.1})$

$= 1 - 1.1(0.904)$

$= 1 - 0.995$

$= 0.005$

24. soln:

$p = \frac{1}{500}$

$n = 10$.

$\lambda = np = \frac{10}{500} = \frac{1}{50}$.

$\lambda = 1/50$

(i). $P(X=0) = \frac{e^{-1/50}(1/50)^0}{0!} = e^{-1/50} = 0.98$.

ii). $P(X=1) = \frac{e^{-1/50}(1/50)^1}{1!} = e^{-1/50}\left(\frac{1}{50}\right) = \frac{0.98}{50} = 0.19$

26.

iii). $P(X=2) = \frac{e^{-1/50}(1/50)^2}{2!} = \frac{e^{-1/50}(0.0004)}{2}$

$= (0.98)(0.0002)$

$= 0.000196.$

(i). No defective packets $= 1000 \times 0.98 = 980$ packets

(ii). One defective packets $= 1000 \times 0.19 = 190$ packets

(iii). two defective packets $= 1000 \times 0.000196 = 0.196$

i.e., Almost no packets with two ~~pa~~ defectives.

Geometric Random Variable:

25. Soln:

$p = 1/10$

$q = 9/10.$

Mean, $E(x) = 1/p = \frac{1}{1/10} = 10.$

$Var(x) = q/p^2 = \frac{9/10}{1/100} = \frac{9}{10} \times 100 = 90.$

26. soln:

$P = 0.9, \quad q = 0.1$

$X = 7.$

X: No. of ~~misses~~ free throws.

$P(X=x) = pq^{x-1}, \quad x = 1, 2, \ldots \Rightarrow$ In general.

$P(X=7) = (0.9)^6 (0.1)^1$

$= (0.531441) \cdot (0.1)$

$= 0.0531441.$

Here since the X is no. of ~~misses~~ free throws,

$P(X=x) = p^{x-1} q.$

$q \rightarrow$ misses

$p \rightarrow$ getting free throws.

27. soln:

$p = 0.7$

$q = 0.3$

i).

$P(X=10) = (0.7)(0.3)^9$

$= (0.7)(0.000019683)$

$= 0.0000137781$

ii). $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= (0.7)(0.3)^0 + (0.7)(0.3)^1 + (0.7)(0.3)^2$

$= 0.7 + (0.7)(0.3) + (0.7)(0.09)$

$= 0.7 + 0.21 + 0.063$

$= 0.973.$

iii). $P(\text{Even no. of shots}) = p(X=2) + P(X=4) + P(X=6) + \ldots$

$$= (0.7)(0.3)^1 + (0.7)(0.3)^3 + (0.7)(0.3)^5 + \ldots$$

$$a = 0.7, \quad r = (0.3)^2$$

$$S = \frac{a}{1-r}$$

$$P(\text{Even no. of shots}) = \frac{0.7}{1-(0.3)^2}$$

$$= \frac{0.7}{1-0.09}$$

$$= \frac{0.7}{0.91} = 0.7692.$$

28. soln:

$$P = 0.05, \quad q = 0.95$$

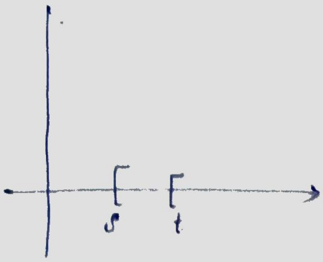
$$P(X=6) = (0.05)(0.95)^5$$

$$= (0.05)(0.7737)$$

$$= 0.0386.$$

29. P.T $P(X>s+t \,/\, X>s) = P(X>t)$

$$P(X>s+t \,/\, X>s) = \frac{P(X>s+t \cap X>s)}{P(X>s)}$$



$$= \frac{P(X>s+t)}{P(X>s)}$$

$X \sim Geo(p)$

$P(x) = pq^{x-1}, \quad x = 1, 2, \dots$

$P(X > k) = P(X \geq k+1)$

$= \sum_{x=k+1}^{\infty} pq^{x-1}$

$= p[q^k + q^{k+1} + q^{k+2} + \dots]$

$= pq^k [1 + q + q^2 + \dots]$

$= pq^k \left(\frac{1}{1-q}\right) = pq^k (1/p)$

$P(X > k) = q^k.$

$\Rightarrow \quad P(X > s+t / X > s) = \frac{q^{s+t}}{q^s} = q^t.$

$\therefore \; P(X > s+t / X > s) = P(X > t).$

30. Soln:

$p = 0.15, \quad q = 0.85$

$P(X = 5) = (0.15)(0.85)^4$

$= (0.15)(0.522)$

$= 0.078.$

Uniform Random Variable.

