6

Further rules for derivatives

Introduction

In many applications involving rates of change and the derivative, the relevant mathematical functions are combinations of polynomials, exponentials, logarithms, and trigonometric functions. Further rules are needed to find the derivatives of such combinations of functions. The second derivative of a function leads to a simple and elegant method of classifying stationary points.

This topic covers the 3 rules needed to find the derivative of any product, quotient or chain of functions, the definition and meaning of the second derivative, higher derivatives, and the use of the Second Derivative Test to classify stationary points.After studying this topic, you should be able to:

- distinguish between products of functions and chains of functions;
- understand and use the product rule to find derivatives;
- understand and use the quotient rule to find derivatives;
- understand and use the chain rule to find derivatives;
- find the second derivative of a given function;
- use the Second Derivative Test to classify stationary points.

6.1 Combinations of functions

1. Sums, differences and quotients of functions are easily recognisable, but care must be taken to distinguish between **products of functions** and **functions of functions**. A function of a function is also known as a **chain of functions**.
2. Any trigonometric, exponential or logarithmic function of a function of x is a chain of functions. Some examples are:

   $\sin(2x-1)$; $\cos(x^3)$; e^{4x+3}; $\ln(2x^2+7x-1)$.

   In each case, several operations have been performed on x, and then the trigonometric, exponential or logarithmic function has been calculated. The order of the operations performed on x is important.

   For instance, in $\sin(2x-1)$, x has been multiplied by 2, then 1 has been subtracted. This can be considered as one set of operations on x. After these operations, the sine has been found, and the result is a **chain of functions**.
   The function $(2x-1)\sin x$, however, is a **product** of $(2x-1)$ and $\sin x$.

Examples

1. Identify the form of the following functions

   (i) $\dfrac{\sin x}{x+1}$ (ii) xe^x (iii) $\ln(3x^2+5x-8)$

   (iv) $(x^3-4)^5$ (v) $\cos(x^2-4x+5)$.

1. (i) $\dfrac{\sin x}{x+1}$ is clearly a quotient (the function $\sin x$ being divided by the function $(x+1)$)

(ii) xe^x is a product (the function x being multiplied by the function e^x)

(iii) $\ln(3x^2+5x-8)$ is a function of a function (the first set of operations on x being the function $(3x^2+5x-8)$, followed by the ln function)

(iv) $(x^3-4)^5$ is a function of a function (the first set of operations on x being the function (x^3-4), followed by raising to the power 5)

(v) $\cos(x^2-4x+5)$ is a function of a function (the first set of operations on x being the function (x^2-4x+5), followed by the cosine function).

Problems

1. Identify the form of the following functions:

(i) $x^3\cos x$ (ii) e^{4x^2-7} (iii) $\dfrac{\ln(x+3)}{x^2+1}$

(iv) $(5x^2-4)^3$ (v) $\ln(x^3-4x+8)$.

Answers

1. (i) Product (ii) Function of a function
(iii) Quotient (iv) Function of a function
(v) Function of a function.

6.2 The product rule

1. The **product rule** can be stated as:

if $y = uv$, then $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Note that, as $\dfrac{dy}{dx}$ is a sum, the order of the two terms involved in the sum can be changed.

2. Before using the product rule, it is necessary to ensure that the function has the form of a product. If so, the two functions multiplied together can be labelled as u and v, respectively. After some practice, the product rule can be used without recourse to labelling u and v.

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

(i)	$y = (2x-1)\sin x$	(ii)	$y = (3x^2-6x+1)e^x$
(iii)	$y = (6x+5)\ln x$	(iv)	$y = x^2\cos x$
(v)	$y = x\sin 2x$	(vi)	$y = e^{3x}\cos x$.

1. (i) For $y = (2x-1)\sin x$,

$u = 2x-1$ and $v = \sin x$

$\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = \cos x$.

By the product rule, $\dfrac{dy}{dx} = (2x-1)\cos x + 2\sin x$.

(ii) For $y = (3x^2-6x+1)e^x$,

$u = 3x^2-6x+1$ and $v = e^x$

$\therefore \dfrac{du}{dx} = 6x-6$ and $\dfrac{dv}{dx} = e^x$.

