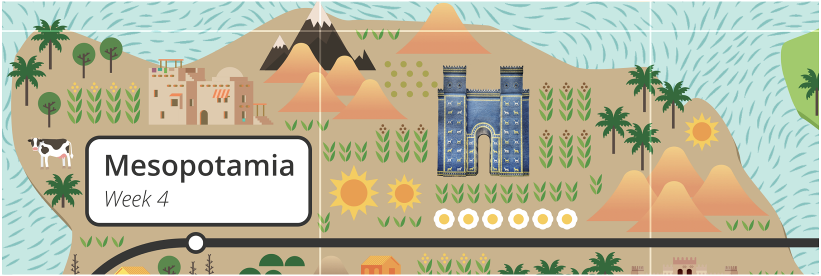
SIT190 - WEEK 4



Overview

- Quadratics
- Factorising
- Solving
- Some special quadratics
- Completing the square
- Quadratic formula
- Sketching Quadratics
- Cubics

Arriving at Mesopotamia

Your band arrives in ancient Mesopotamia (Greek: ‘between rivers’) the region between the Euphrates and Tigris rivers.

- Mesopotamia lay in the fertile crescent and the presence of rich land and plentiful water encouraged agriculture including the development of irrigation and aqueducts.
- The Sumerians had established a number of walled cities by 4000BCE including Uruk, Eridu, Ur, Nippur, Lagash and Kish.

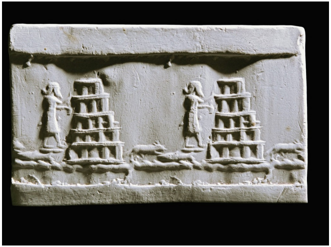


Image: (2nd-1st millennium BCE). Ziggurat and a priest or god. [cylinder seal imprint]. Retrieved from `https://library-artstor-org.ezproxy-b.deakin.edu.au/asset/LESSING_ART_10311440988`

Arriving at Mesopotamia

The Sumerians:

- invented the wheel and the plow,
- were beer brewers (there was a Sumerian goddess, Ninkasi, of brewing),
- were traders with links across sea and land,
- invented cuneiform writing.

Cuneiform Tablets

Cuneiform symbols were pressed into clay tablets using a stylus. The tablets were then baked in the sun or an oven (or possibly in some cases squashed down and used again).

- Archaeologists have found over 500,000 clay tablets and many more are thought to lie buried in the ruins of ancient civilisation.
- 50,000 clay tablets discovered in Ancient Nippur
- Mathematical records mainly from two periods:
  - Early 2nd Millennium BCE (Old Babylonian Age)
  - Late 1st Millennium BCE (Seleucid Period)
- Two types:
  - Table-texts: multiplication tables, weights and measures, reciprocals, etc.
  - Problem texts: Solutions or methods of solution to algebraic and other mathematical problesm

Plimpton 322

One famous tablet is Plimpton 322 from the Old Babylonian period. It was purchased by Plimpton in 1923 from Edgar Banks who claimed it was from a location near Larsa (Tell Senkereh) in Iraq.

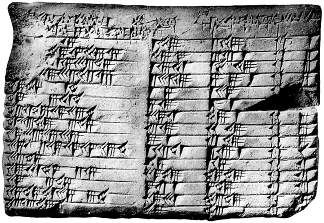
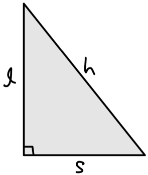


Image: The Babylonian tablet Plimpton 322 by Bill Casselman

Plimpton 322

- Approximately 13cm $\times$ 9cm and 2cm thick.
- Part of original - evidence of modern glue on broken side.
- Four columns on this piece (read right to left):
  - Row number (rows 1-15)
  - Number h
  - Number s
  - (Possibly) Number $(\frac{h}{l})^2$



Right-angled triangle with hypothesis h, shorter side s and longer side l.

Plimpton 322

- First row: (1) 59 00 15, 1 59, 2 49 , 1 (Sexagesimal notation)
- First row: 1.98340..., 119, 169, 1 (Decimal notation)
- Triangle with sides 119, 120, 169 has $(\frac{169}{120})^2 \approx 1.98349\ldots$

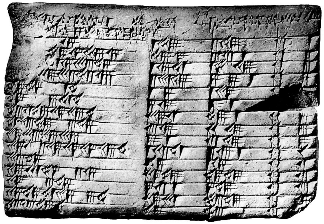


Image: The Babylonian tablet Plimpton 322 by Bill Casselman

Plimpton 322

- Tablet contains a list of integral Pythagorean triples in order of angles of incline (44.76^o to 31.28^o)
- About 1000 years before Pythagoras!
- Some guesses about the columns on the missing part.

