1/10/2020

UNIT- 1
LINEAR ALGEBRA

Basic concepts:

Binary operations group

Any expression which have $A \neq NULL$ $(A \neq \phi)$

$$\boxed{*: A \times A \rightarrow A}$$

$A = \{1,2,3\}$; $B = \{a,b\}$

$A \times B = \{(x,y) / x \in A, y \in B\}$

$= \{(1,a)(1,b)(2,a)(2,b)(3,a)(3,b)\}$

Natural numbers:

$+ : N \times N \rightarrow N$ ✓

$- : N \times N \rightarrow N$ ✗

$\times : N \times N \rightarrow N$ ✓

$\div : N \times N \rightarrow N$ ✗

Integer :

$+ : Z \times Z \rightarrow Z$ ✓

$- : Z \times Z \rightarrow Z$ ✓

$\times : Z \times Z \rightarrow Z$ ✓

$\div : Z \times Z \rightarrow Z$ ✗

Group $(G, *)$

(i) $\forall a,b \in G$

$a \times b \in G$ (Closure)

(ii) $\forall a,b,c \in G$

SEMI GROUP $(a \times b) \times c = a \times (b \times c)$ (Associative)

(iii) $\forall a \in G \; \exists \; e \in G$ uniquely exist

such that

MONOID $a \times e = e \times a = a$

---

satisfies 3 properties

(iv) For each $a \in G \; \exists \; a^{-1} \in G$

such that

$a \times a^{-1} = a^{-1} \times a = e$

(v) $\forall a,b \in G$

$a \times b = b \times a$ (Commutative)

ABELIAN GROUP → satisfies all properties.

$(N, +)$ Natural — iii, iv fails

$(W, +)$ Whole — Additive inverse fails

$(Z, +)$ Int — Group

$(Q, +)$ Rational — Group

$(\mathbb{R}, +)$ Set of Relation — Group

$(C, +)$ Complex — Group

$(N, \cdot)$ Inverse property fails

$(W, \cdot)$ Not a group

$(Z, \cdot)$

$(Q, \cdot)$ ✗ $(Q - \{0\}, \cdot)$ group

$(R - \{0\}, \cdot)$ group

$(C - \{0\}, \cdot)$ group

$\left\{ \begin{array}{l} \text{Addition} - + \\ \text{Scalar multiplication} - \cdot \end{array} \right\}$

28/10/2023

VECTOR SPACE

$(V, +, \cdot)$, $\overset{\text{Field}}{F}(+, \cdot)$ is said to be vector space if it follows following condition (5 properties):

(i) $(V, +)$ is an Abelian group
vector addn

(*) For all $x, y \in V$, $a \in F$

vector Mult (ii) $a(x+y) = a \cdot x + a \cdot y$

(iii) $\underset{\text{scalar}}{1} \cdot x = x$

(iv) $(a+b) \cdot x = a \cdot x + b \cdot x$

(v) $a \cdot (bx) = (ab) \cdot x$

Ring $(R, +, \cdot)$ → Multiplication

Non set relation (Ring)

$(R, +)$ is an Abelian group

Distributive:

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

$a \cdot (b + c) = a \cdot b + a \cdot c$

Commutative Ring:

$(R, \cdot)$

Integral Domain - Any commutative ring, R is said to be integral domain for binary operation $(R, +, \cdot)$ if does not have zero divisor

$a \neq 0$, $b \neq 0$, $ab \neq 0$

Field:

$(R, +, \cdot)$ ∀ non-zero elements has an inversive element.

Field of z

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ → Remainder of 6

$(\mathbb{Z}_6, \cdot) = 2 \cdot 3 = 6 = 0$ (Remainder)

$$\left\{ R = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a, b, c, d \in R \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

INNER PRODUCT SPACE ($\langle \ , \ \rangle$)

Inner product

$|| : M_{n \times n}(\mathbb{R}) \to \mathbb{R}$

V - vector space

F - $\mathbb{R}$ or $\mathbb{C}$ (Real or complex)

$\langle \ , \ \rangle : V \times V \longrightarrow \mathbb{R}$ $\quad x, y, z \in$ … $c \in$ …

satisfy property:

(i) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

(ii) $\langle cx, y \rangle = c \langle x, y \rangle$

$\langle x, cy \rangle = \bar{c} \langle x, y \rangle$

(iii) $\overline{\langle x, y \rangle} = \langle y, x \rangle$

(iv) $\langle x, x \rangle > 0$ if $x \neq 0$

{vector × vector = scalar}

(*) $V = \mathbb{R}^2$, $F = \mathbb{R}$

($\mathbb{R}$ → Field of real number)

$\langle x, y \rangle = \sum x \bar{y}$

Standard inner product:

$\langle (a, b_1), (a_2 b_2) \rangle$

Norm of any vector: $(\underset{\text{Norm of } x}{\|x\|})$

V - Inner product space

$x \in V$

$\|x\| = \sqrt{\langle x, x \rangle}$

$\|x\|^2 = \langle x, x \rangle$

QR decomposition / QR factorisation / QR transportation:

→ Decompose Eigen values / vectors of matrix

$\boxed{A = QR}$ → upper triangular matrix; Q ↙ orthogonal matrix

[LU – Lower △le, upper △le]

$Q^T = Q^{-1}$

• $A = \bar{A}$ {A is square matrix}

$A^TA = AA^T = I$

$A^{-1}A = AA^{-1} = I$

$A^{-1}A = A^TA$

$A^{-1} = A^T$

$$A = \begin{array}{ccc} a_1 & a_2 & a_3 \\ \end{array} \to \text{column name}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$= (a_1 | a_2 | a_3)$

• $u_1 = a_1$, $e_1 = \dfrac{u_1}{\|u_1\|}$ $\left\{\dfrac{\text{vector}}{\text{scalar}} = \text{vector}\right\}$

• $u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$e_2 = \dfrac{u_2}{\|u_2\|}$

• $u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$

• $u_{k+1} = a_{k+1} - \langle a_{k+1}, e_1\rangle e_1 - \langle a_{k+1}, e_2\rangle e_2 - \cdots \langle a_{k+1}, e_k\rangle e_k$

