Task 1: Differentiation

1) Find the derivative of the following functions

a) $y = \frac{4x^5 - 7x^2 + 3x - 8}{x - 3}$

We want to use the quotient rule here, $f'(x) = \frac{vu' - uv'}{v^2}$
$u = 4x^5 - 7x^2 + 3x - 8$
-> $5 \times 4x^{5-1} + 2 \times -7x^{2-1} + 3 \times 1x^{1-1} \cancel{-8}$
$u' = 20x^4 - 14x + 3$

$v = x - 3$
-> $x^{1-1} \cancel{-3}$
$v' = 1$

Expanding $vu' - uv'$
$\frac{(x-3)(20x^4-14x+3)-(4x^5-7x^2+3x-8)(1)}{(x-3)^2}$
-> $(x - 3)(20x^4 - 14x + 3)$
-> $20x^5 - 14x^2 + 3x - 60x^4 + 42x - 9$
-> $-(4x^5 - 7x^2 + 3x - 8)$
-> $-4x^5 + 7x^2 - 3x + 8$

$20x^5 - 14x^2 + 3x - 60x^4 + 42x - 9 - 4x^5 + 7x^2 - 3x + 8$
-> $20x^5 - 14x^2 \cancel{+3x} - 60x^4 + 42x - 9 \cancel{-4x^5} \cancel{+7x^2} \cancel{-3x} \cancel{+8}$
-> $16x^5 - 60x^4 - 7x^2 + 42x - 1$

$(x - 3)^2$ is already simplified, so:

Solution: Derivative of y
$y' = \frac{16x^5-60x^4-7x^2+42x-1}{(x-3)^2}$

b) $y = \ln(3x^3 - x^2 + 2)$

$u = 3x^3 - x^2 + 2$
-> $3 \times 3x^{3-1} - 1 \times 2x^{2-1} \cancel{+2}$
$u' = 9x^2 - 2x$

$v' = \frac{1}{u}$

$u' \times v'$
-> $y' = \frac{9x^2-2x}{3x^3-x^2+2}$

c) $y = e^{3x} \cos(2x - 5)$

This problem employs the product rule, $uv' + vu'$.

We use the $f'(x) = ke^{kx}$ to find the e derivative
$u = e^{3x}$
$u' = 3e^{3x}$

We use $f'(x) = -k\sin(kx)$
$v = \cos(2x - 5)$
$v' = -2\sin(2x - 5)$

Finally, add all the variables together..
$uv' + vu'$
$e^{3x} \times -2\sin(2x-5) + \cos(2x-5) \times 3e^{3x}$
-> $-2e^{3x}\sin(2x-5) + 3e^{3x}\cos(2x-5)$

to get the derivative of y
$y' = e^{3x}(-2\sin(2x-5) + 3\cos(2x-5))$

2) The distance of a particle from a point, O, at time t seconds is given by $s = 2t^3 + 27t^2 + 105t - 50$ metres for $t \geq 0$

s is the displacement in metres, t is time in seconds

a) Find the velocity and acceleration at time t

Velocity (s'), the rate of change of the displacement
$s = 2t^3 + 27t^2 + 105t - 50$
-> $2 \times 3t^{3-1} + 27 \times 2t^{2-1} + 105t^{1-1} \cancel{-50}$
-> $6t^2 + 54t^1 + 105t^0$
$v = 6t^2 + 54t + 105$

Acceleration (s''), the rate of change of the velocity
$6t^2 + 54t + 105$
-> $2 \times 6t^{2-1} + 54 \times 1t^{1-1} \cancel{105}$
$a = 12t + 54$

b) Find the time(s) when the particle has a velocity of zero. Give the displacement & acceleration at these times.

Plan: Set velocity to zero, find t, then plug t into the acceleration and displacement equations

Finding the times (t_1 and t_2)
$v = 6t^2 + 54t + 105$
-> $0 = 6t^2 + 54t + 105$
-> $t = \frac{-(54) \pm \sqrt{(54)^2 - 4(6)(105)}}{2(6)}$
-> $t = \frac{-54 \pm \sqrt{2916 - 2520}}{12}$
-> $t = \frac{-54 \pm \sqrt{396}}{12}$
$t_1^+ = -2.842$s

$t_2^- = -6.158$s
The times are negative so next equations don't exactly make sense as time cannot traditionally be negative, but they are correct.

Acceleration at velocity of 0 m/s^2
$a = 12t_1 + 54$
-> $a_1 = 12(-2.842) + 54$
$a_1 = 19.8997 \text{ m/s}^2$

$a = 12t_2 + 54$
-> $a_2 = 12(-6.158) + 54$
$a_2 = -19.8997 \text{ m/s}^2$

Interestingly but perhaps not coincidentally, the acceleration function reverses the quadratic equation division and $-b$ term.

