Chapter 2

2 Coordinate geometry and matrices

Objectives

- To revise:
  - methods for solving **linear equations**
  - methods for solving **simultaneous linear equations**
  - finding the **distance** between two points
  - finding the **midpoint** of a line segment
  - calculating the **gradient** of a straight line
  - interpreting and using different forms of the **equation of a straight line**
  - finding the **angle of slope** of a straight line
  - determining the gradient of a line **perpendicular** to a given line
  - **matrix** arithmetic.
- To apply a knowledge of **linear functions** to solving problems.

Much of the material presented in this chapter has been covered in Mathematical Methods Units 1 & 2. The chapter provides a framework for revision with worked examples and practice exercises.

There is also a section on the solution of simultaneous linear equations with more than two variables. The use of a CAS calculator to solve such systems of equations is emphasised.

The use of matrices in this course is confined to the description of transformations of the plane, which is covered in Chapter 3. Additional material on matrices is available in the Interactive Textbook.

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2A Linear equations

This section contains exercises in linear equations. The worded problems provide an opportunity to practise the important skill of going from a problem expressed in English to a mathematical formulation of the problem.

Section summary

- An equation is solved by finding the value or values of the variables that would make the statement true.
- A linear equation is one in which the variable is to the first power.
- There are often several different ways to solve a linear equation. The following steps provide some suggestions:
  - Expand brackets and, if the equation involves fractions, multiply through by the lowest common denominator of the terms.
  - Group all of the terms containing a variable on one side of the equation and the terms without the variable on the other side.
- Steps for solving a word problem with a linear equation:
  - Read the question carefully and write down the known information clearly.
  - Identify the unknown quantity that is to be found.
  - Assign a variable to this quantity.
  - Form an expression in terms of x (or the variable being used) and use the other relevant information to form the equation.
  - Solve the equation.
  - Write a sentence answering the initial question.

Exercise 2A

Skillsheet

1 Solve the following linear equations:

a $3x - 4 = 2x + 6$ **b** $8x - 4 = 3x + 1$ **c** $3(2 - x) - 4(3 - 2x) = 14$

d $\dfrac{3x}{4} - 4 = 17$ **e** $6 - 3y = 5y - 62$ **f** $\dfrac{2}{3x - 1} = \dfrac{3}{7}$

g $\dfrac{2x - 1}{3} = \dfrac{x + 1}{4}$ **h** $\dfrac{2(x - 1)}{3} - \dfrac{x + 4}{2} = \dfrac{5}{6}$

i $4y - \dfrac{3y + 4}{2} + \dfrac{1}{3} = \dfrac{5(4 - y)}{3}$ **j** $\dfrac{x + 1}{2x - 1} = \dfrac{3}{4}$

2 Solve each of the following pairs of simultaneous linear equations:

a $x - 4 = y$, $4y - 2x = 8$

b $9x + 4y = 13$, $2x + y = 2$

c $7x = 18 + 3y$, $2x + 5y = 11$

d $5x + 3y = 13$, $7x + 2y = 16$

e $19x + 17y = 0$, $2x - y = 53$

f $\dfrac{x}{5} + \dfrac{y}{2} = 5$, $x - y = 4$

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3 The length of a rectangle is 4 cm more than the width. If the length were to be decreased by 5 cm and the width decreased by 2 cm, the perimeter would be 18 cm. Calculate the dimensions of the rectangle.

4 In a basketball game, a field goal scores two points and a free throw scores one point. John scored 11 points and David 19 points. David scored the same number of free throws as John, but twice as many field goals. How many field goals did each score?

5 The weekly wage, $\$w$, of a sales assistant consists of a fixed amount of \$800 and then \$20 for each unit he sells.

a If he sells n units a week, find a rule for his weekly wage, w, in terms of the number of units sold.

b Find his wage if he sells 30 units.

c How many units does he sell if his weekly wage is \$1620?

6 Water flows into a tank at a rate of 15 litres per minute. At the beginning, the tank contained 250 litres.

a Write an expression for the volume, V litres, of water in the tank at time t minutes.

b How many litres of water are there in the tank after an hour?

c The tank has a capacity of 5000 litres. How long does it take to fill?

7 A tank contains 10 000 litres of water. Water flows out at a rate of 10 litres per minute.

a Write an expression for the volume, V litres, of water in the tank at time t minutes.

b How many litres of water are there in the tank after an hour?

c How long does it take for the tank to empty?

8 An aircraft, used for fire spotting, flies from its base to locate a fire at an unknown distance, x km away. It travels straight to the fire and back, averaging 240 km/h for the outward trip and 320 km/h for the return trip. If the plane was away for 35 minutes, find the distance, x km.

9 A group of hikers is to travel x km by bus at an average speed of 48 km/h to an unknown destination. They then plan to walk back along the same route at an average speed of 4.8 km/h and to arrive back 24 hours after setting out in the bus. If they allow 2 hours for lunch and rest, how far must the bus take them?

10 The cost of hiring diving equipment is \$100 plus \$25 per hour.

a Write a rule which gives the total charge, $\$C$, of hiring the equipment for t hours (assume that parts of hours are paid for proportionately).

b Find the cost of hiring the equipment for:

i 2 hours **ii** 2 hours 30 minutes

c For how many hours can the equipment be hired if the following amounts are available?

i \$375 **ii** \$400

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2B Linear literal equations and simultaneous linear literal equations

A literal equation in x is an equation whose solution will be expressed in terms of pronumerals rather than numbers.

For the equation $2x + 5 = 7$, the solution is the number 1.

For the literal equation $ax + b = c$, the solution is $x = \dfrac{c - b}{a}$.

Literal equations are solved in the same way as numerical equations. Essentially, the literal equation is transposed to make x the subject.

Example 1

Solve the following for x:

a $px - q = r$ **b** $ax + b = cx + d$ **c** $\dfrac{a}{x} = \dfrac{b}{2x} + c$

Solution

a
$$\begin{aligned} px - q &= r \\ px &= r + q \\ \therefore \quad x &= \frac{r + q}{p} \end{aligned}$$

b
$$\begin{aligned} ax + b &= cx + d \\ ax - cx &= d - b \\ x(a - c) &= d - b \\ \therefore \quad x &= \frac{d - b}{a - c} \end{aligned}$$

c Multiply both sides of the equation by $2x$:
$$\begin{aligned} 2a &= b + 2xc \\ 2a - b &= 2xc \\ \therefore \quad x &= \frac{2a - b}{2c} \end{aligned}$$

Simultaneous literal equations are solved by the usual methods of solution of simultaneous equations: substitution and elimination.

Example 2

Solve the following simultaneous equations for x and y:

$$y = ax + c$$
$$y = bx + d$$

Solution

$$\begin{aligned} ax + c &= bx + d \qquad \text{(Equate the two expressions for } y.\text{)} \\ ax - bx &= d - c \\ x(a - b) &= d - c \end{aligned}$$

Thus $\quad x = \dfrac{d - c}{a - b}$

and $\quad y = a\left(\dfrac{d - c}{a - b}\right) + c$

$$= \frac{ad - ac + ac - bc}{a - b} = \frac{ad - bc}{a - b}$$

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Example 3

Solve the simultaneous equations $ax - y = c$ and $x + by = d$ for x and y.

Solution

$$ax - y = c \qquad (1)$$
$$x + by = d \qquad (2)$$

Multiply (1) by b:

$$abx - by = bc \qquad (1')$$

Add (1′) and (2):

$$abx + x = bc + d$$
$$x(ab + 1) = bc + d$$
$$\therefore \quad x = \frac{bc + d}{ab + 1}$$

Using equation (1):

$$y = ax - c$$
$$= a\left(\frac{bc + d}{ab + 1}\right) - c = \frac{ad - c}{ab + 1}$$

Section summary

- An equation for the variable x in which all the coefficients of x, including the constants, are pronumerals is known as a **literal equation**.
- The methods for solving linear literal equations or pairs of simultaneous linear literal equations are exactly the same as when the coefficients are given numbers.