Matrix $Q = (e_1 \;\; e_2 \;\; e_3)$

$$R = \begin{pmatrix} a_1\cdot e_1 & a_1\cdot e_2 & a_1\cdot e_3 \\ 0 & a_2\cdot e_2 & a_2\cdot e_3 \\ 0 & 0 & a_3\cdot e_3 \end{pmatrix}$$

$\underbrace{a_i, e_i}_{\text{vectors}}$

$a_i \cdot e_i$ ↙ Dot product

$A = QR$

$AX = B$

$QRX = B$

premultiply: Q^{-1}

$Q^{-1}QRX = Q^{-1}B$ 3 by 1

$RX = Q^TB$ → Matrix

$Y = Q^TB$ ($\because RX = Y$)

Q: Find the QR decomposition of matrix A

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$\Rightarrow (a_1 \;\; a_2 \;\; a_3)$

* $a_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (1 \;\; 1 \;\; 0)^T$

* $a_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1 \;\; 0 \;\; 1)^T$

* $a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (0 \;\; 1 \;\; 1)^T$

• $u_1 = a_1 = (1 \;\; 1 \;\; 0)$

$e_1 = \dfrac{u_1}{\|u_1\|}$

$\|u_1\| = \sqrt{\langle u_1, u_1\rangle}$

$= \sqrt{1^2 + 1^2 + 0} = \sqrt{2}$

$e_1 = \dfrac{(1 \;\; 1 \;\; 0)}{\sqrt{2}}$

* $e_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T$

• $u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$\langle a_2, e_1\rangle = (1 \;\; 0 \;\; 1)\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$

$= \frac{1}{\sqrt{2}} + 0 + 0 = \frac{1}{\sqrt{2}}$

$= (1 \;\; 0 \;\; 1) - \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} (1 \;\; 1 \;\; 0)$

$= (1 \;\; 0 \;\; 1) - \left(\frac{1}{2} \;\; \frac{1}{2} \;\; 0\right)$

$u_2 = \left(\frac{1}{2} \;\; \frac{-1}{2} \;\; 1\right)$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{\left(\frac{1}{2} \quad \frac{-1}{2} \quad 1\right)}{\sqrt{\langle u_2, u_2\rangle}}$$

$$= \frac{\left(\frac{1}{2} \quad \frac{-1}{2} \quad 1\right)}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}$$

$$e_2 = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$$

$$* \; e_2 = \left(\frac{1}{\sqrt{6}} \quad \frac{-1}{\sqrt{6}} \quad \frac{\sqrt{2}}{\sqrt{3}}\right)^T$$

$$\cdot \; u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$$

$$\langle a_3, e_1\rangle = (0 \quad 1 \quad 1)\left(\tfrac{1}{\sqrt{2}} \quad \tfrac{1}{\sqrt{2}} \quad 0\right)$$

$$= 0 + \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$$

$$\langle a_3, e_2\rangle = (0 \quad 1 \quad 1)\left(\frac{1}{\sqrt{6}} \quad -\frac{1}{\sqrt{6}} \quad \frac{\sqrt{2}}{\sqrt{3}}\right)$$

$$= 0 - \frac{1}{\sqrt{6}} + \frac{\sqrt{2}\times\sqrt{2}}{\sqrt{3}\times\sqrt{2}} = \frac{-1+2}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} = \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2}\right)$$

$$= (0 \quad 1 \quad 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}} \quad 0\right)$$

$$- \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2}\right)\frac{\sqrt{2}}{\sqrt{3}}\left(\frac{1}{2} \quad \frac{-1}{2} \quad 1\right)$$

$$= (0 \quad 1 \quad 1) - \left(\frac{1}{2} \quad \frac{1}{2} \quad 0\right)$$

$$- \left(\frac{1}{6} \quad \frac{-1}{6} \quad \frac{1}{3}\right)$$

$$u_3 = \left(-\frac{2}{3} \quad \frac{2}{3} \quad \frac{2}{3}\right)$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{\left(\frac{-2}{3} \quad \frac{2}{3} \quad \frac{2}{3}\right)}{\sqrt{\frac{4}{9}+\frac{4}{9}+\frac{4}{9}}}$$

$$= \frac{\sqrt{3}}{2}\left(\frac{-2}{3} \quad \frac{2}{3} \quad \frac{2}{3}\right)$$

$$* \; e_3 = \left(\frac{-1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}}\right)^T$$

$$Q = (e_1 \quad e_2 \quad e_3)$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{\sqrt{2}}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$$

$$R = \begin{pmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_3 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+0 & \frac{1}{\sqrt{2}}+0+0 & 0+\frac{1}{\sqrt{2}}+0 \\ 0 & \frac{1}{\sqrt{6}}+0+\frac{\sqrt{2}}{\sqrt{3}} & 0-\frac{1}{\sqrt{6}}+\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & 0 & 0+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{3}} \end{pmatrix}$$

$$R = \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} & \frac{1}{\sqrt{6}} \\ 0 & 0 & 2/\sqrt{3} \end{pmatrix}$$

Q: Find QR decomposition

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = (1 \quad 0 \quad 1)^T$$

$$a_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = (2 \quad 1 \quad 0)^T$$

$$a_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = (0 \quad 1 \quad 1)^T$$

- $u_1 = a_1 = (1 \quad 0 \quad 1)$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{(1 \quad 0 \quad 1)}{\sqrt{1^2+0+1^2}}$$

* $e_1 = \left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right)^T$

- $u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$$\langle a_2, e_1\rangle = (2 \quad 1 \quad 0)\left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{2}{\sqrt{2}} + 0 + 0\right) = \sqrt{2}$$

$$= (2 \quad 1 \quad 0) - \sqrt{2}\left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right) \quad \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= (2 \quad 1 \quad 0) - (1 \quad 0 \quad 1)$$

$$u_2 = (1 \quad 1 \quad -1)$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{(1 \quad 1 \quad -1)}{\sqrt{1^2+1^2+1^2}}$$

* $e_2 = \left(\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad \frac{-1}{\sqrt{3}}\right)$

