4

Trigonometric functions

Introduction

Trigonometry is a branch of mathematics which has been studied since ancient times. Its earliest uses were confined to the exact measurement of lengths and angles. Such applications are still used in surveying, navigation and astronomy. In more recent times its applications include several types of periodic phenomena, including sound waves and electronics.

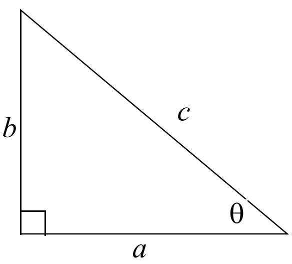
This topic covers trigonometric ratios, right-angled triangles, trigonometric identities, radian measure of angles, and the graphs of the three major trigonometric functions.After studying this topic, you should be able to:

- understand and use the definitions of the three major trigonometric ratios;
- find lengths and angles in right-angled triangles;
- understand the basic trigonometric identity;
- convert angles from degrees to radians and vice versa;
- sketch graphs of the three major trigonometric functions.

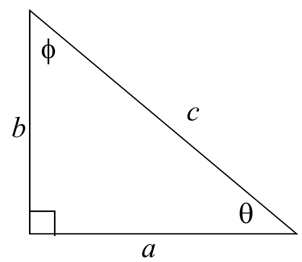
4.1 Trigonometric ratios

1. By definition, in the right-angled triangle shown:

$$\sin\theta = \frac{b}{c}; \qquad \cos\theta = \frac{a}{c}; \qquad \tan\theta = \frac{b}{a}.$$



2. Pythagoras's Theorem ($a^2 + b^2 = c^2$), and the definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$ can be used to find unknown sides and angles in any **right-angled** triangle. Also, the sum of all angles in **any triangle** (whether right-angled or not) must be $180°$.



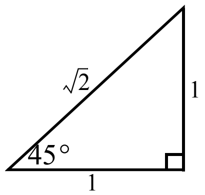
In the right-angled triangle above, this means that

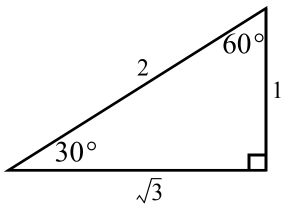
$$\theta + \phi + 90° = 180°$$

$$\therefore \theta + \phi = 90°.$$

3. If a trigonometric ratio is known, the angle can be found using the **inverse trigonometric function**. For instance, if it is known that $\sin\theta = k$, then $\theta = \sin^{-1}k$, read as 'the **inverse sine** of *k*'). Most calculators use the notation $\sin^{-1}k$, though some calculators require using the inverse instruction (normally **inv**), followed by the trigonometric function, e.g. inv sin *k.* (See Example 1(iv)).

4. There are two special right-angled triangles which occur regularly in mathematics. These triangles are called **Standard Triangles**, and should be remembered. The two standard triangles are:



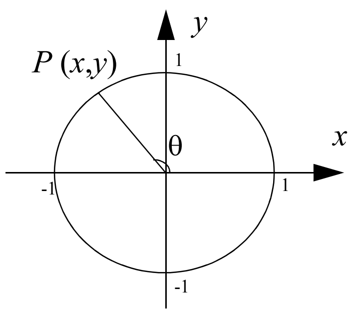


Using these two triangles, all trigonometric ratios of the angles $45°, 30°$, and $60°$ can be found exactly. The relevant trigonometric ratios are shown in the table below:

θ	$30°$	$45°$	$60°$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

5. Note that, in any right-angled triangle, the angle θ cannot be greater than 90^{o}. So the above definitions of $\sin\theta$, $\cos\theta$, and $\tan\theta$ apply only for angles between 0^{o} and 90^{o}. For angles beyond this interval, further definitions are required.

6. The trigonometric ratios can be defined for angles greater than $90°$. This is done by considering points $P(x, y)$ on the unit circle $x^2 + y^2 = 1$, and measuring the angle θ **anti-clockwise from the positive *x* axis** (as shown in the diagram below).



By definition, $\cos\theta = x$ and $\sin\theta = y$. Hence, $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$.

