2

Functions and graphs

Introduction

A function is a mathematical rule describing how one quantity depends on another. The expression of relations between physical quantities as functions provides a means of modelling the real world in mathematical terms.The concept of functions is essential to calculus - the mathematics of motion and change.

This topic covers the definition and the properties of functions, linear functions, sketches of straight lines, the solution of quadratic equations, sketches of quadratic functions, and two special types of quadratic functions.

After studying this topic, you should be able to:

- determine whether a given graph represents a function;
- find the domain of a function;
- find the equation of the straight line connecting two given points;
- sketch a straight line given its equation;
- solve a quadratic equation using factorisation or the Quadratic Formula;
- sketch a quadratic function;
- factorise perfect squares and the difference of two squares

2.1 Properties of functions

1. The formula $y = f(x)$ defines a **function**, provided that **there is exactly one *y* value for each possible *x* value**. So, $y = \frac{1}{x}$ is a function which is defined for all x values except $x = 0$.

2. Given a function $y = f(x)$, the **domain** is the set of **all possible *x* values**, and the **range** is the set of **all possible *y* values**. In general, the range is difficult to determine unless a graph of the function is given. However, the domain can often be found by examining the form of the function, without recourse to the graph.

   Many functions are defined for all values of x, e.g. $y = x^2$ (as **any** real number x can be squared). A function which has a denominator and/or a square root may, however, have a restricted domain. In summary:

   (i) $y = f(x) = \frac{1}{u}$ is not defined when $u = 0$;

   (ii) $y = f(x) = \sqrt{u}$ is defined only for $u \geq 0$ (i.e. u is greater than or equal to 0).

   For instance, $y = f(x) = \frac{1}{x-3}$ is not defined when $x - 3 = 0$, i.e. $x = 3$. So, the domain of the function is all x except $x = 3$.

   Also, $y = f(x) = \sqrt{x+4}$ is defined only for $x + 4 \geq 0$, i.e. $x \geq -4$. So, the domain of the function is $x \geq -4$.

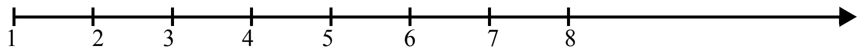
Note: Inequalities, such as $x + 4 \geq 0$ are solved in the same way as equations (in this case by subtracting 4 from both sides).

3. Apart from real numbers, there are several number systems which are frequently used in mathematics.

I Natural Numbers

The counting numbers 1,2,3,4,.... are referred to as the natural numbers. There are infinitely many natural numbers, and the set of all natural numbers is denoted by **N**. In set notation, **N** = {1,2,3,4,5,6,....}

The natural numbers can be represented by equally spaced points on the real number line.

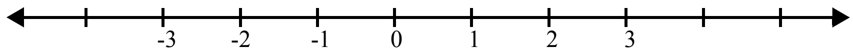


II Integers

The integers consist of the natural numbers, negative whole numbers and zero. The integers are denoted by **Z** or **J**. In set notation,

Z = {.....,-3,-2,-1,0,1,2,3,4,....}

The natural numbers can also be represented by equally spaced points on the real number line.



III Rational Numbers

All fractions can be expressed as one integer divided by another, i.e. $\frac{a}{b}$, where a and b are integers with $b \neq 0$, e.g. $\frac{2}{3}, \frac{-5}{7}, \frac{103}{12}, 4\frac{5}{9}$.

As a fraction is a ratio of integers, the set of all fractions is also known as the set of rational numbers, denoted **Q**.

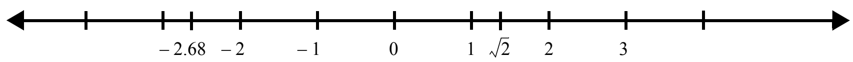
Note: Any integer (and hence any natural number) is also a rational number.

e.g. $3 = \frac{3}{1} = \frac{9}{3} = \frac{-18}{-6}$.

IV Real Numbers

An **irrational number** is a number whose decimal form has infinitely many non-zero digits after the decimal point, but with no systematic repetition. Examples are $\pi = 3.14159265$ ..... and $\sqrt{2} = 1.41421$...... The set of irrationals is denoted **Q'** (the complement of **Q**).

The real numbers, **R**, are made up of the rationals and irrationals. The reals encompass all the number systems above, and can be represented on the usual real number line.



4. The set of **Complex Numbers**, denoted **C**, extends the number system beyond real numbers. For instance, the equation $x^2 = -1$ has no solutions in the real number system (as the square of a real number

cannot be negative), but does have complex number solutions. **Only real numbers (and subsets thereof) are considered in this unit.**

5. There are several ways of specifying intervals on the real number line. These are summarised below.

   If the variable x can lie between 5 and 10 **inclusive**, this can be written as $5 \le x \le 10$, and the corresponding interval on the number line is

$$[5,10]$$

   If x lies between 5 and 10 **exclusive**, this is written as $5 < x < 10$, and the corresponding interval on the number line is

$$(5, 10)$$

   Here, $\le$ means 'is less than or equal to', while $<$ means 'is less than'.

   Similarly, if, say, x lies between 0 and 4, and can be 0 but cannot be 4, this is written as $0 \le x < 4$. The corresponding interval on the number line is [0,4). This is marked on the number line as:

6. If a graph of a function is given, any points of discontinuity can be readily seen. In general, the functions studied in this unit are continuous for all values of x in their domains.
7. If a graph of a relationship between x and y is given, the graph represents a function if and only if any vertical line cuts the graph in at most one point.This is known as the **Vertical Line Test** (VLT).

