26/09/2023

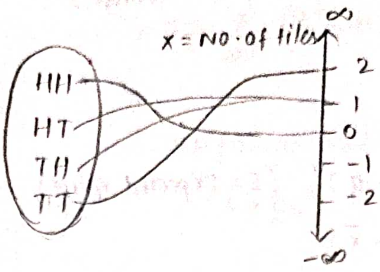
UNIT - II

ONE DIMENSIONAL RANDOM VARIABLES

RANDOM VARIABLE:

$X : S \rightarrow R$ → Random Experiment

↓ Function ↓ sample space



Random Variable

Discrete

- x takes countable infinite or finite values
- probability mass function (PMF): (can be pdf)

X	x_1	x_2	....	x_n
$P(X)$	$p(x_1)$	$p(x_2)$	....	$p(x_n)$

All probability should lie b/w 0 & 1 :

(i) $0 \leq p(x_i) \leq 1$

(ii) $\sum_{i=1}^{n} p(x_i) = 1$

- cumulative distribution frequency (CDF):

$F(X) = P(X \leq x)$

cumulative frequency

$F(0) = P(X \leq 0) = 1/4 \quad (1/4)$

$S = \{HH, HT, TH, TT\}$ $F(1) = P(X \leq 1) = 3/4 \quad (1/4 + 2/4)$

X = No. of tiles

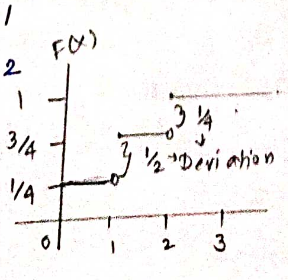
$X = \{0, 1, 2\}$ $F(2) = P(X \leq 2) = 1 \quad (3/4 + 1/4)$

x	0	1	2
$P(x)$	1/4	2/4	1/4

$F(3) = P(X \leq 3) = 1$

$F(-1) = P(X \leq -1) = 0$

$$F(x) = \begin{cases} 0 : & x < 0 \\ 1/4 : & 0 \leq x \leq 1 \\ 3/4 : & 1 \leq x \leq 2 \\ 1 : & x \geq 2 \end{cases}$$



continuous

- Takes in some interval $x \in [\ ]$
- probability density function (PDF):

$f(x) = \ldots\ldots \quad x \in [a, b]$

(i) $f(x) \geq 0 \quad \forall x$

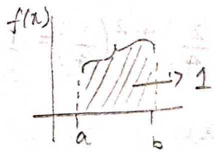
(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

ex: $f(x) = e^{-x}$, $x \geq 0$

$\int_0^{\infty} e^{-x}\,dx$

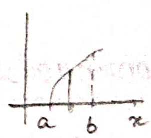
$= \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$

$= -[0 - 1] = 1$



- cumulative distribution frequency (CDF):

$F(X) = P(X \leq x)$



→ for $x \leq a$

$F(X) = 0$

→ for $a \leq x \leq b$

$F(X) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$

→ for $x \geq b$

$F(X) = 1$

ex: $f(x) = e^{-x} \quad 0 \leq x < \infty$

$F(X) = \int_0^{x} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^{x}$

$= -[e^{-x} - 1] = 1 - e^{-x}$

* Derivative of CDF gives PDF: $f(x) = \frac{d}{dx} F(x)$

MOMENTS (μ) & MOMENTS GENERATING FUNCTIONS: (MGF)

Moments (μ)

X

$\Rightarrow \mu_r = E(X-a)^r$ {E = Expected value}

$\mu_r = E(X-\bar{X})^r$

$\boxed{\mu_r' = E(X^r)}$ → Moment above origin

$\mu_1' = E(X)$

$\mu_2' = E(X^2)$

$\mu_3' = E(X^3)$

• Discrete:

$p(x) =$ ____ $x = \dots$

$$\boxed{E(X^r) = \sum x^r p(x)}$$

$$E(X) = \sum_x x p(x)$$

$$E(X) = \sum_{x=0}^{3} x\, p(x)$$

$$= 0\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 2\left(\frac{1}{4}\right) = 1$$

$$E(X^2) = \sum_{x=0}^{3} x^2 p(x)$$

$$= 0\left(\frac{1}{4}\right) + 1\left(\frac{2}{4}\right) + 4\left(\frac{1}{4}\right)$$

• Continuous:

$f(x) =$ ____ $x = \dots$

$$\boxed{E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx}$$

MGF:

$$M_x(t) = E(e^{tx}) = \begin{cases} \sum_x e^{tx} p(x) \\ \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \end{cases}$$

x = Random variable

$$M_x(t) = E(e^{tx})$$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots\right]$$

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots\right]$$

$$= E\left[1 + \frac{t}{1!}E(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots\right]$$

First movement generating from origin

second movement

$$\left[e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right]$$

DISTRIBUTIONS:

Discrete	Continuous
• Binomial	• Uniform
• Poisson	• Exponential
• Geometric	• Gamma
	• Normal

First movement above origin:

$$E(X) = \frac{d}{dt} M_x(t)\Big|_{t=0}$$

Second movement above origin

$$E(X^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0}$$

$$\frac{d^2}{dt^2} = uv' + vu'$$

Binomial Distribution:

$B(n, p)$

The discrete random variable is said to be binomial random variable with parameter n, p if it's probability mass function (PMF) is given by,

$$P(X=x) = p(x) = nC_x \, p^x q^{n-x} \; ; \quad x = 0, 1, 2, \dots n$$

{ x = No. of success, n = no. of trial, p = probability of success }

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

MGF :

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} nC_x p^x q^{n-x}$$

$$= \sum_{x=0}^{n} nC_x (pe^t)^x q^{n-x}$$

{ Binomial expansion

$(a+b)^n = nC_0 a^n b^0 + nC_1 a^{n-1} b^1 + nC_2 a^{n-2} b^2 + \dots + nC_n a^0 b^n$

$= \sum_{i=0}^{n} nC_i a^{n-i} b^i$ $\quad \{\Rightarrow a = q, \; b = pe^t\}$ }

$$\boxed{M_X(t) = (q + pe^t)^n}$$

Mean :

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} \qquad [e^0 = 1]$$

$$= n[q + pe^t]^{n-1}(0 + pe^t) \Big|_{t=0}$$

$$= n[q + pe^0]^{n-1} pe^0 = n(p+q)^{n-1} p \qquad \Big| \; p+q = 1$$

$$= n(1)^{n-1} p$$

$$\boxed{E(X) = np}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= np \frac{d}{dt}[e^t(q+pe^t)^{n-1}] \Big|_{t=0}$$

$$= np[e^t(n-1)(q+pe^t)^{n-2}(0+pe^t) + (q(pe^t)e^t)] \Big|_{t=0}$$

$$= np[(n-1)p + 1]$$

$$= np[np - p + 1]$$

$$\boxed{E(X^2) = n^2p^2 - np^2 + np}$$

Variance:

$$var(X) = E(X^2) - [E(X)]^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$\boxed{var(X) = npq}$$

Poisson Distribution:

$poisson(\lambda)$

A discrete random variable X said to be a poisson random variable with parameter λ then it's PMF is given by,

$$P(X=x) = p(x) = \frac{e^{-\lambda} \lambda^x}{x!} \; ; \quad x = 0, 1, 2, \dots$$

(λ - Expected value)

- Poisson distribution is used when no. of trials is larger.
- Binomial - used for limited.