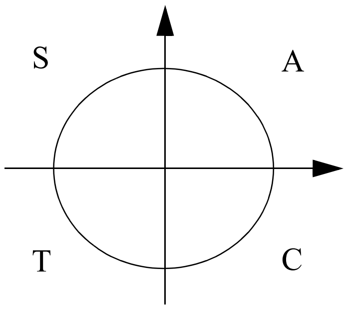
As x and y are the co-ordinates of points on the circle $x^2 + y^2 = 1$,

i.e. $-1 \le x \le 1$ and $-1 \le y \le 1$, and the inequalities
$-1 \le \cos\theta \le 1$ and $-1 \le \sin\theta \le 1$ are always true.

Whenever x is positive, $\cos\theta$ is positive, and whenever y is positive, $\sin\theta$ is positive. Similarly, whenever x is negative, $\cos\theta$ is negative, and whenever y is negative, $\sin\theta$ is negative.

7. There are several ways to remember the signs of the trigonometric ratios in each quadrant.

   One such method is to start in the 1st Quadrant (where both x and y are positive, and move around the circle anti-clockwise. The initial letters of the phrase '**A**ll **S**tations **T**o **C**entral' give the signs of the ratios which are positive in a quadrant.
   Here, **A** means that **a**ll of the ratios are positive (in the 1st quadrant);
   S means that only the **s**ine is positive (in the 2nd quadrant);
   T means that only the **t**angent is positive (in the 3rd quadrant);
   C means that only the **c**osine is positive (in the 4th quadrant);
   Another similar method starts in the 4th quadrant, and moves around the circle anti-clockwise. The letters of the word '**CAST**' then give the correct signs, as shown in the diagram below.



8. **An angle measured clockwise is considered to be a negative angle.** For instance, an angle of $-90°$ represents the same point on the unit circle as an angle of $270°$. In general, adding or subtracting $360°$ to an angle does not change the trigonometric ratio.

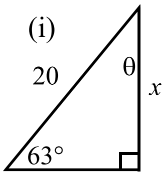
Examples

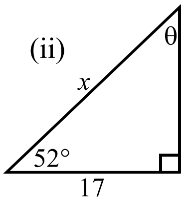
1. Use a calculator to evaluate (to 4 decimal places)

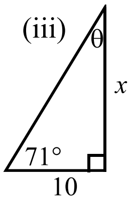
   (i) $\cos 132°$ (ii) $\sin 204°$

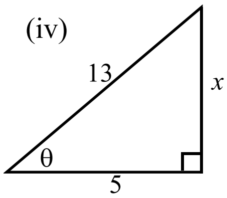
   (iii) $\tan 263°$ (iv) $\cos(-78°)$.

2. In each of the following triangles, find the unknown length, x and angle, θ. Note: round all answers to 1 decimal place.









1. (i) $\cos 132° \approx -0.6691$ (ii) $\sin 204° \approx -0.4067$

   (iii) $\tan 263° \approx 8.1443$ (iv) $\cos(-78°) \approx 0.2079$.

2. (i) As $\theta + 63° = 90°$, $\theta = 90° - 63° = 27°$

   $\therefore \theta = 27°$

   From the diagram, $\sin 63° = \dfrac{x}{20}$.

   $\therefore x = 20\sin 63° \qquad \approx 20 \times 0.8910$

   $\therefore x \approx 17.820 \qquad$ i.e. $x \approx 17.8$.

   (ii) As $\theta + 52° = 90°$, $\theta = 90° - 52° = 38°$

   $\therefore \theta = 38°$

   From the diagram, $\cos 52° = \dfrac{17}{x}$.

   $\therefore x\cos 52° = 17 \qquad \therefore x = \dfrac{17}{\cos 52°}$

   $\therefore x \approx \dfrac{17}{0.6157} \qquad \approx 27.6126$

   $\therefore x \approx 27.6$.

   (iii) As $\theta + 71° = 90°$, $\theta = 90° - 71° = 19°$

   From the diagram, $\tan 71° = \dfrac{x}{10}$.

   $\therefore x = 10\tan 71° \qquad \approx 10 \times 2.9042$

   $\therefore x \approx 29.042 \qquad$ i.e. $x \approx 29.0$.

   (iv) By Pythagoras, $x^2 + 5^2 = 13^2$

   $\therefore x^2 = 13^2 - 5^2 \qquad = 169 - 25 = 144$

$\therefore x = \sqrt{144} = 12$

(the negative square root is not needed, as x is a length).