Exercise 2B

Example 1

1 Solve each of the following for x:

a $ax + n = m$

b $ax + b = bx$

c $\dfrac{ax}{b} + c = 0$

d $px = qx + 5$

e $mx + n = nx - m$

f $\dfrac{1}{x + a} = \dfrac{b}{x}$

g $\dfrac{b}{x - a} = \dfrac{2b}{x + a}$

h $\dfrac{x}{m} + n = \dfrac{x}{n} + m$

i $-b(ax + b) = a(bx - a)$

j $p^2(1 - x) - 2pqx = q^2(1 + x)$

k $\dfrac{x}{a} - 1 = \dfrac{x}{b} + 2$

l $\dfrac{x}{a - b} + \dfrac{2x}{a + b} = \dfrac{1}{a^2 - b^2}$

m $\dfrac{p - qx}{t} + p = \dfrac{qx - t}{p}$

n $\dfrac{1}{x + a} + \dfrac{1}{x + 2a} = \dfrac{2}{x + 3a}$

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Example 2, 3

2 Solve each of the following pairs of simultaneous equations for x and y:

a $ax + y = c$
$x + by = d$

b $ax - by = a^2$
$bx - ay = b^2$

c $ax + by = t$
$ax - by = s$

d $ax + by = a^2 + 2ab - b^2$
$bx + ay = a^2 + b^2$

e $(a + b)x + cy = bc$
$(b + c)y + ax = -ab$

f $3(x - a) - 2(y + a) = 5 - 4a$
$2(x + a) + 3(y - a) = 4a - 1$

3 For each of the following pairs of equations, write s in terms of a only:

a $s = ah$
$h = 2a + 1$

b $s = ah$
$h = a(2 + h)$

c $as = a + h$
$h + ah = 1$

d $as = s + h$
$ah = a + h$

e $s = h^2 + ah$
$h = 3a^2$

f $as = a + 2h$
$h = a - s$

g $s = 2 + ah + h^2$
$h = a - \dfrac{1}{a}$

h $3s - ah = a^2$
$as + 2h = 3a$

4 For the simultaneous equations $ax + by = p$ and $bx - ay = q$, show that $x = \dfrac{ap + bq}{a^2 + b^2}$ and $y = \dfrac{bp - aq}{a^2 + b^2}$.

5 For the simultaneous equations $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$, show that $x = y = \dfrac{ab}{a + b}$.

2C Linear coordinate geometry

In this section we revise the concepts of linear coordinate geometry.

Example 4

A straight line passes through the points $A(-2, 6)$ and $B(4, 7)$. Find:

a the distance AB

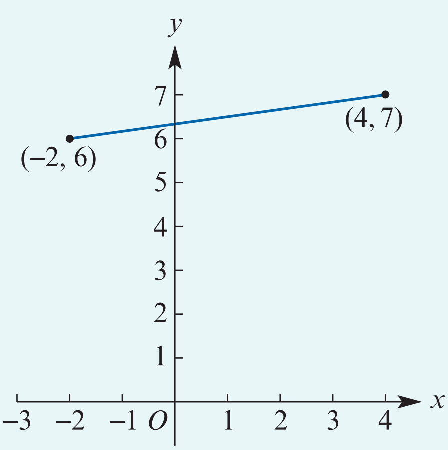
b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.



Solution

a The distance AB is

$$\sqrt{(4 - (-2))^2 + (7 - 6)^2} = \sqrt{37}$$

Explanation

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

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b The midpoint of AB is

$$\left(\frac{-2+4}{2}, \frac{6+7}{2}\right) = \left(1, \frac{13}{2}\right)$$

The line segment joining $A(x_1, y_1)$ and $B(x_2, y_2)$ has midpoint $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.

c The gradient of line AB is

$$\frac{7-6}{4-(-2)} = \frac{1}{6}$$

Gradient

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

d The equation of line AB is

$$y - 6 = \frac{1}{6}(x - (-2))$$

which simplifies to $6y - x - 38 = 0$.

Equation of a straight line passing through a given point (x_1, y_1) and having gradient m is $y - y_1 = m(x - x_1)$.

e Gradient $m = \frac{1}{6}$ and $(x_1, y_1) = (1, 5)$.

The line has equation

$$y - 5 = \frac{1}{6}(x - 1)$$

which simplifies to $6y - x - 29 = 0$.

Parallel lines have the same gradient.

f A perpendicular line has gradient -6.

Thus the equation is

$$y - \frac{13}{2} = -6(x - 1)$$

which simplifies to $2y + 12x - 25 = 0$.

If two straight lines are perpendicular to each other, then the product of their gradients is -1.

Example 5

A fruit and vegetable wholesaler sells 30 kg of hydroponic tomatoes for \$148.50 and sells 55 kg of hydroponic tomatoes for \$247.50. Find a linear model for the cost, $\$C$, of x kg of hydroponic tomatoes. How much would 20 kg of tomatoes cost?

Solution

Let $(x_1, C_1) = (30, 148.5)$ and $(x_2, C_2) = (55, 247.5)$.

The equation of the straight line is given by

$$C - C_1 = m(x - x_1) \quad \text{where } m = \frac{C_2 - C_1}{x_2 - x_1}$$

Now $m = \dfrac{247.5 - 148.5}{55 - 30} = 3.96$ and so

$$C - 148.5 = 3.96(x - 30)$$

Therefore the straight line has equation $C = 3.96x + 29.7$.

Substitute $x = 20$:

$$C = 3.96 \times 20 + 29.7 = 108.9$$

Hence it would cost \$108.90 to buy 20 kg of tomatoes.

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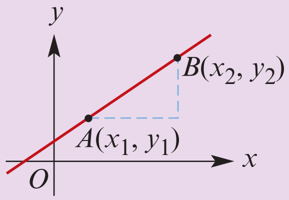
The following is a summary of the material that is assumed to have been covered in Mathematical Methods Units 1 & 2.

Section summary

- **Distance between two points**

  $$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



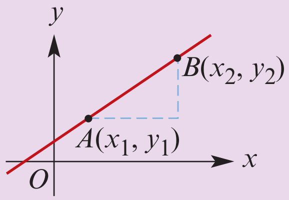
- **Midpoint of a line segment**

  The midpoint of the line segment joining two points (x_1, y_1) and (x_2, y_2) is the point with coordinates

  $$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- **Gradient of a straight line**

  Gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$

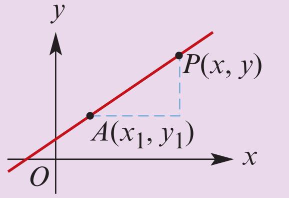


- **Equation of a straight line**
  - Gradient–intercept form: A straight line with gradient m and y-axis intercept c has equation

    $$y = mx + c$$

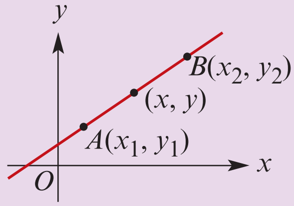
  - The equation of a straight line passing through a given point (x_1, y_1) and having gradient m is

    $$y - y_1 = m(x - x_1)$$



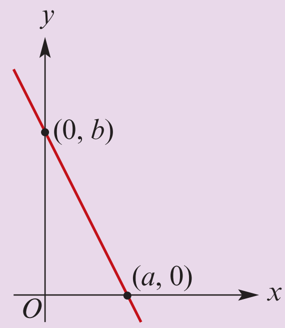
  - The equation of a straight line passing through two given points (x_1, y_1) and (x_2, y_2) is

    $$y - y_1 = m(x - x_1) \quad \text{where } m = \frac{y_2 - y_1}{x_2 - x_1}$$



  - Intercept form: The straight line passing through the two points $(a, 0)$ and $(0, b)$ has equation

    $$\frac{x}{a} + \frac{y}{b} = 1$$



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- **Tangent of the angle of slope**
  For a straight line with gradient m, the angle of slope is found using

  $$m = \tan \theta$$

  where θ is the angle that the line makes with the positive direction of the x-axis.
- **Perpendicular straight lines**
  If two straight lines are perpendicular to each other, the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Exercise 2C

Example 4

Skillsheet

1 A straight line passes through the points $A(-2, 6)$ and $B(4, -7)$. Find:

a the distance AB

b the midpoint of line segment AB

c the gradient of line AB

d the equation of line AB

e the equation of the line parallel to AB which passes through the point $(1, 5)$

f the equation of the line perpendicular to AB which passes through the midpoint of AB.