MGF :

$$M_X(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda} \left[1 + \frac{\lambda e^t}{1!} + \frac{(\lambda e^t)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$\boxed{M_x(t) = e^{\lambda(e^t - 1)}}$$

Mean:

$$E(X) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= e^{\lambda(e^t-1)} \cdot \lambda(e^t - 0) \Big|_{t=0}$$

$$= 1 \cdot \lambda \cdot 1$$

$$\boxed{E(X) = \lambda}$$

$$E(X^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \lambda \frac{d}{dt} e^t \cdot e^{\lambda(e^t-1)} \Big|_{t=0}$$

$$= \lambda[e^t \cdot e^{\lambda(e^t-1)} \cdot \lambda(e^t) + e^{\lambda(e^t-1)} \cdot e^t] \Big|_{t=0}$$

$$= \lambda[\lambda + 1]$$

$$\boxed{E(X^2) = \lambda^2 + \lambda}$$

Variance:

$$var(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{var(X) = \lambda}$$

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Geometric distribution:

A discrete random variable X is said to follow geometric distributions with parameter p (probability of success) if PMF is given by,

$$P(X=x) = p(x) = pq^{x-1} \; ; \; x = 1, 2, 3, \ldots$$

{ x = No. of trials needed to get first success

$X \sim$ Geometric (P) }

(q = not success)

MGF:

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=1}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=1}^{\infty} e^{tx} pq^{x-1}$$

$$= \frac{p}{q} \sum_{x=1}^{\infty} (qe^t)^x$$

{ $\sum_{x=0}^{\infty} e^{tx} pq^x$

$= p \sum_{x=0}^{\infty} (qe^t)^x = p(1 + qe^t + (qe^t)^2 + \cdots)$

[Infinite geometric series: $S_\infty = a + ar + ar^2 + \cdots = \frac{a}{1-r}$ $\because r \neq 1$]

$= p\left(\frac{1}{1-qe^t}\right)$ }

$$= \frac{p}{q}[qe^t + (qe^t)^2 + \cdots]$$

$$= \frac{p}{q} \cdot \frac{qe^t}{1 - qe^t}$$

$$\boxed{M_x(t) = \frac{pe^t}{1 - qe^t}}$$

Mean :

$$E(X) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \frac{(1-qe^t)\cdot pe^t - pe^t(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{(1-q)p - p(-q)}{(1-q)^2} = \frac{p^2+pq}{p^2}$$

$$= \frac{(p+q)p}{p^2} = \frac{p+q}{p} \quad (\because p+q=1)$$

$$\boxed{E(X) = \frac{1}{p}}$$

$$E(X^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)^2}\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pe^t}{(1-qe^t)} \frac{1}{(1-qe^t)}\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{M_x(t)}{1-qe^t}\right)\Big|_{t=0}$$

$$= \frac{(1-qe^t)M_x'(t) - M_x(t)(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{(1-q)\frac{1}{p} - (1)(-q)}{(1-q)^2}$$

$M_x(0) = \frac{p(1)}{1-q(1)} = p/p = 1$

$$\boxed{E(X^2) = \frac{1+q}{p^2}}$$

Variance :

$$var(x) = E(X^2) - [E(X)]^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$\boxed{var(x) = \frac{q}{p^2}}$$

(*) If

$X \to$ No. of failure before I success

$$p(x) = q^x p \qquad x = 0,1,2,3,\ldots$$

MGF :

$$M_x(t) = E(e^{tx})$$

$$= \sum_{x=0}^{\infty} e^{tx} pq^x$$

$$= p\sum_{x=0}^{\infty} (qe^t)^x$$

$$= p(1+qe^t+(qe^t)^2+\ldots)$$

$$\boxed{M_x(t) = p\left(\frac{1}{1-qe^t}\right)} \quad \left(\because S_\infty = \frac{a}{1-r}\right)$$

Mean :

$$E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= p\left(\frac{-1(0-qe^t)}{(1-qe^t)^2}\right)\Big|_{t=0}$$

$$= \frac{pq}{(1-q)^2} = \frac{pq}{p^2}$$

$$\boxed{E(x) = \frac{q}{p}}$$

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{pqe^t}{(1-qe^t)^2}\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left[\frac{p}{(1-qe^t)}\cdot\frac{qe^t}{(1-qe^t)}\right]\Big|_{t=0}$$

$$= \frac{d}{dt}\left(\frac{M_x(t)\,qe^t}{(1-qe^t)}\right)\Big|_{t=0}$$

$$= \frac{(1-qe^t)M_x'(t)qe^t - M_x(t)qe^t(0-qe^t)}{(1-qe^t)^2}$$

$$= \frac{(1-q)(q/p)q - \frac{p}{1-q}(q)(-q)}{(1-q)^2}$$

$$= \frac{\frac{pq^2}{p} + \frac{pq^2}{p}}{(1-q)^2} = \frac{2pq^2}{p(1-q)^2}$$

$$E(x^2) = \frac{2pq^2}{p^3}$$

$$\boxed{E(x^2) = \frac{2q^2}{p^2}}$$

Variance:

$$var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2q^2}{p^2} - \frac{q^2}{p^2}$$

$$\boxed{var(x) = \frac{q^2}{p^2}}$$

Uniform distribution:

A continuous random variable x is said to follow uniform distribution with parameter a, b if it's PDF is given by,

$$x \sim uniform(a, b)$$

$$f(x) = \frac{1}{b-a} \quad ; \ a \le x \le b$$

MGF:

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \int_a^b e^{tx} \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{t(b-a)} [e^{bt} - e^{at}]$$

$$\boxed{M_x(t) = \frac{e^{bt} - e^{at}}{t(b-a)}}$$

Mean:

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_a^b x \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)(b-a)}{2(b-a)}$$

$$\boxed{E(x) = \frac{a+b}{2}}$$

$$E(x^2) = \int_{\infty}^{-\infty} x^2 f(x)\, dx$$

$$= \int_a^b x^2 \left(\frac{1}{b-a}\right) dx$$

$$= \frac{1}{b-a} \left(\frac{x^3}{3}\right)_a^b$$

$$= \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ba + a^2)}{3(b-a)}$$

$$\boxed{E(x^2) = \frac{a^2 + ab + b^2}{3}}$$

variance:

$$var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{a^2 + ab + b^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{4a^2 + 4ab + 4b^2 - 3a^2 - 6ab - 3b^2}{12}$$

$$= \frac{a^2 - 2ab + b^2}{12}$$

$$\boxed{var(x) = \frac{(b-a)^2}{12}}$$

Exponential distribution.

A continuous random variable x is said to follow exponential distribution with parameter λ if it's PDF is given by,

$$x \sim exp(\lambda)$$

$$f(x) = \lambda e^{-\lambda x} \quad ; \ x \ge 0$$

MGF:

$$M_x(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx$$

$$= \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)}\right]_0^{\infty} \quad [e^{-\infty} = 0]$$

$$= \frac{\lambda}{-(\lambda-t)} [0 - 1]$$

$$\boxed{M_x(t) = \frac{\lambda}{\lambda - t}}$$

Mean:

$$E(x) = \frac{d}{dt} M_x(t) \Big|_{t=0}$$

$$= \lambda \left(\frac{-1}{(\lambda-t)^2} (-1)\right)\Big|_{t=0}$$

$$= \frac{\lambda}{\lambda^2}$$

$$\boxed{E(x) = \frac{1}{\lambda}}$$

$$E(x^2) = \frac{d^2}{dt^2} M_x(t) \Big|_{t=0}$$

$$= \lambda \frac{d}{dt} \left(\frac{1}{(\lambda-t)^2}\right)\Big|_{t=0}$$

$$= \lambda \left(\frac{-2}{(\lambda-t)^3} (-1)\right)\Big|_{t=0}$$

$$= \frac{2\lambda}{\lambda^3}$$

$$\boxed{E(x^2) = \frac{2}{\lambda^2}}$$

Variance:

$$var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{var(x) = \frac{1}{\lambda^2}}$$

General Gamma / Erlang distribution:

A continuous random variable x is said to follow erlang distribution with parameter k, λ if it's PDF is given by,

$$x \sim Erlang(k, \lambda) \quad \begin{cases} k = count \\ \lambda = rate \end{cases}$$

