1

Algebra

Introduction

Algebra, the written language of mathematics, is defined in terms of numbers, letters, and mathematical symbols. As with any language, a variety of rules are needed to enhance precision, and to resolve ambiguity. Ultimately, all mathematics uses algebra to describe and clarify relations between variables.

This topic covers algebraic expressions, decimal places, the rules of algebra, the algebra of fractions, and the solution of equations in both one and two variables.

After studying this topic, you should be able to:

- evaluate algebraic expressions for given values of the variables;
- round fractions to a specified number of decimal places;
- understand and apply the rules of algebra;
- understand and apply the rules to simplify operations on fractions;
- re-arrange and solve equations in one variable;
- solve pairs of simultaneous linear equations in two variables.

1.1 Algebraic expressions

1. An algebraic **expression** contains numbers, letters and mathematical symbols. The standard symbols (**operations**) used are the familiar:

   $+$ addition;
   $-$ subtraction;
   $\times$ multiplication;
   $\div$ division;
   and ( ) brackets.

   e.g. $5x - 3$, $2(x + 9)$, $\dfrac{3x + 2}{x}$ are expressions.

   Note that the division of one quantity by another can be denoted in 3 separate ways, e.g. $3 \div 5$, $\dfrac{3}{5}$, and $3/5$ all mean 3 divided by 5.

2. The letters appearing in an expression are called the **variables** (or unknowns). So, x is the variable in the expression $5x - 3$.

3. Expressions can be **evaluated** when the variable has a given value, e.g. when $x = 4$, $5x - 3 = 5 \times 4 - 3 = 20 - 3 = 17$.

4. Expressions may involve more than one variable, e.g. $\dfrac{3(x - 4y)}{2x + 1}$.

Examples

1. Evaluate $5x - 3$ when

   (i) $x = 7$ (ii) $x = 0$ (iii) $x = -3$.

2. Evaluate $\dfrac{10-2x}{x+2y}$ when

   (i) $x = 2$ and $y = 0$ (ii) $x = -3$ and $y = 1$

   (iii) $x = 5$ and $y = -1$ (iv) $x = 1$, and $y = 3$.

1. (i) When $x = 7$, $5x - 3 = 5 \times 7 - 3 = 35 - 3 = 32$.

   (ii) When $x = 0$, $5x - 3 = 5 \times 0 - 3 = 0 - 3 = -3$.

   (iii) When $x = -3$, $5x - 3 = 5 \times -3 - 3 = -15 - 3 = -18$.

2. (i) When $x = 2$ and $y = 0$,

   $$\frac{10-2x}{x+2y} = \frac{10-2\times 2}{2+2\times 0} = \frac{10-4}{2+0} = \frac{6}{2} = 3.$$

   (ii) When $x = -3$ and $y = 1$,

   $$\frac{10-2x}{x+2y} = \frac{10-2\times -3}{-3+2\times 1} = \frac{10+6}{-3+2} = \frac{16}{-1} = -16.$$

   (iii) When $x = 5$ and $y = -1$,

   $$\frac{10-2x}{x+2y} = \frac{10-2\times 5}{5+2\times -1} = \frac{10-10}{5-2} = \frac{0}{3} = 0.$$

   (iv) When $x = 1$ and $y = 3$,

   $$\frac{10-2x}{x+2y} = \frac{10-2\times 1}{1+2\times 3} = \frac{10-2}{1+6} = \frac{8}{7}.$$

Problems

1. Evaluate $6x - 5$ when

   (i) $x = 7$ (ii) $x = 0$ (iii) $x = -3$.

2. Evaluate $\dfrac{16-3x}{y+4}$ when

   (i) $x = 2$ and $y = 6$ (ii) $x = -3$ and $y = 1$

   (iii) $x = 5$ and $y = -1$ (iv) $x = 1$ and $y = 3$.

Answers

1. (i) 37 (ii) –5 (iii) –23.

2. (i) 1 (ii) 5 (iii) $\frac{1}{3}$ (iv) $\frac{13}{7}$.

1.2 Decimal places

1. When a fraction is converted to a decimal, the resulting answer is often an approximation. For instance, a calculator gives the answer to $\frac{254}{21}$ as 12.0952381 .This answer is not exact, and is therefore an approximation.

   To distinguish between an approximate and exact answer, the symbol $\approx$ (read as 'is approximately equal to') is used, e.g. $\frac{254}{21} \approx 12.0952381$. Other symbols, such as $\doteq$ also denote approximations.

2. In approximations, decimals are usually **rounded off** to a specified number of decimal places. The number of places establishes the last digit quoted in the approximation. The procedure is as follows:
   Identify the specified final digit in the decimal. If the next digit is 4 or less, leave the final digit as it is. If, however, the final digit is 5 or more, add 1 to the final digit. Then drop all digits after the specified final digit.

Examples

1. Round $\frac{254}{21}$ to

   (i) 6 decimal places

   (ii) 5 decimal places

   (iii) 3 decimal places

   (iv) 2 decimal places.

2. Round $\frac{17}{29}$ to

   (i) 7 decimal places

   (ii) 4 decimal places

   (iii) 1 decimal place.

1. A calculator gives $\frac{254}{21} \approx 12.0952381$

   (i) 12.095238

   (ii) 12.09524 (next digit is '8' so '3' is rounded up to '4')

   (iii) 12.095

   (iv) 12.10 (the 3rd decimal place is '5' so '9' is rounded up, which means changing two digits.)

2. A calculator gives $\frac{17}{29} \approx 0.58620689$.

   (i) 0.5862069(next digit is '9' so '8' is rounded up to '9')

   (ii) 0.5862

   (iii) 0.6 (the 2nd decimal place is '8' so '5' is rounded up to '6').

Problems

1. Round $\frac{34}{67}$ to

   (i) 6 decimal places

   (ii) 4 decimal places

   (iii) 3 decimal places

   (iv) 2 decimal places.

2. Round $\frac{47}{31}$ to

   (i) 7 decimal places

   (ii) 5 decimal places

   (iii) 1 decimal place.

Answers

1. (i) 0.507463 (ii) 0.5075 (iii) 0.507 (iv) 0.51

2. (i) 1.5161290 (ii) 1.51613 (iii) 1.5

1.3 Rules of algebra

1. Rules of algebra are needed to avoid ambiguity when more than one of the mathematical operations: $+$; $-$; $\times$; $\div$; ( ) is used. Without any rules, both of the following answers could be correct

$$7 + 4 \times 6 = \begin{cases} 11 \times 6 = 66 \\ 7 + 24 = 31 \end{cases}$$

For this reason a precedence rule is required. The rule, BOMDAS (or BODMAS) specifies that operations are performed in the following order:

Brackets
Of
Multiply
Divide
Add
Subtract

Using the BOMDAS rule, $7 + 4 \times 6 = 7 + 24 = 31$, the multiplication ($4 \times 6$) being performed before the addition. If the addition is required to be performed before the multiplication, brackets need to be inserted, i.e. $(7 + 4) \times 6 = 11 \times 6 = 66$.
Note that 'of' is rarely used, and is equivalent to multiplication. For instance, 'one third of six' means $\frac{1}{3} \times 6$.

