Algebraic Description of the Discriminant

Below we provide an algebraic description of the focal locus (discriminant) $\mathcal{F}_{X,\Sigma}$, introduced in section 3.2. Recall the definition of the normal bundle as given in Proposition 3.4:

$$\mathcal{N}_{X,\Sigma} := \{(s,v) \in X \times \mathbb{R}^n : \langle s,v \rangle\}_\Sigma = 0 \quad \forall w \in T_s X\}.$$

We first provide an algorithm for finding $\mathcal{N}_{X,\Sigma}$:

Algorithm 1: Procedure for finding the ideal of $\mathcal{N}_{X,\Sigma}$

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input : polynomials f_i defining the variety X;
teacher vector T;
student vector S;
weight matrix Σ.
output: The ideal of the normal bundle.

1. Find the ideal I_X;
2. Form the Jacobian of I_X, $Jac(X)$;
3. Form the augmented Jacobian, $Jac(X)_{aug}$ by appending a row of weighted difference vectors $\Sigma^{-1}(S-T)$ to $Jac(X)$;
4. Form the ideal consisting of $c \times c$ minors of $Jac(X)_{aug}$ as well as polynomials in I_X;
5. Saturation of this ideal with respect to the singular locus of X is gives the ideal of the normal bundle.

To give the algorithm for finding the ideal of discriminant we first prove a theorem.
Let $F_\mathcal{N} = (f_1, \ldots, f_k)$ be the set of equations defining $\mathcal{N}_{X,\Sigma}$. Consider the $k \times (2n)$ Jacobian, $M := Jac(F_\mathcal{N})$. Note that the kernel of this matrix is the tangent space. We want to show the equivalence of the following two statements:

1. The ramification locus of the normal variety is

$$\{z := (s,t) | rank(Jac_s(F_\mathcal{N})) \neq d\},$$

where $Jac_s(F_\mathcal{N})$ is the submatrix of $Jac(F_\mathcal{N}) =: M_1$ corresponding to student partials and d is the dimension of X, the student variety;

2. The ramification locus of the normal variety $\mathcal{N}_{X,\Sigma}$ is student-teacher pairs where the differential of the projection to teachers, $d\pi_2$, is not onto, ie

$$\{z := (s,t) | Im(d\pi_2) \neq \mathbb{R}^n\}.$$

We show the equivalence of the two statements by means of the following theorem:

Theorem 1. *Let $M = [M_1 | M_2]$ be a $d \times (m+n)$ matrix with linearly independent rows. Then*

$$\pi_2(\ker(M)) \neq \mathbb{R}^n \iff Im(M_1) \neq \mathbb{R}^d.$$

Proof. $\Rightarrow$ We argue by a counterpositive: suppose $Im(M_1) = \mathbb{R}^d$. Then for all $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^m$ with $M_1 x = -M_2 y$. But this is equivalent to $M_1 x + M_2 y = 0$ and $M(x, y) = 0$. Thus $Im(\pi_2(\ker M)) = \mathbb{R}^n$ as desired.
$\Leftarrow$ Suppose $Im(M_1) \neq \mathbb{R}^d$. Then there exists a nonzero vector $h \in \mathbb{R}^d$ such that $h^T M_1 = 0$. Suppose (x, y) is a vector in the kernel of M. We have

$$h^T M(x, y) = h^T (M_1 x + M_2 y) = 0, h^T M_1 x + h^T M_2 y = 0,$$

$$h^T M_2 y = 0,$$

since $h^T M_1 = 0$ by assumption. We cannot have $h^T M_2 = 0$ by the assumption on the independence of the rows of M. Thus y lies on a hyperplane $\langle h^T M_2, \cdot \rangle$ in $\mathbb{R}^n$, ie $Im(d\pi_2) \neq \mathbb{R}^n$. □

In the light of this equivalence we apply the following procedures to find the discriminant:

Algorithm 2: Procedure for finding the ideal of $\mathcal{F}_{X,\Sigma}$

input : The normal bundle $\mathcal{N}_{X,\Sigma}$;
output: The ideal of the discriminant.

1. Form the Jacobian of the normal bundle $\mathcal{N}_{X,\Sigma}$;
2. Take its submatrix with respect to student variables;
3. Form the ideal of $n - c \times n - c$ minors of this submatrix;
4. Saturate this ideal with respect to the singular locus of X;
5. Form the elimination ideal with respect to the student variables, thereby obtaining the ideal of the discriminant.