1 Applications - gradient and sketching

1.1 Maxima and minima

1. (i) $y = -3x^2 + 12x + 2$
   $\frac{dy}{dx} = -6x + 12$ then the stationary point will be the roots of the derivative function.
   $-6x + 12 = 0 \implies x = 2$, for $x = 2$ we have $y = -3 \times 2^2 + 12 \times 2 + 2 = 14$. Now the only stationary point is $(2, 14)$.
   For $x < 2$, e.g. $x = 1$ we get $\frac{dy}{dx}(1) = -6 \times 1 + 12 = 6 > 0$.
   For $x > 2$, e.g. $x = 3$ we get $\frac{dy}{dx}(3) = -6 \times 3 + 12 = -6 < 0$.
   Then the stationary point $(2, 14)$ is a local maximum.

   (ii) $y = x^3 - 6x^2 + 12x + 9$
   $\frac{dy}{dx} = 3x^2 - 12x + 12$ then the stationary point will be the roots of the derivative function.
   $3x^2 - 12x + 12 = 0 \implies 3(x - 2)(x - 2)$, for $x = 2$ we have $y = 2^3 - 6 \times 2^2 + 12 \times 2 + 9 = 17$. Now the only stationary point is $(2, 17)$.
   For $x < 2$, e.g. $x = 1$ we get $\frac{dy}{dx}(1) = 3 - 12 + 12 = 3 > 0$.
   For $x > 2$, e.g. $x = 3$ we get $\frac{dy}{dx}(3) = 3 \times 3^2 - 12 \times 3 + 12 = 3 > 0$.
   Then the stationary point $(2, 17)$ is a horizontal point of inflection.

   (iii) $y = 3x^3 - 9x^2 + 1$
   $\frac{dy}{dx} = 9x^2 - 18x$ then the stationary point will be the roots of the derivative function.
   $9x^2 - 18x = 0 \implies 9x(x - 2)$, for $x = 2$ we have $y = 3 \times 2^3 - 9 \times 2^2 + 1 = -11$ and for $x = 0$ we have $y = 1$. Now the two stationary points are $(2, -11)$ and $(0, 1)$.
   For $x < 2$, e.g. $x = 1$, because 1 is between 0 and 2, we get $\frac{dy}{dx}(1) = 9 - 18 = -9 < 0$.
   For $x > 2$, e.g. $x = 3$ we get $\frac{dy}{dx}(3) = 9 \times 3^2 - 18 \times 3 = 27 > 0$.
   Then the stationary point $(2, -11)$ is a local minimum.
   Now for $x < 0$, e.g. $x = -1$, $\frac{dy}{dx}(1) = 9 + 18 = 27 > 0$ As in $x = 1 > 0$, $\frac{dy}{dx}(1) = 9 - 18 = -9 < 0$ we get $(0, 1)$ is a local maximum.

2. Following the detailed solutions of the previous question, we obtain:

   (i) $y = 5x^2 - 20x + 9$
   $\frac{dy}{dx} = 10x - 20 \implies x = 2$ is the only root. Then the only stationary point is $(2, -11)$ which is a local minimum.

   (ii) $y = 2x^3 - 9x^2 + 12x + 1$
   $\frac{dy}{dx} = 6x^2 - 18x + 12 = 6(x - 1)(x - 2) \implies x = 2$ and $x = 1$ are the roots. Then the stationary points are $(2, 5)$ which is a local minimum, and $(1, 6)$ which is a maximum.

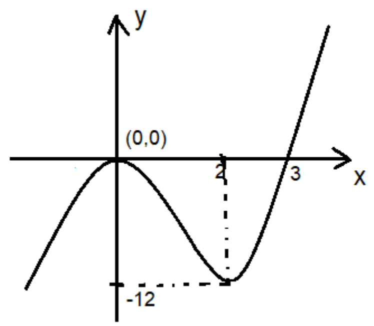
   (iii) $y = x^3 + 3x^2 + 3x + 1$
   $\frac{dy}{dx} = 3x^2 + 6x + 3 = 3(x + 1)^2 \implies x = -1$, is the only root. Then the only stationary point is $(-1, 0)$ which is a horizontal point of inflection.

1.2 Graph sketching

1. For $y = 3x^3 - 9x^2$
   (i) For $x = 0$ we have $y = 0$, then $(0, 0)$ is the x-intercept.
   For $y = 0$ we have $0 = 3x^3 - 9x^2 = 3x^2(x - 3) \implies x = 0$ and $x = 3$ then $(0, 0)$ and $(3, 0)$ are the y-intercepts.

(ii) $\frac{dy}{dx} = 9x^2 - 18x = 9x(x-2)$ then $x = 0$ and $x = 2$ are the roots. Now, the stationary points are $(0,0)$ and $(2,-12)$. Analysing as we do in the Question 1, we obtain that $(0,0)$ is a local maximum, and $(2,-12)$ is a local minimum.

(iii)

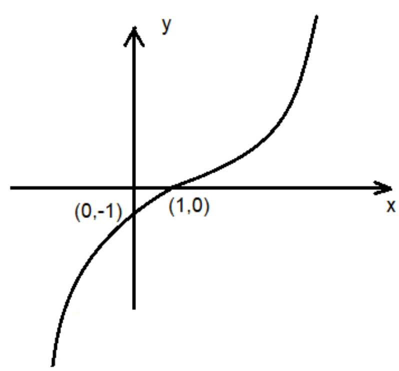


2. For $y = (x-1)^3$

(i) For $x = 0$ we have $y = -1$, then $(0,-1)$ is the x-intercept.
For $y = 0$ we have $x = 1$, then $(1,0)$ is the y-intercept.