By the product rule, $\dfrac{dy}{dx} = (3x^2-6x+1)e^x + (6x-6)e^x$

Taking out the common factor of e^x, and factorising gives

$$\frac{dy}{dx} = (3x^2-6x+1+6x-6)e^x$$

$$\therefore \frac{dy}{dx} = (3x^2-5)e^x.$$

(iii) For $y = (6x+5)\ln x$,

$u = 6x+5$ and $v = \ln x$

$\therefore \dfrac{du}{dx} = 6$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$.

By the product rule, $\dfrac{dy}{dx} = (6x+5)\times\dfrac{1}{x} + 6\ln x$.

$$\therefore \frac{dy}{dx} = \frac{6x+5}{x} + 6\ln x.$$

(iv) For $y = x^2\cos x$,

$u = x^2$ and $v = \cos x$

$$\therefore \frac{du}{dx} = 2x \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x\,.$$

By the product rule, $\dfrac{dy}{dx} = x^2 \times (-\sin x) + 2x\cos x\,.$

$$\therefore \frac{dy}{dx} = -x^2\sin x + 2x\cos x\,.$$

(v) For $y = x\sin 2x$,

$$u = x \qquad \text{and} \qquad v = \sin 2x$$

$$\therefore \frac{du}{dx} = 1 \qquad \text{and} \qquad \frac{dv}{dx} = 2\cos 2x \quad (k = 2)$$

By the product rule, $\dfrac{dy}{dx} = x \times 2\cos 2x + 1\sin 2x\,.$

$$\therefore \frac{dy}{dx} = 2x\cos 2x + \sin 2x\,.$$

(vi) For $y = e^{3x}\cos x$,

$$u = e^{3x} \qquad \text{and} \qquad v = \cos x$$

$$\therefore \frac{du}{dx} = 3e^{3x} \quad (k = 3) \qquad \text{and} \qquad \frac{dv}{dx} = -\sin x\,.$$

By the product rule, $\dfrac{dy}{dx} = e^{3x} \times (-\sin x) + 3e^{3x}\cos x\,.$

$$\therefore \frac{dy}{dx} = e^{3x}(-\sin x + 3\cos x)\,.$$

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i)	$y = x^3\sin x$	(ii)	$y = (6x^2 - 12x + 5)e^x$
(iii)	$y = (7x-4)\ln x$	(iv)	$y = (2x^2 + 3)\cos x$
(v)	$y = x\cos 4x$	(vi)	$y = e^{-x}\sin x\,.$

Answers

1. (i) $y = x^3\sin x$ $\qquad \therefore \dfrac{dy}{dx} = x^3\cos x + 3x^2\cos x$

(ii) $y = (6x^2 - 12x + 5)e^x$ $\qquad \therefore \dfrac{dy}{dx} = (6x^2 - 7)e^x$

(iii) $y = (7x-4)\ln x$ $\qquad \therefore \dfrac{dy}{dx} = \dfrac{7x-4}{x} + 7\ln x$

(iv) $y = (2x^2+3)\cos x \quad \therefore \dfrac{dy}{dx} = -(2x^2+3)\sin x + 4x\cos x$

(v) $y = x\cos 4x \quad \therefore \dfrac{dy}{dx} = -4x\sin 4x + \cos 4x$

(vi) $y = e^{-x}\sin x \quad \therefore \dfrac{dy}{dx} = e^{-x}(\cos x - \sin x)$.

6.3 The quotient rule

1. The **quotient rule** can be stated as:

   if $y = \dfrac{u}{v}$, then $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

   Note that the order of the two terms involved in the top line cannot be changed.

2. When using the quotient rule, the functions on the top and bottom lines can be labelled as u and v, respectively. After some practice, the quotient rule can be used without recourse to labelling u and v.

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = \dfrac{2x-3}{2x+7}$ (ii) $y = \dfrac{e^{4x}}{x+6}$

(iii) $y = \dfrac{x^2-4}{2x^2+1}$ (iv) $y = \dfrac{\ln x}{x^2}$

(v) $y = \dfrac{\sin 2x}{x}$ (vi) $y = \dfrac{\sin x}{\cos x}$.