- $u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$

$$\langle a_3, e_1\rangle = (0 \quad 1 \quad 1)\left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right)$$

$$= 0 + 0 + \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\langle a_3, e_2\rangle = (0 \quad 1 \quad 1)\left(\frac{1}{\sqrt{3}} \quad \frac{1}{\sqrt{3}} \quad -\frac{1}{\sqrt{3}}\right)$$

$$= 0 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} = 0$$

$$= (0 \quad 1 \quad 1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} \quad 0 \quad \frac{1}{\sqrt{2}}\right) - 0$$

$$= (0 \quad 1 \quad 1) - \left(\frac{1}{2} \quad 0 \quad \frac{1}{2}\right)$$

$$u_3 = \left(\frac{-1}{2} \quad 1 \quad \frac{1}{2}\right)$$

$$e_3 = \frac{u_3}{\|u_3\|} = \frac{\left(-\frac{1}{2} \quad 1 \quad \frac{1}{2}\right)}{\sqrt{\frac{1}{4}+1+\frac{1}{4}}}$$

$$= \frac{\sqrt{2}}{\sqrt{3}}\left(\frac{-1}{2} \quad 1 \quad \frac{1}{2}\right)$$

* $e_3 = \left(\frac{-1}{\sqrt{6}} \quad \frac{\sqrt{2}\times\sqrt{2}}{3\times\sqrt{2}} \quad \frac{1}{\sqrt{6}}\right)^T$

* $e_3 = \left(\frac{-1}{\sqrt{6}} \quad \frac{2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)^T$

$$Q = (e_1 \quad e_2 \quad e_3)$$

$$Q = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$R = \begin{pmatrix} a_1 \cdot e_1 & a_2 \cdot e_1 & a_3 \cdot e_1 \\ 0 & a_2 \cdot e_2 & a_2 \cdot e_2 \\ 0 & 0 & a_3 \cdot e_3 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} + 0 + \frac{1}{\sqrt{2}} & \frac{2}{\sqrt{2}} + 0 + 0 & 0 + 0 + \frac{1}{\sqrt{2}} \\ 0 & \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + 0 & 0 + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}} \\ 0 & 0 & 0 + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}} \end{pmatrix}$$

$$R = \begin{pmatrix} \sqrt{2} & \sqrt{2} & 1/\sqrt{2} \\ 0 & \sqrt{3} & 0 \\ 0 & 0 & 3/\sqrt{6} \end{pmatrix}$$

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Q: Find the decomposition matrix

$$A = \begin{pmatrix} -4 & 2 & 2 \\ 3 & -3 & 3 \\ 6 & 6 & 0 \end{pmatrix}$$

$$a_1 = (-4 \quad 3 \quad 6)^T$$

$$a_2 = (2 \quad -3 \quad 6)^T$$

$$a_3 = (2 \quad 3 \quad 0)^T$$

$\bullet\ u_1 = a_1 = (-4 \quad 3 \quad 6)$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{(-4 \quad 3 \quad 6)}{\sqrt{4^2+3^2+6^2}}$$

$$* \ e_1 = \left(\frac{-4}{\sqrt{61}} \quad \frac{3}{\sqrt{61}} \quad \frac{6}{\sqrt{61}}\right)$$

$\bullet\ u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$$\langle a_2, e_1\rangle = (2 \quad -3 \quad 6)\left(\frac{-4}{\sqrt{61}} \quad \frac{3}{\sqrt{61}} \quad \frac{6}{\sqrt{61}}\right)$$

$$= \left(\frac{-8}{\sqrt{61}} - \frac{9}{\sqrt{61}} + \frac{36}{\sqrt{61}}\right)$$

$$= \frac{19}{\sqrt{61}}$$

$$= (2 \quad -3 \quad 6) - \frac{19}{\sqrt{61}}\left(\frac{-4}{\sqrt{61}} \quad \frac{3}{\sqrt{61}} \quad \frac{6}{\sqrt{61}}\right)$$

$$= (2 \quad -3 \quad 6) - \left(\frac{-76}{61} \quad \frac{57}{61} \quad \frac{114}{61}\right)$$

$$= \left(\frac{122+76}{61} \quad \frac{-183-57}{61} \quad \frac{244-114}{61}\right)$$

$$u_2 = \left(\frac{198}{61} \quad \frac{-240}{61} \quad \frac{130}{61}\right)$$

$$e_2 = \frac{u_2}{\|u_2\|} = \frac{\left(\frac{198}{61} \quad \frac{-240}{61} \quad \frac{130}{61}\right)}{\sqrt{\left(\frac{198}{61}\right)^2 + \left(\frac{240}{61}\right)^2 + \left(\frac{130}{61}\right)^2}}$$

$$= \frac{61}{337}\left(\frac{198}{61} \quad \frac{-240}{61} \quad \frac{130}{61}\right)$$

$$* \ e_2 = \left(\frac{198}{337} \quad \frac{-240}{337} \quad \frac{130}{337}\right)$$

$\bullet\ u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$

$$\langle a_3, e_1\rangle = (6 \quad 6 \quad 0)\left(\frac{-4}{\sqrt{61}} \quad \frac{3}{\sqrt{61}} \quad \frac{6}{\sqrt{61}}\right)$$

$$= \frac{-24}{\sqrt{61}} + \frac{18}{\sqrt{61}} = \frac{-6}{\sqrt{61}}$$

$$\langle a_3, e_2\rangle = (6 \quad 6 \quad 0)\left(\frac{198}{337} \quad \frac{-240}{337}\right.$$

$$= \frac{1188 - 1440}{337}$$

$$= -\frac{252}{337}$$

$$= (6 \quad 6 \quad 0) - \left(\frac{-6}{\sqrt{61}}\right)\left(\frac{-4}{\sqrt{61}} \quad \frac{3}{\sqrt{61}} \quad \frac{6}{\sqrt{61}}\right)$$

$$- \left(\frac{-252}{337}\right)\left(\frac{198}{337} \quad \frac{-240}{337} \quad \frac{130}{337}\right)$$

$$= (6 \quad 6 \quad 0) - \left(\frac{24}{61} \quad \frac{-18}{61} \quad \frac{-36}{61}\right)$$

$$A = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$a_1 = (-2 \quad 1 \quad 1)^T$$

$$a_2 = (1 \quad -1 \quad 1)^T$$

$$a_3 = (1 \quad 1 \quad 0)^T$$

$$u_1 = a_1 = (-2 \quad 1 \quad 1)$$

$$e_1 = \frac{u_1}{\|u_1\|} = \frac{(-2 \quad 1 \quad 1)}{\sqrt{2^2+1^2+1^2}}$$

$$* \ e_1 = \left(\frac{-2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)^T$$

