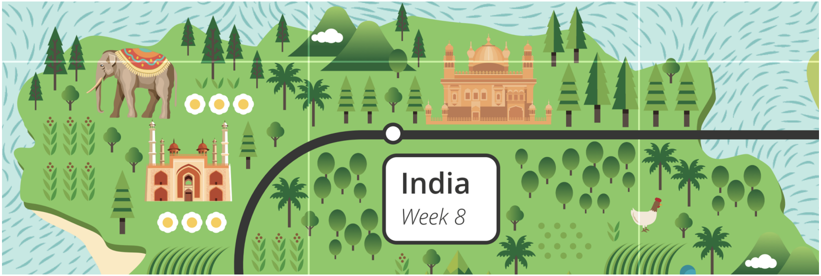
SIT190 - WEEK 8



- Differentiation of functions
  - product rule
  - quotient rule
  - chain rule
- Kinematics

Srinivasa Ramanujan

Srinivasa Ramanujan (1187-1920) was a pure mathematician.

- Untrained genius who became a Fellow at Trinity College, Cambridge University (UK).
- Formula for π
- Method for solving quartics
- Partition functions
- Continued fractions and infinite series

Movie: The Man who knew Infinity. [1]

[1] https://theconversation.com/the-man-who-knew-infinity-inspiration-rigour-and-the-art-of-mathematics-59520

India

Your band has arrived safely in India. In this leg of the journey, we are looking for Ramanujan's missing notebook.

- The real missing notebook was discovered by George Andrews in 1976.
- More than 100 loose pages and > 600 mathematical formulas listed without proofs.

Your band believes that there exists another notebook by Ramanujan and have headed to India to find it.

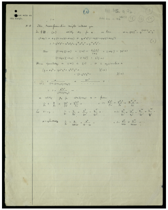


Image: The First Page of the So-called Lost Notebook, Add.ms.a.94, page 1 by Trinity College, Cambridge

Mathematics in India

- Evidence of advanced civilisation in ruins from Mohenjo Daro ($\sim$ 5000 year old).
  - Bathrooms and swimming pools
  - Drains and irrigation systems
  - Writing and counting systems
  - Measurement systems
- Civilisation $\Rightarrow$ Mathematics
- Mathematical works ($\sim$1500BCE) by *Aryans* (from Sanskrit meaning noblemen or owners of land)

Mathematics in India

- Number system
  - Brahami numerals (3rd century BCE) - precursor to Hindu-Arabic numeral system.
  - Hindu numerals
    - Reference to numerals in wrtings of Syrian bishop Severus Sebokht (772AD).
    - Decimal place value notation on an Indian plate (595AD)
    - Notation for 0 in inscription (876AD)
- Mathematician Brahmagupta (628 AD) :
  - How to deal with negative numbers: 'A fortune subtracted from zero is a debt. A debt subtracted from zero is a fortune. ie positive The product of a debt and fortune is a debt. '
  - Rules for solving quadratic equations

Kerala school of astronomy and mathematics

- Founded by Madhava of Sangamagrama (~1300AD)
- Use of mathematical induction
- Concepts such as limit were developed (~1350AD)
- Calculus of polynomial and trigonometrical functions

This news article claims that calculus created in India 250 years before Newton
`https://www.cbc.ca/news/technology/calculus-created-in-india-250-years-before-newton-study-1.632433`

Differentiation

- In Week 7, we found the derivative using the power rule, and some special rules for trigonometric, logarithmic and exponential functions.
- We could use these rules if the function had a scalar factor. For example, to find the derivative of $y = 3x^2$, we can find the derivative of x^2 using the power rule, then multiply by the 3 so $\frac{dy}{dx} = 3 \times 2x = 6x$.
- If a function consists of a number of terms then we can differentiate each term. For example, there are three terms in

$$f(x) = \ln(x) + \sin(2x) - x^3.$$

The derivative is $f'(x) = \frac{1}{x} + 2\cos(2x) - 3x^2$.