1. (i) For $y = \dfrac{2x-3}{2x+7}$,

$u = 2x-3$ and $v = 2x+7$

$\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{2(2x+7)-2(2x-3)}{(2x+7)^2}$.

$$\therefore \frac{dy}{dx} = \frac{4x+14-4x+6}{(2x+7)^2}$$

$$\therefore \frac{dy}{dx} = \frac{20}{(2x+7)^2}.$$

(ii) For $y = \dfrac{e^{4x}}{x+6}$,

$u = e^{4x}$ and $v = x + 6$

$\therefore \dfrac{du}{dx} = 4e^{4x} \quad (k = 4)$ and $\dfrac{dv}{dx} = 1$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{(x+6)(4e^{4x}) - 1(e^{4x})}{(x+6)^2}$.

$$\therefore \frac{dy}{dx} = \frac{[4(x+6)-1]e^{4x}}{(x+6)^2} =$$

$$\therefore \frac{dy}{dx} = \frac{[4x+24-1]e^{4x}}{(x+6)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(4x+23)e^{4x}}{(x+6)^2}.$$

(iii) For $y = \dfrac{x^2-4}{2x^2+1}$,

$u = x^2 - 4$ and $v = 2x^2 + 1$

$\therefore \dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = 4x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{2x(2x^2+1) - 4x(x^2-4)}{(2x^2+1)^2}$

$$\therefore \frac{dy}{dx} = \frac{4x^3 + 2x - 4x^3 + 16x}{(2x^2+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{18x}{(2x^2+1)^2}.$$

(iv) For $y = \dfrac{\ln x}{x^2}$,

$u = \ln x$ and $v = x^2$

$\therefore \dfrac{du}{dx} = \dfrac{1}{x}$ and $\dfrac{dv}{dx} = 2x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{x^2 \times \frac{1}{x} - 2x\ln x}{(x^2)^2}$

$$\therefore \frac{dy}{dx} = \frac{x - 2x\ln x}{x^4} = \frac{x(1 - 2\ln x)}{x^4}$$

$$\therefore \frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}.$$

(v) For $y = \dfrac{\sin 2x}{x}$,

$u = \sin 2x$ and $v = x$

$\therefore \dfrac{du}{dx} = 2\cos 2x \quad (k = 2)$ and $\dfrac{dv}{dx} = 1$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{x(2\cos 2x) - 1\sin 2x}{x^2}$.

$$\therefore \frac{dy}{dx} = \frac{2x\cos 2x - \sin 2x}{x^2}.$$

(vi) For $y = \dfrac{\sin x}{\cos x}$,

$u = \sin x$ and $v = \cos x$

$\therefore \dfrac{du}{dx} = \cos x$ and $\dfrac{dv}{dx} = -\sin x$.

By the quotient rule, $\dfrac{dy}{dx} = \dfrac{\cos x \times \cos x - \sin x \times (-\sin x)}{(\cos x)^2}$.

$$\therefore \frac{dy}{dx} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

By the basic trigonometric identity, $\therefore \dfrac{dy}{dx} = \dfrac{1}{\cos^2 x}$.

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = \dfrac{3x - 7}{7x - 2}$ (ii) $y = \dfrac{e^{2x}}{4x + 1}$

(iii) $y = \dfrac{2x^2 + 3}{x^2 + 1}$ (iv) $y = \dfrac{\ln x}{2x^3}$

(v) $y = \dfrac{\sin 5x}{x}$ (vi) $y = \dfrac{\cos x}{\sin x}$.

Answers

1. (i) $y = \dfrac{3x-7}{7x-2}$ $\therefore \dfrac{dy}{dx} = \dfrac{43}{(7x-2)^2}$

(ii) $y = \dfrac{e^{2x}}{4x+1}$ $\therefore \dfrac{dy}{dx} = \dfrac{(8x-2)e^{2x}}{(4x+1)^2}$

(iii) $y = \dfrac{2x^2+3}{x^2+1}$ $\therefore \dfrac{dy}{dx} = \dfrac{-2x}{(x^2+1)^2}$

(iv) $y = \dfrac{\ln x}{2x^3}$ $\therefore \dfrac{dy}{dx} = \dfrac{1-3\ln x}{2x^4}$

(v) $y = \dfrac{\sin 5x}{x}$ $\therefore \dfrac{dy}{dx} = \dfrac{5x\cos 5x - \sin 5x}{x^2}$

(vi) $y = \dfrac{\cos x}{\sin x}$ $\therefore \dfrac{dy}{dx} = \dfrac{-1}{\sin^2 x}$.