2. Whole number powers of a variable, e.g. x^2, x^3 can be considered as multiplications, as $x^2 = x \times x$, etc. Note the difference between, say,
$-5^2 = -5 \times 5 = -25$, and $(-5)^2 = -5 \times -5 = 25$.

3. The square root, denoted $\sqrt{\ }$, can be considered to act like a bracket. Note the difference between, say,
$\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, and $\sqrt{9 + 16} = \sqrt{25} = 5$.

4. The **distributive law** is used to expand brackets, as follows:
$a(b + c) = ab + ac$, and $a(b - c) = ab - ac$.
Note that the left side (L.S.) in each case is ultimately a product of terms, and the right side (R.S.) is a sum (or difference) of terms. In general, the distributive rule is used to convert products to sums (by **expanding** brackets), or to convert sums to products (called '**factorising**').
When the distributive law is used to expand more than one bracket, all of the required terms must be multiplied, i.e.
$(a + b)(c + d) = a(c + d) + b(c + d)$ (where a multiplies all terms in the second bracket, then b multiplies all terms in the second bracket).
Continuing the expansion, $a(c + d) + b(c + d) = ac + ad + bc + bd$.
In general, if there are 2 terms in each bracket, there will be $2 \times 2 = 4$ terms in the expansion. Similarly, 3 terms in one bracket and 2 terms in the other means $3 \times 2 = 6$ terms in the expansion.

Examples

1. Evaluate

(i) $8 + 4 \times 3$	(ii) $(8 + 4) \times 3$	(iii) $6 \div 3 - 1$
(iv) $8 \div (4 - 2)$	(v) $5 \times 3 - 4$	(vi) $5 \times (3 - 4)$
(vii) $4 - 3^2$	(viii) $4 + (-3)^2$	(ix) $\sqrt{5^2 - 4^2}$.

2. Expand

(i) $2(x-6)$ (ii) $-3(2x+3)$ (iii) $-2(x-7)$

(iv) $-(2x-3y)$ (v) $x(y+9)$ (vi) $x^2(x-5)$.

3. Expand and simplify

(i) $(x+2)(x+3)$ (ii) $(2x+1)(x-3)$

(iii) $(3x-2y)(x-y)$ (iv) $(x+4)(x^2-3x+7)$.

4. Factorise

(i) $5x+10y$ (ii) $xy-5x$ (iii) $3x^2+2x$.

1. (i) $8+4\times 3 = 8+12 = 20$

(ii) $(8+4)\times 3 = 12\times 3 = 36$

(iii) $6\div 3-1 = 2-1 = 1$

(iv) $8\div(4-2) = 8\div 2 = 4$

(v) $5\times 3-4 = 15-4 = 11$

(vi) $5\times(3-4) = 5\times -1 = -5$

(vii) $4-3^2 = 4-9 = -5$

(viii) $4+(-3)^2 = 4+9 = 13$

(ix) $\sqrt{5^2-4^2} = \sqrt{25-16} = \sqrt{9} = 3$.

2. (i) $2(x-6) = 2x-2\times 6 = 2x-12$

(ii) $-3(2x+3) = -3\times 2x+(-3\times 3) = -6x-9$

(iii) $-2(x-7) = -2x-2\times -7 = -2x+14$

(iv) $-(2x-3y) = -2x-(-3y) = -2x+3y$

(v) $x(y+9) = xy+9x$

(vi) $x^2(x-5) = x^3-5x^2$.

3. (i) $(x+2)(x+3) = x(x+3)+2(x+3) = x^2+3x+2x+6$

$= x^2+5x+6$ (after simplifying)

(ii) $(2x+1)(x-3) = 2x(x-3)+1(x-3) = 2x^2-6x+x-3$

$= 2x^2-5x-3$ (after simplifying)

(iii) $(3x-2y)(x-y) = 3x(x-y)-2y(x-y)$

$= 3x^2-3xy-2xy+2y^2$

$= 3x^2-5xy+2y^2$ (after simplifying)

(iv) $(x+4)(x^2-3x+7) = x(x^2-3x+7)+4(x^2-3x+7)$

$= x^3-3x^2+7x+4x^2-12x+28$ (Note the 6 terms)

$= x^3+x^2-5x+28$ (after simplifying).

4. (i) $5x+10y = 5(x+2y)$

(ii) $xy-5x = x(y-5)$

(iii) $3x^2+2x = x(3x+2)$.

Problems

1. Evaluate

(i) $8+5\times 2$ (ii) $(8+5)\times 2$ (iii) $4\div 4-6$
(iv) $4\div(4-6)$ (v) $5\times 6-8$ (vi) $5\times(6-8)$
(vii) $9-4^2$ (viii) $9+(-4)^2$ (ix) $\sqrt{5^2-3^2}$.

2. Expand

(i) $3(x-3)$ (ii) $-2(3x+1)$ (iii) $-5(x-3)$
(iv) $-(4x-5y)$ (v) $2x(y+7)$ (vi) $x^2(3x-7)$.

3. Expand and simplify

(i) $(x+5)(x+3)$ (ii) $(2x+3)(x-2)$
(iii) $(3x-y)(x-2y)$ (iv) $(x+3)(x^2-3x+2)$.

4. Factorise

(i) $3x+9y$ (ii) $2xy-4x$ (iii) $5x^2+9x$.

Answers

1. (i) 18 (ii) 26 (iii) –5
(iv) –2 (v) 22 (vi) –10
(vii) –7 (viii) 25 (ix) 4 .

2. (i) $3x-9$ (ii) $-6x-2$ (iii) $-5x+15$
(iv) $-4x+5y$ (v) $2xy+14x$ (vi) $3x^3-7x^2$.

3. (i) $x^2+8x+15$ (ii) $2x^2-x-6$
(iii) $3x^2-7xy+2y^2$ (iv) x^3-7x+6.

4. (i) $3(x+3y)$ (ii) $2x(y-2)$ (iii) $x(5x+9)$.

1.4 The algebra of fractions

1. There are 4 rules which govern the algebra of fractions. These 4 rules are as follows.

   RULE I. $\dfrac{am}{bm} = \dfrac{a}{b}$ (Cancellation Rule)

   i.e. common factors can be cancelled.

   e.g. $\dfrac{6}{15} = \dfrac{2\times 3}{5\times 3} = \dfrac{2}{5}$, and $\dfrac{7x}{4x} = \dfrac{7}{4}$, but $\dfrac{2x+1}{x+1}$ does not simplify, as their is no factor common to both the top and bottom lines.

2. **RULE II**. $\dfrac{a}{c} + \dfrac{b}{c} = \dfrac{a+b}{c}$ (Addition Rule)

   i.e. fractions must have the same (common) denominator in order to be added. The same rule applies to subtraction of fractions, i.e.

   $$\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}.$$

   e.g. $\dfrac{5}{9} + \dfrac{3}{9} = \dfrac{5+3}{9} = \dfrac{8}{9}$, and $\dfrac{5}{9} - \dfrac{2}{9} = \dfrac{5-2}{9} = \dfrac{3}{9} = \dfrac{1}{3}$.