The challenge is for your team to locate the missing fragment of the Plimpton 322 Tablet.

Gilgamesh

One of the earliest works of literature is the *Epic of Gilgamesh* and consists of poems about Gilgamesh, King of Uruk. In this leg of the quest, we will travel with Gilgamesh and his companion Enkidu to the Cedar Forest. They plan to slay Humbaba the terrible, and have agreed to help us find the tablet if we accompany them on their journey.

Image: (1st half 2nd millennium BCE). Gilgamesh and Enkidu Killing Humbaba, Guardian of the Cedar Forest [Huwawa, der Hüter des Zedernwaldes, wird durch Gilgamesch und Enkidu getötet]. [relief].

Before we begin this journey, we briefly revise quadratics.

Quadratics

A quadratic is a mathematical expression of the form

$$ax^2 + bx + c$$

where a, b, c are real numbers. A quadratic polynomial has degree 2. Here are some examples:

$$x^2 - 4x + 3$$
$$x^2 - 4$$
$$3 - 4x^2$$
$$x^2$$

Note: They all have the term with x^2 and there are no terms with a higher power. The coefficient of this term can be positive or negative, but not zero.

Quadratics

So if a quadratic is a polynomial of degree 2, why is it called a 'quadratic'?

The word *quadratic* comes from the Latin 'quadratus' meaning to make square. Many mathematics problems originally arose from practical problems.

- What is the area of the square with sides of length x?
- What is the area of the square with side of length $x + a$?
- What is the area of the square with sides of length $x - a$?
- What is the area of the rectangle with sides $x - a$ and $x + a$?

Problems of this type produce mathematical expressions involving quadratics. It is not surprising that the ancient Babylonians were interested in solving problems of this type (although because of the practical nature they focused on solutions with positive numbers).

Quadratics from practical problems

One practical problem is to find the lengths of the sides, x and y, of a rectangle with a given area and perimeter.
For example, if a farmer wants to fence an area of 20 square metres and has 18 metres of rope:

$$\text{Area: } xy = 20$$
$$\text{Perimeter: } 2(x+y) = 18 \Rightarrow x + y = 9.$$

Rearranging the first equation we get $y = \frac{20}{x}$ which we substitute into the second equation giving the quadratic equation:

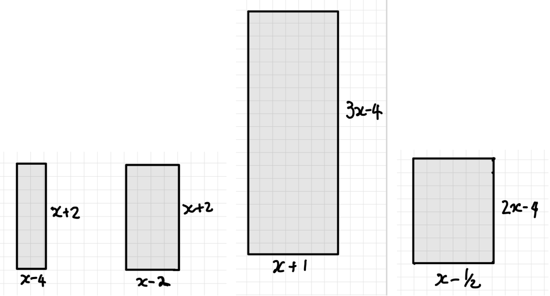
$$x + \frac{20}{x} = 9 \Rightarrow x^2 + 20 = 9x \Rightarrow x^2 - 9x + 20 = 0.$$

Advice for the journey

Before Gilgamesh, Enkidu and your band begins the journey, they must first visit his mother Ninsun for advice for the journey. You wish to hear her advice, but cannot enter by the King's entrance. There are four doors in the passage and your team must identify the correct door to open.

Which door

In order to identify the correct door to enter you must find the one with area $2x^2 - 5x + 2$. The area is the width $\times$ the height of the door. For example, the first door has area $(x-4) \times (x+2)$ sq units.



Four doors - only one is correct

The first band to identify the correct door earns 10 points. The second band earns 5 points.

Reflection

Factorisation of quadratics:

- Historically, many problems with quadratics have arisen from real life problems such as measuring areas of rectangles (e.g. surveying and engineering).
- If $(x+m)(x+n) = x^2 + bx + c$, express b and c in terms of m and n.
- This gives us a method for factorising a quadratic $x^2 + bx + c$
  - Consider all possible m, n, such that $m \times n = c$ and
  - Consider all possibly m, n, such that $m + n = b$.
  - Is there a m and n that satisfies both these requirements?
- Try to use this method to factorise the following expressions:
  - $x^2 - 4x - 5$
  - $x^2 + 5x + 6$

Reflection

Factorisation of quadratics is an important step in solving quadratic equations.

