SIT190 - WEEK 10

The House of Wisdom, Baghdad

Your band has safely arrived in Baghdad.

- You have gathered all your artefacts.
  - Fragments of the Rhind Papyrus
  - Dragon Bones
  - Missing piece of Plimpton Tablet
  - Fragments of the Antikythera Device

The House of Wisdom, Baghdad

You have gathered some tools and techniques along the way.

- Algebra
  - how to simplify mathematical expressions and solve equations
  - how to apply algebra to other mathematical structures eg matrices
- Differentiation
  - how to find the derivative
  - how to use it in graph sketching and to solve problems
- Integration
  - how to 'undo' the derivative
  - how to use it to solve problems

The House of Wisdom, Baghdad

The challenge is now to apply these skills to solve problems using the definite integral.

- Definite Integral
- How to use the definite integral to find the area under a curve
- How to use the definite integral to find the area between two curves

The House of Wisdom, Baghdad

Completing this week's challenges will grant your team admission to the House of Wisdom.

Image: Manuscript with depiction by Yahya ibn Vaseti found in the Maqama of Hariri located at the Bibliotheque Nationale de France. Image depicts a library with pupils in it by Zereshk

The House of Wisdom, Baghdad

- The House of Wisdom was founded in 8CE
- Destroyed in Siege of Baghdad in 1258.
- A library, gathering of knowledge and scholars.
- Mathematician and astronomer, Muhammad ibn Musa al-Khwarizmi (750-850CE).
  - Head of House of Wisdom ($\sim$ 820)
  - Algorithms and Algebra
  - Solving quadratics by completing the square.
  - Geometric proofs.

Muhammad ibn Musa al-Khwarizmi

'Al-Khwārizmī Concerning the Hindu Art of Reckoning'.

- Latin version: Algorithmo de Numero Indoreum (No known surviving Arabic copies)
- Describes the Hindu base-10 number system and how to perform calculations using it (possibly why our number system is sometimes called Arabic numerals or Hindu-Arabic)
- Operations on integers and fractions
- Square roots

Muhammad ibn Musa al-Khwarizmi

'Hisab Al-Jabr w'al Muqabala' (The compendious book on calculation by completion and balancing)

- 'Al-jabr' - restoration or completion, possibly referring to subtracting to remove a term to the other side and so isolate x

$$\text{eg. } x + 4 = 12 \Rightarrow x + 4 - 4 = 12 - 4 \Rightarrow x = 8.$$

- 'Muqabala' - reducing or balancing, possibly for cancelling like terms from both sides of an equation

$$\text{eg. } x + 2y = 4 + 2y => x = 4$$

The Final Quest

“A golden thread paves the way to the end of the road,
Follow it carefully to wisdom’s abode.
Carry each artefact that you have obtained,
Symbols that represent skills you have gained.
A quantity of silver and another of gold,
A precious pearl that must never be sold.
For like a pearl hidden in its unassuming shell,
The knowledge you’ve gained will help you excel.
To enter the house, you must offer exact
Amounts for these treasures; nothing must lack,
An excess is wasteful and will cost you much more,
So find the correct amounts to open the door.”

The Final Quest

The first challenge is to identify the length of the golden thread that paves the way to the House of Wisdom ...

But first we will look at definite integrals.

Definite Integral

If a function f is continuous over the interval $[a, b]$, then the definite integral is

$$\int_a^b f(x)dx = [g(x)]_a^b = g(b) - g(a)$$

For example:

$$\int_1^4 (3x^2 - 2x)dx = [x^3 - x^2]_1^4$$

$$= (4^3 - 4^2) - (1^3 - 1^2) = (64 - 16) - (1 - 1) = 48$$

Definite Integral

Calculate the definite integral $\int_1^4 (x^2 - 4x + 1)dx$ and the indefinite integral $\int (x^2 - 4x + 1)dx$.

What is the difference between the definite integral and the indefinite integral?

Can you explain why we do not need to include the constant c when calculating the definite integral?

Definite Integral

Try these examples with your band:

1. $\int_{-2}^{2}(x^3 - 5x^2 + 4x + 4)dx$
2. $\int_{-2}^{2}(x^3 - 5x^2 + 4x + 4)dx$
3. $\int_{-\pi}^{\pi} \cos(2x)dx$
4. $\int_{0}^{1}(e^{6x})dx$

The Quest

The length of the golden thread is the same value as this integral: $\int_{-2}^{3}(-4x^3 + 12x^2 - 4x + 1)dx$.