   If fractions do not have a common denominator, the cancellation property can be used to form the common denominator before adding (or subtracting) the fractions. For instance, to evaluate $\dfrac{2}{3} + \dfrac{1}{6}$, it is necessary to write $\dfrac{2}{3}$ as $\dfrac{2\times 2}{3\times 2} = \dfrac{4}{6}$ before adding. Then,

   $\dfrac{2}{3} + \dfrac{1}{6} = \dfrac{4}{6} + \dfrac{1}{6} = \dfrac{5}{6}$. Similarly, $\dfrac{x}{2} + \dfrac{3x}{4} = \dfrac{2x}{4} + \dfrac{3x}{4} = \dfrac{5x}{4}$.

   If a common denominator is not readily apparent, it can always be found by multiplying the denominators of the two fractions together. For instance, $\dfrac{2}{7}$ and $\dfrac{4}{5}$ have a common denominator of $7 \times 5 = 35$. So,

   $$\frac{2}{7} + \frac{4}{5} = \frac{2\times 5}{7\times 5} + \frac{4\times 7}{5\times 7} = \frac{10}{35} + \frac{28}{35} = \frac{38}{35}.$$

   Similarly, $\dfrac{2x}{x+1}$ and $\dfrac{x}{2x+1}$ have a common denominator of $(x+1)(2x+1)$.

   $$\text{So, } \frac{2x}{x+1} - \frac{x}{2x+1} = \frac{2x(2x+1)}{(x+1)(2x+1)} - \frac{x(x+1)}{(2x+1)(x+1)}$$

   $$= \frac{2x(2x+1) - x(x+1)}{(x+1)(2x+1)}$$

   $$= \frac{4x^2 + 2x - x^2 - x}{(x+1)(2x+1)}$$

$$= \frac{3x^2 + x}{(x+1)(2x+1)} = \frac{x(3x+1)}{(x+1)(2x+1)}.$$

3. **RULE III**. $\quad \dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{ac}{bd}$ (Multiplication Rule)

   i.e. numerators are multiplied, and denominators are multiplied, there being no need for a common denominator.

   e.g. $\dfrac{3}{5} \times \dfrac{4}{7} = \dfrac{3 \times 4}{5 \times 7} = \dfrac{12}{35}$, and $\dfrac{3}{x} \times \dfrac{2x+7}{3x-5} = \dfrac{3(2x+7)}{x(3x-5)}$.

   Note also, that $4 \times \dfrac{2}{3} = \dfrac{4}{1} \times \dfrac{2}{3} = \dfrac{4 \times 2}{1 \times 3} = \dfrac{8}{3}$, i.e. when multiplying by a whole number, the numerator of the fraction is multiplied by that number, and the denominator remains unchanged.

4. **RULE IV**. $\quad \dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a}{b} \times \dfrac{d}{c} = \dfrac{ad}{bc}$ (Division Rule)

   i.e. the second fraction is inverted, then multiplied by the first fraction.

   e.g. $\dfrac{2}{7} \div \dfrac{3}{5} = \dfrac{2}{7} \times \dfrac{5}{3} = \dfrac{2 \times 5}{7 \times 3} = \dfrac{10}{21}$, and $\dfrac{x}{4} \div \dfrac{2}{3} = \dfrac{x}{4} \times \dfrac{3}{2} = \dfrac{3x}{8}$.

   Similarly, $\dfrac{3/4}{2/3} = \dfrac{3}{4} \times \dfrac{3}{2} = \dfrac{3 \times 3}{4 \times 2} = \dfrac{9}{8}$.

Examples

1. Simplify, i.e.write as a single fraction with no common factors

   (i) $\dfrac{84}{16}$ (ii) $\dfrac{7}{5} - \dfrac{1}{10}$ (iii) $\dfrac{2}{7}\left(\dfrac{5}{4} - \dfrac{3}{8}\right)$

   (iv) $\dfrac{5/9}{5/4}$ (v) $\dfrac{2+3/5}{6-4/5}$ (vi) $\dfrac{7}{9} + \dfrac{8}{11}$.

2. Simplify, i.e.write as a single fraction with no common factors

   (i) $\dfrac{5x(y+3)}{15x^2y}$ (ii) $\dfrac{3xy^2 - 2y}{y - 4xy}$ (iii) $\dfrac{x}{x+3} - \dfrac{3}{4}$

   (iv) $\dfrac{2x-1}{x+3} - 7$ (v) $\dfrac{2x}{2x+5} - \dfrac{4}{2x+1}$

   (vi) $\dfrac{3x-7}{x+3} - \dfrac{3x-4}{x-5}$.

1. (i) $\dfrac{84}{16} = \dfrac{21 \times 4}{4 \times 4} = \dfrac{21}{4}$

   (ii) $\dfrac{7}{5} - \dfrac{1}{10} = \dfrac{7 \times 2}{5 \times 2} - \dfrac{1}{10} = \dfrac{14}{10} - \dfrac{1}{10} = \dfrac{13}{10}$

(iii) $\frac{2}{7}\left(\frac{5}{4}-\frac{3}{8}\right) = \frac{2}{7}\left(\frac{5\times 2}{4\times 2}-\frac{3}{8}\right) = \frac{2}{7}\left(\frac{10}{8}-\frac{3}{8}\right) = \frac{2}{7}\times\frac{7}{8} = \frac{2}{8} = \frac{1}{4}$

(iv) $\frac{5/9}{5/4} = \frac{5}{9}\times\frac{4}{5} = \frac{4}{9}$

(v) $\frac{2+3/5}{6-4/5} = \frac{10/5+3/5}{30/5-4/5} = \frac{13/5}{26/5} = \frac{13}{5}\times\frac{5}{26} = \frac{13}{26} = \frac{1}{2}$

(vi) $\frac{7}{9}+\frac{8}{11} = \frac{7\times 11}{9\times 11}+\frac{8\times 9}{11\times 9} = \frac{77}{99}+\frac{72}{99} = \frac{149}{99}$.

2. (i) $\frac{5x(y+3)}{15x^2y} = \frac{5x(y+3)}{5x(xy)} = \frac{y+3}{xy}$

(ii) $\frac{3xy^2-2y}{y-4xy} = \frac{y(3xy-2)}{y(1-4x)} = \frac{3xy-2}{1-4x}$

(iii)
$$\frac{x}{x+3}-\frac{3}{4} = \frac{4x}{4(x+3)}-\frac{3(x+3)}{4(x+3)} = \frac{4x-3(x+3)}{4(x+3)}$$
$$= \frac{4x-3x-9}{4(x+3)} = \frac{x-9}{4(x+3)}$$

(iv)
$$\frac{2x-1}{x+3}-7 = \frac{2x-1}{x+3}-\frac{7(x+3)}{x+3} = \frac{2x-1-7(x+3)}{x+3}$$
$$= \frac{2x-1-7x-21}{x+3} = \frac{-5x-22}{x+3}$$

(v)
$$\frac{2x}{2x+5}-\frac{4}{2x+1} = \frac{2x(2x+1)}{(2x+5)(2x+1)}-\frac{4(2x+5)}{(2x+1)(2x+5)}$$
$$= \frac{2x(2x+1)-4(2x+5)}{(2x+5)(2x+1)}$$
$$= \frac{4x^2+2x-8x-20}{(2x+5)(2x+1)} = \frac{4x^2-6x-20}{(2x+5)(2x+1)}$$