2 Find the coordinates of M, the midpoint of AB, where A and B have the following coordinates:

a $A(1, 4),\ B(5, 11)$ **b** $A(-6, 4),\ B(1, -8)$ **c** $A(-1, -6),\ B(4, 7)$

3 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-4, 5),\ M(0, 6)$ **b** $X(-1, -4),\ M(2, -3)$

c $X(6, -3),\ M(4, 8)$ **d** $X(2, -3),\ M(0, -6)$

4 Use $y = mx + c$ to sketch the graph of each of the following:

a $y = 3x - 3$ **b** $y = -3x + 4$ **c** $3y + 2x = 12$

d $4x + 6y = 12$ **e** $3y - 6x = 18$ **f** $8x - 4y = 16$

5 Find the equations of the following straight lines:

a gradient $+2$, passing through $(4, 2)$

b gradient -3, passing through $(-3, 4)$

c passing through the points $(1, 3)$ and $(4, 7)$

d passing through the points $(-2, -3)$ and $(2, 5)$

6 Use the intercept method to find the equations of the straight lines passing through:

a $(-3, 0)$ and $(0, 2)$ **b** $(4, 0)$ and $(0, 6)$

c $(-4, 0)$ and $(0, -3)$ **d** $(0, -2)$ and $(6, 0)$

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7 Write the following in intercept form and hence draw their graphs:

a $3x + 6y = 12$ **b** $4y - 3x = 12$ **c** $4y - 2x = 8$ **d** $\frac{3}{2}x - 3y = 9$

Example 5 **8** A printing firm charges \$35 for printing 600 sheets of headed notepaper and \$46 for printing 800 sheets. Find a linear model for the charge, \$$C$, for printing n sheets. How much would they charge for printing 1000 sheets?

9 An electronic bank teller registered \$775 after it had counted 120 notes and \$975 after it had counted 160 notes.

a Find a formula for the sum registered (\$$C$) in terms of the number of notes (n) counted.

b Was there a sum already on the register when counting began?

c If so, how much?

10 Find the distance between each of the following pairs of points:

a $(2, 6),\ (3, 4)$ **b** $(5, 1),\ (6, 2)$ **c** $(-1, 3),\ (4, 5)$

d $(-1, 7),\ (1, -11)$ **e** $(-2, -6),\ (2, -8)$ **f** $(0, 4),\ (3, 0)$

11 **a** Find the equation of the straight line which passes through the point $(1, 6)$ and is:

i parallel to the line with equation $y = 2x + 3$

ii perpendicular to the line with equation $y = 2x + 3$.

b Find the equation of the straight line which passes through the point $(2, 3)$ and is:

i parallel to the line with equation $4x + 2y = 10$

ii perpendicular to the line with equation $4x + 2y = 10$.

12 Find the equation of the line which passes through the point of intersection of the lines $y = x$ and $x + y = 6$ and which is perpendicular to the line with equation $3x + 6y = 12$.

13 The length of the line segment joining $A(2, -1)$ and $B(5, y)$ is 5 units. Find y.

14 The length of the line segment joining $A(2, 6)$ and $B(10, y)$ is 10 units. Find y.

15 The length of the line segment joining $A(2, 8)$ and $B(12, y)$ is 26 units. Find y.

16 Find the equation of the line passing through the point $(-1, 3)$ which is:

a **i** parallel to the line with equation $2x + 5y - 10 = 0$

ii parallel to the line with equation $4x + 5y + 3 = 0$

b **i** perpendicular to the line with equation $2x + 5y - 10 = 0$

ii perpendicular to the line with equation $4x + 5y + 3 = 0$.

17 For each of the following, find the angle that the line joining the given points makes with the positive direction of the x-axis:

a $(-4, 1),\ (4, 6)$ **b** $(2, 3),\ (-4, 6)$ **c** $(5, 1),\ (-1, -8)$ **d** $(-4, 2),\ (2, -8)$

18 Find the acute angle between the lines $y = 2x + 4$ and $y = -3x + 6$.

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19 Given the points $A(a, 3)$, $B(-2, 1)$ and $C(3, 2)$, find the possible values of a if the length of AB is twice the length of BC.

20 Three points have coordinates $A(1, 7)$, $B(7, 5)$ and $C(0, -2)$. Find:

a the equation of the perpendicular bisector of AB

b the point of intersection of this perpendicular bisector and BC.

21 The point (h, k) lies on the line $y = x + 1$ and is 5 units from the point $(0, 2)$. Write down two equations connecting h and k and hence find the possible values of h and k.

22 P and Q are the points of intersection of the line $\frac{y}{2} + \frac{x}{3} = 1$ with the x- and y-axes respectively. The gradient of QR is $\frac{1}{2}$ and the point R has x-coordinate $2a$, where $a > 0$.

a Find the y-coordinate of R in terms of a.

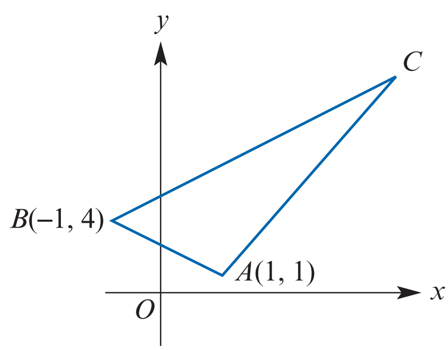
b Find the value of a if the gradient of PR is -2.

23 The figure shows a triangle ABC with $A(1, 1)$ and $B(-1, 4)$. The gradients of AB, AC and BC are $-3m$, $3m$ and m respectively.

a Find the value of m.

b Find the coordinates of C.

c Show that $AC = 2AB$.



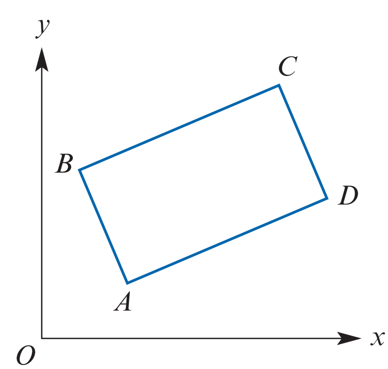
24 In the rectangle $ABCD$, the points A and B are $(4, 2)$ and $(2, 8)$ respectively. Given that the equation of AC is $y = x - 2$, find:

a the equation of BC

b the coordinates of C

c the coordinates of D

d the area of rectangle $ABCD$.



25 $ABCD$ is a parallelogram, with vertices labelled anticlockwise, such that A and C are the points $(-1, 5)$ and $(5, 1)$ respectively.

a Find the coordinates of the midpoint of AC.

b Given that BD is parallel to the line with equation $y + 5x = 2$, find the equation of BD.

c Given that BC is perpendicular to AC, find:

i the equation of BC

ii the coordinates of B

iii the coordinates of D.

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2D Applications of linear functions

In this section, we revise applications of linear functions.

Example 6

There are two possible methods for paying gas bills:

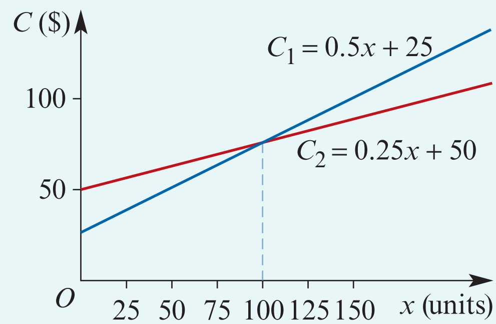
Method A A fixed charge of \$25 per quarter + 50c per unit of gas used
Method B A fixed charge of \$50 per quarter + 25c per unit of gas used

Determine the number of units which must be used before method B becomes cheaper than method A.

Solution

Let C_1 = charge (\$) using method A
C_2 = charge (\$) using method B
x = number of units of gas used

Then $C_1 = 25 + 0.5x$
$C_2 = 50 + 0.25x$



From the graph, we see that method B is cheaper if the number of units exceeds 100.

The solution can also be obtained by solving simultaneous linear equations:

$$C_1 = C_2$$
$$25 + 0.5x = 50 + 0.25x$$
$$0.25x = 25$$
$$x = 100$$

Exercise 2D

Example 6 Skillsheet

1 On a small island two rival taxi firms have the following fare structures:

Firm A Fixed charge of \$1 plus 40 cents per kilometre
Firm B 60 cents per kilometre, no fixed charge

a Find an expression for C_A, the charge of firm A, in terms of n, the number of kilometres travelled, and an expression for C_B, the charge of firm B, in terms of the number of kilometres travelled.

b On the one set of axes, sketch the graphs of the charge of each firm against the number of kilometres travelled.

c Find the distance for which the two firms charge the same amount.

d On a new set of axes, sketch the graph of $D = C_A - C_B$ against n, and explain what this graph represents.

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2 A car journey of 300 km lasts 4 hours. Part of this journey is on a freeway at an average speed of 90 km/h. The rest is on country roads at an average speed of 70 km/h. Let T be the time (in hours) spent on the freeway.

a In terms of T, state the number of hours travelling on country roads.

b **i** State the distance travelled on the freeway in terms of T.

ii State the distance travelled on country roads in terms of T.

c **i** Find T.

ii Find the distance travelled on each type of road.