Finding the displacement at a velocity of 0m/s^2
$s = 2t_1^3 + 27t_1^2 + 105t_1 - 50$
-> $s = 2(-2.842)^3 + 27(-2.842)^2 + 105(-2.842) - 50$
-176.2414 metres

$s = 2t_2^3 + 27t_2^2 + 105t_2 - 50$
-> $s = 2(-6.158)^3 + 27(-6.158)^2 + 105(-6.158) - 50$
$s = -139.7586$ metres

Zero seconds elapsed
$s = 2t^3 + 27t^2 + 105t - 50$
-> $s = 2(0)^3 + 27(0)^2 + 105(0) - 50$
$s = -50$ metres

This implies that -2.8 and -6.2 seconds in the past, the object was displaced negative distances. Granted, this is from the point of observation. At -2.8 seconds, relatively it would have been at 0m displaced, we're simply saying that 0 seconds elapsed is the present, even though at zero seconds, we are -50 metres behind. This question broaches subjects uncomfortable to the fettered mind.

3) Find the stationary points of the function $y = x^3 + \frac{11}{2}x^2 - 4x + 31$

- Use a sign diagram to classify each of the stationary points.
- Use the second derivative test to classify each of the stationary points.
- Explain which method you prefer and why.
  $y = x^3 + \frac{11}{2}x^2 - 4x + 31$
  -> $3x^{3-1} + 2\frac{11}{2}x^{2-1} - 4x^{1-1} \cancel{+31}$
  $y' = 3x^2 + 11x - 4$

Finding y''
-> $3x^2 + 11x - 4$
-> $2 \times 3x^{2-1} + 11x^{1-1} \cancel{-4}$

-> $6x^1 + 11x^0$
$y'' = 6x + 11$

Finding the stationary points (y'' method)

$(x_1, y_1) : (\frac{1}{3}, 30.315)$
$(x_2, y_2) : (-4, 71)$

Working out

$0 = 3x^2 + 11x - 4$
-> $x = \frac{-(11) \pm \sqrt{(11)^2 - 4(3)(-4)}}{2(3)}$
-> $x = \frac{-11 \pm \sqrt{121+48}}{6}$
-> $121 + 48 \rightarrow \sqrt{169}$
-> $x = \frac{-11 \pm 13}{6}$
$x_1^+ = \frac{2}{6} = \frac{1}{3}$
$x_2^- = -\frac{24}{6} = -\frac{4}{1} = -4$

$y_1 = x^3 + \frac{11}{2}x^2 - 4x + 31$
-> $\frac{1}{3}^3 + \frac{11}{2}(\frac{1}{3})^2 - 4(\frac{1}{3}) + 31$
-> $\frac{1}{27} + \frac{11}{2}\frac{1}{9} - \frac{4}{3} + 31$
-> $\frac{1}{27} + \frac{11}{18} - \frac{4}{3} + 31$
-> $\frac{2}{54} + \frac{33}{54} - \frac{72}{54} + 31$
-> $-\frac{37}{54} + \frac{1674}{54}$
-> $\frac{1637}{54}$
$= 30.315$

$y_2 = x^3 + \frac{11}{2}x^2 - 4x + 31$
-> $-4^3 + \frac{11}{2}(-4)^2 - 4(-4) + 31$
-> $-64 + \frac{176}{2} + 16 + 31$
-> $88 - 17$
$y_2 = 71$

Classifying the points

Points:

- $(\frac{1}{3}, 30.315)$
- $(-4, 71)$

Derivative of the derivative test

$y'' = 6x + 11$
-> $6\left(\frac{1}{3}\right) + 11$
-> $2 + 11$
-> 13
As $13 > 0$, point $(\frac{1}{3}, 30.315)$ is a local minimum

$y'' = 6x + 11$
-> $6(-4) + 11$
-> $-24 + 11$
-> -13
As $-13 < 0$, point $(-4, 71)$ is a local maximum

Sign test

$y' = 3x^2 + 11x - 4$

$3(\frac{1}{3} \pm \frac{1}{3})^2 + 11(\frac{1}{3} \pm \frac{1}{3}) - 4$
-> $x = 0 \to 4.6\dot{6}$
-> $x = \frac{2}{3} \to -4$

$y' = 3(-4 \pm 1)^2 + 11(-4 \pm 1) - 4$
-> $x = -5 \to 16$
-> $x = -3 \to -10$

Points	$< x$	$x = 0$	$x >$	**Answer**
$(\frac{1}{3}, 30.315)$	$-$	0	$+$	Local minimum
$(-4, 71)$	$+$	0	$-$	Local maximum

It is a local minimum as the left side is descending downwards $(-)$ into the stationary point (0) which then it ascends $(+)$. Same but reverse for the local maximum: It ascends $(+)$ until the point (0) then it descends $(-)$.

Preferred method explanation

I prefer the y'', double derivative method. The sign method feels untrustworthy and seemingly reliant on trial-and-error. I got the 'wrong answers' multiple times using the sign test even though I was told and had seen multiple resources saying that you can use *any* value below/above x. From my trials, this is not true, and unless otherwise learnt, I will not be using the sign test due to it's unpredictability.
In retrospect after writing this, I believe the sign test will work every time if the function is not a polynomial past quadratics. More than one stationary point means the test will always give a wrong answer.
The y'' method works no matter the amount of points and since it seems to usually be fairly condensed by the time you derive the original function so many times, it is trivial to do so.