6.4 The chain rule

1. The **chain rule** (or function of a function rule) is usually stated as:
   if $y = f(u)$, and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.
   Note that the function *u* is not normally given, but is recognisable as the first set of operations performed on *x*.
2. In most cases, u is either a power of e, or appears in brackets, or appears under a square root sign. For instance, $u = 3x^2 - 1$ in each of:

   $y = e^{3x^2-1}$; $y = \cos(3x^2-1)$; $y = \ln(3x^2-1)$; $y = \sqrt{3x^2-1}$.
3. Note that it is not necessary to use the chain rule to find $\dfrac{dy}{dx}$ when

   $y = \sin 3x$. Using $k = 3$, the derivative is given by $\dfrac{dy}{dx} = 3\cos 3x$.
4. Combinations of the product, quotient and chain rules are needed to find the derivatives of some functions (see Example 2 below).

Examples

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = (3x^2-8)^4$ (ii) $y = e^{4x^5}$

(iii) $y = \ln(5x^4 - 3x^2 - 1)$ (iv) $y = \sqrt{x^2+7}$

(v) $y = \sin(8x^3-5)$ (vi) $y = \cos^3 x$.

2. Find $\frac{dy}{dx}$ for each of the following

(i) $y = x\ln(x^2+1)$ (ii) $y = \sin(xe^x)$.

1. (i) For $y = (3x^2-8)^4$,

let $u = 3x^2 - 8$. $\therefore y = u^4$

$\therefore \frac{du}{dx} = 6x$, and $\frac{dy}{du} = 4u^3$.

By the chain rule, $\frac{dy}{dx} = (6x)\times(4u^3) = 24xu^3$.

As $u = 3x^2-8$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = 24x(3x^2-8)^3.$$

(ii) For $y = e^{4x^5}$,

let $u = 4x^5$. $\therefore y = e^u$

$\therefore \frac{du}{dx} = 20x^4$, and $\frac{dy}{du} = e^u$.

By the chain rule, $\frac{dy}{dx} = 20x^4 \times e^u = 20x^4e^u$.

As $u = 4x^5$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = 20x^4e^{4x^5}.$$

(iii) For $y = \ln(5x^4-3x^2-1)$

let $u = 5x^4-3x^2-1$. $\therefore y = \ln u$

$\therefore \frac{du}{dx} = 20x^3-6x$, and $\frac{dy}{du} = \frac{1}{u}$.

By the chain rule, $\frac{dy}{dx} = (20x^3-6x)\times\frac{1}{u} = \frac{20x^3-6x}{u}$.

As $u = 5x^4-3x^2-1$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = \frac{20x^3-6x}{5x^4-3x^2-1}.$$

(iv) For $y = \sqrt{x^2+7}$, i.e. $y = (x^2+7)^{1/2}$

let $u = x^2+7$. $\therefore y = u^{1/2}$

$\therefore \frac{du}{dx} = 2x$, and $\frac{dy}{du} = \frac{1}{2}u^{-1/2}$.

By the chain rule, $\frac{dy}{dx} = 2x\times\frac{1}{2}u^{-1/2} = xu^{-1/2} = \frac{x}{u^{1/2}}$.

As $u = x^2 + 7$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = \frac{x}{(x^2+7)^{1/2}} = \frac{x}{\sqrt{x^2+7}}.$$

(v) For $y = \sin(8x^3 - 5)$,

let $u = 8x^3 - 5$. $\therefore y = \sin u$

$\therefore \frac{du}{dx} = 24x^2$, and $\frac{dy}{du} = \cos u$.

By the chain rule, $\frac{dy}{dx} = 24x^2 \cos u$.

As $u = 8x^3 - 5$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = 24x^2 \cos(8x^3 - 5).$$

(vi) For $y = \cos^3 x$, i.e. $y = (\cos x)^3$,

let $u = \cos x$. $\therefore y = u^3$

$\therefore \frac{du}{dx} = -\sin x$, and $\frac{dy}{du} = 3u^2$

By the chain rule, $\frac{dy}{dx} = -3u^2 \sin x$.

As $u = \cos x$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = -3\cos^2 x \sin x.$$

2. (i) $y = x\ln(x^2 + 1)$ is a product, so the product rule must be used.

$\therefore u = x$ and $v = \ln(x^2 + 1)$

$\therefore \frac{du}{dx} = 1$ and the chain rule is needed to find $\frac{dv}{dx}$.