3 A farmer measured the quantity of water in a storage tank 20 days after it was filled and found it contained 3000 litres. After a further 15 days it was measured again and found to contain 1200 litres of water. Assume that the amount of water in the tank decreases at a constant rate.

a Find the relation between L, the number of litres of water in the tank, and t, the number of days after the tank was filled.

b How much water does the tank hold when it is full?

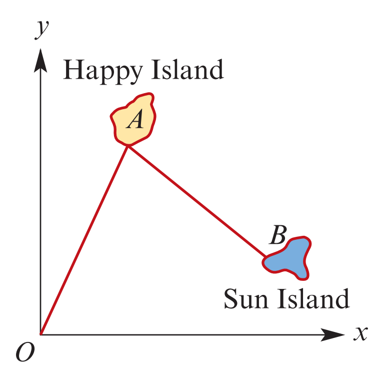
c Sketch the graph of L against t for a suitable domain.

d State this domain.

e How long does it take for the tank to empty?

f At what rate does the water leave the tank?

4 A boat leaves from O to sail to two islands. The boat arrives at a point A on Happy Island with coordinates $(10, 22.5)$, where units are in kilometres.



a Find the equation of the line through points O and A.

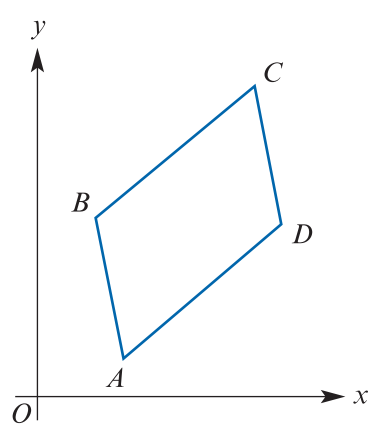
b Find the distance OA to the nearest metre.

The boat arrives at Sun Island at point B. The coordinates of point B are $(23, 9)$.

c Find the equation of line AB.

d A third island lies on the perpendicular bisector of line segment AB. Its port is denoted by C. It is known that the x-coordinate of C is 52. Find the y-coordinate of the point C.

5 $ABCD$ is a parallelogram with vertices $A(2, 2)$, $B(1.5, 4)$ and $C(6, 6)$.



a Find the gradient of:

i line AB **ii** line AD

b Find the equation of:

i line BC **ii** line CD

c Find the equations of the diagonals AC and BD.

d Find the coordinates of the point of intersection of the diagonals.

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6 The triangle ABC is isosceles. The vertices are $A(5, 0)$, $B(13, 0)$ and $C(9, 10)$.

a Find the coordinates of the midpoints M and N of AC and BC respectively.

b Find the equation of the lines:

i AC

ii BC

iii MN

c Find the equations of the lines perpendicular to AC and BC, passing through the points M and N respectively, and find the coordinates of their intersection point.

2E Matrices

This section provides a brief introduction to matrices. In Chapter 3, we will see that the transformations we consider in this course can be determined through matrix arithmetic. We will consider the inverse of a 2×2 matrix only in the context of transformations; this is done in Section 3J. Additional information and exercises on matrices are available in the Interactive Textbook.

▶ Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The following are examples of matrices:

$$\begin{bmatrix} -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The size, or **dimension**, of a matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines).

The dimensions of the above matrices are, in order:

$$2 \times 2, \qquad 2 \times 1, \qquad 3 \times 3, \qquad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

In this book we are only interested in 2×2 matrices and 2×1 matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 2×2 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A general 2×1 matrix may be written as

$$\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

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A matrix is, then, a way of recording a set of numbers, arranged in a particular way. As in Cartesian coordinates, the order of the numbers is significant. Although the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

have the same numbers and the same number of entries, they are different matrices (just as (2, 1) and (1, 2) are the coordinates of different points).

Two matrices **A** and **B** are **equal**, and we can write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same number or entry at corresponding positions.

Addition, subtraction and multiplication by a scalar

Addition is defined for two matrices only when they have the same dimension. In this case, the sum of the two matrices is found by adding corresponding entries.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and $\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} \\ a_{21} + b_{21} \end{bmatrix}$

Subtraction is defined in a similar way: If two matrices have the same dimension, then their difference is found by subtracting corresponding entries.

Example 7

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

It is useful to define **multiplication of a matrix by a real number**. If **A** is an $m \times n$ matrix and k is a real number (also called a **scalar**), then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of **A**. Thus

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

These definitions have the helpful consequence that, if a matrix is added to itself, the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. Similarly, the sum of n matrices each equal to **A** is $n\mathbf{A}$ (where n is a natural number).

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**.

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Example 8

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

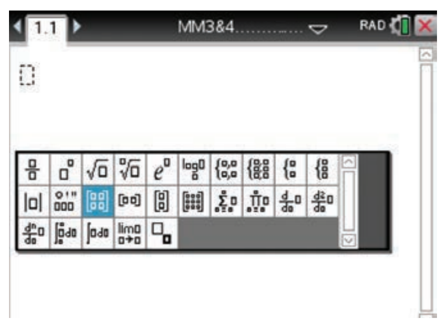
If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned}\mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix}\end{aligned}$$

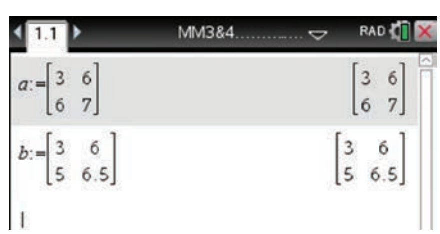
Using the TI-Nspire

The matrix template

- The simplest way to enter a 2×2 matrix is using the 2×2 matrix template as shown. (Access the templates using either or ctrl menu > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.



- Define the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as shown. The assignment symbol := is accessed using ctrl . Use the touchpad arrows to move between the entries of the matrix.
- Define the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$ similarly.



Note: All variables will be changed to lower case.
Alternatively, you can store (ctrl var) the matrices if preferred.

Entering matrices directly

- To enter matrix **A** without using the template, enter the matrix row by row as [[3, 6][6, 7]].

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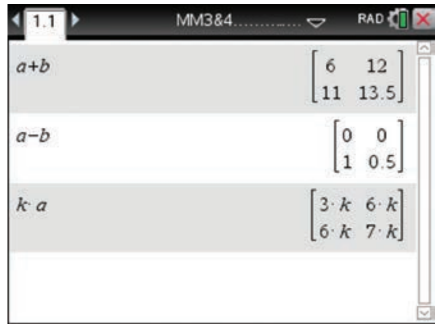
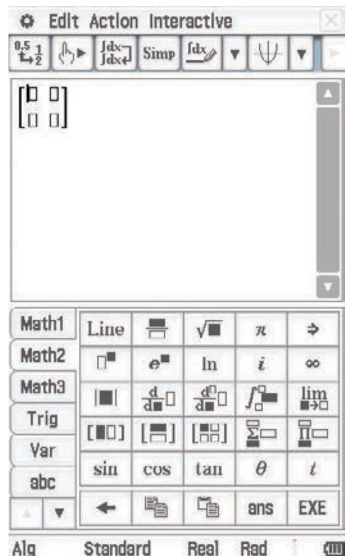
Addition, subtraction and multiplication by a scalar

- Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



Using the Casio ClassPad

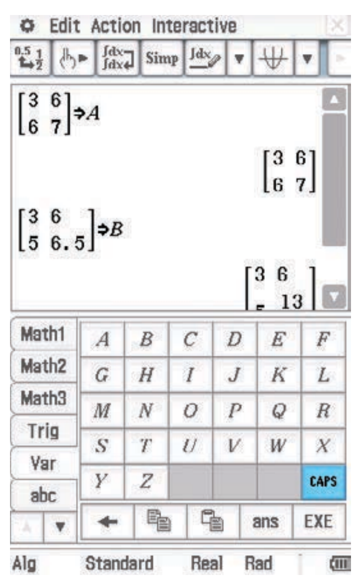
- Matrices are accessed through the Math2 keyboard.
- Select [⊞] and tap on each of the entry boxes to enter the matrix values.

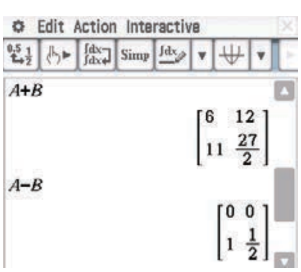


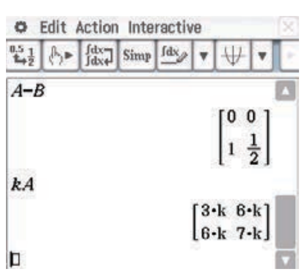
Notes:

- To expand the 2×2 matrix to a 3×3 matrix, tap on the [⊞] button twice.
- To increase the number of rows, tap on the [⊟] button. To increase the number of columns, tap on the [◫] button.