From the diagram, $\cos\theta = \frac{5}{13} \approx 0.3846$, and θ is the inverse cosine of 0.3846, i.e. $\theta = \cos^{-1}0.3846 \approx 67.4°$.

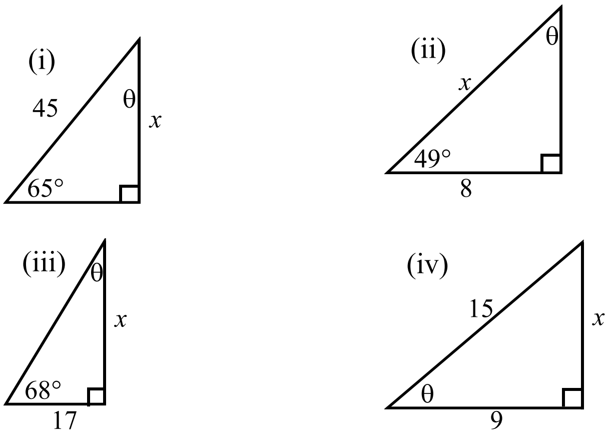
Problems

1. Use a calculator to evaluate (to 4 decimal places)

   (i) $\cos 341°$ (ii) $\sin 111°$

   (iii) $\tan 202°$ (iv) $\sin(-78°)$.

2. In each of the following triangles, find the unknown length, x and angle, θ. Note: round all answers to 1 decimal place.



Answers

1. (i) 0.9455 (ii) 0.9336 (iii) 0.4040 (iv) -0.9781.

2. (i) $\theta = 25°$ and $x = 45\sin 65° \approx 58.9$

   (ii) $\theta = 41°$ and $x = \frac{8}{\cos 49°} \approx 12.2$

   (iii) $\theta = 22°$ and $x = 17\tan 68° \approx 42.1$

   (iv) $x^2 = 15^2 - 9^2 = 144$ $\therefore x = 12$

   and $\cos\theta = \frac{9}{15} = 0.6$ $\therefore \theta \approx 53.1°$.

4.2 Trigonometric identities

1. Many trigonometric identities are derived using the symmetry of the unit circle. Of the symmetry identities, the most important are the following.

   For **any** angle θ,

   $\sin(-\theta) = -\sin\theta$; $\cos(-\theta) = \cos\theta$; $\tan(-\theta) = -\tan\theta$.

2. The **basic trigonometric identity** is

   $$\cos^2\theta + \sin^2\theta = 1$$

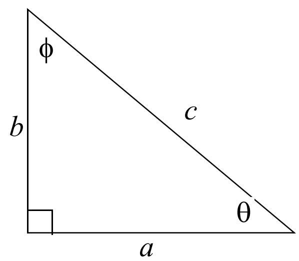
   Note that the value of $\cos^2\theta$ is found by first finding $\cos\theta$, then squaring, i.e. $\cos^2\theta$, pronounced 'cos squared theta', is defined as $(\cos\theta)^2$.

   For instance, $\cos^2 48° = (\cos 48°)^2 \approx (0.6691)^2 \approx 0.4477$.

   Similarly, $\sin^2 48° = (\sin 48°)^2 \approx (0.7431)^2 \approx 0.5523$.

   So, $\cos^2 48° + \sin^2 48° = 0.4477 + 0.5523 = 1$.

3. Another useful set of trigonometric identities can be observed from the right-angled triangle below.



   Since $\theta + \phi = 90°$, the angle ϕ is given by $\phi = 90° - \theta$.

   Using the definitions of $\sin\phi$, $\cos\phi$, and $\tan\phi$,

   $$\sin\phi = \frac{a}{c} = \cos\theta \qquad \therefore \sin(90° - \theta) = \cos\theta$$

   $$\cos\phi = \frac{b}{c} = \sin\theta \qquad \therefore \cos(90° - \theta) = \sin\theta$$

   $$\tan\phi = \frac{a}{b} = \frac{1}{\tan\theta} \qquad \therefore \tan(90° - \theta) = \frac{1}{\tan\theta}$$

Examples

1. Use a calculator to verify that $\cos^2\theta + \sin^2\theta = 1$ for

   (i) $\theta = 289°$ (ii) $\theta = -146°$.