You follow the path for length of the golden thread.

Image: Gold Thread. [Textile; gold thread; textiles; thread, Textile; textiles]. Retrieved from
`https://library.artstor.org/asset/SS37466_37466_42953193`

The next task is to identify the correct quantities of silver and gold. You must have sufficient, but too much would be a waste of valuable resources. For this task, you need to understand how the definite integral can be used to measure area.

Image: Gold Thread. [Textile; gold thread; textiles; thread, Textile; textiles]. Retrieved from
https://library.artstor.org/asset/SS37466_37466_42953193

Area under a curve

- There is a connection between the definite integral and the area under the curve $f(x)$, that is between $f(x)$ and the x-axis).
- There are a few things to watch out for:
  - Sections where $f(x) > 0$ contribute positive values.
  - Sections where $f(x) < 0$ contribute negative values.
  - So the definite integral finds the net signed value.
  - So if $f(x)$ is not always greater than 0 in $[a, b]$ then we need to make some adjustments to find the area.
- In the next few slides, we give some intuition about the relationship between the definite integral and the area. (These slides are extra content and can be skipped if you prefer.)

Estimating the area

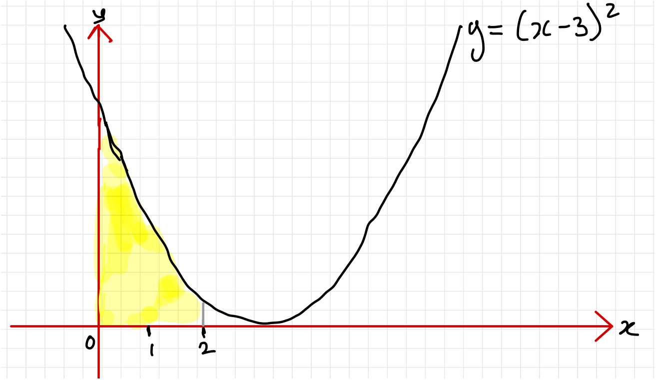
Some areas are easy to calculate:

- Area of a square is l^2 where l is the length of the sides
- Area of a triangle is $\frac{1}{2}bh$ where b is the length of the base and h is the height.
- Area of a rectangle is wh where h is the length and w is the width.

Unfortunately, the area under a curve is often not one of these. Suppose we estimate the area using rectangles.

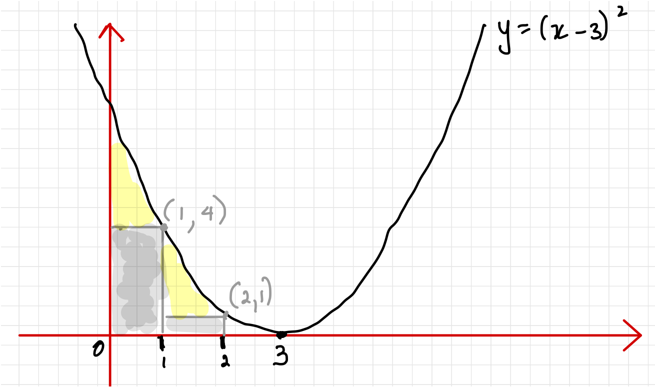
Estimating the area

Suppose, we want to find the area between the curve $y = (x-3)^2$ and the x-axis between $x = 0$ and $x = 2$.



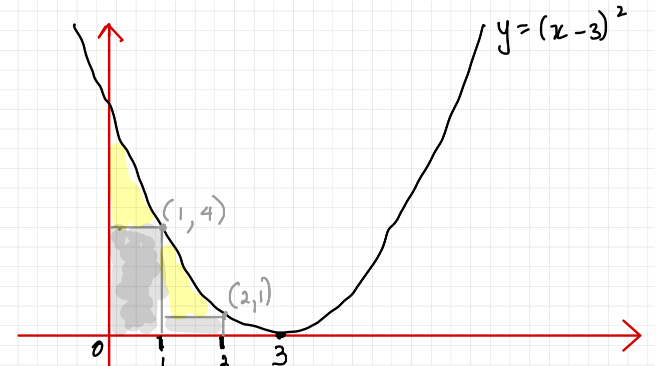
Estimating the area

- We divide the region into rectangles of width 1.
- The first rectangle has height 4, so has area $1 \times 4 = 4$
- The second rectangle has height 2, so has area $1 \times 2 = 2$
- So the total area is approximately $4 + 2 = 6$ square units.