- Matrices can be stored as a variable for later use in operations by selecting the store button ⇒ located in Math1 followed by the variable name (usually a capital letter).
- Once **A** and **B** are defined as shown, the matrices **A** + **B**, **A** − **B** and k**A** can be found. (Use the Var keyboard to enter the variable names.)







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Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous subsection. The definition for multiplication of matrices is less natural. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

$$\begin{aligned} \text{Then} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Example 9

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

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Using the TI-Nspire

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

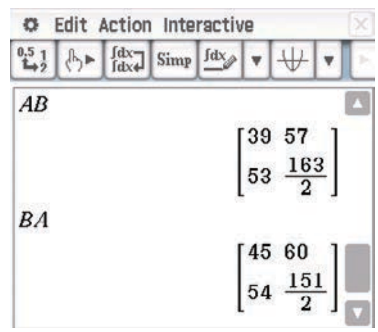
The products **AB** and **BA** are shown.

Using the Casio ClassPad

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

The products **AB** and **BA** are shown.



A matrix with the same number of rows and columns is called a **square matrix**.

For 2×2 matrices, the **identity matrix** is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix has the property that $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for any 2×2 matrix $\mathbf{A}$.

In general, for the family of $n \times n$ matrices, the multiplicative identity $\mathbf{I}$ is the matrix that has ones in the 'top left' to 'bottom right' diagonal and has zeroes in all other positions.

Section summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same number or entry at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is defined in a similar way.

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- If $\mathbf{A}$ is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is defined to be an $m \times n$ matrix whose entries are k times the corresponding entries of $\mathbf{A}$.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

- If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Exercise 2E

Example 7

1 Find:

a $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -2 & 3 \end{bmatrix}$ **b** $\begin{bmatrix} 4 \\ -2 \end{bmatrix} - \begin{bmatrix} 6 \\ -2 \end{bmatrix}$

Example 8

2 If $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -4 \\ -3 & 6 \end{bmatrix}$, find the matrix $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Example 9

3 For $\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, find $\mathbf{AB}$.

4 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} -2 & -2 \\ 3 & 2 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{BA}$ **d** $\mathbf{A} - \mathbf{B}$
e $k\mathbf{A}$ **f** $2\mathbf{A} + 3\mathbf{B}$ **g** $\mathbf{A} - 2\mathbf{B}$

5 $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ -3 & -3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ 5 & 1 \end{bmatrix}$

Calculate:

a $2\mathbf{A}$ **b** $3\mathbf{B}$ **c** $2\mathbf{A} + 3\mathbf{B}$ **d** $3\mathbf{B} - 2\mathbf{A}$

6 $\mathbf{P} = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} -1 & -4 \\ 5 & 0 \end{bmatrix}$, $\mathbf{R} = \begin{bmatrix} 0 & -4 \\ 1 & 1 \end{bmatrix}$

Calculate:

a $\mathbf{P} + \mathbf{Q}$ **b** $\mathbf{P} + 3\mathbf{Q}$ **c** $2\mathbf{P} - \mathbf{Q} + \mathbf{R}$

7 If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -3 & -4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -5 \\ -2 & 1 \end{bmatrix}$, find matrices $\mathbf{X}$ and $\mathbf{Y}$ such that $2\mathbf{A} - 3\mathbf{X} = \mathbf{B}$ and $3\mathbf{A} + 2\mathbf{Y} = 2\mathbf{B}$.

8 If $\mathbf{X} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
find the products $\mathbf{AX}$, $\mathbf{BX}$, $\mathbf{IX}$, $\mathbf{AI}$, $\mathbf{IB}$, $\mathbf{AB}$, $\mathbf{BA}$, $\mathbf{A}^2$ and $\mathbf{B}^2$.

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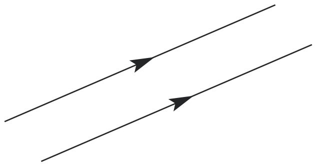
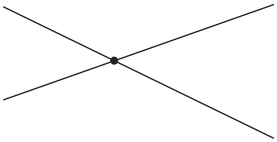
2F The geometry of simultaneous linear equations with two variables

Two distinct straight lines are either parallel or meet at a point.





There are three cases for a system of two linear equations with two variables.

	Example	Solutions	Geometry
Case 1	$2x + y = 5$ $x - y = 4$	Unique solution: $x = 3,\ y = -1$	Two lines meeting at a point
Case 2	$2x + y = 5$ $2x + y = 7$	No solutions	Distinct parallel lines
Case 3	$2x + y = 5$ $4x + 2y = 10$	Infinitely many solutions	Two copies of the same line

Example 10

Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 24$ have no solution.

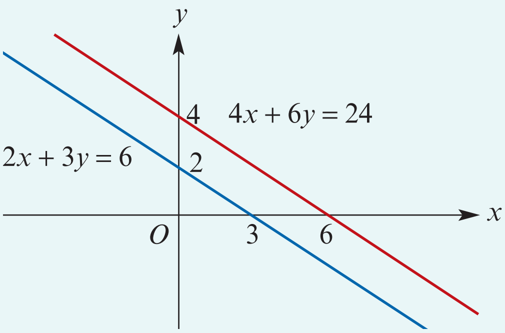
Solution

First write the two equations in the form $y = mx + c$. They become

$y = -\frac{2}{3}x + 2$ and $y = -\frac{2}{3}x + 4$

Both lines have gradient $-\frac{2}{3}$. The y-axis intercepts are 2 and 4 respectively.

The equations have no solution as they correspond to distinct parallel lines.



Example 11

The simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Solution

The two lines coincide, and so the solutions are all points on this line. We make use of a third variable λ as the parameter. If $y = \lambda$, then $x = \dfrac{6 - 3\lambda}{2}$. The points on the line are all points of the form $\left(\dfrac{6 - 3\lambda}{2}, \lambda\right)$.

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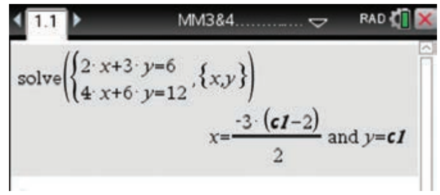
Using the TI-Nspire

Simultaneous equations can be solved in a **Calculator** application.

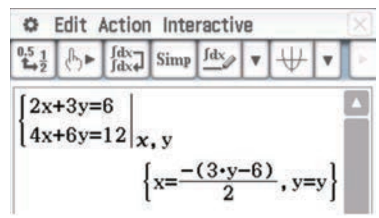


- Use (menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**.
- Complete the pop-up screen.

The solution to this system of equations is given by the calculator as shown. The variable $c1$ takes the place of λ.

Using the Casio ClassPad

To solve the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 12$:



- Open the (Math1) keyboard.
- Select the simultaneous equations icon.
- Enter the two equations into the two lines and type x, y in the bottom-right square to indicate the variables.
- Select (EXE).

Choose $y = \lambda$ to obtain the solution $x = \dfrac{6 - 3\lambda}{2}$, $y = \lambda$ where λ is any real number.

Example 12

Consider the simultaneous linear equations

$$(m-2)x + y = 2 \qquad \text{and} \qquad mx + 2y = k$$

Find the values of m and k such that the system of equations has:

a a unique solution **b** no solution **c** infinitely many solutions.

Solution

Use a CAS calculator to find the solution:

$$x = \frac{4-k}{m-4} \quad \text{and} \quad y = \frac{k(m-2)-2m}{m-4}, \qquad \text{for } m \neq 4$$

a There is a unique solution if $m \neq 4$ and k is any real number.

b If $m = 4$, the equations become

$$2x + y = 2 \quad \text{and} \quad 4x + 2y = k$$

There is no solution if $m = 4$ and $k \neq 4$.

c If $m = 4$ and $k = 4$, there are infinitely many solutions as the equations are the same.

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Section summary

There are three cases for a system of two linear equations in two variables:

- unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
- infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
- no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.

Exercise 2F

1 Solve each of the following pairs of simultaneous linear equations:

a $3x + 2y = 6$, $x - y = 7$ **b** $2x + 6y = 0$, $y - x = 2$ **c** $4x - 2y = 7$, $5x + 7y = 1$ **d** $2x - y = 6$, $4x - 7y = 5$

2 For each of the following, state whether there is no solution, one solution or infinitely many solutions:

a $3x + 2y = 6$, $3x - 2y = 12$ **b** $x + 2y = 6$, $2x + 4y = 12$ **c** $x - 2y = 3$, $2x - 4y = 12$

Example 10 **3** Explain why the simultaneous equations $2x + 3y = 6$ and $4x + 6y = 10$ have no solution.