$\bullet\ u_2 = a_2 - \langle a_2, e_1\rangle e_1$

$$= (1 \quad -1 \quad 1) + \frac{2}{\sqrt{6}}\left(\frac{-2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)$$

$$\langle a_2, e_1\rangle = (1 \quad -1 \quad 1)\left(-2/\sqrt{6} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)$$

$$= \frac{-2}{\sqrt{6}} - \frac{2}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{-2}{\sqrt{6}}$$

$$= (1 \quad -1 \quad 1) + \left(-4/6 \quad 2/6 \quad 2/6\right)$$

$$u_2 = \left(\frac{2}{6} \quad \frac{-4}{6} \quad \frac{8}{6}\right) = \left(\frac{1}{3} \quad -\frac{2}{3} \quad \frac{4}{3}\right)$$

$e_2 = \dfrac{u_2}{\|u_2\|} = \dfrac{\left(\frac{1}{3} \quad \frac{-2}{3} \quad \frac{4}{3}\right)}{\sqrt{\frac{1}{9}+\frac{4}{9}+\frac{16}{9}}}$

$= \dfrac{3}{\sqrt{21}} \left(\frac{1}{3} \quad \frac{-2}{3} \quad \frac{4}{3}\right)$

* $e_2 = \left(\frac{1}{\sqrt{21}} \quad \frac{-2}{\sqrt{21}} \quad \frac{4}{\sqrt{21}}\right)^T$

• $u_3 = a_3 - \langle a_3, e_1\rangle e_1 - \langle a_3, e_2\rangle e_2$

$\langle a_3, e_1\rangle = (1 \quad 1 \quad 0)\left(\frac{-2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)$

$= \frac{-2}{\sqrt{6}} + \frac{1}{\sqrt{6}} = \frac{-1}{\sqrt{6}}$

$\langle a_3, e_2\rangle = (1 \quad 1 \quad 0)\left(\frac{1}{\sqrt{21}} \quad \frac{-2}{\sqrt{21}} \quad \frac{4}{\sqrt{21}}\right)$

$= \frac{1}{\sqrt{21}} - \frac{2}{\sqrt{21}} + 0 = \frac{-1}{\sqrt{21}}$

$= (1 \quad 1 \quad 0) + \frac{1}{\sqrt{6}}\left(\frac{-2}{\sqrt{6}} \quad \frac{1}{\sqrt{6}} \quad \frac{1}{\sqrt{6}}\right)$

$+ \frac{1}{\sqrt{21}}\left(\frac{1}{\sqrt{21}} \quad \frac{-2}{\sqrt{21}} \quad \frac{4}{\sqrt{21}}\right)$

$= (1 \quad 1 \quad 0) + \left(\frac{-2}{6} \quad \frac{1}{6} \quad \frac{1}{6}\right)$

$+ \left(\frac{1}{21} \quad \frac{-2}{21} \quad \frac{4}{21}\right)$

$= \left(1 - \frac{2}{6} + \frac{1}{21} \quad 1 + \frac{1}{6} - \frac{2}{21} \quad 0 + \frac{1}{6} + \frac{4}{21}\right)$

$= \left(\frac{126-42+6}{126} \quad \frac{126+21-12}{126} \quad \frac{0+21+24}{126}\right)$

$u_3 = \left(\frac{90}{126} \quad \frac{135}{126} \quad \frac{45}{126}\right)$

$e_3 = \dfrac{u_3}{\|u_3\|} = \dfrac{\left(\frac{90}{126} \quad \frac{135}{126} \quad \frac{45}{126}\right)}{\sqrt{\frac{90^2+135^2+45^2}{126^2}}}$

$= \frac{126}{168}\left(\frac{90}{126} \quad \frac{135}{126} \quad \frac{45}{126}\right)$

* $e_3 = \left(\frac{90}{168} \quad \frac{135}{168} \quad \frac{45}{168}\right)^T$

$Q = (e_1 \quad e_2 \quad e_3)$

$= \begin{pmatrix} -2/\sqrt{6} & 1/\sqrt{21} & 90/168 \\ 1/\sqrt{6} & -2/\sqrt{21} & 135/168 \\ 1/\sqrt{6} & 4/\sqrt{21} & 45/168 \end{pmatrix}$

× wrong

Singular value decomposition (SVD)

singular value - Eigen value (or) characteristic value

○ How to find singular value for square matrix?

* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times2}$

Non zero column vector matrix

scalar

$\boxed{\underset{2\times2}{A}\ \underset{2\times1}{X} = \lambda \underset{2\times1}{X}}$ → solve ①

λ is eigen value of A

X is corresponding eigen vector

A - can have 2 eigen values

② Characteristic Eqn:

$$\boxed{\lambda^2 - S_1\lambda + S_2 = 0}$$

$S_1 = a_{11} + a_{22}$

$S_2 = |A|$ (Determinant)

* $A_{3\times3}$ Matrix:

$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\boxed{\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0}$$

$S_1 = a_{11} + a_{22} + a_{33}$

$$S_2 = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$S_3 = |A|$

$\Rightarrow$ 3 Eigen values: $\lambda_1, \lambda_2, \lambda_3$

$AX = \lambda X$

$AX - \lambda X = 0$

$(A - \lambda I)X = 0$

$|A - \lambda I| = 0$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$X_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad X_2 = (\ ) \quad X_3 = (\ )$

(*) $\boxed{A = U \Sigma V^T}$

Here, U = orthogonal matrix

Σ = Diagonal matrix

V^T = Transpose

⊙ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Find SVD:

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$AA^T = \underset{2\times2}{(\ )}\ \underset{2\times2}{(\ )} = \underset{2\times2}{(\ )}$

$\lambda_1 = \sigma_1^2 \ ; \ \lambda_2 = \sigma_2^2$

u_1 = Normalised eigen vector of X_1

u_2 = Normalised eigen vector of X_2

$U = \begin{pmatrix} u_1 & u_2 \\ | & | \end{pmatrix}$

$\Sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$

$A^TA \Rightarrow v_1, v_2$ (Eigen vectors)

$V = \begin{pmatrix} v_1 & v_2 \\ | & | \end{pmatrix}$ { λ, σ - scalar

(*) For 2x2 Matrix,

$\sigma_1 v_1 = A^T u_1 \Rightarrow v_1 = \frac{1}{\sigma_1} A^T u_1$

$\sigma_2 v_2 = A^T u_2 \Rightarrow v_2 = \frac{1}{\sigma_2} A^T u_2$

$\sigma_1 u_1 = A v_1$

$\sigma_2 u_2 = A v_2$

⊙ Normalised Eigen Vector:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{a}{\sqrt{a^2+b^2+c^2}} \\ \frac{b}{\sqrt{a^2+b^2+c^2}} \\ \frac{c}{\sqrt{a^2+b^2+c^2}} \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{14} \\ 2/\sqrt{14} \\ 3/\sqrt{14} \end{pmatrix}$$

Q: Find SVD of matrix $A = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}$
①

$$A = U \Sigma V^T$$

row × column

$$C = AA^T = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 16+16 & -12+12 \\ -12+12 & 9+9 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 32 & 0 \\ 0 & 18 \end{pmatrix} = C$$

If AA^T is diagonal matrix then

eigen values: $\lambda_1 = 32$ $\lambda_2 = 18$

$\sigma_1^2 = 32$ $\sigma_2^2 = 18$

$\sigma_1 = \sqrt{32}$ $\sigma_2 = \sqrt{18}$

$$\Sigma = \begin{pmatrix} \sqrt{32} & 0 \\ 0 & \sqrt{18} \end{pmatrix} = \begin{pmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix}$$

To find Eigen vectors,

solve: $(C - \lambda I)X = 0$

$$\left[\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \lambda I \right] \left[C = \begin{pmatrix} 32 & 0 \\ 0 & 18 \end{pmatrix} \right]$$

$$\begin{pmatrix} 32-\lambda & 0 \\ 0 & 18-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

Homogenous eqns:

$$\left.\begin{aligned} (32-\lambda)x_1 + 0x_2 &= 0 \\ 0x_1 + (18-\lambda)x_2 &= 0 \end{aligned}\right\} \text{——①}$$

· when $\lambda = 32$

$$① \Rightarrow \left.\begin{aligned} 0x_1 + 0x_2 &= 0 \\ 0x_1 - 14x_2 &= 0 \end{aligned}\right\}$$

$x_2 = 0$

$x_1 = k$, $k \in \mathbb{R}$ [k - such a way we get normalised vector]

Let $k = 1$,

$$X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

· when $\lambda = 18$

$$① \Rightarrow \left.\begin{aligned} 14x_1 + 0x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}\right\}$$

$x_1 = 0$

$x_2 = t$, $t \in \mathbb{R}$

Let $t = 1$,

$$X_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\therefore u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ {Normalised Eigen vector of X_1}

$u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ {NEV of X_2}

$$U = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$v_1 = \frac{1}{\sigma_1} A^T u_1$$

$$= \frac{1}{4\sqrt{2}} \begin{pmatrix} 4 & -3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{4\sqrt{2}} \begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A^T u_2$$

$$= \frac{1}{3\sqrt{2}} \begin{pmatrix} 4 & -3 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{3\sqrt{2}} \begin{pmatrix} -3 \\ 3 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$\begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 4\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{pmatrix} 4+0 & 4+0 \\ 0-3 & 0+3 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ -3 & 3 \end{pmatrix}$$

② Find SVD of matrix $A = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix}$

$$C = AA^T = \begin{pmatrix} 3 & 0 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 12 \\ 12 & 41 \end{pmatrix}$$ (Not diagonal matrix)

The characteristic eqn:

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$$S_1 = 9 + 41 = 50$$

$$S_2 = |A| = 369 - 144 = 225$$

$$\lambda^2 - 50\lambda + 255 = 0$$

+	×
-50	225

$\boxed{-45, -5}$

$$(\lambda - 45)(\lambda - 5) = 0$$

$$\lambda = 45, 5$$

$$\sigma_1^2 = \lambda_1 = 45 \Rightarrow \sigma_1 = \sqrt{45} = 3\sqrt{5}$$

$$\sigma_2^2 = \lambda_2 = 5 \Rightarrow \sigma_2 = \sqrt{5}$$

$$\Sigma = \begin{pmatrix} 3\sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$$

To find eigen vectors:

$$(C - \lambda I)X = 0$$

$$\begin{pmatrix} 9-\lambda & 12 \\ 12 & 41-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (9-\lambda)x_1 + 12x_2 &= 0 \\ 12x_1 + (41-\lambda)x_2 &= 0 \end{aligned}\right\} \text{—①}$$

• When $\lambda = 45$

① $\Rightarrow$ $-36x_1 + 12x_2 = 0$ $\times -\frac{1}{12}$

$12x_1 - 4x_2 = 0$ $\times \frac{1}{4}$

$$3x_1 - x_2 = 0$$

$$3x_1 - x_2 = 0$$

$$\boxed{3x_1 = x_2}$$

Let $x_1 = k$

$x_2 = 3k$

$X_1 = \begin{pmatrix} k \\ 3k \end{pmatrix}, k \in R$

Let $k = 1$

$$X_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

• When $\lambda = 5$

① $\Rightarrow$ $4x_1 + 12x_2 = 0$ $\times \frac{1}{4}$

$12x_1 + 36x_2 = 0$ $\times \frac{1}{12}$

$$x_1 + 3x_2 = 0$$

$$x_1 + 3x_2 = 0$$

$$\boxed{x_1 = -3x_2}$$

Let $x_2 = t$

$x_1 = -3t$

$X_2 = \begin{pmatrix} -3t \\ t \end{pmatrix}, t \in R$

Let $t = 1$

$$X_2 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{pmatrix} \qquad u_2 = \begin{pmatrix} \frac{-3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{-3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

$$v_1 = \frac{1}{\sigma_1} A^T u_1$$

$$= \frac{1}{3\sqrt{5}} \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix}$$

$$v_1 = \frac{1}{3\sqrt{5}} \begin{pmatrix} 15/\sqrt{10} \\ 15/\sqrt{10} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A^T u_2$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 3 & 4 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -5/\sqrt{10} \\ 5/\sqrt{10} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

