Algebraic Description of the Discriminant

Below we provide an algebraic description of the ramification locus of the endpoint map, introduced in section 3.2. Let $F = (f_1, \ldots, f_k)$ be the set of equations defining the normal bundle of teachers t and critical students s (ie equations of the student variety X along with minors of the augmented Jacobian). Consider the $k \times (2n)$ Jacobian, $M := Jac(F)$. Note that the kernel of this matrix is the tangent space. We want to show the equivalence of the following two statements:

1. The ramification locus of the normal variety is
$$\{z := (s,t) | rank(Jac_s(F)) \neq d\},$$
where $Jac_s(F)$ is the submatrix of $Jac(F) =: M_1$ corresponding to student partials and d is the dimension of X, the student variety;

2. The ramification locus of the normal variety is student-teacher pairs where the differential of the projection to teachers, $d\pi_2$, is not onto, ie
$$\{z := (s,t) | Im(d\pi_2) \neq \mathbb{R}^n\}.$$

Note that the second formulation is consistent with the endpoint map view laid out in section 4.2. We show the equivalence of the two statements by means of the following theorem:

Theorem 1. *Let $M = [M_1 | M_2]$ be a $d \times (m+n)$ matrix with linearly independent rows. Then*
$$\pi_2(\ker(M)) \neq \mathbb{R}^n \iff Im(M_1) \neq \mathbb{R}^d.$$

Proof. $\Rightarrow$ We argue by a counterpositive: suppose $Im(M_1) = \mathbb{R}^d$. Then for all $y \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^m$ with $M_1 x = -M_2 y$. But this is equivalent to $M_1 x + M_2 y = 0$ and $M(x,y) = 0$. Thus $Im(\pi_2(\ker M)) = \mathbb{R}^n$ as desired.
$\Leftarrow$ Suppose $Im(M_1) \neq \mathbb{R}^d$. Then there exists a nonzero vector $h \in \mathbb{R}^d$ such that $h^T M_1 = 0$. Suppose (x,y) is a vector in the kernel of M. We have
$$h^T M(x,y) = h^T(M_1 x + M_2 y) = 0, h^T M_1 x + h^T M_2 y = 0,$$
$$h^T M_2 y = 0,$$
since $h^T M_1 = 0$ by assumption. We cannot have $h^T M_2 = 0$ by the assumption on the independence of the rows of M. Thus y lies on a hyperplane $\langle h^T M_2, \cdot \rangle$ in $\mathbb{R}^n$, ie $Im(d\pi_2) \neq \mathbb{R}^n$. $\square$

In the light of this equivalence we apply the following procedures to find the discriminant:

Algorithm 1: Procedure for finding the normal bundle variety

input : polynomials f_i defining the variety X;
teacher vector T;
student vector S;
weight matrix M.

output: The ideal of the normal bundle

1. Find the ideal I_X;
2. Form the Jacobian of I_X, $Jac(X)$;
3. Form the augmented Jacobian, $Jac(X)_{aug}$ by appending a row of weighted difference vectors $M^{-1}(S-T)$ to $Jac(X)$;
4. Form the ideal consisting of $c \times c$ minors of $Jac(X)_{aug}$ as well as polynomials in I_X;
5. Saturation of this ideal with respect to the singular locus of X is gives the ideal of the normal bundle.

Algorithm 2: Procedure for finding the discriminant of the endpoint map

input : The normal bundle

output: The ideal of the discriminant.

1. Form the Jacobian of the normal bundle;
2. Take its submatrix with respect to student variables;
3. Form the ideal of $n-c \times n-c$ minors of this submatrix;
4. Saturate this ideal with respect to the singular locus of X;
5. Form the elimination ideal with respect to the student variables, thereby obtaining the ideal of the discriminant.