Example 11 **4** The simultaneous equations $x - y = 6$ and $2x - 2y = 12$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Example 12 **5** Find the value of m for which the simultaneous equations

$$3x + my = 5$$
$$(m + 2)x + 5y = m$$

a have infinitely many solutions
b have no solution.

6 Find the value of m for which the simultaneous equations

$$(m + 3)x + my = 12$$
$$(m - 1)x + (m - 3)y = 7$$

have no solution.

7 Consider the simultaneous equations

$$mx + 2y = 8$$
$$4x - (2 - m)y = 2m$$

a Find the values of m for which there are:
  i no solutions
  ii infinitely many solutions.
b Solve the equations in terms of m, for suitable values of m.

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8 **a** Solve the simultaneous equations $2x - 3y = 4$ and $x + ky = 2$, where k is a constant.

b Find the value of k for which there is not a unique solution.

9 Find the values of b and c for which the equations $x + 5y = 4$ and $2x + by = c$ have:

a a unique solution

b an infinite set of solutions

c no solution.

2G Simultaneous linear equations with more than two variables

Consider the general system of three linear equations in three unknowns:

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

In this section we look at how to solve such systems of simultaneous equations. In some cases, this can be done easily by elimination, as shown in Examples 13 and 14. In these cases, you could be expected to find the solution by hand. We will see that in some cases using a calculator is the best choice.

Example 13

Solve the following system of three equations in three unknowns:

$$2x + y + z = -1 \qquad (1)$$
$$3y + 4z = -7 \qquad (2)$$
$$6x + z = 8 \qquad (3)$$

Solution

Subtract (1) from (3):

$$4x - y = 9 \qquad (4)$$

Subtract (2) from $4 \times (3)$:

$$24x - 3y = 39$$
$$8x - y = 13 \qquad (5)$$

Subtract (4) from (5) to obtain $4x = 4$. Hence $x = 1$.

Substitute in (4) to find $y = -5$, and substitute in (3) to find $z = 2$.

Explanation

The aim is first to eliminate z and obtain two simultaneous equations in x and y only.

Having obtained equations (4) and (5), we solve for x and y. Then substitute to find z.

It should be noted that, just as for two equations in two unknowns, there is a geometric interpretation for three equations in three unknowns. There is only a unique solution if the three equations represent three planes intersecting at a point.

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Example 14

Solve the following simultaneous linear equations for x, y and z:

$$x - y + z = 6, \qquad 2x + z = 4, \qquad 3x + 2y - z = 6$$

Solution

$$\begin{aligned} x - y + z &= 6 \qquad (1) \\ 2x + z &= 4 \qquad (2) \\ 3x + 2y - z &= 6 \qquad (3) \end{aligned}$$

Eliminate z to find two simultaneous equations in x and y:

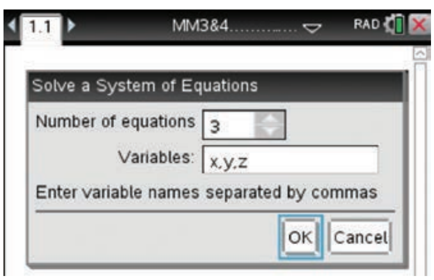
$$\begin{aligned} x + y &= -2 \qquad (4) \qquad \text{subtracted (1) from (2)} \\ 5x + 2y &= 10 \qquad (5) \qquad \text{added (2) to (3)} \end{aligned}$$

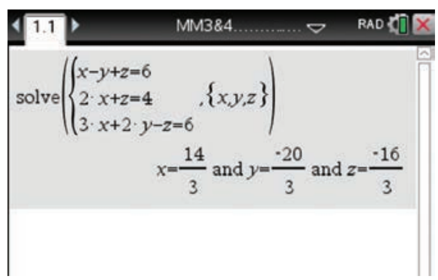
Solve to find $x = \frac{14}{3}$, $y = -\frac{20}{3}$, $z = -\frac{16}{3}$.

A CAS calculator can be used to solve a system of three equations in the same way as for solving two simultaneous equations.

Using the TI-Nspire

Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) as shown.

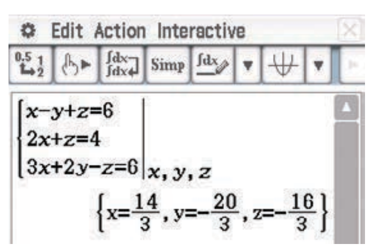




Note: The result could also be obtained using:
solve($x - y + z = 6$ and $2x + z = 4$ and $3x + 2y - z = 6$, $\{x, y, z\}$)

Using the Casio ClassPad

- From the Math1 keyboard, tap twice to create a template for three simultaneous equations.
- Enter the equations using the Var keyboard.



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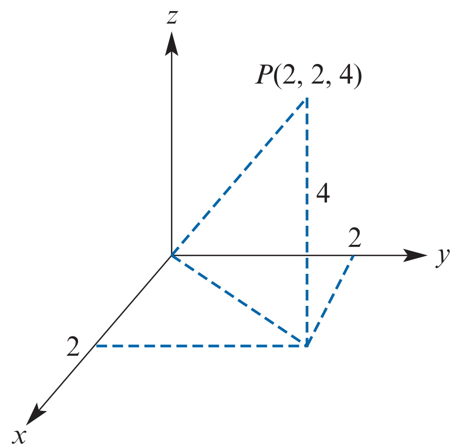
As a linear equation in two variables defines a line, a linear equation in three variables defines a plane.

The coordinate axes in three dimensions are drawn as shown. The point $P(2, 2, 4)$ is marked.



An equation of the form

$$ax + by + cz = d$$

defines a plane. As an example, we will look at the plane

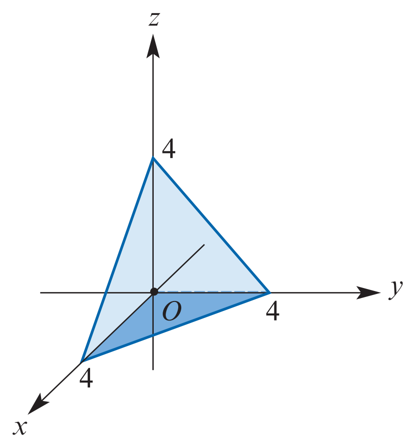
$$x + y + z = 4$$

We get some idea of how the graph sits by considering

- $x = 0, \; y = 0, \; z = 4$
- $x = 0, \; y = 4, \; z = 0$
- $x = 4, \; y = 0, \; z = 0$

and plotting these three points.

This results in being able to sketch the plane $x + y + z = 4$ as shown opposite.



The solution of simultaneous linear equations in three variables can correspond to:

- a point
- a line
- a plane

There also may be no solution. The situations are as shown in the following diagrams. Examples 13 and 14 provide examples of three planes intersecting at a point (Diagram 1).

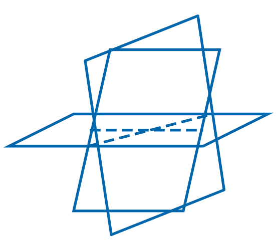


Diagram 1:
Intersection at a point

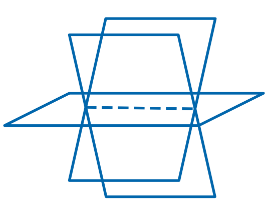


Diagram 2:
Intersection in a line

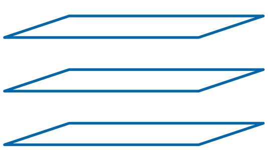


Diagram 3:
No intersection

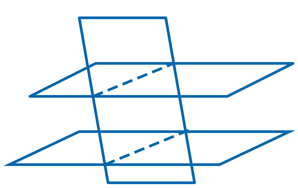


Diagram 4:
No common intersection

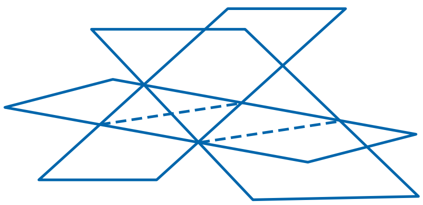


Diagram 5:
No common intersection

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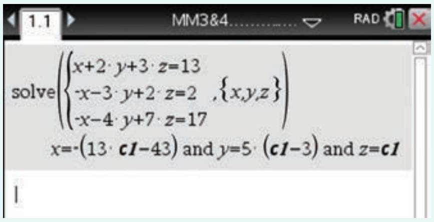
Example 15

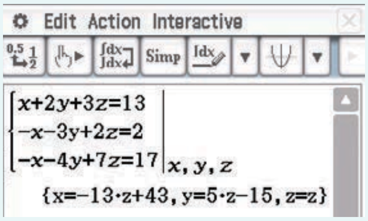
The simultaneous equations $x + 2y + 3z = 13$, $-x - 3y + 2z = 2$ and $-x - 4y + 7z = 17$ have infinitely many solutions. Describe these solutions through the use of a parameter.