Let $w = x^2 + 1$ (as u has been used earlier) $\therefore v = \ln w$

$\therefore \frac{dw}{dx} = 2x$ and $\frac{dv}{dw} = \frac{1}{w}$

By the chain rule $\frac{dv}{dx} = \frac{dv}{dw} \times \frac{dw}{dx} = (2x) \times \frac{1}{w} = \frac{2x}{w} = \frac{2x}{x^2+1}$

i.e. $u = x$ and $v = \ln(x^2 + 1)$

$$\therefore \frac{du}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = \frac{2x}{x^2+1}$$

By the product rule, $\dfrac{dy}{dx} = x \times \dfrac{2x}{x^2+1} + 1\ln(x^2+1)$.

$$\therefore \frac{dy}{dx} = \frac{2x^2}{x^2+1} + \ln(x^2+1).$$

(ii) $y = \sin(xe^x)$ is a function of a function, so the chain rule must be used.

let $u = xe^x$ $\qquad \therefore y = \sin u$

$\therefore \dfrac{du}{dx}$ is found using the product rule, and $\dfrac{dy}{du} = \cos u$.

To find $\dfrac{du}{dx}$,

let $w = x$ (as u has been used), and $v = e^x$.

$$\therefore \frac{dw}{dx} = 1 \quad \text{and} \quad \frac{dv}{dx} = e^x.$$

Now, $\dfrac{du}{dx} = w\dfrac{dv}{dx} + v\dfrac{dw}{dx}$

$$\therefore \frac{du}{dx} = xe^x + e^x = (x+1)e^x.$$

i.e. $\quad u = xe^x \qquad y = \sin u$

$$\therefore \frac{du}{dx} = (x+1)e^x \quad \text{and} \quad \frac{dy}{du} = \cos u.$$

By the chain rule, $\dfrac{dy}{dx} = (x+1)e^x\cos u$.

As $u = xe^x$, expressing the final answer in terms of x only,

$$\frac{dy}{dx} = (x+1)e^x\cos(xe^x).$$

Problems

1. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = (2x^3-5)^8$ $\qquad$ (ii) $y = e^{4\sin x}$

(iii) $y = \ln(3x^3-5x^2-1)$ $\qquad$ (iv) $y = \sqrt{2x^3+9}$

(v) $y = \cos(4x^2-x+3)$ $\qquad$ (vi) $y = \sin^3 x$.

2. Find $\dfrac{dy}{dx}$ for each of the following

(i) $y = xe^{x^2+3}$ (ii) $y = (4 + x\ln x)^5$.

Answers

1. (i) $y = (2x^3 - 5)^8$ $\therefore \dfrac{dy}{dx} = 48x^2(2x^3 - 5)^7$

(ii) $y = e^{4\sin x}$ $\therefore \dfrac{dy}{dx} = 4\cos x\, e^{4\sin x}$

(iii) $y = \ln(3x^3 - 5x^2 - 1)$ $\therefore \dfrac{dy}{dx} = \dfrac{9x^2 - 10x}{3x^3 - 5x^2 - 1}$

(iv) $y = \sqrt{2x^3 + 9}$ $\therefore \dfrac{dy}{dx} = \dfrac{3x^2}{\sqrt{2x^3 + 9}}$

(v) $y = \cos(4x^2 - x + 3)$ $\therefore \dfrac{dy}{dx} = -(8x - 1)\sin(4x^2 - x + 3)$

(vi) $y = \sin^3 x$ $\therefore \dfrac{dy}{dx} = 3\sin^2 x \cos x$.

2. (i) $y = xe^{x^2+3}$ $\therefore \dfrac{dy}{dx} = (2x^2 + 1)e^{x^2+3}$

(ii) $y = (4 + x\ln x)^5$ $\therefore \dfrac{dy}{dx} = 5(1 + \ln x)(4 + x\ln x)^4$.

6.5 Higher derivatives

1. Given the function $y = f(x)$, the second derivative is the derivative of the derivative, and is generally denoted by $\dfrac{d^2y}{dx^2}$ or $f''(x)$ or y''.

   Successive differentiations produce the third, and fourth derivatives, and the process can be continued indefinitely. All derivatives beyond the first derivative are called higher derivatives.

2. For motion in a straight line, the first derivative represents velocity, and the second derivative represents acceleration. Physical meanings can be attached to other derivatives. In particular, the fourth derivative is used in the equations describing the bending of beams.

3. The Second Derivative Test can be used to determine local maxima and minima, and is stated below.

   Given the function $y = f(x)$,

   if $\dfrac{dy}{dx} = 0$ when $x = c$, then $x = c$ provides:

(i) a local maximum if $\frac{d^2y}{dx^2} < 0$;

(ii) a local minimum if $\frac{d^2y}{dx^2} > 0$;

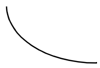
(iii) no conclusion if $\frac{d^2y}{dx^2} = 0$ (and the First Derivative test must be used instead).