Examples

1. Evaluate

   (i) $f(-1)$ (ii) $f(0)$ (iii) $f(3)$

   for each of the following functions:

   (a) $f(x) = \dfrac{4}{3x+2}$ (b) $f(x) = \sqrt{4x+4}$

2. Find the domain for each of the following functions:

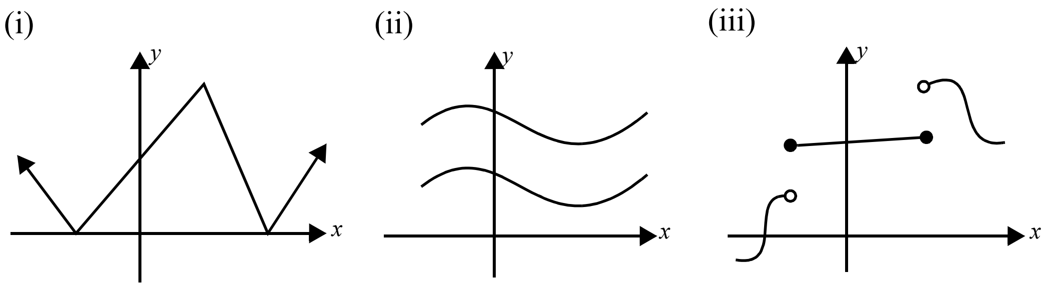
   (i) $f(x) = \dfrac{3}{2x-1}$ (ii) $f(x) = \sqrt{4x+5}$

   (iii) $f(x) = \dfrac{3}{x^2+1}$ (iv) $f(x) = \sqrt{x^2+7}$.

3. Mark each of the following intervals on a number line

   (i) $(-2, 1]$ (ii) $-2 \le x < 4$.

4. Determine whether each of the following graphs represents a function



1. (a) For $f(x) = \dfrac{4}{3x+2}$,

(i) $f(-1) = \dfrac{4}{-3+2} = \dfrac{4}{-1} = -4$

(ii) $f(0) = \dfrac{4}{0+2} = \dfrac{4}{2} = 2$

(iii) $f(3) = \dfrac{4}{9+2} = \dfrac{4}{11}$.

(b) For $f(x) = \sqrt{4x+4}$,

(i) $f(-1) = \sqrt{-4+4} = \sqrt{0} = 0$

(ii) $f(0) = \sqrt{0+4} = \sqrt{4} = 2$

(iii) $f(3) = \sqrt{12+4} = \sqrt{16} = 4$.

2. (i) $f(x) = \dfrac{3}{2x-1}$ is not defined when $2x - 1 = 0$, i.e. $2x = 1$, i.e. $x = \dfrac{1}{2}$. So, the domain of the function is all x except $x = \dfrac{1}{2}$.

(ii) $f(x) = \sqrt{4x+5}$ is defined only for $4x + 5 \geq 0$, i.e. $4x \geq -5$. i.e. $x \geq \dfrac{-5}{4}$. So, the domain of the function is $x \geq \dfrac{-5}{4}$.

(iii) $f(x) = \dfrac{3}{x^2+1}$ is defined for all values of x, as the denominator can never be 0 (as $x^2 + 1 \geq 1$). So, the domain of the function is all x.

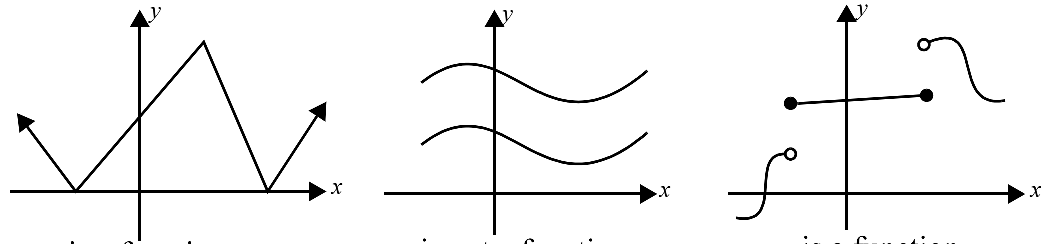
(iv) $f(x) = \sqrt{x^2+7}$ is defined for all values of x, as $x^2 + 7 \geq 7$. So, the domain of the function is all x.

3. (i) $(-2, 1]$;

(ii) $-2 \leq x < 4$.

4. By the Vertical Line Test (V.L.T),

(i) (ii) (iii)



(i) is a function. (ii) is not a function. (iii) is a function.

Problems

1. Evaluate

   (i) $f(-1)$ (ii) $f(4)$ (iii) $f(7)$

   for each of the following functions:

   (a) $f(x) = \dfrac{5}{x^2+2}$ (b) $f(x) = \sqrt{3x+4}$

2. Find the domain for each of the following functions

   (i) $f(x) = \dfrac{9}{5x-2}$ (ii) $f(x) = \sqrt{4x+1}$

   (iii) $f(x) = 3x^2 + 7$ (iv) $f(x) = \sqrt{4x^2+1}$.

Answers

1. (a) (i) $\dfrac{5}{3}$; (ii) $\dfrac{5}{18}$; (iii) $\dfrac{5}{51}$.

1. (b) (i) 1 ; (ii) 4 ; (iii) 5

2. (i) All x except $x = \dfrac{2}{5}$; (ii) $x \geq \dfrac{-1}{4}$

   (iii) All x; (iv) All x.

2.2 Linear functions

1. A **linear function** has a straight line graph. Any straight line (except a vertical line) can be represented by the equation

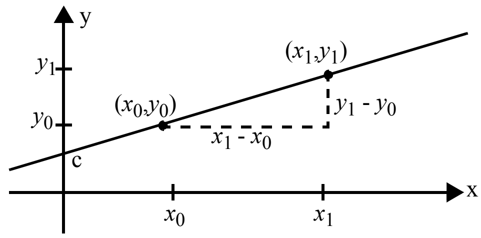
$$y = f(x) = mx + c \qquad (1)$$

where m is the **gradient (or slope)** of the line and c is the **y-intercept**.

Note that a vertical line has the equation $x = k$, where k is a constant. A vertical line does not represent a function as, for $x = k$, there are infinitely many y values on the graph.

2. The gradient m measures the steepness of the slope of the line. To calculate m, any two points on the line are required. Labelling the two points as (x_0, y_0) and (x_1, y_1) in the diagram below gives:

$$\boxed{m = \frac{rise}{run} = \frac{y_1 - y_0}{x_1 - x_0}}$$



A positive gradient indicates that, as x runs from left to right, the line tends upwards. A negative gradient means that the line tends downwards, while $m = 0$ means that the line is horizontal.

Once m has been found, the value of c can be determined by substituting the coordinates of any point on the line into (1).