Differentiation

But the case is not so simple when we have any of the following:

- A function multiplied by a function. eg. $f(x) = x^2 \sin(x)$.
- A function divided by a function. eg. $f(x) = \frac{x^2}{\sin(x)}$ or $f(x) = x^2 \div \sin(x)$
- A function of a function. eg $f(x) = \sin(e^{3x})$.

The following functions are the product of two functions u & v.

Complete the table below:

Function $y = u \times v$	u	v
$y = e^{3x} \ln(x)$	e^{3x}	$\ln(x)$
$y = \sin(x) \cos(x)$		
$y = (x^3 - 2x^2 + 4)(x^4 - 2x)$		
$y = (x^2 - 3x) \ln(x)$		
$y = (e^{3x} - 2)(e^{3x} + 2)$		

Product Rule

Product Rule If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

1. Identify the factors u and v
2. Find $\frac{du}{dx}$ and $\frac{dv}{dx}$
3. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

For example: $y = e^{3x}\ln(x)$

1. Let $u = e^{3x}$ and $v = \ln(x)$
2. $\frac{du}{dx} = 3e^{3x}$ and $\frac{dv}{dx} = \frac{1}{x}$
3. $\frac{dy}{dx} = e^{3x} \times \frac{1}{x} + \ln(x) \times 3e^{3x} = e^{3x}(\frac{1}{x} + 3\ln(x))$

Product Rule

Product Rule If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$.

1. Identify the factors u and v
2. Find $\frac{du}{dx}$ and $\frac{dv}{dx}$
3. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Can your band find the derivatives of the functions:

1. $y = \sin(x)\cos(x)$
2. $y = (x^3 - 2x^2 + 4)(x^4 - 2x)$
3. $y = (x^2 - 3x)\ln(x)$
4. $y = (e^{3x} - 2)(e^{3x} + 2)$

10 points for the band who completes this task first.

Reflection

- When can you use the product rule?
- The derivatives of some of the examples on the previous slide, could be found without using the product rule.
  - Is it easier to use the product rule for these functions?

Quotient Rule

The following functions $y = \frac{u}{v}$ are the quotient of two functions u & v.

Complete the table below:

Function $y = \frac{u}{v}$	u	v
$y = \frac{2\sin(x)}{\cos(x)}$	$2\sin(x)$	$\cos(x)$
$y = \frac{e^{3x}}{e^{6x}}$		
$y = \frac{x^3-4}{x^4+1}$		
$y = \frac{\ln(x)}{x^2+1}$		
$y = \frac{e^{3x}-2}{e^{3x}+2}$		

Quotient Rule

Quotient Rule If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

1. Identify the factors u and v
2. Find $\frac{du}{dx}$ and $\frac{dv}{dx}$
3. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

For example: $y = \frac{2\sin(x)}{\cos(x)}$

1. Let $u = 2\sin(x)$ and $v = \cos(x)$
2. $\frac{du}{dx} = 2\cos(x)$ and $\frac{dv}{dx} = -\sin(x)$
3. $\frac{dy}{dx} = \frac{\cos(x) \times 2\cos(x) - 2\sin(x) \times (-\sin(x))}{\cos^2(x)}$
   $= \frac{2\cos^2(x) + 2\sin^2(x)}{\cos^2(x)} = \frac{2(\cos^2(x) + \sin^2(x))}{\cos^2(x)} = \frac{2}{\cos^2(x)}$

Quotient Rule

Quotient Rule If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

1. Identify the factors u and v
2. Find $\frac{du}{dx}$ and $\frac{dv}{dx}$
3. $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Can your band find the derivatives of the functions:

1. $y = \dfrac{e^{3x}}{e^{6x}}$
2. $y = \dfrac{x^3 - 4}{x^4 + 1}$
3. $y = \dfrac{\ln(x)}{x^2 + 1}$
4. $y = \dfrac{e^{3x} - 2}{e^{3x} + 2}$

10 points for the band who completes this task first.

Reflection

- When can you use the quotient rule?
- Consider the function $y = \frac{\ln(x)}{x^2}$ and the function $y = x^{-2}\ln(x)$
  - Find the derivative for these functions using the quotient and product rules respectively.
  - Reflect on the results - does it suggest how you could manipulate a function to make finding the derivative easier?
- Did you notice that we can find the derivative of $f(x) = \tan(x)$ using the quotient rule?