4. The sign (positive or negative) of the second derivative $\frac{d^2y}{dx^2}$ determines the concavity (curvature) of the curve $y = f(x)$, as follows:

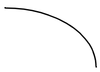
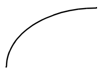
   if $\frac{d^2y}{dx^2} > 0$ at a point, the curve is concave up at that point;

   if $\frac{d^2y}{dx^2} < 0$ at a point, the curve is concave down at that point

   A **concave up** curve is **cup-shaped**

   i.e.  and  are both concave up curves.

   A **concave down** curve is **frown-shaped**

   i.e.  and  are both concave down curves.

   For the parabola $y = ax^2 + bx + c$,

   $$\frac{dy}{dx} = 2ax + b \qquad \text{and} \qquad \frac{d^2y}{dx^2} = 2a.$$

   So, if $a > 0$, $\frac{d^2y}{dx^2} = 2a > 0$, and the parabola is concave up.

   Also, if $a < 0$, $\frac{d^2y}{dx^2} = 2a < 0$, and the parabola is concave down.

   These results agree with those used in sketching parabolas in Chapter 2 of this Study Guide.

5. A point of inflection occurs when the concavity changes from up to down (or down to up). Hence, a point of inflection occurs at $x = c$ if $\frac{d^2y}{dx^2} = 0$ at $x = c$, and $\frac{d^2y}{dx^2}$ has different signs either side of $x = c$.

Examples

1. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ for each of the following:

(i) $y = 2x^3 - 4x^2 - 5x + 9$ (ii) $y = (x-2)e^{2x}$

(iii) $y = x\ln x$ (iv) $y = 8\sqrt{x} - \cos 3x$.

2. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, or a local minimum using the Second Derivative Test.

(i) $y = 3x^3 - 9x^2 + 1$; (ii) $y = x^4 - 8x^2 + 10$

1. (i) $y = 2x^3 - 4x^2 - 5x + 9$

$$\therefore \frac{dy}{dx} = 6x^2 - 8x - 5$$

$$\therefore \frac{d^2y}{dx^2} = 12x - 8 .$$

(ii) $y = (x-2)e^{2x}$ (a product)

let $u = x - 2$ and $v = e^{2x}$

$\therefore \dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = 2e^{2x}$.

By the product rule, $\dfrac{dy}{dx} = (x-2) \times (2e^{2x}) + 1e^{2x} = (2x - 4 + 1)e^{2x}$.

$\therefore \dfrac{dy}{dx} = (2x-3)e^{2x}$ (a product)

let $u = 2x - 3$ and $v = e^{2x}$

$\therefore \dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2e^{2x}$.

By the product rule,

$$\frac{d^2y}{dx^2} = (2x-3) \times (2e^{2x}) + 2e^{2x} = (4x - 6 + 2)e^{2x} .$$

$$\therefore \frac{d^2y}{dx^2} = (4x-4)e^{2x} .$$

(iii) $y = x\ln x$ (a product)

let $u = x$ and $v = \ln x$

$\therefore \dfrac{du}{dx} = 1$ and $\dfrac{dv}{dx} = \dfrac{1}{x}$.

By the product rule, $\dfrac{dy}{dx} = x \times \dfrac{1}{x} + 1\ln x$.

$$\therefore \frac{dy}{dx} = 1 + \ln x$$

$$\therefore \frac{d^2y}{dx^2} = \frac{1}{x}.$$

(iv) $y = 8\sqrt{x} - \cos 3x$ $\therefore y = 8x^{1/2} - \cos 3x$

$$\therefore \frac{dy}{dx} = 8 \times \frac{1}{2}x^{-1/2} - (-3\sin 3x)$$

$$\therefore \frac{dy}{dx} = 4x^{-1/2} + 3\sin 3x$$

$$\therefore \frac{d^2y}{dx^2} = 4 \times \left(\frac{-1}{2}x^{-3/2}\right) + 3(3\cos 3x)$$

$$\therefore \frac{d^2y}{dx^2} = -2x^{-3/2} + 9\cos 3x$$

2. (i) $y = 3x^3 - 9x^2 + 1$

$$\therefore \frac{dy}{dx} = 9x^2 - 18x$$

For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

$x^2 - 2x = 0$. Factorising gives

$x^2 - 2x = x(x-2) = 0$

$\therefore x = 0$, and $x = 2$ (2 stationary points)

When $x = 0$, $y = 0 - 0 + 1 = 1$

So, one stationary point is $(0, 1)$.