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \quad ; \ x > 0$$

GAMMA INTEGRAL:

Gamma integral for any parameter n,

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \quad , n > 0$$

Properties:

(i) $\Gamma(1/2) = \sqrt{\pi}$

(ii) $\Gamma(1) = 1$

(iii) $\Gamma(n+1) = n\Gamma(n) = n!$

Ex: $\Gamma 5 = 4\Gamma(4) = 4 \times 3\,\Gamma(3) = 4 \times 3 \times 2\,\Gamma(2)$
$= 4 \times 3 \times 2 \times 1\,(\Gamma 1)$
$= 4!$

(iv) $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

(v) $\int_0^\infty x^{n-1} e^{-ax}\,dx = \frac{\Gamma(n)}{a^n}$

MGF:

$$M_X(t) = E(e^{tX})$$

$$= \int_0^\infty \frac{e^{tx} \lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}\,dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \int_0^\infty x^{k-1} e^{-(\lambda-t)x}\,dx$$

$$= \frac{\lambda^k}{\Gamma(k)} \frac{\Gamma(k)}{(\lambda-t)^k} = \frac{\lambda^k}{(\lambda-t)^k}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^k = \left(\frac{\lambda-t}{\lambda}\right)^{-k}$$

$$\boxed{M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-k}}$$

Mean:

$$E(X) = \frac{d}{dt} M_X(t)\Big|_{t=0}$$

$$= \lambda^k \frac{d}{dt}\left[\frac{1}{(\lambda-t)^k}\right]\Big|_{t=0}$$

$$= \lambda^k \left[\frac{-k\ (-1)}{(\lambda-t)^{k+1}}\right]\Big|_{t=0}$$

$$= \frac{k\cdot\lambda^k}{\lambda^k\cdot\lambda}$$

$$\boxed{E(X) = \frac{k}{\lambda}}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t)\Big|_{t=0}$$

$$= k\cdot\lambda^k \frac{d}{dt}\left[\frac{1}{(\lambda-t)^{k+1}}\right]\Big|_{t=0}$$

$$= k\cdot\lambda^k \left[\frac{-(k+1)}{(\lambda-t)^{k+2}}(-1)\right]\Big|_{t=0}$$

$$= \frac{k(k+1)\lambda^k}{\lambda^k\cdot\lambda^2}$$

$$\boxed{E(X^2) = \frac{k(k+1)}{\lambda^2}}$$

Variance:

$$var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{k(k+1)}{\lambda^2} - \frac{k^2}{\lambda^2}$$

$$\boxed{var(X) = \frac{k}{\lambda^2}}$$

Gamma distribution: (one parameter)

$X \sim Gamma(\lambda)$

$\lambda = 1,\ k = \lambda$

$$f(x) = \frac{e^{-x} x^{\lambda-1}}{\Gamma(\lambda)}\ ;\ x > 0$$

MGF:

$$\boxed{M_X(t) = (1-t)^{-\lambda}}$$

Mean:

$$\boxed{E(X) = \lambda}$$

$$\boxed{E(X^2) = \lambda^2 + \lambda}$$

Variance:

$$var(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{var(X) = \lambda}$$

Normal distribution:

$X \sim N(\mu, \sigma^2)$

A continuous random variable X is said to follow normal distribution with parameter μ, σ if its PDF is given by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2};$$

$-\infty < x < \infty$

$-\infty < \mu < \infty$

$\sigma^2 > 0$

$\left\{ \begin{array}{l} \mu = \text{Mean} \\ \sigma = \text{variance} \end{array} \right\}$

MGF:

$$M_X(t) = E(e^{tx})$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} \cdot e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

put $z = \frac{x-\mu}{\sigma} \Rightarrow z\sigma = x - \mu$

$x = z\sigma + \mu$

$dx = \sigma dz$

$$M_X(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(z\sigma+\mu)} e^{-1/2 z^2} \sigma dz$$

$$= \frac{\sigma}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z + 2t\mu)} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2(z^2 - 2t\sigma z)} \cdot e^{t\mu} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-1/2[(z-t\sigma)^2 - t^2\sigma^2]} dz$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{z-t\sigma}{\sqrt{2}}\right)^2} dz$$

put $u = \frac{z - t\sigma}{\sqrt{2}}$

$\sqrt{2} du = dz$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-u^2} \sqrt{2} du$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{\pi}} \times 2\left(\int_0^{\infty} e^{-u^2} du\right)$$

$$= \frac{e^{\mu t + \frac{t^2\sigma^2}{2}}}{\sqrt{\pi}} \times 2 \times \frac{\sqrt{\pi}}{2}$$

$$\boxed{M_X(t) = e^{\mu t + \frac{t^2\sigma^2}{2}}}$$

Mean:

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0}$$

$$= e^{\mu t + \frac{t^2\sigma^2}{2}} (\mu + \sigma^2 t) \Big|_{t=0}$$

$$= e^0(\mu + 0)$$

$$\boxed{E(X) = \mu}$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0}$$

$$= M_X(t)(\sigma^2) + (\mu + \sigma^2 t) M_X'(t) \Big|_{t=}$$

$$\boxed{E(X^2) = \sigma^2 + \mu^2}$$

Variance:

$$var(X) = \sigma^2 + \mu^2 - \mu^2$$

$$\boxed{var(X) = \sigma^2}$$

10/10/2023

* Poisson distribution is a limiting case of binomial distribution

As $n \to \infty$

Binomial $\Rightarrow$ poisson

$$nC_x p^x q^{n-x} \xrightarrow{n \to \infty} \frac{e^{-\lambda} \lambda^x}{x!}$$

Equating their mean:

$$np = \lambda$$

$\therefore$ probability, $\boxed{p = \frac{\lambda}{n}}$

* $\lim_{n \to \infty} nC_x p^x q^{n-x}$ $\left[nCr = \frac{n!}{(n-r)!\,r!} \right]$

$$= \lim_{n \to \infty} \frac{n(n-1)(n-2)\cdots[n-(x-1)]}{1 \times 2 \times 3 \times \cdots x \cdot x} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}$$

$$= \lim_{n \to \infty} \frac{n \cdot n\left(1-\frac{1}{n}\right) n\left(1-\frac{2}{n}\right)\cdots n\left(1-\frac{(x-1)}{n}\right) \frac{\lambda^x}{n^x} \times \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}}{x!}$$

$$= \lim_{n \to \infty} \frac{n^x \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right) \frac{\lambda^x}{n^x} \cdot \times \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}}{x!}$$

$$= \lim_{n \to \infty} \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{(x-1)}{n}\right) \lambda^x \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}}{x!}$$

$$= \frac{1}{x!} \lambda^x \lim_{n \to \infty} \left(1-\frac{\lambda}{n}\right)^n \cdot \lim_{n \to \infty} \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$\left(\because \lim_{n \to \infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda} \right)$$

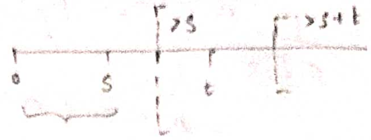
$$= \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$ Poisson is limiting case of binomial