3. If the linear function $y = f(x) = mx + c$ is given, the corresponding graph can be sketched by finding any two points which satisfy the equation, and connecting them with a straight line. The simplest points to use are the **y-intercept** (where $x = 0$), and the **x-intercept** (where $y = 0$).

Examples

1. Find the equation of the straight line

   (i) passing through the points $(-2, 4)$ and $(4, 1)$.

   (ii) passing through the point $(-2, -3)$ with gradient 2.

2. Sketch $y = 3x - 6$.

1.(i) $m = \dfrac{\text{rise}}{\text{run}} = \dfrac{1-4}{4-(-2)} = \dfrac{1-4}{4+2} = \dfrac{-3}{6} = \dfrac{-1}{2}$

So, the equation has the form $y = \dfrac{-1}{2}x + c$ (1)

Since (4,1) is a point on the line, when $x = 4$, $y = 1$, substituting in (1) gives

$1 = \dfrac{-1}{2} \times 4 + c$

$\therefore 1 = -2 + c \qquad \therefore c = 3$

Hence, the required equation is $y = \frac{-1}{2}x + 3$, i.e. $y = \frac{-x}{2} + 3$.

Note: The answer here can be verified by checking that the point $(-2, 4)$ satisfies the equation, i.e. when $x = -2$, $y = \frac{-(-2)}{2} + 3 = 1 + 3 = 4$, as required.

1. (ii) Given $m = 2$, the equation has the form $y = 2x + c$ (2)

Since $(-2, -3)$ is a point on the line, when $x = -2$, $y = -3$, substituting in (2) gives

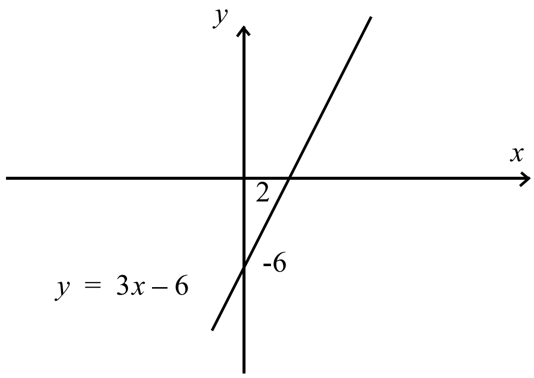
$-3 = 2 \times -2 + c \qquad \therefore -3 = -4 + c \qquad \therefore c = 1$

Hence, the required equation is $y = 2x + 1$

2. For $y = 3x - 6$, when $x = 0, y = -6$ (y-intercept).

Also, when $y = 0$, $3x - 6 = 0$

$\therefore 3x = 6 \qquad \therefore x = 2$ (x-intercept). Hence the sketch



Problems

1. Find the equation of the straight line
   (i) passing through the points $(-2, 4)$ and $(-1, 1)$.
   (ii) passing through the point $(-2, 4)$ with gradient -5.
2. Sketch $y = 4 - x$.

Answers

1.(i) $m = \frac{\text{rise}}{\text{run}} = \frac{1-4}{-1-(-2)} = \frac{1-4}{-1+2} = \frac{-3}{1} = -3$

So, the equation has the form $y = -3x + c$ (1)

Since is $(-1, 1)$ a point on the line, when $x = -1$, $y = 1$, substituting in (1) gives

$1 = (-3) \times (-1) + c$

$\therefore 1 = 3 + c$ $\therefore c = -2$

Hence, the required equation is $y = -3x - 2$.

1. (ii) Given $m = -5$, the equation has the form $y = -5x + c$ (2)

Since $(-2, 4)$ is a point on the line, when $x = -2$, $y = 4$, substituting in (2) gives

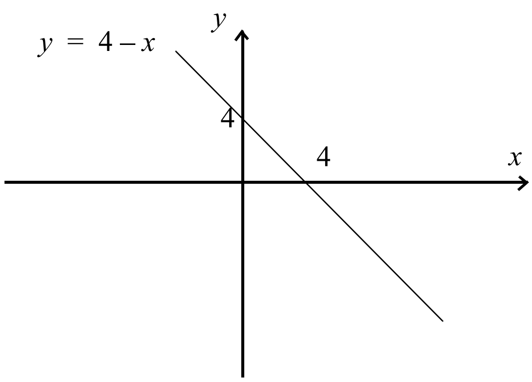
$4 = (-5) \times (-2) + c$ $\therefore 4 = 10 + c$

$\therefore c = -6$

Hence, the required equation is $y = -5x - 6$.

2. For $y = 4 - x$, when $x = 0$, $y = 4$, and when $y = 0$, $x = 4$.

Hence the sketch



2.3 Quadratic functions

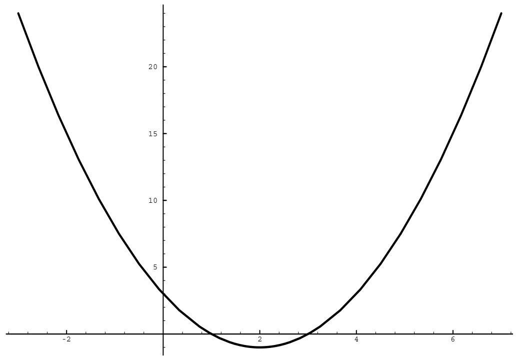
1. A **quadratic function** can be written in the form

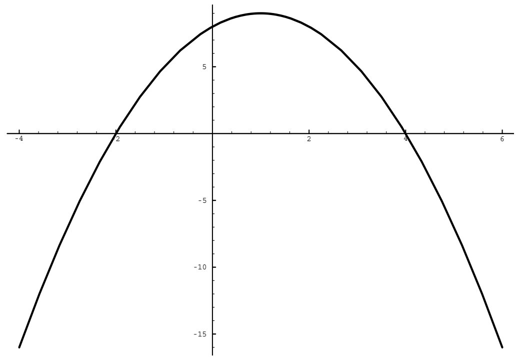
$y = f(x) = ax^2 + bx + c$, where a, b, and c are constants with $a \neq 0$

Note that, if $a = 0$, the equation becomes $y = bx + c$, and is then a linear function.