(ii) $\frac{dy}{dx} = 3(x-1)^2$ then $x = 1$ is the root. Now, the stationary points are $(1,0)$ which is a horizontal point of inflection.

(iii)



1.3 Second derivative

1. (i) $y = 2x^3 - 4x^2 - 5x + 9$, Here $\frac{dy}{dx} = 6x^2 - 8x - 5$, then $\frac{d^2y}{dx^2} = 12x - 8$.

(ii) $y = (x-2)e^{2x}$, Here
$\frac{dy}{dx} = (x-2)'e^{2x} + (x-2)(e^{2x})' = e^{2x} + 2(x-2)e^{2x} = e^{2x}(1 + 2(x-2)) = e^{2x}(2x-1)$.
And $\frac{d^2y}{dx^2} = 2(e^{2x})'(2x-1) + 2e^{2x}(2x-1)' =$
$= 4e^{2x}(2x-1) + 4e^{2x} = e^{2x}(4(2x-1)+4) = 8xe^{2x}$

(iii) $y = x\ln x$, Here $\frac{dy}{dx} = x'\ln x + x(\ln x)' = \ln x + x\frac{1}{x} = \ln x + 1$.
$\frac{d^2y}{dx^2} = \frac{1}{x}$.

(iv) $y = 8\sqrt{x} - \cos(3x)$, Here $\frac{dy}{dx} = 8\frac{1}{2\sqrt{x}} + 3\sin(3x) = \frac{4}{\sqrt{x}} + 3\sin(3x)$.
Now $\frac{d^2y}{dx^2} = 4(x^{-\frac{1}{2}})' + 3(\sin(3x))' = -\frac{4}{2}x^{-3/2} + 9\cos(3x) = -\frac{2}{\sqrt{x^3}} + 9\cos(3x)$.

2. (i) $y = 3x^3 - 9x^2 + 1$, here $\frac{dy}{dx} = 9x^2 - 18x = 9x(x-2)$ where $x = 0$ and $x = 2$ are the roots. Then the points $(0,0)$ and $(2,-11)$ are the stationary points. Now, $\frac{d^2y}{dx^2} = 18x - 18$, when evaluated on the stationary points we have:
$\frac{d^2y}{dx^2}(0) = -18 < 0$ then $(0,0)$ is a local maximum.
$\frac{d^2y}{dx^2}(2) = 18 > 0$ then $(2,-11)$ is a local minimum.

(ii) $y = x^4 - 8x^2 + 10$, here $\frac{dy}{dx} = 4x^3 - 16x = 4x(x^2-4) = 4x(x-2)(x+2)$, where $x = 0$, $x = 2$ and $x = -2$ are the roots of the first derivative. Then, $(0,10)$, $(2,-6)$ and $(-2,-6)$ are the stationary points. The second derivative is $\frac{d^2y}{dx^2} = 12x^2 - 16 =$ when evaluated on the stationary points we have:
$\frac{d^2y}{dx^2}(0) = -16 < 0$, then $(0,10)$ is a local maximum.
$\frac{d^2y}{dx^2}(2) = 32 > 0$, then $(2,-11)$ is a local minimum.
$\frac{d^2y}{dx^2}(-2) = 32 > 0$, then $(-2,-11)$ is a local minimum.

3. (i) $y = x^4 - 4x^3 - 7x + 7$, here $\frac{dy}{dx} = 4x^3 - 12x^2 - 7$ and $\frac{d^2y}{dx^2} = 12x^2 - 24x$.

(ii) $y = (4x-3)e^x$, here $\frac{dy}{dx} = (4x-3)'e^x + (4x-3)(e^x)' = 4e^x + (4x-3)e^x$ and $\frac{d^2y}{dx^2} = 4(e^x)' + ((4x-3)e^x)'$ as calculated before, we have $\frac{d^2y}{dx^2} = 4e^x + 4e^x + (4x-3)e^x = e^x(4+4+4x-3) = (4x+5)e^x$.

(iii) $y = x\sin x$, here $\frac{dy}{dx} = x'\sin x + x(\sin x)' = \sin x + x\cos x$. And $\frac{d^2y}{dx^2} = \cos x + x'\cos x + x(\cos x)' = \cos x + \cos x - x\sin x = 2\cos x - x\sin x$.

(iv) $y = 16x^{3/4} - 4\ln x$, here $\frac{dy}{dx} = 16 \times \frac{3}{4}x^{3/4-1} - 4\frac{1}{x} = 12x^{-\frac{1}{4}} - \frac{4}{x}$.

And $\frac{d^2y}{dx^2} = -\frac{12}{4}x^{-1/4-1} + \frac{4}{x^2} = -3x^{-\frac{5}{4}} + \frac{4}{x^2}$.

4. (i) $y = 2x^3 + 9x^2 - 24x$, here $\frac{dy}{dx} = 6x^2 + 18x - 24 = 6(x^2 + 3x - 4) = 6(x+4)(x-1)$. The stationary points are $(-4,112)$ and $(1,-13)$. Now $\frac{d^2y}{dx^2} = 12x + 18$, then
$\frac{d^2y}{dx^2}(-4) = -30 < 0$, then $(-4,112)$ is a local maximum.
$\frac{d^2y}{dx^2}(1) = 30 > 0$, then $(1,-13)$ is a local minimum.

(ii) $y = x^3 + 3x^2 - 4$, here $\frac{dy}{dx} = 3x^2 + 6x = 3x(x+2)$. The stationary point are $(0,-4)$ and $(-2,0)$. Now $\frac{d^2y}{dx^2} = 6x + 6$, then
$\frac{d^2y}{dx^2}(0) = 6 > 0$, then $(0,-4)$ is a local minimum.
$\frac{d^2y}{dx^2}(-2) = -6 < 0$, then $-2, 0$ is a local maximum.