Which Rule?

For each of the following functions give the rule your band would use to find the derivative:
Product Rule; **Q**uotient Rule; or a **D**ifferent rule.
Note: we will deal with some of the **D** in subsequent slides.

1. $y = (x^2 - 4x)\sin(\frac{x}{2})$
2. $y = \sin(x^2 - 4x + 7)$
3. $y = \frac{cos(4x)}{x^{-3}}$
4. $f(x) = (x^{10} - 4x + 3)(x^{10} - 4x + 5)$
5. $f(x) = \frac{x^2-1}{x-1}$
6. $f(x) = e^{x^2-3x+1}$
7. $y = \frac{\ln(3x)}{2x^2+1}$

Chain Rule

Consider the following functions. Can you identify part of the function that if you replaced with another variable, for example u, then the new function would be easy to differentiate? For example, $y = \sin(x^{-3} + 2x)$ looks difficult, but replacing $x^{-3} + 2x$ with u would give $y = \sin(u)$.

Function	Replace	New Function
$y = e^{x^2-3x+1}$	$x^2 - 3x + 1$	$y = e^x$
$y = \ln(x^2 - 3x + 2)$		
$y = \sin(x^3 - 4x + 1)$		
$y = \cos(2x + 4)$		
$y = \sqrt{x^2 - 4}$		

Chain Rule

- Replacing parts of the function to find a function we know how to differentiate is the basis of the Chain Rule.
- In the previous table, we replaced part of the expression with x, but in the Chain Rule we replace it with u.
- The replacement should always result in something easier to differentiate.

Chain Rule

Chain Rule If $y = f(u)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1. Identify the function u
2. Re-express your function in terms of u
3. Find $\frac{dy}{du}$ and $\frac{du}{dx}$
4. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
5. Substitute u into

For example: $y = e^{x^2-3x+1}$

1. Let $u = x^2 - 3x + 1$.
2. Then $y = e^u$.
3. $\frac{dy}{du} = e^u$ and $\frac{du}{dx} = 2x - 3$
4. $\frac{dy}{dx} = e^u \times (2x - 3)$
5. $\frac{dy}{dx} = e^{x^2-3x+1} \times (2x - 3) = (2x - 3)e^{x^2-3x+1}$

Chain Rule

Chain Rule If $y = f(u)$ then $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

1. Identify the function u
2. Re-express your function in terms of u
3. Find $\frac{dy}{du}$ and $\frac{du}{dx}$
4. $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$
5. Substitute u into

Can your band find the derivatives of the functions:

1. $y = \ln(x^2 - 3x + 2)$
2. $y = \sin(x^3 - 4x + 1)$
3. $y = \cos(2x + 4)$
4. $y = \sqrt{x^2 - 4}$

10 points for the band who completes this task first.

Combinations of Product, Quotient and Chain

Sometimes you may have to combine more than one rule to find the derivative.

Can your band identify the rules to use to find the derivatives of these functions:

1. $y = \sin(3x)\ln(3x^2 + 2x)$
2. $y = \dfrac{\cos(x^2 - 3x + 2)}{x}$

Combinations of Product, Quotient and Chain

An example: $y = (x^2 - 3x + 5)\sin(x^3 + 1)$
Applying the Product Rule with:

$$u = (x^2 - 3x + 5) \text{ and } v = \sin(x^3 + 1)$$

We can easily find $\frac{du}{dx} = 2x - 3$ but $\frac{dv}{dx}$ requires the Chain Rule.
Applying the Chain Rule[2] to $v = \sin(x^3 + 1)$, we let $w = x^3 + 1$ which gives $v = \sin(w)$.

$$\frac{dv}{dw} = \cos(w) \text{ and } \frac{dw}{dx} = 3x^2$$

The Chain Rule gives

$$\frac{dv}{dx} = \frac{dv}{dw}\frac{dw}{dx} = \cos(w)3x^2 = 3x^2\cos(x^3 + 1)$$

Returning to the Product Rule

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} = 3x^2(x^2 - 3x + 5)\cos(x^3 + 1) + (2x - 3)\sin(x^3 + 1)$$

[2] We use w instead of u since u has already been used in the problem

Combinations of Product, Quotient and Chain

Find the derivatives of these functions:

1. $y = \sin(3x)\ln(3x^2+2x)$
2. $y = e^{4x}\cos(x^2-3x+2)$

Applications - Kinematics

Derivatives can help solve problems involving rates of change. Our example looks at the displacement of an object from a given point.