1. Rearrange the equation so that one side is 0.
2. Factorise the quadratic on the other side.
3. The quadratic is zero, if at least one of the factors is zero, so each factor separably for x.

Solve the following expressions for x:

1. $x^2 - 7x - 44 = 0$
2. $x^2 + 8x - 13 = 7$

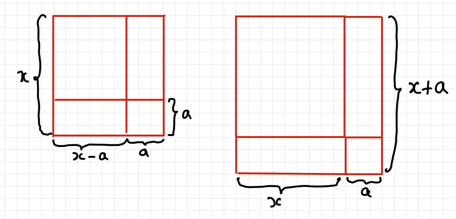
Advice for the journey

Ninsun wishes Gilgamesh well on his journey. She asks Enkidu to protect him on the journey, placing her amulet around his neck. Your band is intrigued by the amulet which is a curious design - a square with a smaller square in one corner. The amulet is purported to have magic powers, but you realise that it is in fact it is something better than magic ...

Magic or Mathematics

You recognise that the amulet represents some common quadratic factorisations:

$$(x-a)^2 = x^2 - 2ax + a^2 \text{ and } (x+a)^2 = x^2 + 2ax + a^2$$



Can you map the areas of the 2 squares and 2 rectangles that make up each larger square to the terms in the expanded quadratic?

Magic or Mathematics

Your band wishes to stay close to Gilgamesh and Enkidu as the journey may be dangerous. They agree to take you if you beat Enkidu in a mathematical duel. The winner is the one who can factorise the following quadratics. Fortunately, you can use the expressions on the previous slide, and Enkidu does not realise the power in his amulet.

Yes, people did duel to win supremacy in solving polynomial equations. For example, Tartaglia had a duel with Fior on cubic polynomials see *https://euro-math-soc.eu/review/secret-formula*

Magic or Mathematics

The duel requires your band to factorise the following quadratics:

$$x^2 - 6x + 9$$
$$x^2 - 10x + 25$$
$$4x^2 + 32x + 64$$
$$x^2 + x + \frac{1}{4}$$

The first band to successfully factorise these quadratics earns their team 10 points.

Reflection

Before your band commences on the journey. We note that we have some new items that you could add to your treasure chest. These will help you factorise quadratics.

- Step by step method for factorising quadratics of the form $x^2 + bx + c = (x + m)(x + n)$
- Formulas $(x \pm a)^2 = x^2 \pm 2ax + a^2$

Factorising is the first step to use when solving most quadratic equations.

Completing the square

- Unfortunately, not every quadratic can be expressed in the form $(x \pm a)^2$.
- The technique of completing the square can be used to express any quadratic as a square term plus or minus a constant term.
- The steps are as follows:
  1. If the coefficient of x^2 is not 1, then divide by this coefficient, and complete the square of the resulting quadratic.
     eg. $2x^2 - 4x + 8 = 2(x^2 - 2x + 2)$
  2. Identify the coefficient b of the x term eg. $x^2 - 2x + 2$ has $b = -2$.
  3. Adding $+\frac{b^2}{4} - \frac{b^2}{4}$ to the quadratic. eg
     $2(x^2 - 2x + 2) = 2(x^2 - 2x + 2 + 1 - 1)$
  4. Completing the square, i.e. $x^2 + bx + c = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$.
     eg $2(x^2 - 2x + 2 + 1 - 1) = 2((x - 1)^2 + 1)$

Completing the square

Complete the square of the following quadratics:

1. $x^2 - 4x + 8$
2. $x^2 + 3x + 2$
3. $2x^2 + 4x - 6$
4. $x^2 - 25$
5. $x^2 - 4x + 4$

10 points for the first band that successfully completes this activity.

Reflection

- Which of the quadratics was the easiest to complete the square?
- When is the non-squared part positive? and when is it negative?
- The Ancient Babylonians were interested in cases where solutions of $ax^2 + bx + c = 0$ were positive values of x.
- Find the solutions of the following equations by first completing the square:
  - $x^2 - 25 = 0$.
  - $x^2 - 4x + 8 = 0$
  - $x^2 + 3x + 2 = 0$
  - $2x^2 + 4x - 6 = 0$
  - $x^2 - 4x + 4 = 0$
- Explain why some of these quadratics have no real solutions.