31. Soln:

$f(x) = \frac{1}{b-a}, \quad a \leq x \leq b$

for $x < a$,

$$f(x) = 0$$

for $a \leq x \leq b$,

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{x} f(x)\,dx$$

$$= 0 + \int_{a}^{x} \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[x\right]_a^x$$

$$F(x) = \frac{x-a}{b-a}$$

for $x > b$,

$$F(x) = 1$$

32. Soln:

$$f(x) = \frac{1}{2a}, \quad -a \leq x \leq a.$$

$$P(X \geq 1) = 1/3$$

$$\int_{1}^{a} \frac{1}{2a}\,dx = 1/3$$

$$\frac{1}{2a}\left[x\right]_1^a = 1/3$$

$$\frac{a-1}{2a} = \frac{1}{3}$$

$$\frac{1}{2} - \frac{1}{2a} = \frac{1}{3}.$$

$$\frac{1}{2a} = \frac{1}{2} - \frac{1}{3}.$$

$$\frac{1}{2a} = \frac{3-2}{6}$$

$$\frac{1}{2a} = \frac{1}{6}$$

$$\Rightarrow \boxed{a = 3.}$$

33. Soln:

$$f(x) = \frac{1}{30}, \quad 0 \le x \le 30.$$

i). P(less than 6 minutes) $= P(9 \le x \le 15) + P(24 \le X \le 30)$

$$= \frac{1}{30}\left[\int_{9}^{15} x\,dx + \int_{24}^{30} x\,dx\right]$$

$$= \frac{1}{30}\left[(15-9) + (30-24)\right]$$

$$= \frac{2}{5} = 0.4.$$

ii). p(more than 10 minutes) $= P(0 \le x \le 5) + p(15 \le x \le 20)$

$$= \frac{1}{30}\int_{0}^{5} dx + \frac{1}{30}\int_{15}^{20} dx$$

$$= \frac{1}{30}(5) + \frac{1}{30}(5)$$

$$= \frac{1}{3} = 0.333.$$

Exponential Random variable:

34. soln:

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

$$\lambda = 4,$$

i). $P(x \geq 2) = \int_2^{\infty} 4 e^{-4x} \, dx$

$$= 4 \left(\frac{e^{-4x}}{-4} \right)_2^{\infty}$$

$$= -[e^{-\infty} - e^{-8}]$$

$$= -(e^{-8})$$

$$= 0.000335.$$

ii). $P(X \leq 3) = \int_0^3 4e^{-4x} \, dx$

$$= 4 \left[\frac{e^{-4x}}{-4} \right]_0^3$$

$$= -(e^{-12} - e^{0}).$$

$$= e^{0} - e^{-12}$$

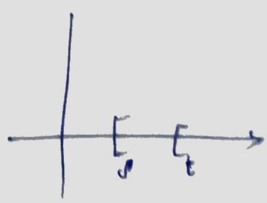
$$= 1 - 0.00000614$$

$$= 0.99999385$$

35. soln:

$$P(X > s+t / X > s) = P(X > t).$$

$$P(X > s+t / X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)}$$



$$= \frac{P(X > s+t)}{P(X > s)}$$

$X \sim exp(\lambda)$

$f(x) = \lambda e^{-\lambda x}, \quad x > 0$

$P(X > k) = \int_k^\infty \lambda e^{-\lambda x} \, dx$

$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_k^\infty$

$= -[0 - e^{-\lambda k}]$

$P(X > k) = e^{-\lambda k}$

$\Rightarrow \quad \frac{P(X > s+t)}{P(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t}$