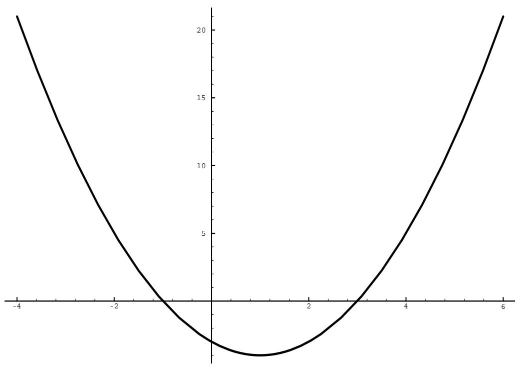
2. The graphs of these functions are **parabolas**, and have a 'cup' shape or a 'frown' shape , as can be seen in the diagrams below.

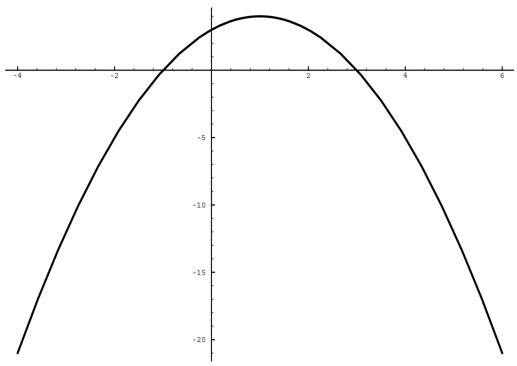
$y = x^2 - 4x + 3$ $y = -x^2 + 2x + 8$



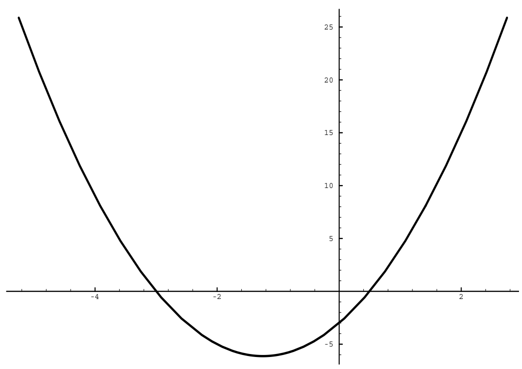


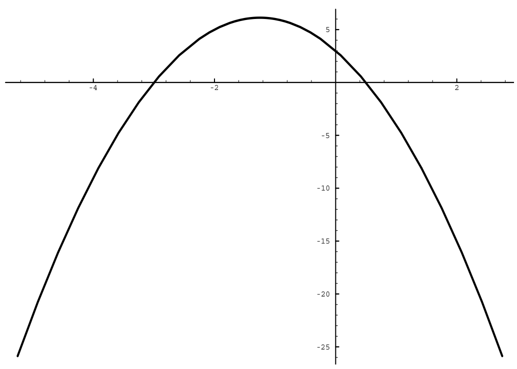
$y = x^2 - 2x - 3$ $\qquad$ $y = -x^2 + 2x + 3$





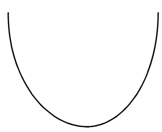
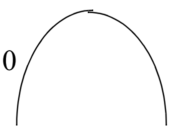
$y = 2x^2 + 5x - 3$ $\qquad$ $y = -2x^2 - 5x + 3$





3. The sketches above indicate that the quadratic function
   $y = ax^2 + bx + c$ is:

   'cup' shaped if $a > 0$; $\qquad$ 'frown' shaped if $a < 0$

   
   $a > 0$ $\qquad$ $a < 0$
   

   Clearly, the graph has a lowest point (minimum) if $a > 0$, and a highest point (maximum) if $a < 0$. The graph is then symmetric about a vertical line drawn through this lowest (or highest) point.

4. The y-intercept for a quadratic function is found by setting $x = 0$ in the equation, i.e. when $x = 0$, $y = c$.

   The x-intercepts for a quadratic function are found by setting $y = 0$ in the equation, i.e. when $y = 0$, $ax^2 + bx + c = 0$.

   Solving this quadratic equation allows any quadratic function to be sketched without relying on plotting points. The 2 methods of solution are covered in Chapter 2.4 below.

Examples

1. For each of the following quadratic functions:

   (i) identify a, b, and c

   (ii) find the y-intercept

   (iii) determine whether the graph is 'cup' shaped or 'frown' shaped.

(a)	$y = 7x^2 - 3x + 8$	(b)	$y = 4x - 3x^2$
(c)	$y = 3 + 5x^2$	(d)	$y = -6x^2 + 3 - x$

1. (a)	(i)		$a = 7, b = -3, c = 8$.
	(ii)		When $x = 0$, $y = 8$
	(iii)		As $a = 7 > 0$, curve is 'cup' shaped.
(b)	(i)		$a = -3, b = 4, c = 0$.
	(ii)		When $x = 0$, $y = 0$
	(iii)		As $a = -3 < 0$, curve is 'frown' shaped.
(c)	(i)		$a = 5, b = 0, c = 3$.
	(ii)		When $x = 0$, $y = 3$
	(iii)		As $a = 5 > 0$, curve is 'cup' shaped.
(d)	(i)		$a = -6, b = 3, c = -1$.
	(ii)		When $x = 0$, $y = -1$
	(iii)		As $a = -6 < 0$, curve is 'frown' shaped.

Problems

1. For each of the following quadratic functions

   (i) identify a, b, and c

   (ii) find the y-intercept

(iii) determine whether the graph is 'cup' shaped or 'frown' shaped.

(a) $y = 9x^2 + 8$ (b) $y = 4 - 3x^2$

(c) $y = 3x + 5x^2$ (d) $y = -6x^2 - 1 - x$

Answers

1. (a) (i) $a = 9, b = 0, c = 8$.
   - (ii) When $x = 0$, $y = 8$
   - (iii) As $a = 9 > 0$, curve is 'cup' shaped.

   (b) (i) $a = -3, b = 0, c = 4$.
   - (ii) When $x = 0$, $y = 4$
   - (iii) As $a = -3 < 0$, curve is 'frown' shaped.

   (c) (i) $a = 5, b = 3, c = 0$.
   - (ii) When $x = 0$, $y = 0$
   - (iii) As $a = 5 > 0$, curve is 'cup' shaped.

   (d) (i) $a = -6, b = -1, c = 3$.
   - (ii) When $x = 0$, $y = 3$
   - (iii) As $a = -6 < 0$, curve is 'frown' shaped.