Memoryless Property:

$$\begin{cases} X \sim \text{Geometric}(p) \\ X \sim \text{EXP}(\lambda) \end{cases}$$

$$P(X > s+t \,/\, X > s) = P(X > t)$$

Let this be expiry date



Proof:

$$P(X > s+t \,/\, X > s) = \frac{P(X > s+t \cap X > s)}{P(X > s)}$$

$$= \frac{P(X > s+t)}{P(X > s)}$$

$$= \frac{q^{s+t}}{q^s} = \frac{q^s \cdot q^t}{q^s} = q^t$$

$$= P(X > t)$$

* $X \sim \text{Geo}(p)$

$$P(x) = pq^{x-1} \quad ; x = 1, 2, \ldots$$

$$P(X > k) = P(X \geq k+1)$$

$$= \sum_{x=k+1}^{\infty} pq^{x-1}$$

$$= p\left[q^k + q^{k+1} + q^{k+2} + \cdots\right]$$

$$= pq^k\left[1 + q + q^2 + \cdots\right]$$

$$= pq^k \left[\frac{1}{1-q}\right] = \frac{pq^k}{p}$$

$$\boxed{P(X > k) = q^k}$$

* $X \sim \text{Exp}(\lambda)$

$$f(x) = \lambda e^{-\lambda x} \quad ; x > 0$$

$$P(X > k) = \int_k^{\infty} \lambda e^{-\lambda x} \, dx$$

$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_k^{\infty}$

$= -[0 - e^{-\lambda k}]$

$\boxed{P(X>k) = e^{-\lambda k}}$

$\frac{P(X>s+t)}{P(X>s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}}$

$= \frac{e^{-\lambda s} \cdot e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t}$

$= P(X>t)$ //

PROBLEMS:

① $M_X(t) = \frac{0.4e^t}{1-0.6e^t}$

$X \sim Geo(P)$ $\left[M_X(t) = \frac{pe^t}{1-qe^t}\right]$

$p = 0.4$ $q = 0.6$

Mean: $E(X) = \frac{1}{p} = \frac{1}{0.4} = \frac{10}{4}$

Mean $= 2.5$

② $f(x) = \begin{cases} ke^{-x} & ; x>0 \\ 0 & ; \text{otherwise } (x<0) \end{cases}$ // continuous

$\int_{-\infty}^{\infty} f(x)\, dx = 1$

$\int_{-\infty}^{\infty} ke^{-x}\, dx = 1$ $[e^{-\infty} = 0; e^{0} = 1]$

$k\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = -k[0-1] = 1$

$\boxed{k=1}$

$f(x) = \begin{cases} e^{-x} & ; x>0 \\ 0 & ; \text{otherwise} \end{cases}$

$f(x) = e^{-x}$; $x>0$

$X \sim exp(\lambda)$ $[f(x) = \lambda e^{-\lambda x}]$

$\lambda = 1$

Mean: $E(X) = \frac{1}{\lambda} = \frac{1}{1}$

Mean $= 1$

(or) $E(X) = \int_0^{\infty} x f(x)\, dx$

$= \int_0^{\infty} x e^{-x}\, dx$

③ $P(x) = \frac{3}{4}\left(\frac{1}{4}\right)^x$; $x = 0, 1, 2, \dots$

$\sum_x p(x) = \sum_{x=0}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^x$

$= \frac{3}{4}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \dots\right]$

$= \frac{3}{4}\left[\frac{1}{1-1/4}\right] = \frac{3}{4}\left[\frac{4}{3}\right]$

$\sum_x p(x) = 1 = p(x)$ //

④ $M_X(t) = \frac{2}{2-t}$

$X \sim exp(\lambda)$

$\left[M_X(t) = \frac{\lambda}{\lambda - t}\right]$

$\therefore \lambda = 2$

$var(X) = \frac{1}{\lambda^2} = \frac{1}{4}$

⑤ $F_X(x) = \begin{cases} 0 & ; x<0 \\ 1-e^{-x/5} & ; x \geq 0 \end{cases}$

CDF

$f(x) = \frac{d}{dx}(1-e^{-x/5}) = 0 - \left(-\frac{1}{5}e^{-x/5}\right)$

PDF

$= \frac{1}{5}e^{-1/5 x}$

$f(x) = \begin{cases} 0 & ; x<0 \\ \frac{1}{5}e^{-1/5 x} & ; x \geq 0 \end{cases}$

$X \sim exp(\lambda)$

$[PDF, f(x) = \lambda e^{-\lambda x}]$ $\therefore \lambda = 1/5$

Expected lifetime,

$$E(X) = \frac{1}{\lambda} = \frac{1}{1/5}$$

$$E(X) = 5$$

(6) $f(x) = cx^2 \quad ; 0 < x < 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$c\int_0^1 x^2\,dx = 1$$

$$c\left[\frac{x^3}{3}\right]_0^1 = 1$$

$$\frac{c}{3}[1-0] = 1$$

$$\boxed{c = 3}$$

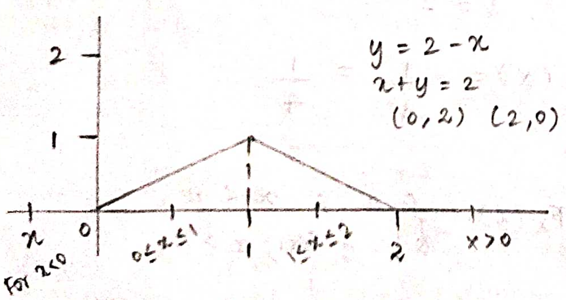
(7) $f(x) = \frac{1}{3} e^{-x/3} \quad ; x > 0$

$X \sim \exp(\lambda)$

$\lambda = 1/3$

Mean, $E(X) = \frac{1}{\lambda} = \frac{1}{1/3} = 3$

(8) $f(x) = \begin{cases} x & ; 0 \le x \le 1 \\ 2-x & ; 1 \le x \le 2 \end{cases}$



$F(x) = P(X = x)$

(i) For $x < 0$

$F(x) = 0$

(ii) For $0 \le x \le 1$

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \int_0^x x\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^x$$

$$F(x) = \frac{x^2}{2} \quad ; 0 \le x \le 1$$

(iii) For $1 \le x \le 2$

$$F(x) = \int_0^1 f(x)\,dx + \int_1^x (2-x)\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \left[2x - \frac{x^2}{2} - \frac{3}{2}\right]$$

$$F(x) = 2x - \frac{x^2}{2} - 1 \quad ; 1 \le x \le 2$$

(iv) For $x > 2$

$F(x) = 1$

(9) $p(x) = \frac{3}{4}\left(\frac{1}{4}\right)^{x-1} \quad ; x = 1, 2, 3, \ldots$

$X \sim \text{Geo}(p) \Rightarrow p = \frac{3}{4},\ q = \frac{1}{4}$

$[P(x) = pq^{x-1}]$

Mean, $E(X) = \frac{1}{p} = \frac{1}{3/4} = 4/3$

variance, $var(x) = \frac{q}{p^2} = \frac{1/4}{9/16}$

$$= \frac{1}{4} \times \frac{16}{9}$$

$$var(x) = \frac{4}{9}$$

$p(x>4 / x>2)$

$= p(x>2)$

$= p(x \geq 3)$

$= \sum_{x=3}^{\infty} \frac{3}{4}\left(\frac{1}{4}\right)^{x-1}$

$= \frac{3}{4}\left[\left(\frac{1}{4}\right)^2+\left(\frac{1}{4}\right)^3+\cdots\right]$

$a=\frac{1}{4} \quad r=\frac{1}{4}$

$= \frac{3}{4}\left[\frac{1/4}{1-1/4}\right] = \frac{3}{4}\left[\frac{1}{4}\times\frac{4}{3}\right]$