HW:

3 $A = \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix}$

Least Square Approximation

- Regression line
- Method of least square

(x_1, y_1) (x_2, y_2)
(x_3, y_3), (x_4, y_4)

$y = ax + b$ (Linear eqn)

$1 \times b + ax = y$

$$\begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ 1 & x_4 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$$

$$A \quad X = B$$

$X = A^{-1} B$ (Many matrix it is failure)

$\Rightarrow AX = B$

$(A^T A) X = A^T B$

premultiply by $(A^T A)^{-1}$

$(A^T A)^{-1} (A^T A) X = (A^T A)^{-1} A^T B$

$$\boxed{X = (A^T A)^{-1} A^T B}$$

Eg 1: Find least square approximation for line using points: (1,3) (2,4) (5,5) (6,10)

$y = ax + b$

$1 \cdot b + ax = y$ $\left[\begin{pmatrix} 1 & x \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \right]$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix} \cdot \begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$$

$$A X = B$$

where $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}$ $X = \begin{pmatrix} b \\ a \end{pmatrix}$

$B = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 10 \end{pmatrix}$

$$A^T A = \underset{2\times4}{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 5 & 6 \end{pmatrix}} \underset{4\times2}{\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 5 \\ 1 & 6 \end{pmatrix}}$$

$$A^T A = \underset{2\times2}{\begin{pmatrix} 4 & 14 \\ 14 & 66 \end{pmatrix}}$$

$$(A^TA)^{-1} = \frac{1}{|A^TA|} \text{ Adjoint } (A^TA)$$

$$= \frac{1}{68}\begin{pmatrix}66 & -14\\ -14 & 4\end{pmatrix}$$

$$X = (A^TA)^{-1}A^TB$$

$$= \frac{1}{68}\begin{pmatrix}66 & -14\\ -14 & 4\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&2&5&6\end{pmatrix}\begin{pmatrix}3\\4\\5\\10\end{pmatrix}$$

$$= \frac{1}{68}\begin{pmatrix}52&38&-4&-18\\-10&-6&6&10\end{pmatrix}\begin{pmatrix}3\\4\\5\\10\end{pmatrix}$$

$$\begin{pmatrix}b\\a\end{pmatrix} = \frac{1}{68}\begin{pmatrix}108\\76\end{pmatrix}$$

$$b = \frac{108}{68} = \frac{27}{17}, \quad a = \frac{76}{68} = \frac{19}{17}$$

$$\therefore\ y = \left(\frac{19}{17}\right)x + \frac{27}{17}$$

② (1, 3) (3, 9) (5, 15) (6, 18)

$y = ax + b$ $\qquad y = 3x + 0$

$1.b + ax = y$

$$\begin{pmatrix}1&1\\1&3\\1&5\\1&6\end{pmatrix}\begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}3\\9\\15\\18\end{pmatrix}$$

$$A \quad X \quad = \quad B$$

$$A = \begin{pmatrix}1&1\\1&3\\1&5\\1&6\end{pmatrix} \qquad X = \begin{pmatrix}b\\a\end{pmatrix}$$

$$B = \begin{pmatrix}3\\9\\15\\18\end{pmatrix}$$

$$A^TA = \begin{pmatrix}1&1&1&1\\1&3&5&6\end{pmatrix}\begin{pmatrix}1&1\\1&3\\1&5\\1&6\end{pmatrix}$$

$$A^TA = \begin{pmatrix}4&15\\15&71\end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{|A^TA|} \text{ Adj } (A^TA)$$

$$(A^TA)^{-1} = \frac{1}{59}\begin{pmatrix}71&-15\\-15&4\end{pmatrix}$$

$$X = (A^TA)^{-1}A^TB$$

$$= \frac{1}{59}\begin{pmatrix}71&-15\\-15&4\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&3&5&6\end{pmatrix}\begin{pmatrix}3\\9\\15\\18\end{pmatrix}$$

$$= \frac{1}{59}\begin{pmatrix}56&26&-4&-19\\-11&-3&5&9\end{pmatrix}\begin{pmatrix}3\\9\\15\\18\end{pmatrix}$$

$$\therefore\ X = \frac{1}{59}\begin{pmatrix}0\\177\end{pmatrix} = \begin{pmatrix}b\\a\end{pmatrix}$$

$$b = 0, \quad a = \frac{177}{59} = 3$$

$y = ax + b$

$\therefore\ y = 3x + 0$

HW

③ (1, 2) (2, 3) (3, 5) (4, 7)

④ (1, 1) (2, 3) (3, 4) (4, 5)

H/W:

③ $y = ax + b$

$1.b + ax = y$

$$\begin{pmatrix}1&1\\1&2\\1&3\\1&4\end{pmatrix}\begin{pmatrix}b\\a\end{pmatrix} = \begin{pmatrix}2\\3\\5\\7\end{pmatrix}$$

$$A \quad X \quad = \quad B$$

$$A^TA = \begin{pmatrix}1&1&1&1\\1&2&3&4\end{pmatrix}\begin{pmatrix}1&1\\1&2\\1&3\\1&4\end{pmatrix}$$

$$A^TA = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{|A^TA|} Adj(A^TA)$$

$$(A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix}$$

$$X = (A^TA)^{-1}A^TB$$

$$= \frac{1}{20}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \end{pmatrix}$$

$$X = \frac{1}{20}\begin{pmatrix} 0 \\ 34 \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$b = 0 \qquad a = \frac{34}{20} = \frac{17}{10}$$