2.4 Quadratic equations

1. There are 2 methods of solving the **quadratic equation**

$$ax^2 + bx + c = 0$$

The first method, **factorising**, should only be used when $a = 1$, i.e. when the quadratic equation has the form $x^2 + bx + c = 0$.

The factorisation method relies on writing the quadratic expression

$x^2 + bx + c$ (1) as the product of 2 bracketed terms, i.e.

$x^2 + bx + c = (x + m)(x + n)$,

where the constants m and n are to be found.

Since $(x + m)(x + n) = x(x + n) + m(x + n)$

$$= x^2 + nx + mx + mn$$

$$= x^2 + (n + m)x + mn \qquad (2)$$

Comparing (1) and (2), it can be seen that the numbers m and n must satisfy both of the equations:

$mn = c$ and $n + m = b$.

In summary, $x^2 + bx + c = (x + m)(x + n)$, where the numbers ***m* and *n* have a product equal to *c*, and a sum equal to *b***.

2. Once the quadratic expression $x^2 + bx + c$ has been factorised, the quadratic equation $x^2 + bx + c = 0$ can be solved immediately. The factorised form gives

$$(x + m)(x + n) = 0$$

So, either $x + m = 0$, or $x + n = 0$

Hence, the solutions are $x = -m$, and $x = -n$

3. Before factorising and solving, the quadratic equation must have the form $x^2 + bx + c = 0$,

i.e. **the quadratic expression must be on the L.S, and the R.S. must be zero**.

For instance, the solutions to $x^2 + 8x + 8 = 1$ are found by first subtracting 1 to give $x^2 + 8x + 7 = 0$, then factorising and solving.

4. The **Quadratic Formula** is the second method of solving the quadratic equation $ax^2 + bx + c = 0$.

The Quadratic Formula provides the solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

and **should be remembered**. It is used (in most cases) when $a \neq 1$.

As is the case with the factorising method, **the quadratic expression must be on the L.S, and the R.S. must be zero**.

5. The expression $b^2 - 4ac$ completely determines the number of solutions to the equation $ax^2 + bx + c = 0$.

Such an equation has either 2 solutions, 1 solution, or no solution.

If $b^2 - 4ac > 0$, there are 2 real solutions.

If $b^2 - 4ac = 0$, there is 1 real solution.

If $b^2 - 4ac < 0$, there are no real solutions.

Examples

1. Factorise, and solve the following quadratic equations

(i) $x^2 + 8x + 12 = 0$ (ii) $x^2 - 8x + 7 = 0$

(iii) $x^2 - 3x - 10 = 0$ (iv) $x^2 + 9x - 10 = 0$

(v) $x^2 + 8x + 8 = 1$ (vi) $x^2 = 4x - 4$.

2. Use the Quadratic Formula to solve the following quadratic equations

(i) $4x^2 - 8x + 3 = 0$ (ii) $6x^2 + x - 2 = 0$

(iii) $3x^2 - 12x + 9 = 0$ (iv) $2x^2 - x + 6 = 1$.

1. (i) $x^2 + 8x + 12 = 0$

[Here, $mn = 12$ (so m and n have the same sign)

and $n + m = 8$ (so m and n are both positive)

Possibilities for m and n are: 1 and 12; 2 and 6; 3 and 4.

Of these, 2 and 6 satisfy both equations.]

$\therefore x^2 + 8x + 12 = (x+2)(x+6) = 0$

So, there are 2 solutions: $x = -2$, and $x = -6$.

(ii) $x^2 - 8x + 7 = 0$

[Here, $mn = 7$ (so m and n have the same sign)

and $n + m = -8$ (so m and n are both negative)

Possibilities for m and n are: -1 and -7 only.]

$\therefore x^2 - 8x + 7 = (x-1)(x-7) = 0$

So, there are 2 solutions: $x = 1$, and $x = 7$.

(iii) $x^2 - 3x - 10 = 0$

[Here, $mn = -10$ (so m and n are of opposite sign)

and $n + m = -3$ (so m and n have a negative sum)

Possibilities for m and n are: 1 and -10; 2 and -5.

Of these, 2 and -5 satisfy both equations.]

$\therefore x^2 - 3x - 10 = (x+2)(x-5) = 0$

So, there are 2 solutions: $x = -2$, and $x = 5$.

(iv) $x^2 + 9x - 10 = 0$

[Here, $mn = -10$ (so m and n are of opposite sign)

and $n + m = 9$ (so m and n have a positive sum)

Possibilities for m and n are: -1 and 10; -2 and 5.

Of these, -1 and 10 satisfy both equations.]

$\therefore x^2 + 9x - 10 = (x-1)(x+10) = 0$

So, there are 2 solutions: $x = 1$, and $x = -10$.

(v) $x^2 + 8x + 8 = 1$

$\therefore x^2 + 8x + 7 = 0$ (correct form)

[Here, $mn = 7$ (so m and n have the same sign)

and $n + m = 8$ (so m and n are both positive)

Possibilities for m and n are: 1 and 7 only]

$\therefore x^2 + 8x + 7 = (x+1)(x+7) = 0$

So, there are 2 solutions: $x = -1$, and $x = -7$.

(vi) $x^2 = 4x - 4$

$\therefore x^2 - 4x + 4 = 0$ (correct form)

[Here, $mn = 4$ (so m and n have the same sign)

and $n + m = -4$ (so m and n are both negative)

Possibilities for m and n are: -1 and -4; -2 and -2.

Of these, -2 and -2 satisfy both equations.]

$\therefore x^2 - 4x + 4 = (x-2)(x-2) = 0$

So, there is one solution: $x = 2$.

2. (i) $4x^2 - 8x + 3 = 0$ $(a = 4, b = -8, c = 3)$

$$\therefore x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4 \times 4 \times 3}}{2 \times 4}$$

$$\therefore x = \frac{8 \pm \sqrt{64 - 48}}{8} = \frac{8 \pm \sqrt{16}}{8} = \frac{8 \pm 4}{8}$$

$$\therefore x = \frac{12}{8} = \frac{3}{2}, \text{ and } x = \frac{4}{8} = \frac{1}{2}.$$

So, there are 2 solutions: $x = \frac{3}{2}$, and $x = \frac{1}{2}$.