$p(x\geq 3) = \frac{1}{4}$

(10) $f(x) = \begin{cases} 2e^{-2x} & ; x \geq 0 \\ 0 & ; x<0 \end{cases}$

(i) $X \sim Exp(\lambda) \quad [\lambda e^{-\lambda x}]$

$\lambda = 2$

$E(X) = \frac{1}{\lambda} = \frac{1}{2}$

(ii) $var(X) = \frac{1}{\lambda^2} = \frac{1}{4}$

(iii) $P(X\leq 3) = \int_0^3 2e^{-2x}\,dx$

$= 2\left[\frac{e^{-2x}}{-2}\right]_0^3$

$= -[e^{-6} - e^0] = -[e^{-6}-1]$

$P(X\leq 3) = 1-e^{-6}$

(12) $X \sim exp(\lambda)$

$\lambda = 1/10$

$f(x) = \lambda e^{-\lambda x}$

$f(x) = \frac{1}{10}e^{-1/10\,x}$

(a) $P(X<5)$

$= \int_0^5 \frac{1}{10}e^{-1/10\,x}\,dx$

$= \frac{1}{10}\left[\frac{e^{-1/10\,x}}{-1/10}\right]_0^5$

$= -[e^{-0.5} - 1]$

$= 1-e^{-0.5}$

(b) $P(5<X<10)$

$= \int_5^{10} \frac{1}{10}e^{-1/10\,x}\,dx$

$= \frac{1}{10}\left[\frac{e^{-1/10\,x}}{-1/10}\right]_5^{10}$

$= -[e^{-1} - e^{-0.5}]$

$= e^{-0.5}-e^{-1}$

(13) $n=5$

$p=\frac{1}{4} \quad q=\frac{3}{4}$

$X \sim Binomial(n,p)$

$p(x) = nC_x p^x q^{n-x} \; ; x=0,1,2,\dots,n$

(a) $P(X=3)$

$\cancel{= 5C_3 p^{(1/4)}}$

$= 5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2$

$= \frac{5\times4\times3}{3\times2\times1}\left(\frac{1}{64}\right)\left(\frac{9}{16}\right)$

$= 0.0878$

(b) $P(X\geq 3) = P(X=3)+P(X=4)+P(X=5)$

$= 5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + 5C_4\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right)^1$

$+ 5C_5\left(\frac{1}{4}\right)^5\left(\frac{3}{4}\right)^0$

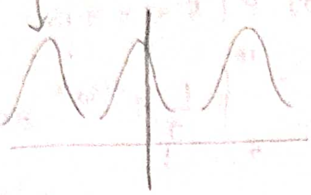
(c) $P(X<3) = 1 - P(X \geq 3)$

$= 1 -$

$=$

(*) $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$z = \frac{x-\mu}{\sigma}$

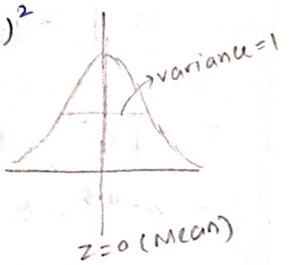


STANDARD NORMAL

$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(z)^2}$

$SN(0, 1)$

Mean Variance

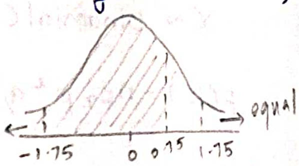


(14) $\mu = 45$ min

$\sigma = 12$ min

$X \sim N(\mu, \sigma^2)$

$Z = \frac{x-\mu}{\sigma}$ (standard normal)



(i) $P(24 < X < 54)$

$= P\left(\frac{24-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{54-\mu}{\sigma}\right)$

$= P\left(\frac{24-45}{12} < z < \frac{54-45}{12}\right)$

$= P(-1.75 < z < 0.75)$

$= P(z < 0.75) - [1 - P(z < 1.75)]$

$= 0.7734 - [1 - 0.9599]$

$= 0.7734 + 0.9599 - 1$

$= 0.7333$

$\Rightarrow 200 \times 0.7333 = 146$

(ii) $P(X > 39)$

$= P\left(\frac{X-\mu}{\sigma} > \frac{39-\mu}{\sigma}\right)$

$= P\left(z > \frac{39-45}{12}\right)$

$= P(z > -0.5)$

$= P(z < 0.5)$

$= 0.6915$

17/10/2023

(15) $\lambda = \frac{2021}{2021} = 1$

$X \sim$ poisson (λ),

$p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$; $x = 0, 1, 2, \ldots$

$P(X \geq 4) = 1 - P(X < 4)$

$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[\frac{e^{-1}(1)^0}{0!} + \frac{e^{-1}(1)^1}{1!} + \frac{e^{-1}(1)^2}{2!} + \frac{e^{-1}(1)^3}{3!}\right]$

$= 1 - e^{-1}\left[1 + 1 + \frac{1}{2} + \frac{1}{6}\right]$

$= 1 - e^{-1}\left[\frac{16}{6}\right]$

(16) $F(x) = \begin{cases} 0 & ; x < -2 \\ 0.25x + 0.5 & ; -2 \leq x \leq 2 \\ 1 & ; x \geq 2 \end{cases}$

$F(x) = P(X \leq x)$

(i) $P(X < 1.8)$

$= P(X \leq 1.8) = F(1.8)$

$= 0.25(1.8) + 0.5$

$=$

(ii) $P(X > -1.5)$

$= 1 - P(X \leq -1.5)$

$= 1 - F(-1.5)$

$= 1 - [0.25(-1.5) + 0.5]$

$=$

(iii) $P(X < -2)$

$= P(X \leq -2)$

$= F(-2) = 0$

(17) $X \sim B(n, p)$

$n = 7$, $p = 0.7$, $q = 0.3$

$p(x) = nC_x \, p^x q^{n-x}$; $x = 0, 1, 2, \dots n$

i) $P(X = 5) = 7C_5 (0.7)^5 (0.3)^2$

$= 0.3176$

ii) $P(X \geq 5) = P(X=5) + P(X=6) + P(X=7)$

$= 0.3176 + 7C_6 (0.7)^6 (0.3)^1 + 7C_7 (0.7)^7 (0.3)^0$

$=$

iii) $P(X < 5) = 1 - P(X \geq 5)$

$= 1 -$

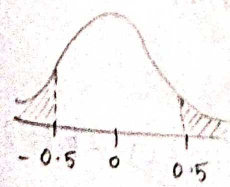
$=$

(18) $\mu = 12$, $\sigma = 0.2$

$X \sim N(\mu, \sigma^2)$

$P(X < 11.9) = P\left(\frac{X-\mu}{\sigma} < \frac{11.9-\mu}{\sigma}\right)$

$= P\left(Z < \frac{11.9-12}{0.2}\right)$



$= P(Z < -0.5)$

$= 1 - P(Z < 0.5)$

$= 1 - 0.6915 = 0.3085$

$P(11.8 < X < 11.9) = P\left(\frac{11.8-12}{0.2} < Z < \frac{11.9-12}{0.2}\right)$

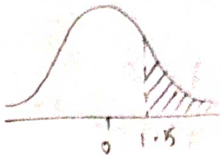
$= P(-1 < Z < -0.5)$

$= P(Z < 1) - P(Z < 0.5)$

$= 0.8413 - 0.6915$

$= 0.1498$

$P(X > 12.3) = P\left(Z > \frac{12.3-12}{0.2}\right)$



$= P(Z > 1.5)$

$= 1 - P(Z < 1.5)$

$= 1 - 0.9332$

$= 0.0668$

(20) $X \sim \exp(\lambda)$

Mean = 3000 hrs

$\frac{1}{\lambda} = 3$ (in 1000 of hrs)

$\lambda = \frac{1}{3}$

$f(x) = \lambda e^{-\lambda x}$; $x > 0$

(i) $P(X > 3) = \int_3^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx$

$= \frac{1}{3} \left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} \right]_3^\infty$

$= -[0 - e^{-1}]$

$= e^{-1}$

(ii) $P(2 \leq X \leq 3) = \int_2^3 \frac{1}{3} e^{-\frac{1}{3}x} dx$

$= \frac{1}{3} \left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}} \right]_2^3 = -[e^{-1} - e^{-0.667}]$

$= e^{-0.667} - e^{-1}$

(iii) $P(X>3.5 / X>2.5) = P(X>1)$

$$= \int_1^\infty \frac{1}{3} e^{-\frac{1}{3}x} dx = \frac{1}{3}\left[\frac{e^{-\frac{1}{3}x}}{-\frac{1}{3}}\right]_1^\infty$$

$$= -[0 - e^{-0.333}] = e^{-0.333}$$

(21) $F(x) = \begin{cases} 1 - e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