Bablyonians and quadratics

Babylonians mainly considered quadratics of the form:

$$x^2 + bx = c \text{ and } x^2 - bx = c$$

where b and c were positive numbers (but not necessarily integers).

They found the positive root of $x^2 + bx = c$ using
$x = \sqrt{(\frac{b}{2})^2 + c} - \frac{b}{2}$,
and
the positive root of $x^2 - bx = c$ using $x = \sqrt{(\frac{b}{2})^2 + c} + \frac{b}{2}$.
These calculations were essentially the same as the formula we use today for finding roots, namely,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Bablyonians and quadratics

There was a third form used by the Babylonians:

$$x^2 + c = bx$$

which arose from simultaneous equations: $xy = c$ and $x + y = b$.
The Babylonians did not have a calculator, but they did have cuneiform tablets containing mathematical tables for finding squares and cubes, and reciprocals.
Multiplying by reciprocals was used instead of long division. For example, to solve the problem

$$\frac{5}{9}x = 100,$$

you would look up the reciprocal of $\frac{5}{9}$ and then multiply both sides by the reciprocal,

$$x = \frac{1}{5/9} \times 100$$

All that was necessary was a table of reciprocals. We still have their reciprocal tables going up to the reciprocals of numbers up to several billion.

Band Activity

Randomly pick 3 cards from a pack of cards (Ace is 1, Jack is 10, Queen is 11 and King is 12). Use these cards to form a quadratic $ax^2 + bx + c$ as follows:

- The first card gives you the value of a. If the card is red negate the value.
- The second card gives you the value of b If the card is red negate the value.
- The third card gives you the value of c If the card is red negate the value.

For example: 2 of Hearts, 3 of Diamonds and 10 of Spades gives you the quadratic $-2x^2 - 3x + 10$.
Use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the equation $ax^2 + bx + c = 0$.

Reflection

- Was your equation easy or hard to solve?
- What made it difficult, or easy?
- How many solutions did it have, and why?

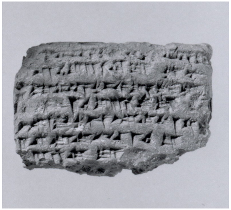


Image: (ca. 604-562 B.C.). Cuneiform tablet: account of barley and date disbursements, Ebabbar archive. [Cuneiform tablet, Clay-Tablets-Inscribed]. Retrieved from `https://library-artstor-org.ezproxy-b.deakin.edu.au/asset/SS7731421_7731421_11816453`

Ready to Go

At this point, your band is prepared to start the trip. In preparation for this quest, you are able to do the following:

- Factorise quadratics
- Complete the square
- Solve quadratic equations using the quadratic formula

We are now ready to face this week's challenge - sketching quadratics. We will travel with Gilgamesh and Enkidu who plan to conquer Humbaba the terrible. Maybe graph sketching is your own dragon, and we encourage you to conquer it this week, or maybe you are a competent Magi skilled in this art ready to guide your team to success. In either case, you have something to give and something to learn on this quest.

The Road to the Forest

The elders counsel Gilgamesh not to trust too much in his own strength but to value the support and protection of this companion Enkidu. Enkidu knows the path to the forest and the pitfalls along the road.

Enkidu will help you along the path but you will need to help him along the way.

Image: (c. 2000-1500 BCE). Demon, called Humbaba. [sculpture]. Retrieved from https://library-artstor-org.ezproxy-b.deakin.edu.au/asset/LESSING_ART_10311440572

Cedar Forest

You enter the cedar forest and come to a fork along the way. There are three paths, but only one is safe to traverse. Each path is labelled by a quadratic:

$$\begin{aligned} \text{Path 1: } & y = -3x^2 + 6x - 3 \\ \text{Path 2: } & y = 16 - 4x^2 \\ \text{Path 3: } & y = x^2 - 6x + 7 \end{aligned}$$

The shape of the graph of a quadratic is either a smile-shape or a sad-shape. The safe path is the one that is a smile-shaped quadratic.

Your band must identify the correct path to continue safely.