(ii) $6x^2 + x - 2 = 0$; $(a = 6, b = 1, c = -2)$

$$\therefore x = \frac{-1 \pm \sqrt{1^2 - 4 \times 6 \times (-2)}}{2 \times 6}$$

$$\therefore x = \frac{-1 \pm \sqrt{1+48}}{12} = \frac{-1 \pm \sqrt{49}}{12} = \frac{-1 \pm 7}{12}$$

$$\therefore x = \frac{6}{12} = \frac{1}{2}, \text{ and } x = \frac{-8}{12} = \frac{-2}{3}.$$

So, there are 2 solutions: $x = \frac{1}{2}$, and $x = \frac{-2}{3}$.

(iii) $3x^2 - 12x + 9 = 0 \quad (a = 3, b = -12, c = 9)$

$$\therefore x = \frac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 9}}{2 \times 3}$$

$$\therefore x = \frac{12 \pm \sqrt{144 - 108}}{6} = \frac{12 \pm \sqrt{36}}{6} = \frac{12 \pm 6}{6}$$

$$\therefore x = \frac{18}{6} = 3, \text{ and } x = \frac{6}{6} = 1.$$

So, there are 2 solutions: $x = 3$ and $x = 1$.

NOTE: The original equation can be divided by 3 to give $x^2 - 4x + 3 = 0$.

As the coefficient of x is now 1, the quadratic can be factorised, i.e.

$$x^2 - 4x + 3 = (x-3)(x-1) = 0$$

Hence, again the solutions are $x = 3$ and $x = 1$.

(iv) $2x^2 - x + 6 = 1$

$\therefore 2x^2 - x + 5 = 0$ (correct form)

$(a = 2, b = -1, c = 5)$

$$\therefore x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$\therefore x = \frac{1 \pm \sqrt{1-40}}{4} = \frac{1 \pm \sqrt{-39}}{4}$$

Since $\sqrt{-39}$ is not a real number, there is no solution.

Problems

1. Factorise, and solve the following quadratic equations

(i) $x^2 + 13x + 12 = 0$ (ii) $x^2 - 15x + 26 = 0$

(iii) $x^2 - 3x - 4 = 0$ (iv) $x^2 + 13x - 14 = 0$

(v) $x^2 + 6x + 10 = 1$ (vi) $x^2 = 6x - 5$.

2. Use the Quadratic Formula to solve the following quadratic equations

(i) $4x^2 + 4x + 1 = 0$ (ii) $5x^2 + 4x - 1 = 0$

(iii) $3x^2 - 2x - 1 = 0$ (iv) $2x^2 + 3 = 3x$.

Answers

1. (i) $x = -1$ and $x = -12$. (ii) $x = 2$ and $x = 13$.

(iii) $x = -1$ and $x = 4$. (iv) $x = -14$ and $x = 1$.

(v) $x = -3$. (vi) $x = 1$, and $x = 5$.

2. (i) $x = \frac{-1}{2}$. (ii) $x = \frac{1}{5}$, and $x = -1$.

(iii) $x = \frac{-1}{3}$ and $x = 1$. (iv) No real solutions.

2.5 Sketching quadratic functions

1. Whether obtained by factorising or the Quadratic Formula, the solutions of the quadratic equation $ax^2 + bx + c = 0$ completely determine the x-intercepts of the quadratic function $y = ax^2 + bx + c$.

   In conjunction with the y-intercepts, the ‘cup’ or ‘frown’ shape, and the axis of symmetry, the x-intercepts can be used to sketch any quadratic function quickly. **All quadratics can be sketched without either plotting points or using a graphics package or calculator.**

2. The x value of the axis of symmetry of the parabola

$$y = ax^2 + bx + c$$

   is the **average value of the x-intercepts**. In general, since the x-intercepts are given by the Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

   these intercepts can be labelled as

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The average of x_1 and x_2 is thus

$$\frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}\right)$$

$$= \frac{1}{2}\left(\frac{-b-b}{2a}\right) = \frac{1}{2}\left(\frac{-2b}{2a}\right) = \frac{-b}{2a}.$$

Hence, the axis of symmetry occurs at $x = \frac{-b}{2a}$.

3. To sketch the quadratic function $y = ax^2 + bx + c$:

I **Find the y-intercept** (occurs when $x = 0$)

II **Find the x-intercept** (occurs when $y = 0$)

III **Note the sign of *a*** ($a > 0$: 'cup', and $a < 0$: 'frown')

IV **Find the co-ordinates of the highest or lowest point**

(occurs when $x = \frac{-b}{2a}$)

Examples

1. For each of the following quadratic functions

(i) find the y-intercept

(ii) find the x-intercepts

(iii) determine whether the curve has a highest or lowest point

(iv) find the co-ordinates of the highest or lowest point

(v) sketch the curve.

(a) $y = x^2 - 2x - 15$ (b) $y = -x^2 + 8x - 16$

(a) $y = x^2 - 2x - 15$

(i) When $x = 0$, $y = -15$

(ii) When $y = 0$, $x^2 - 2x - 15 = 0$

Factorising gives $x^2 - 2x - 15 = (x+3)(x-5) = 0$

$\therefore x = -3$, and $x = 5$. (two x-intercepts)

(iii) As $a = 1$ ($a > 0$), the curve is 'cup' shaped, and has a lowest point.

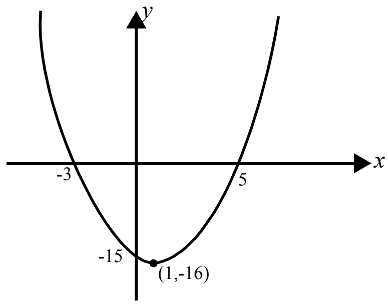
(iv) As $a = 1$, $b = -2$, and $c = -15$, for the axis of symmetry, $x = \frac{-b}{2a} = \frac{-(-2)}{2 \times 1} = \frac{2}{2} = 1$.

(alternatively, the average of the x-intercepts is $\frac{-3+5}{2} = \frac{2}{2} = 1$)

When $x = 1$, $y = 1 - 2 - 15 = -16$.