(vi)
$$\frac{3x-7}{x+3}-\frac{3x-4}{x-5} = \frac{(3x-7)(x-5)}{(x+3)(x-5)}-\frac{(3x-4)(x+3)}{(x-5)(x+3)}$$
$$= \frac{(3x-7)(x-5)-(3x-4)(x+3)}{(x+3)(x-5)}$$
$$= \frac{3x^2-15x-7x+35-(3x^2+9x-4x-12)}{(x+3)(x-5)}$$
$$= \frac{3x^2-22x+35-(3x^2+5x-12)}{(x+3)(x-5)}$$
$$= \frac{3x^2-22x+35-3x^2-5x+12}{(x+3)(x-5)}$$
$$= \frac{-27x+47}{(x+3)(x-5)}.$$

Problems

1. Simplify, i.e.write as a single fraction with no common factors

(i) $\dfrac{15}{95}$ (ii) $\dfrac{7}{8}-\dfrac{1}{4}$ (iii) $\dfrac{5}{2}\left(\dfrac{4}{5}-\dfrac{2}{3}\right)$

(iv) $\dfrac{6/11}{3/2}$ (v) $\dfrac{2+5/8}{2-7/8}$ (vi) $\dfrac{7}{11}+\dfrac{3}{5}$.

2. Simplify, i.e.write as a single fraction with no common factors

(i) $\dfrac{4x(3y+8)}{8xy^2}$ (ii) $\dfrac{7xy^2-2xy}{3xy-4x^2y}$ (iii) $\dfrac{5x}{x+1}-\dfrac{6}{5}$

(iv) $\dfrac{3x-1}{x+6}-3$ (v) $\dfrac{x}{2x+5}-\dfrac{2}{2x+7}$

(vi) $\dfrac{2x-7}{x+3}-\dfrac{2x+5}{x-5}$.

Answers

1. (i) $\dfrac{3}{19}$ (ii) $\dfrac{5}{8}$ (iii) $\dfrac{1}{3}$

(iv) $\dfrac{4}{11}$ (v) $\dfrac{7}{3}$ (vi) $\dfrac{68}{55}$.

2. (i) $\dfrac{3y+8}{2y^2}$ (ii) $\dfrac{xy(7y-2)}{xy(3-4x)} = \dfrac{7y-2}{3-4x}$

(iii) $\dfrac{25x}{5(x+1)}-\dfrac{6(x+1)}{5(x+1)} = \dfrac{19x-6}{5(x+1)}$

(iv) $\dfrac{3x-1}{x+6}-\dfrac{3(x+6)}{x+6} = \dfrac{-19}{x+6}$

(v) $$\frac{x}{2x+5}-\frac{2}{2x+7} = \frac{x(2x+7)}{(2x+5)(2x+7)}-\frac{2(2x+5)}{(2x+7)(2x+5)}$$

$$= \frac{x(2x+7)-2(2x+5)}{(2x+5)(2x+7)}$$

$$= \frac{2x^2+7x-4x-10}{(2x+5)(2x+7)} = \frac{2x^2+3x-10}{(2x+5)(2x+7)}$$

(vi) $$\frac{2x-7}{x+3}-\frac{2x+5}{x-5} = \frac{(2x-7)(x-5)}{(x+3)(x-5)}-\frac{(2x+5)(x+3)}{(x-5)(x+3)}$$

$$= \frac{(2x-7)(x-5)-(2x+5)(x+3)}{(x+3)(x-5)}$$

$$= \frac{2x^2 - 10x - 7x + 35 - (2x^2 + 6x + 5x + 15)}{(x+3)(x-5)}$$

$$= \frac{2x^2 - 17x + 35 - 2x^2 - 11x - 15}{(x+3)(x-5)}$$

$$= \frac{-28x + 20}{(x+3)(x-5)}.$$

1.5 Equations

1. When two expressions are equal for certain values of a variable (normally x), the result is an equation, e.g. $5x - 7 = 4x + 1$.
   This equation is true for one value of x only; $x = 8$. This can be checked by evaluating both the left side (L.S.), and the right side (R.S.) of the equation when $x = 8$, i.e.

   L.S. $= 5 \times 8 - 7 = 40 - 7 = 33$, and
   R.S. $= 4 \times 8 + 1 = 32 + 1 = 33$.

   This means that **the equation** $5x - 7 = 4x + 1$ **has the solution** $x = 8$.

2. **The solution to an equation can be found by performing the same operations on both the L.S. and the R.S. of the equation.** The aim is to isolate the variable x on the L.S. For the above equation,

$$5x - 7 = 4x + 1$$

$$\therefore 5x = 4x + 1 + 7 \qquad \text{(adding 7 to both sides)}$$

$$\therefore 5x = 4x + 8 \qquad \text{(simplifying)}$$

$$\therefore 5x - 4x = 8 \qquad \text{(subtracting } 4x \text{ from both sides)}$$

$$\therefore x = 8 \qquad \text{(simplifying)}$$

   Note that all terms not involving x are removed form the L.S. (by adding 7), and then all x terms are removed from the R.S. (by subtracting $4x$).

3. The process of isolating x on the L.S.can be pictured as 'undoing' what has happened to x. For instance, to solve the equation $\frac{4x+3}{5} = 7$, note that x has been:
   multiplied by 4, 3 has been added, then the result has been divided by 5.
   To isolate x, these operations are 'undone' in reverse order, i.e.
   first multiply by 5, then subtract 3, then divide by 4.

   So, $\frac{4x+3}{5} = 7$

$$\therefore 4x + 3 = 35 \qquad [\ \times 5]$$

$$\therefore 4x = 32 \qquad [\ -3]$$

$$\therefore x = \frac{32}{4} = 8 \qquad [\ \div 4].$$

Note that 'undoing' an operation is the same as performing the inverse operation. The common operations, and their inverses are shown in the following table.

OPERATION	INVERSE
Addition	Subtraction
Subtraction	Addition
Multiplication	Division
Division	Multiplication

4. When solving an equation involving squares and/or square roots, the following rules should be used. For any $a \geq 0$:

   If $x^2 = a$, then $x = \pm\sqrt{a}$
   (the inverse of the square is $\pm$ the square root);

   If $\sqrt{x} = a$, then $x = a^2$
   (the inverse of the square root is the square).

   Note that, for any $a > 0$, there are **two solutions** to the equation $x^2 = a$. For instance, if $x^2 = 9$, then $x = \pm\sqrt{9} = \pm 3$. So, both $x = 3$ and $x = -3$ are solutions. This is a consequence of the fact that the square of a negative number is positive.
   However, for any $a \geq 0$, there is only **one solution** to the equation $\sqrt{x} = a$. For instance, if $\sqrt{x} = 4$, then $x = 4^2 = 16$. This is a consequence of the fact that x cannot be negative if $\sqrt{x}$ exists.

5. An equation containing 2 (or more) variables can also be 'solved' for one of the variables, i.e. one variable can be made **the subject** of the equation by isolating it on the L.S. For instance, the equation $\frac{x-2y}{3} = 2y+4$ can be re-arranged to isolate x on the L.S, as follows.