Do you think that this is a good estimate of the area under this curve?

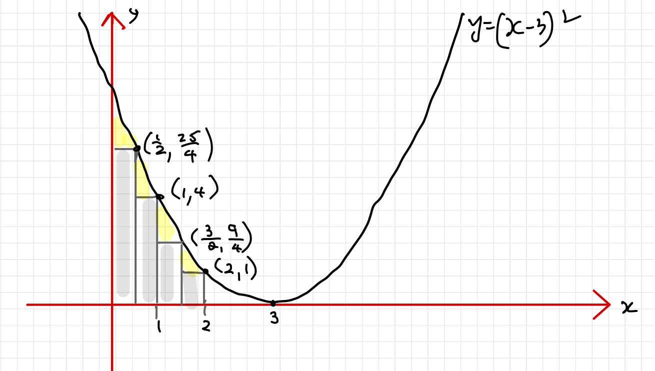
We used the point on the right-hand side for the height of the rectangle in this example. What would happen if we had chosen the left-hand side?



Estimating the area

Suppose we estimate with four rectangles instead of two.

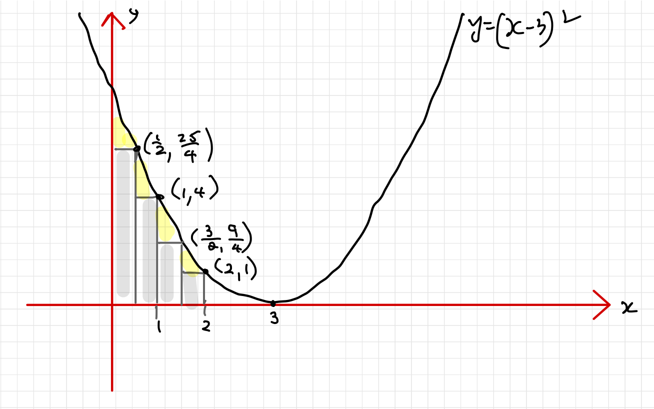
- We divide the region into 4 rectangles of width $\frac{1}{2}$.
- The first rectangle has height $\frac{25}{4}$, so has area $\frac{1}{2} \times \frac{25}{4} = \frac{25}{8}$
- The second rectangle has height 4, so has area $\frac{1}{2} \times 4 =$ _____
- The third rectangle has height $\frac{9}{4}$, so has area _____
- The second rectangle has height 1, so has area _____
- So the total area is approximately _____ square units.



Estimating the area

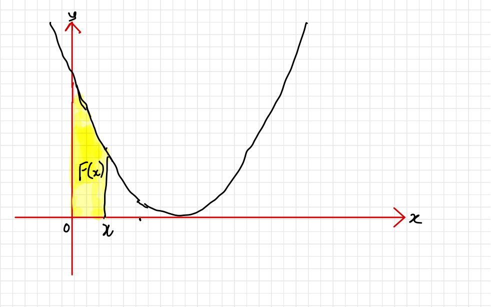
Do you think that this is a better estimate of the area under this curve?

What could you do to get a better estimate?



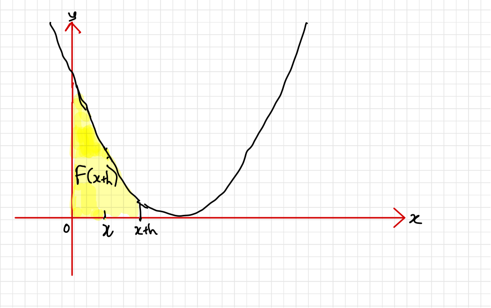
Area

- Thinner rectangles give more accurate estimates.
- So let's investigate what happens with a rectangle of width h.
- Let $F(x)$ denote the area of the region between $f(x)$ and the x-axis between $x = 0$ and some x.