CDF

$f(x) = \frac{d}{dx}(F(x))$

$f(x) = \begin{cases} 2e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

PDF

$X \sim \exp(\lambda)$, $\lambda = 2$

$f(x) = \lambda e^{-\lambda x}$

$$P(X>2) = 1 - P(X \leq 2)$$
$$= 1 - F(2)$$
$$= 1 - [1 - e^{-4}]$$
$$P(X>2) = e^{-4}$$

$$var(X) = \frac{1}{\lambda^2} = \frac{1}{4}$$

(22) $f(x) = \begin{cases} \frac{x^3}{4} & ; 0 < x < c \\ 0 & ; \text{otherwise} \end{cases}$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$\frac{1}{4}\int_0^c x^3 dx = 1$$

$$\frac{1}{4}\left[\frac{x^4}{4}\right]_0^c = 1$$

$$\frac{c^4}{16} = 1$$

$$c^4 = 16 \Rightarrow c = \sqrt[4]{16}$$

$$c = 2$$

(25) $f(x) = \begin{cases} 0 & , x < 0.5 \\ ke^{-2(x-0.5)} & , x \geq 0.5 \end{cases}$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$k\int_{0.5}^{\infty} e^{-2x} \cdot e^1 \, dx = 1$$

$$ke\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{\infty} = 1$$

$$\frac{ke}{-2}[0 - e^{-1}] = 1$$

$$\frac{k}{2} = 1$$

$$\boxed{k = 2}$$

CDF:

- For $x < 0.5$

  $F(x) = 0$

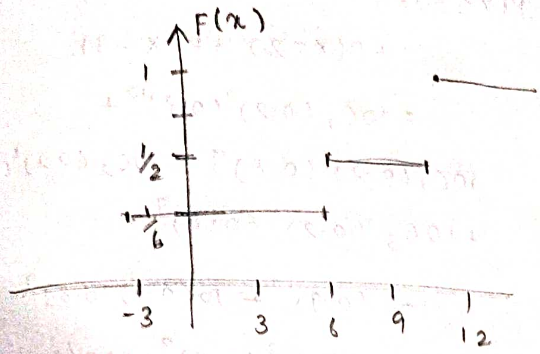
- For $x \geq 0.5$

$$F(x) = \int_{0.5}^{x} 2e^{-2x} \cdot e \, dx$$
$$= 2e\left[\frac{e^{-2x}}{-2}\right]_{0.5}^{x}$$
$$= -e\left[e^{-2x} - e^{-1}\right]$$
$$= 1 - e^{-2x+1}$$

(i) $P(X \leq 1.5) = F(1.5)$
$$= 1 - e^{-2(1.5)+1}$$
$$= 1 - e^{-2}$$

(ii) $P(1.2 < X < 2.4) = F(2.4) - F(1.2)$
$$= 1 - e^{-2(2.4)+1} - [1 - e^{-2(1.2)+1}]$$
$$= e^{-1.4} - e^{-3.8}$$

(26) $F(x) = \begin{cases} 0 & ; x < -3 \\ 1/6 & ; -3 \le x < 6 \\ 1/2 & ; 6 \le x < 10 \\ 1 & ; x \ge 10 \end{cases}$



- $P(X \le 4) = F(4) = 1/6$
- $P(-5 < X < 4) = F(4) - F(-5)$

$= \frac{1}{6} - 0$

$= \frac{1}{6}$

PMF :

x	-3	6	10
$p(x)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

(28) $X \sim unif(-3, 3)$

$f(x) = \frac{1}{b-a} = \frac{1}{6} \; ; -3 < x < 3$

(i) $P(|X| < 2) = P(-2 < X < 2)$

$= \int_{-2}^{2} \frac{1}{6} dx = \frac{1}{6} [x]_{-2}^{2}$

$= \frac{1}{6} [2 - (-2)] = \frac{2}{3}$

(ii) $E(X) = \frac{a+b}{2}$

$= \frac{-3+3}{2} = 0$

(iii) $P(|X-2| < 2)$

$= P(-2 < X - 2 < 2)$

$= P(0 < X < 4)$

$= \int_{0}^{3} \frac{1}{6} dx = \frac{1}{6} [x]_0^3$

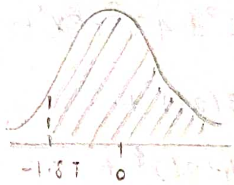
$= \frac{3}{6} = \frac{1}{2}$

(29) $X \sim N(\mu, \sigma^2)$

$\mu = 5, \sigma = 0.8$

(i) $P(X > 3.5) = P\left(Z > \frac{3.5 - 5}{0.8}\right)$

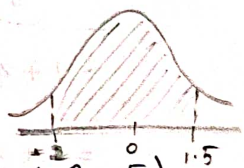
$= P(Z > -1.875)$



$= P(Z < 1.87)$

$= 0.9693$

(ii) $P(3.4 < X < 6.2)$



$= P\left(\frac{3.4 - 5}{0.8} < Z < \frac{6.2 - 5}{0.8}\right)$

$= P(-2 < Z < 1.5)$

$= P(Z < 1.5) - [1 - P(Z < 2)]$

$= 0.9332 - 1 + 0.9772$

$= 0.9104$

(30) $f(x) = \begin{cases} \frac{x}{4} e^{-x/2} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$

$M_X(t) = E(e^{tx})$

$= \int_0^{\infty} e^{tx} \frac{x}{4} e^{-x/2} dx$

$$= \frac{1}{4}\int_0^\infty x e^{-(1/2 - t)x}\, dx$$

$\int u\,dv = uv - \int v\,du$

$= uv - u'v_1 + u''v_2 - u'''v_3 + \cdots$

$$= \frac{1}{4}\left[x\left(\frac{e^{-(\frac{1}{2}-t)x}}{-(\frac{1}{2}-t)}\right) - 1\left(\frac{e^{-(1/2-t)x}}{(1/2-t)^2}\right)\right]_0^\infty$$

$$= \frac{1}{4}\left[0 - \left(0 - \frac{1}{(\frac{1}{2}-t)^2}\right)\right]$$

$$= \frac{1}{4}\left[\frac{4}{(1-2t)^2}\right] = \frac{1}{(1-2t)^2}$$

$$M_x(t) = \frac{1}{(1-2t)^2} = (1-2t)^{-2}$$

$(1-x)^2 = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$

$M_x(t) = 1 + 2(2t) + 3(2t)^2 + 4(2t)^3 + 5(2t)^4 + \cdots$

$= 1 + 4t + 12t^2 + 32t^3 + 80t^4 + \cdots$

$= 1 + 4\frac{t}{1!} + 12\frac{t^2}{2!}\times 2! + 32\frac{t^3}{3!}\times 3! + 80\frac{t^4}{4!}\times 4!$

I II III IV

First 4 moment about origin:

$E(X) = 4$

$E(X^2) = 12 \times 2! = 24$

$E(X^3) = 32 \times 3! = 192$

$E(X^4) = 80 \times 4! = 1920$

(34) $p = 0.2 \quad q = 0.8$

$n = 10$

$p(x) = nC_x\, p^x q^{n-x} \;;\; x = 0, 1, 2, 3 \ldots$

$P(X \le 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$= 10C_0(0.2)^0(0.8)^{10} + 10C_1(0.2)^1(0.8)^9 + 10C_2(0.2)^2(0.8)^8 + 10C_3(0.2)^3(0.8)^7$