   $x - 2y = 3(2y+4) \qquad [\ \times 3]$

   $\therefore x - 2y = 6y + 12 \qquad$ (simplifying)

   $\therefore x = 8y + 12 \qquad [\ +2y].$

6. In any equation, 2 expressions are equal, so the L.S. and R.S. may be switched at any time (since $a = b$ and $b = a$ are regarded as identical

equations). For instance, the equation $y = 3x - 1$ can be re-arranged to isolate x as follows:

$$3x - 1 = y \qquad \text{(switching sides)}$$

$$\therefore 3x = y + 1 \qquad [\ +1]$$

$$\therefore x = \frac{y+1}{3} \qquad [\ \div 3]$$

Examples

1. Solve the following equations for x

(i) $\dfrac{4x-1}{5} = 3$ (ii) $\dfrac{x}{5} + 2x - 9 = 2$

(iii) $\dfrac{7x-4}{2x+1} = 5$ (iv) $\sqrt{2x^2 - 17} = 9$

(v) $3x^2 + 7 = 19$ (vi) $\dfrac{2x-9}{x^2+3} = 0$.

2. Solve the following equations for x, i.e. make x the subject of the formula

(i) $3x + 2y = 10$ (ii) $y = \sqrt{5x+7}$

(iii) $2x^2 - 3 = 5y$ (iv) $y = \sqrt{x^2 - 8}$

(v) $y = \dfrac{2x+3}{x}$ (vi) $y = \dfrac{3x}{x-7}$.

3. Given $y = \dfrac{3x+5}{x}$

(i) make x the subject of the formula

(ii) find x when $y = 4$.

1. (i) $\dfrac{4x-1}{5} = 3$

$$\therefore 4x - 1 = 15 \qquad [\ \times 5]$$

$$\therefore 4x = 16 \qquad [\ +1]$$

$$\therefore x = \frac{16}{4} = 4 \qquad [\ \div 4]$$

i.e. $x = 4$

(ii) $\dfrac{x}{5} + 2x - 9 = 2$

$$\therefore x + 5(2x - 9) = 10 \qquad [\ \times 5]$$

$\therefore x + 10x - 45 = 10$ (simplifying)

$\therefore 11x - 45 = 10$ (simplifying)

$\therefore 11x = 55$ $[\ +45]$

$\therefore x = \frac{55}{11} = 5$ $[\ \div 11]$

i.e. $x = 5$

(iii) $\frac{7x-4}{2x+1} = 5$

$\therefore 7x - 4 = 5(2x+1)$ $[\ \times(2x+1)]$

$\therefore 7x - 4 = 10x + 5$ (simplifying)

$\therefore 7x = 10x + 9$ $[\ +4]$

$\therefore -3x = 9$ $[\ -10x]$

$\therefore x = \frac{9}{-3} = -3$ $[\ \div(-3)]$

i.e. $x = -3$

(iv) $\sqrt{2x^2 - 17} = 9$

$\therefore 2x^2 - 17 = 9^2$ (squaring)

$\therefore 2x^2 - 17 = 81$ (simplifying)

$\therefore 2x^2 = 98$ $[\ +17]$

$\therefore x^2 = 49$ $[\ \div 2]$

$\therefore x = \pm\sqrt{49} = \pm 7$ $[\pm\sqrt{\ }]$

i.e. $x = \pm 7$ (2 solutions)

(v) $3x^2 + 7 = 19$

$\therefore 3x^2 = 12$ $[\ -7]$

$\therefore x^2 = \frac{12}{3} = 4$ $[\ \div 3]$

$\therefore x = \pm\sqrt{4} = \pm 2$ $[\pm\sqrt{\ }]$

i.e. $x = \pm 2$ (2 solutions)

(vi) $\frac{2x-9}{x^2+3} = 0$

$\therefore 2x - 9 = 0(x^2 + 3)$ $\quad [\ \times (x^2 + 3)]$

$\therefore 2x - 9 = 0$ $\quad$ (simplifying)

$\therefore 2x = 9$ $\quad [\ +9]$

$\therefore x = \dfrac{9}{2}$ $\quad [\ \div 2]$

i.e. $x = \dfrac{9}{2}$.

2. (i) $3x + 2y = 10$

$\therefore 3x = 10 - 2y$ $\quad [\ -2y]$

$\therefore x = \dfrac{10 - 2y}{3}$ $\quad [\ \div 3]$

i.e. $x = \dfrac{10 - 2y}{3}$

(ii) $y = \sqrt{5x + 7}$

$\therefore y^2 = 5x + 7$ $\quad$ (squaring)

$\therefore 5x + 7 = y^2$ $\quad$ (switching sides)

$\therefore 5x = y^2 - 7$ $\quad [\ -7]$

$\therefore x = \dfrac{y^2 - 7}{5}$ $\quad [\ \div 5]$

i.e. $x = \dfrac{y^2 - 7}{5}$

(iii) $2x^2 - 3 = 5y$

$\therefore 2x^2 = 5y + 3$ $\quad [\ +3]$

$\therefore x^2 = \dfrac{5y + 3}{2}$ $\quad [\ \div 2]$

$\therefore x = \pm\sqrt{\dfrac{5y + 3}{2}}$ $\quad [\pm\sqrt{\ }\ ]$

i.e. $x = \pm\sqrt{\dfrac{5y + 3}{2}}$

(iv) $y = \sqrt{x^2 - 8}$

$\therefore y^2 = x^2 - 8$ $\quad$ (squaring)

$\therefore x^2 - 8 = y^2$ $\quad$ (switching sides)

$$\therefore x^2 = y^2 + 8 \qquad [\;+8]$$

$$\therefore x = \pm\sqrt{y^2+8} \qquad [\pm\sqrt{\;}]$$

$$\text{i.e. } x = \pm\sqrt{y^2+8}$$

(v) $$y = \frac{2x+3}{x}$$

$$\therefore xy = 2x+3 \qquad [\;\times x]$$

$$\therefore xy - 2x = 3 \qquad [\;-2x]$$

$$\therefore x(y-2) = 3 \qquad \text{(factorising)}$$

$$\therefore x = \frac{3}{y-2} \qquad [\;\div(y-2)]$$

$$\text{i.e. } x = \frac{3}{y-2}$$

(vi) $$y = \frac{3x}{x-7}$$

$$\therefore y(x-7) = 3x \qquad [\;\times(x-7)]$$

$$\therefore xy - 7y = 3x \qquad \text{(expanding)}$$

$$\therefore xy - 3x - 7y = 0 \qquad [\;-3x]$$

$$\therefore xy - 3x = 7y \qquad [\;+7y]$$

$$\therefore x(y-3) = 7y \qquad \text{(factorising)}$$

$$\therefore x = \frac{7y}{y-3} \qquad [\;\div(y-3)]$$

$$\text{i.e. } x = \frac{7y}{y-3}.$$

3. (i) $$y = \frac{3x+5}{x}$$

$$\therefore xy = 3x+5 \qquad [\;\times x]$$

$$\therefore xy - 3x = 5 \qquad [\;-3x]$$

$$\therefore x(y-3) = 5 \qquad \text{(factorising)}$$

$$\therefore x = \frac{5}{y-3} \qquad [\;\div(y-3)]$$

$$\text{i.e. } x = \frac{5}{y-3}$$

(ii) When $y = 4$, $x = \dfrac{5}{4-3} = \dfrac{5}{1} = 5$.