Solution

The point $(-9, 5, 4)$ satisfies all three equations, but it is certainly not the only solution.

We can use a CAS calculator to find all the solutions in terms of a parameter λ.





Let $z = \lambda$. Then $x = 43 - 13\lambda$ and $y = 5\lambda - 15$.

For example, if $\lambda = 4$, then $x = -9$, $y = 5$ and $z = 4$.

Note that, as z increases by 1, x decreases by 13 and y increases by 5. All of the points that satisfy the equations lie on a straight line. This is the situation shown in Diagram 2.

Section summary

- A system of simultaneous linear equations in three or more variables can sometimes be solved by hand using elimination (see Example 13). In other cases, using a calculator is the best choice.
- The solution of simultaneous linear equations in three variables can correspond to a point, a line or a plane. There may also be no solution.

Exercise 2G

Example 13, 14

1 Solve each of the following systems of simultaneous equations:

a $2x + 3y - z = 12$
$2y + z = 7$
$2y - z = 5$

b $x + 2y + 3z = 13$
$-x - y + 2z = 2$
$-x + 3y + 4z = 26$

c $x + y = 5$
$y + z = 7$
$z + x = 12$

d $x - y - z = 0$
$5x + 20z = 50$
$10y - 20z = 30$

Example 15

2 Consider the simultaneous equations $x + 2y - 3z = 4$ and $x + y + z = 6$.

a Subtract the second equation from the first to find y in terms of z.

b Let $z = \lambda$. Solve the equations to give the solution in terms of λ.

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3 Consider the simultaneous equations

$$x + 2y + 3z = 13 \qquad (1)$$
$$-x - 3y + 2z = 2 \qquad (2)$$
$$-x - 4y + 7z = 17 \qquad (3)$$

a Add equation (2) to equation (1) and subtract equation (2) from equation (3).

b Comment on the equations obtained in part a.

c Let $z = \lambda$ and find y in terms of λ.

d Substitute for z and y in terms of λ in equation (1) to find x in terms of λ.

4 Solve each of the following pairs of simultaneous equations, giving your answer in terms of a parameter λ. Use the technique introduced in Question 2.

a $x - y + z = 4$
$-x + y + z = 6$

b $2x - y + z = 6$
$x - z = 3$

c $4x - 2y + z = 6$
$x + y + z = 4$

5 The system of equations

$$x + y + z + w = 4$$
$$x + 3y + 3z = 2$$
$$x + y + 2z - w = 6$$

has infinitely many solutions. Describe this family of solutions and give the unique solution when $w = 6$.

6 Find all solutions for each of the following systems of equations:

a $3x - y + z = 4$
$x + 2y - z = 2$
$-x + y - z = -2$

b $x - y - z = 0$
$3y + 3z = -5$

c $2x - y + z = 0$
$y + 2z = 2$

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Chapter summary

Spreadsheet AS Nrich

Coordinate geometry

- The **distance** between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- The **midpoint** of the line segment joining (x_1, y_1) and (x_2, y_2) is the point with coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

- The **gradient** of the straight line joining two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

- Different forms for the equation of a straight line:

$y = mx + c$	where m is the gradient and c is the y-axis intercept
$y - y_1 = m(x - x_1)$	where m is the gradient and (x_1, y_1) is a point on the line
$\frac{x}{a} + \frac{y}{b} = 1$	where $(a, 0)$ and $(0, b)$ are the axis intercepts

- For a straight line with gradient m, the **angle of slope** is found using

$$m = \tan\theta$$

where θ is the angle that the line makes with the positive direction of the x-axis.

- If two straight lines are **perpendicular** to each other, the product of their gradients is -1, i.e. $m_1 m_2 = -1$. (Unless one line is vertical and the other horizontal.)

Matrices

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns
  - they have the same entry at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Subtraction is defined in a similar way.

- If **A** is an $m \times n$ matrix and k is a real number, then $k\mathbf{A}$ is defined to be an $m \times n$ matrix whose entries are k times the corresponding entries of **A**.

$$k\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}$$

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- If **A** is an $m \times n$ matrix and **B** is an $n \times r$ matrix, then the product **AB** is the $m \times r$ matrix whose entries are determined as follows:

  To find the entry in row i and column j of **AB**, single out row i in matrix **A** and column j in matrix **B**. Multiply the corresponding entries from the row and column and then add up the resulting products.

  The product **AB** is defined only if the number of columns of **A** is the same as the number of rows of **B**.
- The $n \times n$ identity matrix **I** has the property that $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for each $n \times n$ matrix **A**.

Simultaneous equations

- There are three cases for a system of two linear equations in two variables:
  - unique solution (lines intersect at a point), e.g. $y = 2x + 3$ and $y = 3x + 3$
  - infinitely many solutions (lines coincide), e.g. $y = 2x + 3$ and $2y = 4x + 6$
  - no solution (lines are parallel), e.g. $y = 2x + 3$ and $y = 2x + 4$.
- The solution of simultaneous linear equations in three variables can correspond to a point, a line or a plane. There may also be no solution.

Technology-free questions

1 Solve the following linear equations:

a $3x - 2 = 4x + 6$ **b** $\dfrac{x+1}{2x-1} = \dfrac{4}{3}$ **c** $\dfrac{3x}{5} - 7 = 11$ **d** $\dfrac{2x+1}{5} = \dfrac{x-1}{2}$

2 Solve each of the following pairs of simultaneous linear equations:

a $y = x + 4$
$5y + 2x = 6$

b $\dfrac{x}{4} - \dfrac{y}{3} = 2$
$y - x = 5$

3 Solve each of the following for x:

a $bx - n = m$ **b** $b - cx = bx$ **c** $\dfrac{cx}{d} - c = 0$ **d** $px = qx - 6$

e $mx - n = nx + m$ **f** $\dfrac{1}{x-a} = \dfrac{a}{x}$

4 Sketch the graphs of the relations:

a $3y + 2x = 5$ **b** $x - y = 6$ **c** $\dfrac{x}{2} + \dfrac{y}{3} = 1$

5 **a** Find the equation of the straight line which passes through $(1, 3)$ and has gradient -2.

b Find the equation of the straight line which passes through $(1, 4)$ and $(3, 8)$.

c Find the equation of the straight line which is perpendicular to the line with equation $y = -2x + 6$ and which passes through the point $(1, 1)$.

d Find the equation of the straight line which is parallel to the line with equation $y = 6 - 2x$ and which passes through the point $(1, 1)$.

6 Find the distance between the points with coordinates $(-1, 6)$ and $(2, 4)$.

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7 Find the midpoint of the line segment AB joining the points $A(4, 6)$ and $B(-2, 8)$.

8 If M is the midpoint of XY, find the coordinates of Y when X and M have the following coordinates:

a $X(-6, 2)$, $M(8, 3)$ **b** $X(-1, -4)$, $M(2, -8)$

9 For the matrices $\mathbf{A} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, find:

a $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{AB}$ **c** $\mathbf{AC}$ **d** $\mathbf{BC}$ **e** $3\mathbf{C}$
f $\mathbf{BA}$ **g** $\mathbf{A} - \mathbf{B}$ **h** $k\mathbf{A}$ **i** $2\mathbf{A} + 3\mathbf{B}$ **j** $\mathbf{A} - 2\mathbf{B}$

10 The length of the line segment joining $A(5, 12)$ and $B(10, y)$ is 13 units. Find y.

11 Consider the simultaneous linear equations

$$\begin{aligned} mx - 4y &= m + 3 \\ 4x + (m + 10)y &= -2 \end{aligned}$$

where m is a real constant.

a Find the value of m for which there are infinitely many solutions.

b Find the values of m for which there is a unique solution.