Cedar Forest

Your band successfully identifies the correct path and you travel along the path. Gilgamesh is warned in dreams of the dangers ahead: falling mountains, thunderstorms, wild animals and a thunderbird that breathes fire. However, he heeds the elder's counsel and looks for the place where the mountain opens for safe crossing.
The point of crossing is the y-intercept of the function
$y = -4x^2 - 3x + 7$.
Can your band identify the crossing point and cross safely before any of the dreams become a reality?

Cedar Forest

You manage to identify the crossing point, and head deeper into the forest. The sky becomes darker and the mountains begin to quake as you get closer to Humbaba. Enkidu requests your help in identifying his exact location. The location is identified as the point where both of these quadratics intercept the x-axis.

$$2x^2 - 9x + 4$$

$$4x^2 - 15x - 4$$

The first band to identify this point will earn their team 10 points. The second band to identify this point will earn their team 5 points.

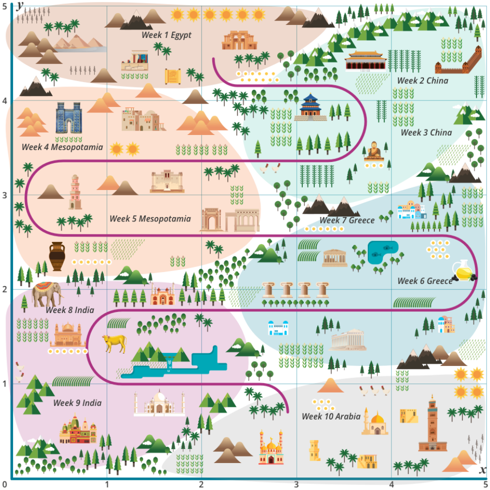
Humbaba is destroyed

You are able to identify the location of Humbaba. Gilgamesh and Enkidu prepare to battle Humbaba, but your role in their journey is over. They thank you for your companionship on their journey and give you a small cuneiform tablet with clues on how to complete your quest to find the missing cuneiform fragment.
They build you a small raft from the cedars of the forest and advise you that the you must leave them at the point on the map corresponding to the turning point of the following quadratic:

$$f(x) = x^2 - \frac{x}{2} + \frac{49}{16}.$$

.
The first band to identify this point will earn their team 10 points. The second band to identify this point will earn their team 5 points.

The journey continues



Your team finds the location on the map, and with your tablet of clues continues to search for the missing cuneiform fragment.

Reflection

The challenges along the forest path have included the key elements to sketching quadratics namely:

- Identifying the shape of the parabola (hint: coefficient of the x^2 term)
- Finding the y intercept (hint: coefficient of the constant term)
- Finding the x intercept(s), if they exist. (hint: factorise or use the quadratic formula.)
- Finding the turning or stationary point of the graph. (hint: completing the square. The shape of the graph will help you decide if the turning point is a local maxima or local minima)

Before we finish this week's workshop, we will apply these elements to sketch some graphs.

Graph 1

$$y = -x^2 + 9x - 20$$

Shape:
y-intercept:
x-intercepts:
Discriminant:
Stationary Point:
Type:

Graph 2

$$y = 4 - 16x^2$$

Shape:
y-intercept:
x-intercepts:
Discriminant:
Stationary Point:
Type:

Graph 3

$$y = 4x^2 - 12x + 9$$

Shape:
y-intercept:
x-intercepts:
Discriminant:
Stationary Point:
Type:

Graph 4

$$y = x^2 - 5x + 7$$

Shape:
y-intercept:
x-intercepts:
Discriminant:
Stationary Point:
Type:

Reflection

On this week's journey, you gained some tools for handling quadratic functions. Some of these generalise to other functions. For example, one Babylonian tablet contained values for n^3, n^2 and $n^3 + n^2$ for integer $n \in [1, 30]$. With the tools you have looked at in this workshop, you can factorise and find the roots of the cubic polynomial

$$x^3 + x^2.$$

- Given your skills with quadratics, what type of cubic polynomials are you able to factorise and find intercepts for?

Reflection

The Babylonians were solving quadratic expressions were over 3600 years ago, motivated by real problems that occurred in their daily lives. Many of these problems involved surveying and engineering. Later in this unit, we will see how we can apply mathematical skills to maximise or minimise quantities arising from these type of problems. We will also be able to identify stationary points in polynomials of higher degree.

Reference: R. McMillan (1984). Babylonian Quadratics. The Mathematics Teacher, 77(1), 63-65. Retrieved July 26, 2020, www.jstor.org/stable/27963851)