Problems

1. Solve the following equations for x

(i) $\dfrac{7x-2}{9} = 6$ (ii) $\dfrac{x}{3} - 2x - 4 = 6$

(iii) $\dfrac{3x-1}{2x-4} = 2$ (iv) $\sqrt{5x-11} = 7$

(v) $2x^2 + 5 = 77$ (vi) $\dfrac{2x+5}{x^2+8} = 0$.

2. Solve the following equations for x, i.e. make x the subject of the formula

(i) $8x - 3y = 10$ (ii) $y = \sqrt{9x+2}$

(iii) $5x^2 - 1 = 7y$ (iv) $y = \sqrt{x^2+8}$

(v) $y = \dfrac{x+4}{2x}$ (vi) $y = \dfrac{3x-1}{x-5}$.

3. Given $y = \dfrac{7x+4}{2x}$

(i) make x the subject of the formula

(ii) find x when $y = 3$.

Answers

1. (i) $\dfrac{7x-2}{9} = 6$

$\therefore 7x - 2 = 54$ $[\ \times 9]$

$\therefore 7x = 56$ $[\ +2]$

$\therefore x = \dfrac{56}{7} = 8$ $[\ \div 7]$

i.e. $x = 8$

(ii) $\dfrac{x}{3} - 2x - 4 = 6$

$\therefore x + 3(-2x - 4) = 18$ $[\ \times 3]$

$\therefore x - 6x - 12 = 18$ (simplifying)

$\therefore -5x - 12 = 18$ (simplifying)

$\therefore -5x = 30$ $[\ +12]$

$\therefore x = \dfrac{30}{-5} = -6$ $[\ \div(-5)]$

i.e. $x = -6$

(iii) $\dfrac{3x-1}{2x-4} = 2$

$\therefore 3x - 1 = 2(2x-4)$ $[\ \times(2x-4)]$

$\therefore 3x - 1 = 4x - 8$ (simplifying)

$\therefore 3x = 4x - 7$ $[\ +1]$

$\therefore -x = -7$ $[\ -4x]$

$\therefore x = \dfrac{-7}{-1} = 7$ $[\ \div(-1)]$

i.e. $x = 7$

(iv) $\sqrt{5x-11} = 7$

$\therefore 5x - 11 = 7^2$ (squaring)

$\therefore 5x - 11 = 49$ (simplifying)

$\therefore 5x = 60$ $[\ +11]$

$\therefore x = \dfrac{60}{5} = 12$ $[\ \div 5]$

i.e. $x = 12$

(v) $2x^2 + 5 = 77$

$\therefore 2x^2 = 72$ $[\ -5]$

$\therefore x^2 = \dfrac{72}{2} = 36$ $[\ \div 2]$

$\therefore x = \pm\sqrt{36} = \pm 6$ $[\pm\sqrt{\ }\ ]$

i.e. $x = \pm 6$ (2 solutions)

(vi) $\dfrac{2x+5}{x^2+8} = 0$

$\therefore 2x + 5 = 0(x^2+8)$ $[\ \times(x^2+8)]$

$\therefore 2x + 5 = 0$ (simplifying)

$\therefore 2x = -5 \qquad [\ -5]$

$\therefore x = \dfrac{-5}{2} \qquad [\ \div 2]$

i.e. $x = \dfrac{-5}{2}$.

2. (i) $8x - 3y = 10$

$\therefore 8x = 10 + 3y \qquad [\ +3y]$

$\therefore x = \dfrac{10+3y}{8} \qquad [\ \div 8]$

i.e. $x = \dfrac{10+3y}{8}$

(ii) $y = \sqrt{9x+2}$

$\therefore y^2 = 9x + 2$ (squaring)

$\therefore 9x + 2 = y^2$ (switching sides)

$\therefore 9x = y^2 - 2 \qquad [\ -2]$

$\therefore x = \dfrac{y^2-2}{9} \qquad [\ \div 9]$

i.e. $x = \dfrac{y^2-2}{9}$

(iii) $5x^2 - 1 = 7y$

$\therefore 5x^2 = 7y + 1 \qquad [\ +1]$

$\therefore x^2 = \dfrac{7y+1}{5} \qquad [\ \div 5]$

$\therefore x = \pm\sqrt{\dfrac{7y+1}{5}} \qquad [\pm\sqrt{\ }\ ]$

i.e. $x = \pm\sqrt{\dfrac{7y+1}{5}}$

(iv) $y = \sqrt{x^2+8}$

$\therefore y^2 = x^2 + 8$ (squaring)

$\therefore x^2 + 8 = y^2$ (switching sides)

$\therefore x^2 = y^2 - 8 \qquad [\ -8]$

$\therefore x = \pm\sqrt{y^2-8} \qquad [\pm\sqrt{\ }\ ]$

i.e. $x = \pm\sqrt{y^2 - 8}$

(v) $y = \dfrac{x+4}{2x}$

$\therefore 2xy = x + 4$ $[\ \times 2x]$

$\therefore 2xy - x = 4$ $[\ -x]$

$\therefore x(2y-1) = 4$ (factorising)

$\therefore x = \dfrac{4}{2y-1}$ $[\ \div(2y-1)]$

i.e. $x = \dfrac{4}{2y-1}$

(vi) $y = \dfrac{3x-1}{x-5}$

$\therefore y(x-5) = 3x-1$ $[\ \times(x-5)]$

$\therefore xy - 5y = 3x - 1$ (expanding)

$\therefore xy - 3x - 5y = -1$ $[\ -3x]$

$\therefore xy - 3x = 5y - 1$ $[\ +5y]$

$\therefore x(y-3) = 5y-1$ (factorising)

$\therefore x = \dfrac{5y-1}{y-3}$ $[\ \div(y-3)]$

i.e. $x = \dfrac{5y-1}{y-3}$.

3. (i) $y = \dfrac{7x+4}{2x}$

$\therefore 2xy = 7x + 4$ $[\ \times 2x]$

$\therefore 2xy - 7x = 4$ $[\ -7x]$

$\therefore x(2y-7) = 4$ (factorising)

$\therefore x = \dfrac{4}{2y-7}$ $[\ \div(2y-7)]$

i.e. $x = \dfrac{4}{2y-7}$

(ii) When $y = 3$, $x = \dfrac{4}{6-7} = \dfrac{4}{-1} = -4$.

1.6 Simultaneous equations

1. A **linear equation** in 2 variables (normally x and y) has the form $Ax + By = D$, where A, B and D are constants. A linear equation is so called because its graph is **a straight line**. This can be seen mathematically by re-arranging the equation $Ax + By = D$ as follows:

$$By = -Ax + D \qquad [\ -Ax]$$

$$\therefore y = \frac{-Ax + D}{B} \qquad [\ \div B]$$

$$\therefore y = \left(\frac{-A}{B}\right)x + \frac{D}{B}$$, which is of the form of the straight line

$$y = mx + c$$, with $m = \frac{-A}{B}$ and $c = \frac{D}{B}$.