$$y = ax + b$$

$$\therefore y = \frac{17}{10}x + 0$$

④ $y = ax + b$

$$1 \cdot b + ax = y$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

$$A \qquad X \quad = \quad B$$

$$A^TA = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{20}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix}$$

$$X = (A^TA)^{-1}A^TB$$

$$= \frac{1}{20}\begin{pmatrix} 30 & -10 \\ -10 & 4 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

$$= \frac{1}{20}\begin{pmatrix} 20 & 10 & 0 & -10 \\ -6 & -2 & 2 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \end{pmatrix}$$

$$X = \frac{1}{20}\begin{pmatrix} 0 \\ 26 \end{pmatrix} = \begin{pmatrix} b \\ a \end{pmatrix}$$

$$b = 0, \quad a = \frac{26}{20} = \frac{13}{10}$$

$$\therefore y = \frac{13}{10}x + 0$$

SVD

③ $A = \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix}$

$$C = AA^T = \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -4 & 1 \\ -7 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 65 & -32 \\ -32 & 17 \end{pmatrix}$$

The characteristic eqn is

$$\lambda^2 - S_1\lambda + S_2 = 0$$

$$S_1 = 65 + 17 = 82$$

$$S_2 = |A| = 81$$

$$\lambda^2 - 82\lambda + 81 = 0$$

+	×
-82	81

$-81, -1$

$$\lambda^2 - 81\lambda - \lambda + 81 = 0$$

$$(\lambda - 81)(\lambda - 1) = 0$$

$$\lambda = 81, 1$$

$$\sigma_1^2 = \lambda_1 = 81 \Rightarrow \sigma_1 = \sqrt{81} = 9$$

$$\sigma_2^2 = \lambda_2 = 1 \Rightarrow \sigma_2 = \sqrt{1} = 1$$

$$\Sigma' = \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix}$$

To find eigen vectors:

$$(C - \lambda I)\, x = 0$$

$$\begin{pmatrix} 65-\lambda & -32 \\ -32 & 17-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\left.\begin{aligned} (65-\lambda)x_1 - 32x_2 &= 0 \\ -32x_1 + (17-\lambda)x_2 &= 0 \end{aligned}\right\} \rightarrow ①$$

- when $\lambda = 81$

$$① \Rightarrow -16x_1 - 32x_2 = 0 \quad \times \tfrac{1}{-16}$$

$$-32x_1 - 64x_2 = 0 \quad \times \tfrac{1}{-32}$$

$$x_1 + 2x_2 = 0$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -2x_2$$

Let $x_2 = k$

$x_1 = -2k$

$$X_1 = \begin{pmatrix} -2k \\ k \end{pmatrix}, \quad k \in \mathbb{R}$$

Let $k = 1$

$$X_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

- when $\lambda = 1$

$$① \Rightarrow 64x_1 - 32x_2 = 0 \quad \times \tfrac{1}{32}$$

$$-32x_1 + 16x_2 = 0 \quad \times \tfrac{1}{-16}$$

$$2x_1 - x_2 = 0$$

$$2x_1 - x_2 = 0$$

$$x_2 = 2x_1$$

Let $x_1 = t$

$x_2 = 2t$

$$X_2 = \begin{pmatrix} t \\ 2t \end{pmatrix}, \quad t \in \mathbb{R}$$

Let $t = 1$

$$X_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$u_1 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$v_1 = \frac{1}{\sigma_1} A^T u_1$$

$$= \frac{1}{9} \begin{pmatrix} -4 & 1 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$$v_1 = \frac{1}{9} \begin{pmatrix} 9/\sqrt{5} \\ 18/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$v_2 = \frac{1}{\sigma_2} A^T u_2$$

$$= \frac{1}{1} \begin{pmatrix} -4 & 1 \\ -7 & 4 \end{pmatrix} \begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{-2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$V^T = \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$A = U \Sigma V^T$$

$$= \begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} 9 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-18}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{9}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-20}{\cancel{\sqrt{5}}} & \frac{-35}{\cancel{\sqrt{5}}} \\ \frac{5}{\cancel{\sqrt{5}}} & \frac{20}{\cancel{\sqrt{5}}} \end{pmatrix}$$

$$= \begin{pmatrix} -4 & -7 \\ 1 & 4 \end{pmatrix} = A$$

04/11/2023

Pseudo Inverse of a Matrix

$$A^{-1} = \frac{1}{|A|} \text{Adj}(A)$$

Then, $A_{n\times n} \to A^{-1}_{n\times n}$

Generalised Inverse of Matrix:

Let $A \in \mathbb{R}^{m\times n}$ / $M_{m\times n}(\mathbb{R})$

we call the matrix $G \in \mathbb{R}^{n\times m}$ a generalised inverse of matrix A if it satisfies the condition:

$$\boxed{AGA = A}$$

$\underbrace{\underbrace{\underbrace{m\times n \quad n\times m}_{m\times m} \quad m\times n}_{m\times n}}$ $\quad m\times n$

REMARK:

For a ~~generalise~~ general matrix $A \in \mathbb{R}^{m\times n}$ it's generalised inverse, G always exist but might not be unique.

More than one generalised inverse is possible for A.

Eg 1: Let $A = (1 \quad 2) \in \mathbb{R}^{1\times 2}$

Find the generalised inverse of A.

Let $G = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times 1}$

$$AGA = A$$

$$(1 \quad 2)\begin{pmatrix} x \\ y \end{pmatrix}(1 \quad 2) = (1 \quad 2) \quad (1)$$

$$(x+2y)(1 \quad 2) = (1 \quad 2)$$

This is possible only if $x + 2y = 1$

$G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad G = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\quad G = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

THEOREM

Let $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \in \mathbb{R}^{m\times n}$

be a matrix of rank r and $A_{11} \in \mathbb{R}^{r\times r}$.