$= (0.8)^{10} + 10(0.2)(0.8)^9 + 45(0.2)^2(0.8)^8 + 120(0.2)^3(0.8)^7$

$P(X \le 3) = 0.8791$

$P(X \le 3) \times 200 \approx 176$

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(35) $X \sim \exp(\lambda)$

$\lambda = \frac{1}{2}$

$f(x) = \lambda e^{-\lambda x} \;;\; x > 0$

$$P(X > 2) = \int_2^\infty \frac{1}{2} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_2^\infty$$

$$= \frac{1}{2} \times \frac{-2}{1}\left[0 - e^{-1}\right]$$

$P(X > 2) = e^{-1}$

$P(X > 10 / X > 9) = P(X > 1)$

$$= \int_1^\infty \frac{1}{2} e^{-\frac{1}{2}x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-\frac{1}{2}x}}{-\frac{1}{2}}\right]_1^\infty$$

$$= -\left[0 - e^{-0.5}\right]$$

$$= e^{-0.5}$$

(36) $X \sim \text{Geometric}(p)$

$p = 0.8 \quad q = 0.2$

$p(x) = pq^{x-1} \; ; \; x = 1, 2, \ldots$

(i) $P(X=4) = (0.8)(0.2)^4$

$= 0.00128$

(ii) $P(X<4) = P(X=1) + P(X=2) + P(X=3)$

$= 0.8(0.2)^0 + 0.8(0.2)^1 + 0.8(0.2)^2$

$= 0.8[1 + 0.2 + 0.04]$

$= 0.8[1.24]$

$= 0.992$

(37) $x \sim \text{unif}(a,b) \; ; \; 0 \le x \le 30$

$f(x) = \frac{1}{b-a} = \frac{1}{30-0} = \frac{1}{30}$

P(A passenger has to wait for more than 5 min)

$= P(0 < X < 15) + P(20 < X < 30)$

$= \int_5^{15} \frac{1}{30} dx + \int_{20}^{30} \frac{1}{30} dx$

$= \frac{1}{30} [x]_0^{15} + \frac{1}{30} [x]_{20}^{30}$

$= \frac{15}{30} + \frac{10}{30} = \frac{25}{30}$

$= \frac{5}{6} = 0.834$

(38) $X \sim \exp(\lambda)$

$\lambda = \frac{1}{6}, \; f(x) = \lambda e^{-\lambda x} \; ; \; x > 0$

$n = 6$

$P(X \ge 9)$

$Y \sim B(n, p) \Rightarrow n = 6$

$p = P(X \ge 9)$

$= \int_9^\infty \frac{1}{6} e^{-\frac{1}{6}x} dx$

$= \frac{1}{6} \left[\frac{e^{-\frac{1}{6}x}}{-1/6} \right]_9^\infty$

$= -[0 - e^{-1.5}]$

$= e^{-1.5} = 0.2231$

$p = 0.2231, \; q = 0.7769$

$P(Y \ge 2)$

$= 1 - [P(X=0) + P(X=1)]$

$= 1 - [6C_0 (0.2231)^0 (0.7769)^6 + 6C_1 (0.2231)^1 (0.7769)^5]$

$= 1 - [(0.7769)^6 + 6(0.2231)(0.7769)^5$

$= 1 -$

$=$

(40) $X = \left(\frac{1}{10000}\right)^2 x \, e^{-\frac{1}{10000}x}$

$Y = X - 20000$

$k = 2, \; \lambda = \frac{1}{10000}$

$Y \sim \text{Gamma}(k, \lambda)$

$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)} \; ; \; x > 0$

P(The stock is insufficient)

$= P(X > 30000)$

$= P(X - 20000 > 30000 - 20000)$

$= P(Y > 10000)$

$= \int_{10000}^\infty \left(\frac{1}{10000}\right)^2 \underset{u}{x} \, \underbrace{e^{-\frac{1}{10000}x} dx}_{dv}$

$$= \left(\frac{1}{10000}\right)^2 [uv - u'v_1 + u''v_2 - \cdots]$$

$$= \left(\frac{1}{10000}\right)^2 \left[x\left(\frac{e^{-\frac{1}{10000}x}}{\left(\frac{-1}{10000}\right)}\right) - 1\left(\frac{e^{-\frac{1}{10000}x}}{\left(\frac{1}{10000}\right)^2}\right)\right]_{10000}^{\infty}$$

$$= \left(\frac{1}{10000}\right)^2 [0 - (10000)^2 e^{-1} - (10000)^2 e^{-1}]$$

$$= \left(\frac{1}{10000}\right)^2 [2e^{-1}(10000)^2]$$

$$= 2e^{-1}$$

(41) $X \sim N(\mu, \sigma^2)$

$\mu = 1000$, $\sigma = 200$

(i) $P(X < 700)$

$= P\left(z < \frac{700-1000}{200}\right)$

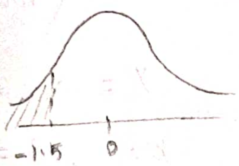
$= P(Z < -1.5)$

$= P(z > 1.5)$

$= 1 - P(z < 1.5)$

$= 1 - 0.9332$

$= 0.0668$



The no. of lamps fails in less than 700 hrs $= 0.0668 \times 2000$

$= 133$

(ii) $P(x < k) = \frac{10}{100}$

$P\left(\frac{x-\mu}{\sigma} < \frac{k-\mu}{\sigma}\right) = 0.1$

$P\left(z < \frac{k-1000}{200}\right) = 0.1$

$P\left(z > \frac{1000-k}{200}\right) = 0.1$

$1 - P\left(z < \frac{1000-k}{200}\right) = 0.1$

$0.9 = P\left(z < \frac{1000-k}{200}\right)$

$\frac{1000-k}{200} = 1.28$

$1000 - k = 1.28 \times 200$

$1000 - k = 256$

$k = 1000 - 256$

$k = 744$

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(42) $X \sim N(\mu, \sigma^2)$

$\mu = 662$, $\sigma = 32$

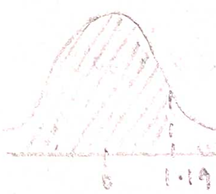
(i) $P(X > 700) = P\left(z > \frac{700-662}{32}\right)$

$= P(z > 1.19)$

$= 1 - P(z < 1.19)$

$= 1 - 0.8830$

$= 0.117$



The no. of plots gives the yield of more than 700kg is

$1000 \times 0.117 \approx 117$

(ii) $P(X < 650) = P\left(z < \frac{650-662}{32}\right)$

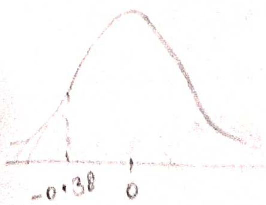
$= P(z < -0.38)$

$= P(z > 0.38)$

$= 1 - P(z < 0.38)$

$= 1 - 0.6480$

$= 0.352$



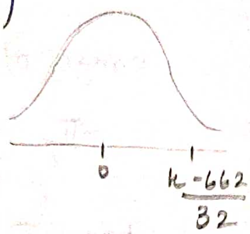
(iii) $P(X \geq k) = \frac{100}{1000}$

$P\left(\frac{X-\mu}{\sigma} \geq \frac{k-\mu}{\sigma}\right) = 0.1$

$P\left(Z \geq \frac{k-662}{32}\right) = 0.1$

$1 - P\left(Z < \frac{k-662}{32}\right) = 0.1$

$0.9 = P\left(Z < \frac{k-662}{32}\right)$



$\frac{k-662}{32} = 1.28$

$k - 662 = 1.28 \times 32$

$k = 662 + 1.28 \times 3$

$k = 702.96 \approx 703$

(43) $X \sim unif(-a, a) \quad ; a > 0$

$P(|X| < 1) = P(|X| > 1)$

$P(|X| < 1) = 1 - P(|X| < 1)$

$2P(|X| < 1) = 1$

$P(|X| < 1) = \frac{1}{2}$

$P(-1 < x < 1) = \frac{1}{2}$

$\int_{-1}^{1} f(x)\,dx = \frac{1}{2}$

$\Rightarrow f(x) = \frac{1}{b-a} = \frac{1}{a-(-a)} = \frac{1}{2a}$

$\int_{-1}^{1} \frac{1}{2a}\,dx = \frac{1}{2}$

$\frac{1}{a}[x]_{-1}^{1} = 1$

$\frac{1}{a}[2] = 1$

$a = 2$

Function of random variable :