12 Solve the following simultaneous equations. (You will need to use a parameter.)

a $$\begin{aligned} 2x - 3y + z &= 6 \\ -2x + 3y + z &= 8 \end{aligned}$$

b $$\begin{aligned} x - z + y &= 6 \\ 2x + z &= 4 \end{aligned}$$

Multiple-choice questions

1 A straight line has gradient $-\frac{1}{2}$ and passes through $(1, 4)$. The equation of the line is

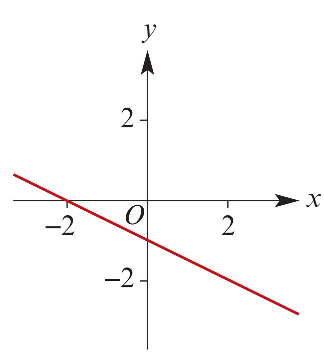
A $y = x + 4$ **B** $y = 2x + 2$ **C** $y = 2x + 4$
D $y = -\dfrac{1}{2}x + 4$ **E** $y = -\dfrac{1}{2}x + \dfrac{9}{2}$

2 The line $y = -2x + 4$ passes through a point $(a, 3)$. The value of a is

A $-\dfrac{1}{2}$ **B** 2 **C** $-\dfrac{7}{2}$ **D** -2 **E** $\dfrac{1}{2}$

3 The gradient of a line that is perpendicular to the line shown could be

A 1 **B** $\frac{1}{2}$ **C** $-\frac{1}{2}$
D 2 **E** -2



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Review

4 The coordinates of the midpoint of AB, where A has coordinates $(1, 7)$ and B has coordinates $(-3, 10)$, are

A $(-2, 3)$ **B** $(-1, 8)$ **C** $(-1, 8.5)$ **D** $(-1, 3)$ **E** $(-2, 8.5)$

5 The solution of the two simultaneous equations $ax - 5by = 11$ and $4ax + 10by = 2$ for x and y, in terms of a and b, is

A $x = -\frac{10}{a},\ y = -\frac{21}{5b}$ **B** $x = \frac{4}{a},\ y = -\frac{4}{5b}$ **C** $x = \frac{13}{5a},\ y = -\frac{42}{25b}$

D $x = \frac{13}{2a},\ y = -\frac{9}{10b}$ **E** $x = -\frac{3}{a},\ y = -\frac{14}{5b}$

6 The gradient of the line passing through $(3, -2)$ and $(-1, 10)$ is

A -3 **B** -2 **C** $-\frac{1}{3}$ **D** 4 **E** 3

7 If two lines $-2x + y - 3 = 0$ and $ax - 3y + 4 = 0$ are parallel, then a equals

A 6 **B** 2 **C** $\frac{1}{3}$ **D** $\frac{2}{3}$ **E** -6

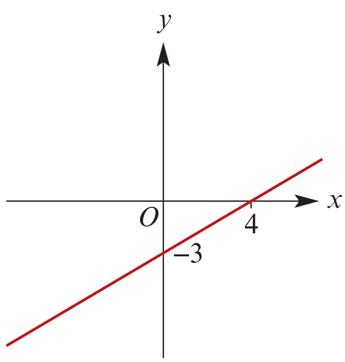
8 A straight line passes through $(-1, -2)$ and $(3, 10)$. The equation of the line is

A $y = 3x - 1$ **B** $y = 3x - 4$ **C** $y = 3x + 1$ **D** $y = \frac{1}{3}x + 9$ **E** $y = 4x - 2$

9 The length of the line segment connecting $(1, 4)$ and $(5, -2)$ is

A 10 **B** $2\sqrt{13}$ **C** 12 **D** 50 **E** $2\sqrt{5}$

10 The function with graph as shown has the rule



A $f(x) = 3x - 3$

B $f(x) = -\frac{3}{4}x - 3$

C $f(x) = \frac{3}{4}x - 3$

D $f(x) = \frac{4}{3}x - 3$

E $f(x) = 4x - 4$

11 The pair of simultaneous linear equations

$$bx + 3y = 0$$
$$4x + (b + 1)y = 0$$

where b is a real constant, has infinitely many solutions for

A $b \in \mathbb{R}$ **B** $b \in \{-3, 4\}$ **C** $b \in \mathbb{R} \setminus \{-3, 4\}$

D $b \in \{-4, 3\}$ **E** $b \in \mathbb{R} \setminus \{-4, 3\}$

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12 The simultaneous equations

$$(a-1)x + 5y = 7$$
$$3x + (a-3)y = a$$

have a unique solution for

A $a \in \mathbb{R} \setminus \{6, -2\}$ **B** $a \in \mathbb{R} \setminus \{0\}$ **C** $a \in \mathbb{R} \setminus \{6\}$
D $a = 6$ **E** $a = -2$

13 The midpoint of the line segment joining $(0, -6)$ and $(4, d)$ is

A $\left(-2, \dfrac{d+6}{2}\right)$ **B** $\left(2, \dfrac{d+6}{2}\right)$ **C** $\left(\dfrac{d+6}{2}, 2\right)$
D $\left(2, \dfrac{d-6}{2}\right)$ **E** $\dfrac{d+6}{4}$

14 The gradient of a line perpendicular to the line through $(3, 0)$ and $(0, -6)$ is

A $\dfrac{1}{2}$ **B** -2 **C** $-\dfrac{1}{2}$ **D** 2 **E** 6

Extended-response questions

1 A firm manufacturing jackets finds that it is capable of producing 100 jackets per day, but it can only sell all of these if the charge to wholesalers is no more than \$50 per jacket. On the other hand, at the current price of \$75 per jacket, only 50 can be sold per day.
Assume that the graph of price, $\$P$, against number sold per day, N, is a straight line.

a Sketch the graph of P against N.
b Find the equation of the straight line.
c Use the equation to find:
  i the price at which 88 jackets per day could be sold
  ii the number of jackets that should be manufactured to sell at \$60 each.

2 A new town was built 10 years ago to house the workers of a woollen mill established in a remote country area. Three years after the town was built, it had a population of 12 000 people. Business in the wool trade steadily grew, and eight years after the town was built the population had swelled to 19 240.

a Assuming the population growth can be modelled by a linear relationship, find a suitable relation for the population, p, in terms of t, the number of years since the town was built.
b Sketch the graph of p against t, and interpret the p-axis intercept.
c Find the current population of the town.
d Calculate the average rate of growth of the town.

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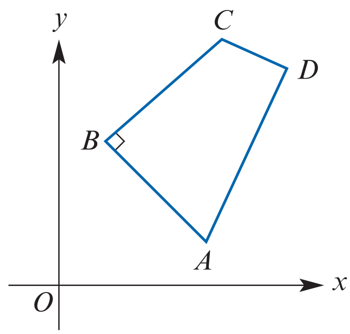
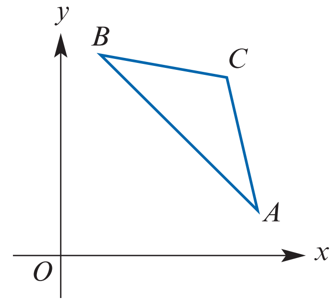
Review

3 $ABCD$ is a quadrilateral with angle ABC a right angle. The point D lies on the perpendicular bisector of AB. The coordinates of A and B are $(7, 2)$ and $(2, 5)$ respectively. The equation of line AD is $y = 4x - 26$.



- **a** Find the equation of the perpendicular bisector of line segment AB.
- **b** Find the coordinates of point D.
- **c** Find the gradient of line BC.
- **d** Find the value of the second coordinate c of the point $C(8, c)$.
- **e** Find the area of quadrilateral $ABCD$.

4 Triangle ABC is isosceles with $BC = AC$. The coordinates of the vertices are $A(6, 1)$ and $B(2, 8)$.



- **a** Find the equation of the perpendicular bisector of AB.
- **b** If the x-coordinate of C is 3.5, find the y-coordinate of C.
- **c** Find the length of AB.
- **d** Find the area of triangle ABC.

5 If $A = (-4, 6)$ and $B = (6, -7)$, find:

- **a** the coordinates of the midpoint of AB
- **b** the length of AB
- **c** the distance between A and B
- **d** the equation of AB
- **e** the equation of the perpendicular bisector of AB
- **f** the coordinates of the point P on the line segment AB such that $AP : PB = 3 : 1$
- **g** the coordinates of the point P on the line AB such that $AP : AB = 3 : 1$ and P is closer to point B than to point A.

6 A chemical manufacturer has an order for 500 litres of a 25% acid solution (i.e. 25% by volume is acid). Solutions of 30% and 18% are available in stock.

- **a** How much acid is required to produce 500 litres of 25% acid solution?
- **b** The manufacturer wishes to make up the 500 litres from a mixture of 30% and 18% solutions.
  Let x denote the amount of 30% solution required.
  Let y denote the amount of 18% solution required.
  Use simultaneous equations in x and y to determine the amount of each solution required.

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