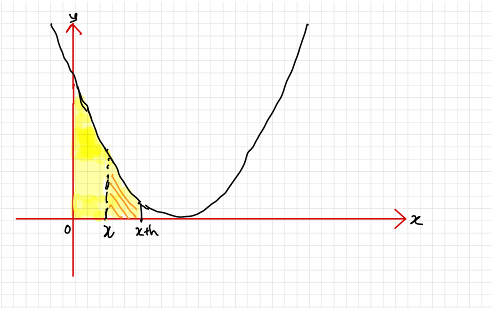
Area

- Then $F(x+h)$ is the area of the region between $f(x)$ and the x-axis between $x = 0$ and $x + h$



Area

- The area of the region between $f(x)$ and the x-axis between x and $x + h$ is $F(x + h) - F(x)$

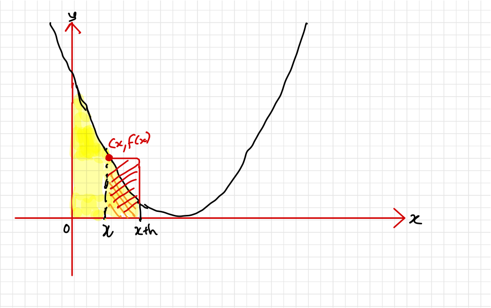


Area

- Suppose we estimate this region with a rectangle of width h and height $f(x)$ (the left hand height).
- The area of this rectangle is $hf(x)$.

$$\text{So } F(x+h) - F(x) \approx hf(x)$$

$$\text{or } \frac{F(x+h) - F(x)}{h} \approx f(x) \text{ which seems familiar.}$$



Area

What happens as h becomes arbitrarily small?

$$\lim_{h \Rightarrow 0} \frac{F(x+h) - F(x)}{h} = F'(x) = f(x)$$

Now

$$\begin{aligned} F'(x) &= f(x) \\ \Rightarrow \int F'(x)dx &= \int f(x)dx \\ \Rightarrow F(x) &= \int f(x)dx \\ \text{Area } &= \int f(x)dx \end{aligned}$$

which gives a relationship between the area and the definite integral.

Definite Integral and Area Under a Curve

More formally, if $f(x)$ is continuous for the interval $[a, b]$ then the definite integral $\int_a^b f(x)dx$ gives the **net signed area** of the regions between the curve $f(x)$ and the x-axis between $x = a$ and $x = b$.

Definite Integral and Area Under a Curve

1. Sketch the graph $y = 2x$
2. Calculate the following definite integrals:
   2.1 $\int_0^3 2x \; dx$
   2.2 $\int_{-2}^0 2x \; dx$
   2.3 $\int_{-2}^3 2x \; dx$
3. The areas of the regions enclosed by the curve $f(x)$ and the x-axis between $x = -2$ and $x = 3$ can be found using the areas of triangles. Calculate these areas and compare them to the definite integrals that your band found.

Reflection

When does the definite integral give the area?

- If $f(x) > 0$ over the interval $[a, b]$ then $\int_a^b f(x)dx$ is the ______________.

- If $f(x) < 0$ over the interval $[a, b]$ then $\int_a^b f(x)dx$ is the ______________.

- If $f(x)$ crosses the x axis at at least one point in the interval $[a, b]$ then $\int_a^b f(x)dx$ is the ______________.

How can we use the definite integral to find the area when the curve crosses the x-axis during the interval of interest?

Different types of question

Be careful that you answer the question being asked. Consider the following two questions:

1. Find $\int_{-\pi}^{\pi} \sin(x) dx$
2. Find the area under the curve $y = \sin(x)$ between $x = -\pi$ and $x = \pi$.

Can your band explain the difference between these two questions?

Examples

Find the following:

1. $\int_0^6 (x^2 - 4x - 3)dx$
2. $\int_1^4 (9 - x^2)$
3. $\int_1^4 (x^3 - 3x^2 + 2x)dx$
4. $\int_1^e (\frac{1}{x})dx$

Finding Area

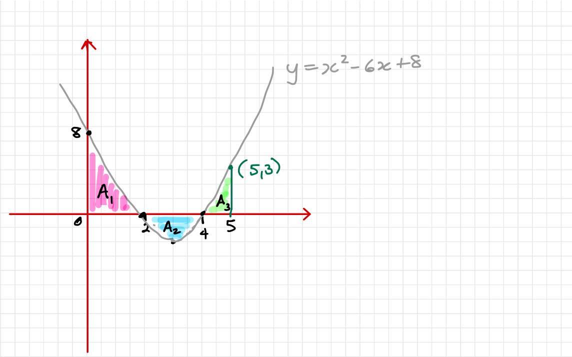
To find the area between the curve $f(x)$ and the x-axis between $x = a$ and $x = b$:

1. Sketch the graph $f(x)$
2. Divide the interval $[a, b]$ into smaller intervals corresponding to where $f(x) > 0$ and $f(x) < 0$.
3. Find the definite integral for each 'piece'
4. Add the values where $f(x) > 0$ and subtract the values where $f(x) < 0$.