So, the lowest point is $(1, -16)$.

(v)



NOTE: The only points required to be shown on the graph are the x and y-intercepts, and the lowest (or highest) point.

(b) $y = -x^2 + 8x - 16$

(i) When $x = 0$, $y = -16$

(ii) When $y = 0$, $-x^2 + 8x - 16 = 0$

Multiplying by -1 gives $x^2 - 8x + 16 = 0$

Factorising gives $x^2 - 8x + 16 = (x-4)(x-4) = 0$

$\therefore x = 4$. (one x-intercept)

(iii) As $a = -1$ ($a < 0$), the curve is 'frown' shaped, and has a highest point.

(iv) As $a = -1$, $b = 8$, and $c = -16$,

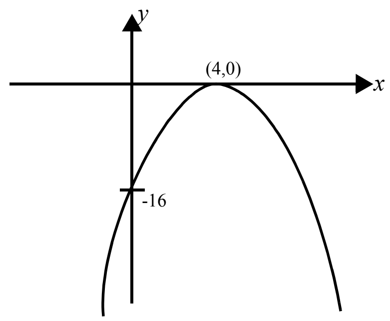
for the axis of symmetry, $x = \frac{-b}{2a} = \frac{-8}{2 \times (-1)} = \frac{-8}{-2} = 4$.

(alternatively, the average of the x-intercepts is $\frac{4+4}{2} = \frac{8}{2} = 4$)

When $x = 4$, $y = 0$ (x-intercept).

So, the highest point is $(4, 0)$.

(v)



Problems

1. For each of the following quadratic functions
   - (i) find the y-intercept
   - (ii) find the x-intercepts
   - (iii) determine whether the curve has a highest or lowest point
   - (iv) find the co-ordinates of the highest or lowest point
   - (v) sketch the curve.

   (a) $y = x^2 - 3x - 4$ (b) $y = -x^2 + 8x - 12$

Answers

1. (a) (i) $y = -4$.

   (ii) $x = -1$, and $x = 4$. (two x-intercepts)

   (iii) As $a = 1 > 0$, curve is 'cup' shaped, and has a lowest point

   (iv) The lowest point is $(3/2, -25/4)$.

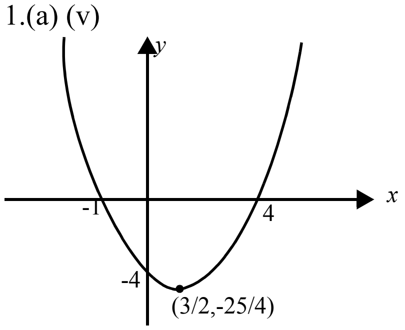
1. (b) (i) $y = -12$

   (ii) $x = 2$, and $x = 6$ (two x-intercepts)

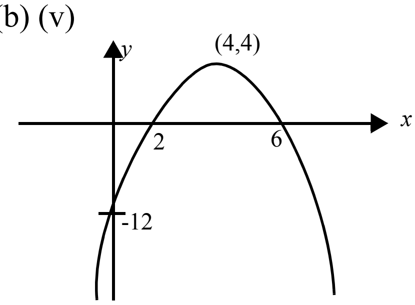
   (iii) As $a = -1 < 0$, curve is 'frown' shaped, and has a highest point

   (iv) The highest point is $(4, 4)$.

1.(a) (v)



1.(b) (v)



2.6 Special quadratic forms

1. For quadratics, factorisation can be simplified in 2 special cases; **perfect squares** and **the difference of two squares**.

   I Perfect squares

   Since $(a+b)^2 = (a+b)(a+b) = a(a+b)+b(a+b)$

   $= a^2+ab+ab+b^2 \qquad = a^2+2ab+b^2$,

   the perfect square expansion is

   $$(a+b)^2 = a^2+2ab+b^2 .$$

   Similarly, $\qquad (a-b)^2 = a^2-2ab+b^2 .$

   Note that there are 3 terms in the perfect square expansion. The first and third terms are simply the squares of each term in the original brackets, while the second term is twice the product of the terms in the original brackets.

   For instance, $(x+3)^2 = x^2+2\times x\times 3+3^2 = x^2+6x+9$,

   and $\qquad (x-5)^2 = x^2+2\times x\times(-5)+(-5)^2 = x^2-10x+25$.

   II Difference of two squares

   Since $(a+b)(a-b) = a(a-b)+b(a-b)$

   $= a^2-ab+ab-b^2 \qquad = a^2-b^2$,

   the difference of two squares expansion is

   $$(a+b)(a-b) = a^2-b^2 .$$

   Note that there are 2 terms in the difference of two squares expansion. The R.S. is the difference of the square of the first term in each of the original brackets and the square of the second term in each of the original brackets.

   For instance, $(x+4)(x-4) = x^2-4^2 = x^2-16$,

   and $\qquad (6+x)(6-x) = 6^2-x^2 = 36-x^2$.

   A difference of two squares is the simplest quadratic form to factorise.

   For instance, $x^2-64 = x^2-8^2 = (x+8)(x-8)$.

   Note, however, that the sum of two squares a^2+b^2 does not factorise.

2. Many quadratics can be written as a difference of two squares using the technique known as **completing the square**. The technique relies on the following result:

   $$\left(x+\frac{b}{2a}\right)^2 = x^2+2\times x\times\frac{b}{2a}+\left(\frac{b}{2a}\right)^2 \qquad \text{(perfect square)}$$

$$\therefore \left(x+\frac{b}{2a}\right)^2 = x^2+\frac{bx}{a}+\left(\frac{b}{2a}\right)^2$$

$$\therefore \left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 = x^2+\frac{bx}{a} \qquad \left[\ -\left(\frac{b}{2a}\right)^2\right]$$

Switching the L.S. and the R.S. gives

$$x^2+\frac{bx}{a} = \left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 \qquad \text{(a difference of two squares)}$$

The above result can be used to complete the square for the quadratic expression ax^2+bx+c as follows:

COMPLETING THE SQUARE

STEP I Take out the co-efficient a as a factor

i.e. $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right]$.