1. (i) For $\theta = 289°$, $\cos^2 289° = (\cos 289°)^2 \approx (0.3256)^2 \approx 0.1060$.

   Similarly, $\sin^2 289° = (\sin 289°)^2 \approx (-0.9455)^2 \approx 0.8940$.

   So, $\cos^2 289° + \sin^2 289° = 0.1060 + 0.8940 = 1$.

   (ii) For $\theta = -146°$,
   $\cos^2(-146°) = [\cos(-146°)]^2 \approx (-0.8290)^2 \approx 0.6873$.
   Similarly,
   $\sin^2(-146°) = [\sin(-146°)]^2 \approx (-0.5592)^2 \approx 0.3127$.

   So, $\cos^2(-146°) + \sin^2(-146°) = 0.6873 + 0.3127 = 1$.

Problems

1. Use a calculator to verify that $\cos^2\theta + \sin^2\theta = 1$ for $\theta = 190°$.

Answers

1. As $\cos 190° \approx -0.9844$ and $\sin 190° \approx -0.1736$,
   $(-0.9844)^2 + (-0.1736)^2 = 0.96985 + 0.03015 = 1$.

4.3 Radian measure

1. Almost all mathematical applications use angles measured in radians, rather than degrees.**In particular, the calculus of trigonometric functions is always performed in radians.**

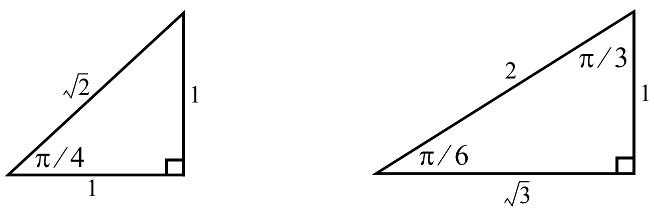
2. To convert angles from degrees to radians, multiply by $\frac{\pi}{180}$.

   To convert angles from radians to degrees, multiply by $\frac{180}{\pi}$.

3. If the units of an angle are not stated, the angle is, by convention, measured in radians. For instance, $\cos 2$ means $\cos 2^R$. Hence

   $\cos 2 = -0.4161$. Note, however, that $\cos 2° = 0.9994$.

4. When angles are measured in radians, the two standard triangles become:



The relevant trigonometric ratios in radians are shown in the table below:

θ	$\frac{\pi}{6}^R$	$\frac{\pi}{4}^R$	$\frac{\pi}{3}^R$
$\sin\theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$
$\cos\theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$
$\tan\theta$	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$

Examples

1. Express in radians in terms of π

   (i) $18°$ (ii) $20°$ (iii) $35°$.

2. Express in degrees

   (i) $\frac{7\pi}{10}$ (ii) $\frac{7\pi}{4}$ (iii) $\frac{2\pi}{15}$.

1. (i) $18° = \frac{18 \times \pi}{180} = \frac{\pi}{10}$ (ii) $20° = \frac{20 \times \pi}{180} = \frac{\pi}{9}$

   (iii) $35° = \frac{35 \times \pi}{180} = \frac{5 \times 7\pi}{5 \times 36} = \frac{7\pi}{36}$.

2. (i) $\frac{7\pi}{10} = \frac{7\pi \times 180°}{10\pi} = 7 \times 18° = 126°$

   (ii) $\frac{7\pi}{4} = \frac{7\pi \times 180°}{4\pi} = 7 \times 45° = 315°$

   (iii) $\frac{2\pi}{15} = \frac{2\pi \times 180°}{15\pi} = 2 \times 12° = 24°$

Problems

1. Express in radians in terms of π

   (i) $54°$ (ii) $100°$ (iii) $135°$.

2. Express in degrees

   (i) $\frac{\pi}{5}$ (ii) $\frac{2\pi}{3}$ (iii) $\frac{7\pi}{9}$.

Answers

1. (i) $\frac{3\pi}{10}$ (ii) $\frac{5\pi}{9}$ (iii) $\frac{3\pi}{4}$.

2. (i) $36°$ (ii) $120°$ (iii) $140°$.