Finding Area - an example

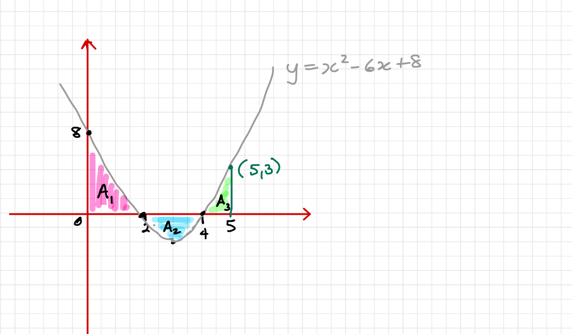
To find the area between the curve $y = x^2 - 6x + 8$ and the x-axis between $x = 0$ and $x = 5$:

1. The graph $y = x^2 - 6x + 8$ has x-intercepts $(2, 0)$ and $(4, 0)$.
2. Divide the interval $[0, 5]$ is broken into smaller intervals: $[0, 2]$, $[2, 4]$ and $[4, 5]$. The corresponding regions are are labelled A_1, A_2 and A_3 on the graph.



Finding Area - an example

3. The definite integrals of each piece are:
   - $A_1 = \int_0^2 (x^2 - 6x + 8)dx = [\frac{x^3}{3} - 3x^2 + 8x]_0^2$
     $= (\frac{8}{3} - 12 + 16) - 0 = 6\frac{2}{3}$
   - $A_2 = \int_2^4 (x^2 - 6x + 8)dx = [\frac{x^3}{3} - 3x^2 + 8x]_2^4$
     $= (\frac{64}{3} - 48 + 32) - (\frac{8}{3} - 12 + 16) = -1\frac{1}{3}$
   - $A_3 = \int_4^5 (x^2 - 6x + 8)dx = [\frac{x^3}{3} - 3x^2 + 8x]_4^5$
     $= (\frac{125}{3} - 75 + 40) - (\frac{64}{3} - 48 + 32) = 1\frac{1}{3}$
4. Total area $= A_1 - A_2 + A_3 = 6\frac{2}{3} + 1\frac{1}{3} + 1\frac{1}{3} = 9\frac{1}{3}$ square units.



Finding Area

Find the area bounded by the x-axis and the function $y = -x^2 + 8x - 15$ between $x = 1$ and $x = 6$.

The solution to this problem is also the amount of silver (in grams) that you require.

Finding Area

Find the area bounded by the x-axis and the function $y = \sin(2x)$ between $x = 0$ and $x = \pi$.

The solution to this problem is also the amount of gold (in grams) that you require.

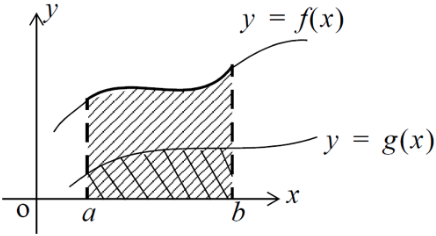
Pearl of Knowlege

Your band has the required silver and gold, and you must now find the pearl. A trader at the market offers to give you a priceless pearl providing you prove your worth by identifying the area of the top of the shell in which it lies.

But before you can do this, we will consider the area between two curves.

Area between two curves

- If $f(x) \geq g(x)$ in the interval $a \leq x \leq b$ then the area between these curves in this interval is $\int_a^b (f(x) - g(x))dx$
- When this is not the case, we need to break the interval into pieces and work out the area of each piece.



Area between two curves

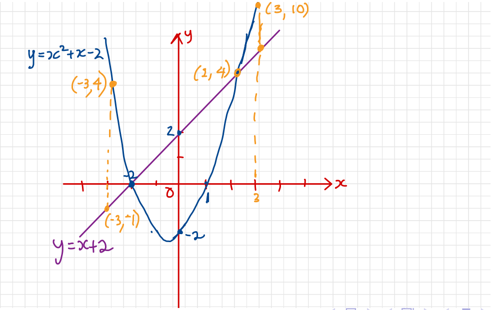
To find the area between $f(x)$
and $g(x)$ for $a \leq x \leq b$ you must do the following:

1. Sketch the graphs $f(x)$ and $g(x)$.
2. Find the points
   that they intersect within the interval $a \leq x \leq b$.
3. Use these points to
   partition the interval $[a, b]$ into smaller intervals.
4. For each of these intervals determine the 'top' curve and the bottom 'curve'. Find the definite integral of 'top'-'bottom' in that interval.
5. Sum these values.