STEP II Replace $x^2+\frac{bx}{a}$ by $\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2$

i.e. $ax^2+bx+c = a\left[x^2+\frac{bx}{a}+\frac{c}{a}\right] = a\left[\left(x+\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$

STEP III Simplify the terms $-\left(\frac{b}{2a}\right)^2+\frac{c}{a}$

i.e. $-\left(\frac{b}{2a}\right)^2+\frac{c}{a} = \frac{-b^2}{4a^2}+\frac{c}{a} \qquad = \frac{-b^2+4ac}{4a^2}$

i.e. $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a^2}\right]$

STEP IV Multiply through by a

i.e. $ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2+\frac{-b^2+4ac}{4a}$

Note: The important part of completing the square occurs in Step II, and if $a = 1$, the process is much simpler, as shown in the following example.

For x^2+6x-1, Step I is unnecessary, and in Step II the first two terms, i.e. x^2+6x are written as $(x+3)^2-3^2$.

Note that the term with *x* in the brackets is half the co-efficient of *x* in the original quadratic (here, half of +6 is +3). **This term is then squared, and subtracted** (here, -3^2).

Hence, $x^2+6x-1 = (x+3)^2-3^2-1$

$= (x+3)^2-9-1 \qquad = (x+3)^2-10,$

and the process is complete.

3. The **Quadratic Formula** introduced in Chapter 2.4 is derived using the technique for completing the square. The equation obtained in Step IV above is

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a}.$$

As L.S. = R.S. in the above equation, the solutions to the quadratic equation

$$ax^2 + bx + c = 0$$

are the same as the solutions to the equation

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a} = 0.$$

Solving for x gives

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} \qquad \left[\ + \frac{b^2 - 4ac}{4a}\right]$$

$$\therefore \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad [\ \div a]$$

$$\therefore x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \qquad [\pm\sqrt{\ }\,]$$

$$\therefore x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{(simplifying)}$$

$$\therefore x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \left[\ - \frac{b}{2a}\right]$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(simplifying),}$$

which is the required Quadratic Formula.

Examples

1. Expand, and simplify

(i) $(x-9)^2$ (ii) $(x+2)^2$

(iii) $(x+10)(x-10)$ (iv) $(2x+1)(2x-1)$

2. Factorise

(i) $x^2 - 49$ (ii) $4x^2 - 81$ (iii) $121 - x^2$

3. Complete the square for the following quadratic expressions

(i) $x^2 + 10x$ (ii) $x^2 - 14x$ (iii) $x^2 + 8x - 3$

(iv) $x^2 - 5x + 2$ (v) $2x^2 + 6x - 1$.

1. (i) $(x-9)^2 = x^2 - 2 \times 9 \times x + (-9)^2$

$= x^2 - 18x + 81$.

(ii) $(x+2)^2 = x^2 + 2 \times 2 \times x + 2^2$

$= x^2 + 4x + 4$

(iii) $(x+10)(x-10) = x^2 - 10^2 = x^2 - 100$

(iv) $(2x+1)(2x-1) = (2x)^2 - 1^2 = 4x^2 - 1$.

2. (i) $x^2 - 49 = x^2 - 7^2 = (x+7)(x-7)$

(ii) $4x^2 - 81 = (2x)^2 - 9^2 = (2x+9)(2x-9)$

(iii) $121 - x^2 = 11^2 - x^2 = (11+x)(11-x)$.

3. (i) For $x^2 + 10x$, since $\frac{10}{2} = 5$,

$x^2 + 10x = (x+5)^2 - 5^2 \qquad = (x+5)^2 - 25$

(ii) For $x^2 - 14x$, since $\frac{-14}{2} = -7$,

$x^2 - 14x = (x-7)^2 - 7^2 \qquad = (x-7)^2 - 49$

(iii) For $x^2 + 8x - 3$, since $\frac{8}{2} = 4$,

$x^2 + 8x - 3 = (x+4)^2 - 4^2 - 3 \qquad = (x+4)^2 - 16 - 3$

$x^2 + 8x - 3 = (x+4)^2 - 19$

(iv) For $x^2 - 5x + 2$, since $\frac{-5}{2}$ doesn't simplify,

$x^2 - 5x + 2 = \left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 2 \qquad = \left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + 2$

$= \left(x - \frac{5}{2}\right)^2 + \frac{8-25}{4} \qquad = \left(x - \frac{5}{2}\right)^2 - \frac{17}{4}$

(v) For $2x^2 + 6x - 1$, first taking out a factor of 2 gives

$2x^2 + 6x - 1 = 2\left[x^2 + 3x - \frac{1}{2}\right]$

Now, since $\frac{3}{2}$ doesn't simplify,

$$2\left[x^2+3x-\frac{1}{2}\right] = 2\left[\left(x+\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-\frac{1}{2}\right]$$

$$= 2\left[\left(x+\frac{3}{2}\right)^2-\frac{9}{4}-\frac{1}{2}\right] \qquad = 2\left[\left(x+\frac{3}{2}\right)^2-\frac{9}{4}-\frac{2}{4}\right]$$

$$= 2\left[\left(x+\frac{3}{2}\right)^2-\frac{11}{4}\right] \qquad = 2\left(x+\frac{3}{2}\right)^2-\frac{11}{2}$$

Problems

1. Expand, and simplify

   (i) $(x-12)^2$ (ii) $(2x+3)^2$

   (iii) $(x+12)(x-12)$ (iv) $(3x+1)(3x-1)$

2. Factorise

   (i) x^2-1 (ii) $9x^2-16$ (iii) $169-x^2$

3. Complete the square for the following quadratic expressions

   (i) x^2+12x (ii) x^2-16x (iii) x^2+4x-7

   (iv) $x^2-7x+12$ (v) $2x^2+8x-1$.

Answers

1. (i) $x^2-24x+144$ (ii) $4x^2+12x+9$

   (iii) x^2-144 (iv) $9x^2-1$.

2. (i) $(x+1)(x-1)$ (ii) $(3x+4)(3x-4)$

   (iii) $(13+x)(13-x)$.

3. (i) $(x+6)^2-36$ (ii) $(x-8)^2-64$

   (iii) $(x+2)^2-11$ (iv) $\left(x-\frac{7}{2}\right)^2-\frac{1}{4}$

   (v) $2(x+2)^2-9$.