$\Rightarrow \quad P(X > s+t / X > s) = P(X > t).$

36. Soln:

$\lambda = 1/2, \quad f(x) = \lambda e^{-\lambda x}, \quad x > 0.$

(a) $P(X > 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x} \, dx$

$= \frac{1}{2} \left[\frac{e^{-1/2 x}}{-1/2} \right]_2^\infty$

$= -[e^{-\infty} - e^{-1}]$

$= e^{-1}$

$= 0.3678$

(b). $P(X > 10+9 / X > 9) = P(X > 1)$

$= \int_1^\infty \frac{1}{2} e^{-x/2} \, dx$

$$= \frac{1}{2}\left[\frac{e^{-x/2}}{-1/2}\right]_1^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1/2}\right]$$

$$= e^{-1/2} \quad = 0.60653.$$

37. **soln:**

$\lambda = 3.$

(i). $P(X \geq 0.5) = \int_{0.5}^{\infty} 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_{0.5}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-1.5}\right]$$

$$= e^{-1.5} \quad = 0.223$$

(ii). $P(X < 1/6) = \int_{0}^{1/6} 3e^{-3x}\,dx$

$$= 3\left[\frac{e^{-3x}}{-3}\right]_0^{1/6}$$

$$= -\left[e^{-1/2} - e^{0}\right]$$

$$= 1 - e^{-1/2}$$

$$= 1 - 0.6065$$

$$= 0.3935$$

38. A

Gamma Random Variable:

38. Soln:

$\lambda = 1/2$,

$k = 3$, $x = 12$.

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}, \quad x > 0$$

$$P(X \geq 12) = \int_{12}^{\infty} \frac{(1/2)^3 \cdot x^2 e^{-1/2 x}}{\Gamma(3)} dx.$$

$$= \frac{1}{8} \cdot \frac{1}{2!} \int_{12}^{\infty} x^2 \cdot e^{-1/2 x} dx$$

$u = x^2$ $\quad dv = e^{-1/2 x} dx$

$\quad\quad v = \frac{e^{-1/2 x}}{-1/2}$

$u' = 2x$

$u'' = 2$ $\quad v_1 = \frac{e^{-1/2 x}}{1/4}$

$u''' = 0$ $\quad v_2 = \frac{e^{-1/2 x}}{-1/8}$.

$$= \frac{1}{16} \left[-2x^2 e^{-1/2 x} - 8x e^{-1/2 x} - 16 e^{-1/2 x} \right]_{12}^{\infty}$$

$$= \frac{1}{16} \left[0 - 0 - 0 + 2(144) e^{-6} - 8(12) e^{-6} - 16 e^{-6} \right]$$

$$= \frac{1}{16} \left[288 e^{-6} - 96 e^{-6} - 16 e^{-6} \right]$$

$$= \frac{1}{16} (176 e^{-6})$$

$$= 11 e^{-6}$$

$$= 0.027.$$

39. Soln:

$\lambda = 1/2$, $k = 2$.

$$f(x) = \frac{\lambda^k \cdot x^{k-1} e^{-\lambda x}}{\Gamma(k)},$$

$$P(X>5) = \int_5^\infty \frac{(1/2)^2 (x)^1 e^{-1/2 x}}{\Gamma(2)} dx$$

$$= \frac{1}{4}\int_5^\infty x e^{-1/2 x} dx.$$

$$= \frac{1}{4}\left[-2x e^{-x/2} - 4e^{-1/2 x}\right]_5^\infty$$

$$= \frac{1}{4}\left[0 - 0 + 10 e^{-5/2} - 4e^{-5/2}\right]$$

$$= \frac{1}{4}\left[6 e^{-5/2}\right]$$

$$= 2/3 \times e^{-5/2} = (0.66)(0.08208)$$

$$= 0.053.$$

$dv = e^{-1/2 x} dx$

$u = x \quad v = \frac{e^{-1/2 x}}{-1/2}$

$u' = 1$

$v_1 = \frac{e^{-1/2 x}}{1/4}$

$\int u\,dv = uv - u'v_1 + \ldots$