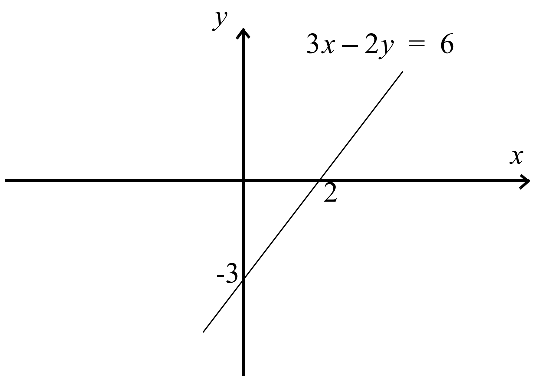
A linear equation can be sketched by finding any two points on the line, and connecting them with a straight line. In general, this is done by simply finding the y value when $x = 0$ (the y-intercept), and finding the x-value when $y = 0$ (the x-intercept).

For instance, given the equation $3x - 2y = 6$:

when $x = 0$, $-2y = 6$ $\quad \therefore y = -3$ (the y-intercept);

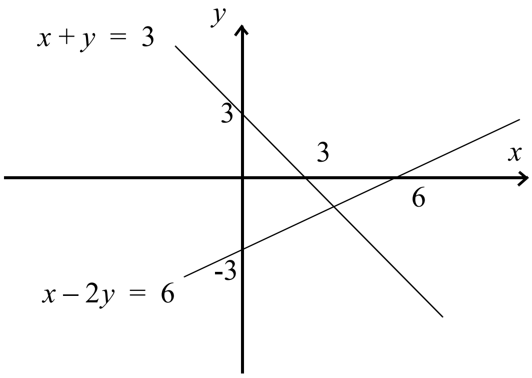
when $y = 0$, $3x = 6$ $\quad \therefore x = 2$ (the x-intercept).

So, the sketch is as follows:



2. A linear equation has infinitely many solutions, and each solution is a single point on the straight line graph of the equation. Some solutions of the equation $3x - 2y = 6$ are $x = 1$ and $y = \frac{-3}{2}$, $x = 4$ and $y = 3$, $x = -2$ and $y = -6$, etc.

Consider 2 linear equations in 2 variables, e.g.
$x - 2y = 6$ and $x + y = 3$.
Each equation has infinitely many solutions, and has a straight line graph. The graphs are sketched below.

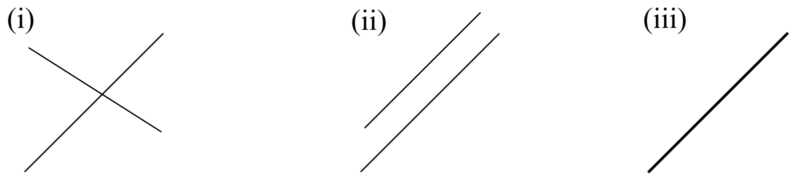


It can be seen that the graphs intersect. The point of intersection indicates the particular values of x and y which are on both straight lines simultaneously. These particular values of x and y represent the **solution to the two simultaneous equations**. From the sketch, the x co-ordinate of the solution is somewhere between 3 and 6. The corresponding y co-ordinate of the solution is between 0 and -3.

3. Any pair of simultaneous linear equations in 2 variables has either:

   (i) exactly one solution (the 2 lines intersect at a single point);
   (ii) no solution (the 2 lines are parallel, and hence don't intersect);
   (iii) infinitely many solutions (the 2 lines are identical).

   The 3 cases are shown in the diagrams below.



In all cases, the solution(s) can be found using either substitution or elimination, as described below. Both methods are used to find the solution to the equations sketched in note 2 above, i.e.

$x - 2y = 6$ (1) and $x + y = 3$ (2)

4. **Substitution**

I **Make one variable the subject of one of the equations**
e.g. from (1), $x = 2y + 6$

II **Substitute for this variable in the other equation**
e.g. in (2), x can be replaced by $2y + 6$.
So, (2) becomes $2y + 6 + y = 3$

III **Solve this equation to find the solution for one variable**
e.g. since $3y + 6 = 3$
$\therefore 3y = -3$ $[\ -6]$
$\therefore y = \frac{-3}{3} = -1$ $[\ \div 3]$ So, $y = -1$.

IV **Substitute the answer found in III into the equation obtained in I to find the solution for the remaining variable**
e.g. since $y = -1$ (from III), and $x = 2y + 6$ (from I),
$x = -2 + 6 = 4$.

So, the solution is $x = 4$ and $y = -1$, i.e. the co-ordinates of the solution are $(4, -1)$.

Note that the solution here is consistent with the solution anticipated in note 2 above.

5. **Elimination**

I Eliminate one variable by adding (or subtracting) a multiple of one equation to (or from) the other equation

e.g. for $x - 2y = 6$ (1) and $x + y = 3$ (2), the co-efficient of x in each equation is 1. So, x can be eliminated by subtracting one equation from the other, i.e. (2) subtract (1) gives

$x + y - (x - 2y) = 3 - 6$. Simplifying gives

$x + y - x + 2y = -3 \qquad \therefore 3y = -3$

II Solve this equation to find the solution for one variable

e.g. since $3y = -3$, $y = \dfrac{-3}{3} \qquad \therefore y = -1$

III Substitute the answer found in II into either of the original equations to find the solution for the remaining variable

e.g. since $y = -1$, using (1) gives $x - 2(-1) = 6$

$\therefore x + 2 = 6 \qquad \therefore x = 6 - 2 = 4$

So, again, the solution is $x = 4$ and $y = -1$, i.e. the co-ordinates of the solution are $(4, -1)$.

In the elimination method, the first step is the most important. **The co-efficients of one variable must be 'matched' before adding or subtracting**. For instance, as the co-efficients of y are -2 and 1, y could be eliminated by multiplying (2) by 2, as follows.

As $x - 2y = 6$ (1) and $x + y = 3$ (2)

$(2) \times 2$ gives $2x + 2y = 6$ (3)

Now, (1) adding (1) and (3) gives $x - 2y + 2x + 2y = 6 + 6$

Simplifying gives $3x = 12 \qquad \therefore x = \dfrac{12}{3} = 4$

Using (2), $4 + y = 3 \qquad \therefore y = 3 - 4 = -1$

So, again, the solution is $x = 4$ and $y = -1$.

6. The solution to a pair of simultaneous equations can be checked by substituting the answers into the original equations. In the example above, when $x = 4$ and $y = -1$, the L.S. of (1) becomes

$4 - (-2) = 4 + 2 = 6$, = R.S, as required.

Similarly, the L.S. of (2) becomes

$4 + (-1) = 4 - 1 = 3$, as required.

Examples

1. Solve the following equations for x and y by

(a) elimination (b) substitution.

(i) $2x + 3y = -1$ (1) and $x - 4y = 16$ (2)

(ii) $4x + 3y = 8$ (1) and $5x - y = 10$. (2).