(*) X

Mass func

x	-1	0	1
$p(x)$	$1/6$	$2/6$	$3/6$

$Y = 2X + 1$

y	-1	1	3
$P(y)$	$1/6$	$2/6$	$3/6$

$P(Y=-1) = P(2X+1=-1)$
$= P(X=-1) = \frac{1}{6}$

$P(Y=1) = P(2X+1=1)$
$= P(X=0) = 2/6$

$P(Y=2) = P(2X+1=3)$
$= P(X=1) = 3/6$

(*) $Y = |X|$

y	0	1
$p(y)$	$\frac{2}{6}$	$\frac{4}{6}$

$P(Y=0) = P(|X|=0)$
$= P(X=0) = \frac{2}{6}$

$P(Y=1) = P(|X|=1)$
$= P(X=-1) + P(X=1)$
$= \frac{4}{6}$

(*) X

$f(x) = e^{-x} \quad ; x > 0$

$Y = 2 \times 1 + 1 = 1$

$$\left\{ \begin{array}{l} y - 1 = 2x \\ x = \frac{y-1}{2} \end{array} \right.$$

$$\boxed{f(y) = f(x) \frac{dx}{dy}} = e^{-x} \cdot \frac{1}{2}$$

$$f(y) = \frac{1}{2} e^{-\left(\frac{y-1}{2}\right)} \quad ; y > 1$$

$Y = \tan X$

$x = \tan^{-1}(y)$

$$\frac{dx}{dy} = \frac{1}{1+y^2}$$

$$f(y) = f(x) \left| \frac{dx}{dy} \right|$$

$$= \frac{1}{\pi} \frac{1}{1+y^2}$$

Range of y:

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\tan\left(-\frac{\pi}{2}\right) < \tan x < \tan \frac{\pi}{2}$$

$$-\infty < \tan x < \infty$$

Exercise:

(32) x

$$f(x) = \left\{ \begin{array}{ll} 2x & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{array} \right.$$

$$y = 8x^3$$

$$f(y) = f(x) \left| \frac{dx}{dy} \right|$$

$$y = 8x^3 \Rightarrow x = \frac{1}{2} y^{1/3}$$

$$\frac{dx}{dy} = \frac{1}{2} \cdot \frac{1}{3} y^{1/3 - 1}$$

$$= \frac{1}{6} y^{-2/3}$$

$$f(y) = 2x \cdot \frac{1}{6} y^{-2/3}$$

$$= \frac{1}{3} \cdot \frac{1}{2} y^{1/3} y^{-2/3}$$

$$= \frac{1}{6} y^{-1/3}$$

$$f(y) = \frac{1}{6y^{1/3}} \quad ; 0 < y < 8$$

Range of y

$0 < x < 1$

$0 < \frac{1}{2} y^{1/3} < 1$

$0 < y^{1/3} < 2$

$0 < y < 8$

(23) $X \sim \text{unif}\left(-\pi/2, \pi/2\right)$

$$f(x) = \frac{1}{b-a} = \frac{1}{\pi/2 + \pi/2}$$

$$= \frac{1}{\pi} \quad ; -\frac{\pi}{2} < x < \frac{\pi}{2}$$

(11)

x	1	2	3	4	5	6	7
$p(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

$$\because \sum_x p(x) = 1$$

$$10k^2 + 9k - 1 = 0$$

$$10k^2 + 10k - k - 1 = 0$$

$$10k(k+1) - 1(k+1) = 0$$

$$(10k - 1)(k+1) = 0$$

$$\boxed{k = \frac{1}{10}} \qquad k = -1$$

$\hookrightarrow$ Not possible

$$F(x) = P(X \le x)$$

$$F(1) = P(X \le 1) = k = \frac{1}{10}$$

$$F(2) = P(X \le 2) = k + 2k$$

$$= 3k = \frac{3}{10}$$

$$F(3) = P(X \le 3) = k + 2k + 2k$$

$$= 5k = \frac{5}{10}$$

$F(4) = 8k = \frac{8}{10}$

$F(5) = 8k + k^2 = \frac{80}{100} + \frac{1}{100} = \frac{81}{100}$

$F(6) = 8k + 3k^2 = \frac{80}{100} + \frac{3}{100} = \frac{83}{100}$

$F(7) = 1$

$P(X \le k) > \frac{1}{2}$

$P(X \le 1) = \frac{1}{10} \not> \frac{1}{2}$

$P(X \le 2) = \frac{3}{10} \not> \frac{1}{2}$

$P(X \le 3) = 5/10 \not> 1/2$

$P(X \le 4) = \frac{8}{10} > \frac{1}{2}$

$P(X \le 5) = \frac{81}{100} > \frac{1}{2}$

$P(X \le 6) = \frac{83}{100} > \frac{1}{2}$

$P(X \le 7) = 1 > \frac{1}{2}$

The min value of k : 4

24, 31, 33

24) $n = 5$ $p = 0.4$ $q = 0.6$

$P(x) = nC_x p^x q^{n-x}$; $x = 0, 1, 2, 3, 4, 5, ..$

(i) $P(X = 0)$

$= nC_x p^x q^{n-x}$

$= 5C_0 p^0 (0.6)^5$

$= (0.6)^5$

$= 0.07776$

(ii) $P(X < 2) = P(X=0) + P(X=1) + P(X=2)$

$= (0.6)^5 + 5C_1 (0.4)^1 (0.6)^4$

$+ 5C_2 (0.4)^2 (0.6)^3$

$= 0.41860$

(iii) $P(X > 3) = P(X=3) + P(X=4) + P(X=5)$

$= 5C_3 (0.4)^3 (0.6)^2 + 5C_4$

$(0.4)^4 (0.6)^1 + 5C_5 (0.4)^5 (0.$

$= 0.08704$

31) $F(x) = \begin{cases} 0 & ; x \le 0 \\ c(1-e^{-x}) & ; x > 0 \end{cases}$

$f(x) = c \frac{d}{dx}(1-e^{-x})$

$= c\left(0 - (-e^{-x})\right)$

(i) $f(x) = ce^{-x}$

$\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_0^{\infty} ce^{-x}\,dx = 1$

$c\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$

$-c[0-1] = 1$

$\boxed{c = 1}$

(ii) $P(1 \le x \le 2)$

$f(x) = \begin{cases} 0 & ; x \le 0 \\ e^{-x} & ; x > 0 \end{cases}$

$X \sim \exp(\lambda)$ $\lambda = 1$

$P(1 \le X \le 2) = \int_1^2 e^{-x}\,dx$

$= -[e^{-x}]_1^2$

$= -[e^{-2} - e^{-1}] = e^{-1} - e^{-2}$

3) $X \sim poisson(\lambda)$

$$p(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!} \quad ; x = 0, 1, 2, \cdots$$

$$\lambda = \frac{3}{5} = 0.6$$

$$P(X=0) = \frac{e^{-0.6}(0.6)^0}{0!} = e^{-0.6}$$

$$P(X=0) \times 100 = e^{-0.6} \times 100$$

$$=$$

$$P(X=1) = \frac{e^{-0.6}(0.6)^1}{1!} = 0.6\,e^{-0.6}$$

$$P(X=1) \times 100 = 0.6\,e^{-0.6} \times 100$$

$$=$$

$$P(X=2) = \frac{e^{-0.6}(0.6)^2}{2!} =$$

$$P(X=2) \times 100 =$$

$$=$$

$$P(X=3) = \frac{e^{-0.6}(0.6)^2}{3!} =$$

$$P(X=3) \times 100 =$$

$$=$$

$$P(X > 3) = 1 - [P(X < 3)]$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$$

$$= 1 - [$$