If A_{11} is invertible then,

$$G = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{R}^{n\times m}$$

- A_{mn} – Minors of matrix in school
  here, $A_{mn} \to$ Submatrix
- Rank – got after doing some elementary operations
  $\to$ Minimum non zero rows

Eg 2: Find the generalised inverse of matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$A \approx \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$A \approx \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$\text{rank}(A) = 2$

$$A_{11} = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$$

$$|A_{11}| = 5 - 8 = -3 \neq 0$$

$\therefore A_{11}$ is invertible

$$A_{11}^{-1} = \frac{1}{-3}\begin{pmatrix} 5 & -2 \\ -4 & 1 \end{pmatrix}$$

$$A_{11}^{-1} = \begin{pmatrix} -5/3 & 2/3 \\ 4/3 & -1/3 \end{pmatrix}$$

$$G = \begin{pmatrix} A_{11}^{-1} & 0 \\ 0 & 0 \end{pmatrix}$$

$$G = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AGA \Rightarrow \quad AG = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$AGA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = A$$

THEOREM

Consider a linear system $AX = b$ suppose $b \in col(A)$ such that the system is consistent (there exist one/more soln [No soln - inconsistent]).

Let G be generalised inverse of A then $\underbrace{X^*}_{soln} = Gb$ is a particular solution to a system

Eg 3: consider linear system, $AX = b$

where $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

$b = \begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix}$. Find the particular soln

$$X^* = Gb$$

$$X^* = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 6 \\ 15 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{30}{3} + \frac{30}{3} + 0 \\ \frac{24}{3} - \frac{15}{3} + 0 \\ 0 + 0 + 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

Pseudo Inverse

Let $A \in \mathbb{R}^{m \times n}$. we call the matrix $B \in \mathbb{R}^{n \times m}$ is pseudo inverse of A if it satisfies following condition:

(i) $ABA = A$ { B is generalised inverse of A }

(ii) $BAB = B$ { A is generalised inverse of B }

(iii) $(AB)^T = AB$

(iv) $(BA)^T = BA$

{ if $A^T = A$ then A is symmetric matrix ∴ AB, BA - symmetric matrix. }

REMARK:

For any matrix,

① $A \in \mathbb{R}^{m \times n}$ the pseudo Inverse exists & is $\boxed{\text{unique}}$

② A pseudo inverse also called as Moore - Penrose inverse

③ Pseudo inverse of matrix A is denoted by A^{+} (Pseudo inverse of A)

Eg 4 : Find the pseudo inverse of matrix $A = \begin{pmatrix} 1 & 2 \end{pmatrix} \in \mathbb{R}^{1\times 2}$

condn 1) Let $B = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^{2\times 1}$

$$ABA = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$ABA = (x+2y)\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \end{pmatrix} = A$$

$$\Rightarrow \boxed{x + 2y = 1} \text{ — ①}$$

condn 2)

$$\underset{\substack{2\times1\ 1\times2\\ 2\times2\ 2\times1 \\ 2\times1}}{BAB} = \begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} x \\ y \end{pmatrix}(x+2y) = \begin{pmatrix} x \\ y \end{pmatrix} = B$$

$$\Rightarrow x + 2y = 1$$

condn 3)

$$AB = \begin{pmatrix} 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x + 2y$$

$$(AB)^T = (x+2y) = (AB =)(x+2y)$$

condn 4)

$$BA = \begin{pmatrix} x \\ y \end{pmatrix}\begin{pmatrix} 1 & 2 \end{pmatrix} = \begin{pmatrix} x & 2x \\ y & 2y \end{pmatrix}$$

$$(BA)^T = \begin{pmatrix} x & y \\ 2x & 2y \end{pmatrix} = \begin{pmatrix} x & 2x \\ y & 2y \end{pmatrix} = BA$$

$$\Rightarrow \boxed{y = 2x} \text{ — ②}$$

using ①, ②

$$x + 2y = 1$$

$$x + 2(2x) = 1 \Rightarrow 5x = 1$$

$$x = \frac{1}{5}, \quad y = \frac{2}{5}$$

$$B = \begin{pmatrix} \frac{1}{5} \\ \frac{2}{5} \end{pmatrix} = A^{+}$$

Eg 5: Consider matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

check whether the generalised inverse $G = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is pseudo inverse of A or not.

condn 1) $AGA = A$ (satisfies)
as G is generalised inverse

condn 2) $GAG = G$

$$GA = \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$GAG = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -5/3 & 2/3 & 0 \\ 4/3 & -1/3 & 0 \\ 0 & 0 & 0 \end{pmatrix} = G$$

condn 3)

$$GA = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$(GA)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 0 \end{pmatrix} \neq GA$$

$\therefore$ G is not a pseudo inverse of A

THEOREM

Let $A \in \mathbb{R}^{m\times n}$ be any matrix with full column rank (i.e., $\text{rank}(A) = n \le m$) then pseudo inverse of A is (m = row, n = column)

$$\boxed{A^{+} = (A^TA)^{-1}A^T}$$

Eg 6: Find the pseudo inverse of matrix $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}_{3\times 2}$.

Clearly rank(A) = 2 ≤ 3

$$A^+ = (A^TA)^{-1} A^T$$

$$A^TA = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$A^TA = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$(A^TA)^{-1} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix}$$

$$A^+ = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$A^+ = \frac{1}{3}\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \end{pmatrix}$$

THEOREM

Let $A \in \mathbb{R}^{m\times n}$ be a diagonal matrix i.e, all of its entries are zero except sum of those along its diagonal. Then pseudo inverse of A is another diagonal matrix $B \in \mathbb{R}^{n\times m}$ such that,

$$b_{ii} = \begin{cases} \frac{1}{a_{ii}}, & a_{ii} \neq 0 \\ 0, & a_{ii} = 0 \end{cases}$$

Eg 7: Find pseudo inverse of matrix $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}_{3\times 2}$

Atleast one $a_{ii} \neq 0$

Here, $a_{22} \neq 0$

$$A^+ = B_{2\times 3} = \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix}$$

Condn 1: ABA = A

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{AB} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$\begin{bmatrix} 2\times 2 \\ 2\times 3 \\ \hline 2\times 3 \end{bmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} = A$$

condn 2: BAB = B

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{BA} \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1/3 \\ 0 & 0 \end{pmatrix} = B$$

condn 3:

$$(AB)^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = AB$$

condn 4:

$$(BA)^T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = BA$$