4.4 Trigonometric functions and graphs

1. The graphs of the three major trigonometric functions:

   $y = \sin x$; $y = \cos x$; and $y = \tan x$ are sketched in note 3 below.

2. The graphs of $y = \sin x$ and $y = \cos x$ are similar in many ways.

   Both

   (a) have a wave shape,

   (b) are defined and continuous for all values of x,

   (c) repeat at intervals of 2π,

   (d) are restricted to the interval $-1 \le y \le 1$.

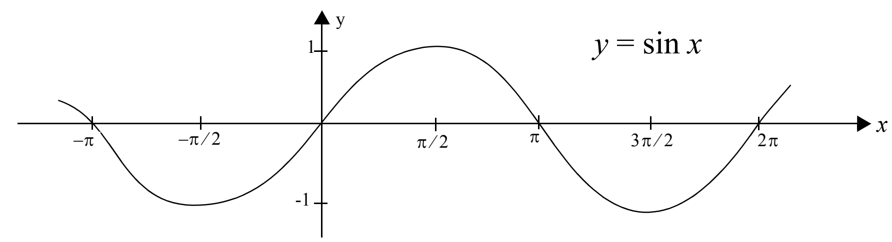
3. The graph of $y = \tan x$

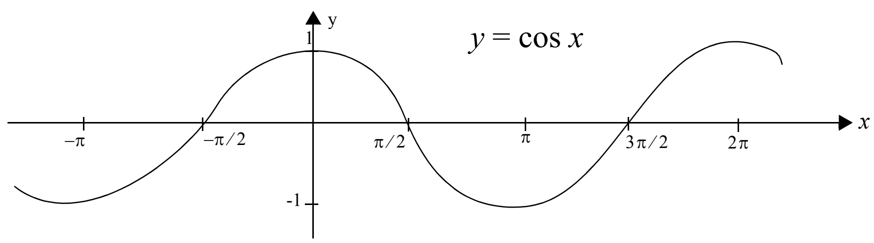
   (a) does not have a wave shape,

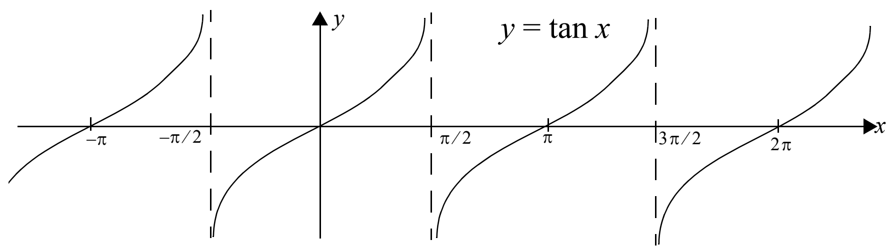
   (b) is defined for all values of x except when x is an odd multiple of $\frac{\pi}{2}$,

   (c) repeats at intervals of π,

   (d) can take on all possible y values, i.e. $-\infty < y < \infty$







Examples

1. Verify that $\sin 2x \neq 2\sin x$ for

   (i) $x = 1.2$ (ii) $x = -0.4$.

1. (i) For $x = 1.2$, $2x = 2.4$

   $\therefore \sin 2x = \sin 2.4 \approx 0.6755$

   Similarly, $2\sin x = 2\sin 1.2 = 2 \times 0.9320 \approx 1.8641$

   (ii) For $x = -0.4$, $2x = -0.8$

   $\therefore \sin 2x = \sin(-0.8) \approx -0.7174$

   Similarly, $2\sin x = 2\sin(-0.4) = 2 \times (-0.3894) \approx -0.7788$.

Problems

1. Verify that $\cos 2x \neq 2\cos x$ for

   (i) $x = 1$ (ii) $x = -0.6$.

Answers

1. (i) For $x = 1$, $2x = 2$

   $\therefore \cos 2x = \cos 2 \approx -0.4161$,

   whereas $2\cos x = 2\cos 1 = 2 \times 0.5403 \approx 1.0806$.

   (ii) For $x = -0.6$, $2x = -1.2$

   $\therefore \cos 2x = \cos(-1.2) \approx 0.3624$,

   whereas $2\cos x = 2\cos(-0.6) = 2 \times 0.8253 \approx 1.6507$.