An Example

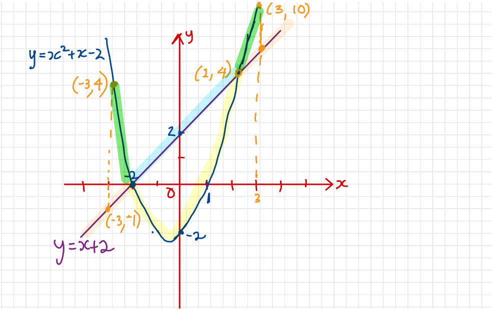
Find the area between $f(x) = x + 2$ and $g(x) = x^2 + x - 2$ for $-3 \leq x \leq 3$.

1. The line $g(x)$ has intercepts $(-2, 0)$ and $(0, 2)$, and the parabola $f(x)$ has intercepts $(-2, 0)$, $(1, 0)$ and $(0, -2)$.
2. These graphs intersect when $f(x) = g(x)$ that is $x + 2 = x^2 + x - 2 \Rightarrow x^2 - 4 = 0 \Rightarrow x = \pm 2$. When $x = 2$, $f(x) = g(x) = 4$ and when $x = -2$, $f(x) = g(x) = 0$, so we have the points $(2, 4)$ and $(-2, 0)$.



An Example

3. We partition the interval $[-3, 3]$ into $[-3, -2]$, $[-2, 2]$ and $[2, 3]$.
4. The parabola is the top curve in $[-3, -2]$ and $[2, 3]$, and the line is the top curve in $[-2, 2]$.
   The line is the bottom curve in $[-3, -2]$ and $[2, 3]$, and the parabola is the bottom curve in $[-2, 2]$.



An Example

4. We calculate the definite integrals for each interval:
   - $A_1 = \int_{-3}^{-2}((x^2 + x - 2) - (x + 2))dx = \int_{-3}^{-2}(x^2 - 4)dx$
     $= \left[\frac{x^3}{3} - 4x\right]_{-3}^{-2} = (\frac{-8}{3} + 8) - (-9 + 12) = 2\frac{1}{3}$
   - $A_2 = \int_{-2}^{2}((x + 2) - (x^2 + x - 2))dx = \int_{-2}^{2}(4 - x^2)dx =$
     $\left[4x - \frac{x^3}{3}\right]_{-2}^{2} = (8 - \frac{8}{3}) - (-8 + \frac{8}{3}) = 10\frac{2}{3}$
   - $A_3 = \int_{2}^{3}((x^2 + x - 2) - (x + 2))dx = \int_{2}^{3}(x^2 - 4)dx$
     $= \left[\frac{x^3}{3} - 4x\right]_{2}^{3} = (9 - 12) - (\frac{8}{3} - 8) = 2\frac{1}{3}$
5. We sum these values to obtain the area
   $A = A_1 + A_2 + A_3 = 2\frac{1}{3} + 10\frac{2}{3} + 2\frac{1}{3} = 15\frac{1}{3}$ square units.

Area between two curves

Can your band find the following areas:

1. The area between $f(x) = 4$ and $g(x) = x^2$ for $-2 \leq x \leq 2$.
   Remember $f(x) = 4$ is the horizontal line that has intercept $(0, 4)$.
2. The area between $f(x) = 3 - x$ and $g(x) = x - 3$ for $0 \leq x \leq 4$.

Pearl of Knowledge

Your band must now find the area of the top of the shell in which the precious pearl lies.
The top of the shell is the shape enclosed between the two curves $f(x) = -x^2 + 4x + 5$ and $g(x) = x^2 - 3x + 5$. Can you find the correct interval and so find the area of the shell's top?

Entering the House of Wisdom

Your band correctly identifies the area, and you gain the priceless pearl of knowledge. Your offerings are accepted and you are granted entry into the House of Wisdom.

- Congratulations - you have finished the quest.
- More importantly you have shown resilience and resourcefulness and hopefully achieved your goals.
- The question is - is the quest really finished?
  or are there more dragons to slay, more mountains to conquer, more ways to apply knowledge gained?

"In the broad light of day mathematicians check their equations and their proofs, leaving no stone unturned in their search for rigour. But, at night, under the full moon, they dream, they float among the stars and wonder at the miracle of the heavens. They are inspired. Without dreams there is no art, no mathematics, no life." Michael Atiya (Notices of the AMS, 2010)