When $x = 2$, $y = 3 \times 8 - 9 \times 4 + 1 = 24 - 36 + 1 = -11$

So, the other stationary point is $(2, -11)$.

Now, $\dfrac{d^2y}{dx^2} = 18x - 18 = 18(x-1)$.

For the stationary point $(0, 1)$, i.e. when $x = 0$,

$\dfrac{d^2y}{dx^2} = 0 - 18 = -18 < 0$, so $(0, 1)$ is a local maximum.

For the stationary point $(2, -11)$, i.e. when $x = 2$,

$$\frac{d^2y}{dx^2} = 36 - 18 = 18 > 0\text{, so } (2, -11) \text{ is a local minimum.}$$

(ii) $y = x^4 - 8x^2 + 10$

$$\therefore \frac{dy}{dx} = 4x^3 - 16x$$

For stationary points, $4x^3 - 16x = 0$.

Factorising gives $4x^3 - 16x = 4x(x^2 - 4) = 4x(x+2)(x-2) = 0$.

$\therefore x = 0$, $x = -2$, and $x = 2$ (3 stationary points)

When $x = 0$, $y = 0 - 0 + 10 = 10$

So, one stationary point is $(0, 10)$.

When $x = -2$, $y = (-2)^4 - 8 \times (-2)^2 + 10 = 16 - 32 + 10 = -6$

So, the second stationary point is $(-2, -6)$.

When $x = 2$, $y = 2^4 - 8 \times 2^2 + 10 = 16 - 32 + 10 = -6$.

So, the third stationary point is $(2, -6)$.

Now, $\dfrac{d^2y}{dx^2} = 12x^2 - 16$.

For the stationary point $(0, 10)$, i.e. when $x = 0$,

$$\frac{d^2y}{dx^2} = 0 - 16 = -16 < 0,$$

so $(0, 10)$ is a local maximum.

For the stationary point $(-2, -6)$, i.e. when $x = -2$,

$$\frac{d^2y}{dx^2} = 12 \times (-2)^2 - 16 = 48 - 16 = 32 > 0,$$

so $(2, -11)$ is a local minimum.

For the stationary point $(2, -6)$, i.e. when $x = 2$,

$$\frac{d^2y}{dx^2} = 12 \times 2^2 - 16 = 48 - 16 = 32 > 0,$$

so $(2, -11)$ is a local minimum.

Problems

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following

   (i) $y = x^4 - 4x^3 - 7x + 7$ (ii) $y = (4x-3)e^x$

   (iii) $y = x\sin x$ (iv) $y = 16x^{3/4} - 4\ln x$.

2. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, or a local minimum using the Second Derivative Test.

   (i) $y = 2x^3 + 9x^2 - 24x$ (ii) $y = x^3 + 3x^2 - 4$.

Answers

1. (i) $\frac{dy}{dx} = 4x^3 - 12x^2 - 7$ and $\frac{d^2y}{dx^2} = 12x^2 - 24x$

   (ii) $\frac{dy}{dx} = (4x+1)e^x$ and $\frac{d^2y}{dx^2} = (4x+5)e^x$

   (iii) $\frac{dy}{dx} = x\cos x + \sin x$ and $\frac{d^2y}{dx^2} = 2\cos x - x\sin x$

   (iv) $\frac{dy}{dx} = 12x^{-1/4} - \frac{4}{x}$ and $\frac{d^2y}{dx^2} = -3x^{-5/4} + \frac{4}{x^2}$

2. (i) For $y = 2x^3 + 9x^2 - 24x$,

   $\frac{dy}{dx} = 6x^2 + 18x - 24$, and $\frac{d^2y}{dx^2} = 12x + 18$.

   The stationary points are: $(1, -13)$ (a local minimum), and $(-4, 112)$ (a local maximum).

   (ii) For $y = x^3 + 3x^2 - 4$,

   $\frac{dy}{dx} = 3x^2 + 6x$, and $\frac{d^2y}{dx^2} = 6x + 6$.

   The stationary points are: $(0, -4)$ (a local minimum), and $(-2, 0)$ (a local maximum).