2. Use elimination to show that there is no solution to the equations

(i) $-2x + 4y = 6$ (1) and $x - 2y = 1$. (2)

3. Use elimination to show that there are infinitely many solutions to the equations

(i) $2x + 4y = -12$ (1) and $x + 2y = -6$. (2)

1.(i)(a) Comparing $2x + 3y = -1$ (1) and $x - 4y = 16$ (2), it can be seen that the x co-efficients can be matched by multiplying (2) by 2, i.e. $2x - 8y = 32$ (3).
Now, (1) subtract (3) gives $2x + 3y - (2x - 8y) = -1 - 32$.

Simplifying gives $2x + 3y - 2x + 8y = -33$

$$\therefore 11y = -33 \qquad \therefore y = \frac{-33}{11} = -3.$$

When $y = -3$, (2) becomes $x - 4(-3) = 16$

$$\therefore x + 12 = 16 \qquad \therefore x = 16 - 12 = 4.$$

So, the solution is $x = 4$ and $y = -3$.

(b) From (2), $x = 4y + 16$. Substituting in (1) gives

$$2(4y + 16) + 3y = -1$$

$$\therefore 8y + 32 + 3y = -1$$

$$\therefore 11y + 32 = -1$$

$$\therefore 11y = -1 - 32 = -33$$

$$\therefore y = \frac{-33}{11} = -3.$$ Substituting in $x = 4y + 16$ gives

$$x = 4(-3) + 16 = -12 + 16 = 4$$

So, again, the solution is $x = 4$ and $y = -3$.

1.(ii)(a) Comparing $4x + 3y = 8$ (1) and $5x - y = 10$ (2), it can be seen that the y co-efficients can be matched by multiplying (2) by 3, i.e. $15x - 3y = 30$ (3).
Now, adding (1) and (3) gives $4x + 3y + 15x - 3y = 8 + 30$.

Simplifying gives $19x = 38 \qquad \therefore x = \frac{38}{19} = 2$

When $x = 2$, (2) becomes $10 - y = 10$

$$\therefore -y = 0 \qquad \therefore y = 0.$$

So, the solution is $x = 2$ and $y = 0$.

(b) From (2), $-y = 10 - 5x \qquad \therefore y = -10 + 5x$.

Substituting in (1) gives

$$4x + 3(-10 + 5x) = 8$$

$$\therefore 4x - 30 + 15x = 8$$

$$\therefore 19x = 8 + 30 = 38$$

$$\therefore x = \frac{38}{19} = 2 \qquad \text{Substituting in } y = -10 + 5x \text{ gives}$$

$$y = -10 + 10 = 0$$

So, again, the solution is $x = 2$ and $y = 0$.

2. Comparing $-2x + 4y = 6$ (1) and $x - 2y = 1$ (2), it can be seen that the x co-efficients can be matched by multiplying (2) by 2, i.e. $2x - 4y = 2$ (3).

   Now, adding (1) and (3) gives $-2x + 4y + 2x - 4y = 6 + 2$.

   Simplifying gives $0 = 8$ (which is impossible).

   Hence, there is no solution (and the equations represent 2 parallel lines).

3. Comparing $2x + 4y = -12$ (1) and $x + 2y = -6$ (2), it can be seen that the x co-efficients can be matched by multiplying (2) by 2, i.e. $2x + 4y = -12$ (3).

   Now, subtracting (1) from (3) gives
   $$2x + 4y - (2x + 4y) = -12 - (-12).$$

   Simplifying gives $2x + 4y - 2x - 4y = -12 + 12 \qquad \therefore 0 = 0$ (which is always true).

   Hence, there are infinitely many solutions (and the equations represent 2 identical lines).

Problems

1. Solve the following equations for x and y by

   (a) elimination (b) substitution.

   (i) $x + 5y = 4$ (1) and $2x + 7y = 2$ (2);

   (ii) $4x + 3y = 4$ (1) and $3x - y = 16$. (2).

2. Use elimination to show that there is no solution to the equations

   (i) $-3x + 6y = 7$ (1) and $x - 2y = -1$. (2)

3. Use elimination to show that there are infinitely many solutions to the equations

   (i) $x + 4y = 7$ (1) and $2x + 8y = 14$. (2)

Answers

1. (i) (a) Comparing $x + 5y = 4$ (1) and $2x + 7y = 2$ (2), it can be seen that the x co-efficients can be matched by multiplying (1) by 2, i.e. $2x + 10y = 8$ (3).
Now, (3) subtract (2) gives $2x + 10y - (2x + 7y) = 8 - 2$.

Simplifying gives $2x + 10y - 2x - 7y = 6$

$$\therefore 3y = 6 \qquad \therefore y = \frac{6}{3} = 2.$$

When $y = 2$, (1) becomes $x + 10 = 4$

$$\therefore x = 4 - 10 = -6.$$

So, the solution is $x = -6$, and $y = 2$.

(b) From (1), $x = 4 - 5y$. Substituting in (2) gives

$$2(4 - 5y) + 7y = 2$$

$$\therefore 8 - 10y + 7y = 2$$

$$\therefore -3y + 8 = 2$$

$$\therefore -3y = 2 - 8 = -6$$

$$\therefore y = \frac{-6}{-3} = 2.$$ Substituting in $x = 4 - 5y$ gives

$$x = 4 - 10 = -6$$

So, again, the solution is $x = -6$, and $y = 2$.

1. (ii) (a) Comparing $4x + 3y = 4$ (1) and $3x - y = 16$ (2), it can be seen that the y co-efficients can be matched by multiplying (2) by 3, i.e. $9x - 3y = 48$ (3).
Now, adding (1) and (3) gives $4x + 3y + 9x - 3y = 4 + 48$.

Simplifying gives $13x = 52$ $\qquad \therefore x = \frac{52}{13} = 4$

When $x = 4$, (2) becomes $12 - y = 16$

$$\therefore -y = 16 - 12 = 4 \qquad \therefore y = -4.$$

So, the solution is $x = 4$, and $y = -4$.

(b) From (2), $-y = 16 - 3x$ $\qquad \therefore y = -16 + 3x$.

Substituting in (1) gives

$$4x + 3(-16 + 3x) = 4$$

$$\therefore 4x - 48 + 9x = 4$$

$$\therefore 13x = 4 + 48 = 52$$

$$\therefore x = \frac{52}{13} = 4 \qquad \text{Substituting in } y = -16 + 3x \text{ gives}$$

$$y = -16 + 12 = -4$$

So, again, the solution is $x = 4$, and $y = -4$.

2. Comparing $-3x + 6y = 7$ (1) and $x - 2y = -1$ (2), it can be seen that the x co-efficients can be matched by multiplying (2) by 3, i.e. $3x - 6y = -3$ (3).

   Now, adding (1) and (3) gives $-3x + 6y + 3x - 6y = 7 - 3$.

   Simplifying gives $0 = 4$ (which is impossible).

   Hence, there is no solution (and the equations represent 2 parallel lines).

3. Comparing $x + 4y = 7$ (1) and $2x + 8y = 14$ (2), it can be seen that the x co-efficients can be matched by multiplying (1) by 2, i.e. $2x + 8y = 14$ (3).

   Now, subtracting (2) from (3) gives
   $$2x + 8y - (2x + 8y) = 14 - 14.$$

   Simplifying gives $2x + 8y - 2x - 8y = 0$ $\quad \therefore 0 = 0$ (which is always true).

   Hence, there are infinitely many solutions (and the equations represent 2 identical lines).