- Displacement s is distance of the object from a given point.
- Time t
- Velocity v is rate of change in displacement with respect to time, that is, $v = \frac{ds}{dt}$
- Acceleration a is rate of change in velocity with respect to time, that is, $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Applications - Kinematics

An Example:

- Your car is $s = 24t - 2t^2 - t^3$ km from home at time $t \in [0, 4]$
- Time t is measured in hours.
- Velocity $v = \frac{ds}{dt} = 24 - 4t - 3t^2$ km/hr
- Acceleration $a = \frac{dv}{dt} = -4 - 6t$ km$/hr^2$

1. How far from home were you at time $t = 0$?
2. What was your velocity at time $t = 0$?
3. What was your velocity and acceleration at time $t = 1$ hour?
4. A policeman claimed that you were driving at 80km/h. Is this true?

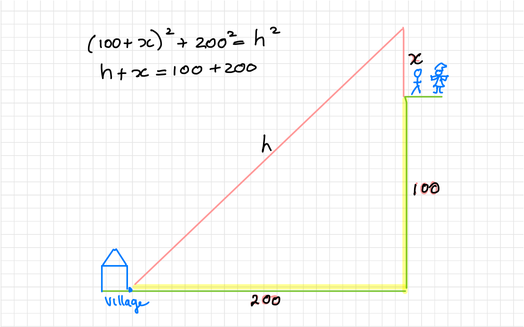
Applications - Kinematics

An Example:

- Your car is $s = 24t - 2t^2 - t^3$ km from home at time $t \in [0, 4]$
- Time t is measured in hour.
- Velocity $v = \frac{ds}{dt} = 24 - 4t - 3t^2$ km/hr
- Acceleration $a = \frac{dv}{dt} = -4 - 6t$ km$/hr^2$

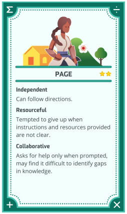
1. When $t = 0$, $s = 24 \times 0 - 2 \times 0^2 - 0^3 = 0$ km from home.
2. When $t = 0$, $v = \frac{ds}{dt} = 24 - 0 - 0^2 = 24$ km/hr
3. When $t = 1$, $v = 24 - 4 - 3 = 17$ km/hr and $a = -4 - 6 = -10$ km$/hr^2$
4. No $v = 24 - 4t - 3t^2 = 80 \Rightarrow 3t^2 + 4t + 56 = 0$ has no solutions as discriminant is $-656 < 0$.

A Hindu Mathematics Problem:
A page and a wizard are at the top of a cliff which is 100 metres high. The base of the cliff is 200 metres from the village. The page climbs down the cliff and walks to the village. The wizard flies up x metres and then directly to the village. The distance travelled by the page and the wizard is the same. How high did the wizard fly?



A Hindu Mathematics Problem:

You can answer this problem using the skills that you have gained in algebra, simultaneous equations and trigonometry. But on this part of our journey we are looking at differentiation . . .



The Wizard and The Page

A slightly different problem.

- The displacement of the page from the village at time t seconds is $s_p = 5000 - 5t,\ t \in [0, 1000]$.
- The displacement of the wizard from the village at time t seconds is $s_w = -t^2 + 490t + 5000,\ t \in [0, 500]$.

1. Who arrives at the village first?
2. What time does the wizard arrive at the village?
3. What is the maximum distance of the wizard from the village?
4. How far is the page from the village when the wizard arrives?
5. Find the velocity and acceleration of the wizard and page.

A Hindu Mathematics Problem - the Remarkable Snake

- The remarkable snake is 80 metres long. It begins to descend into a hole.
- Each $\frac{5}{14}$ of a day it descents $\frac{15}{2}$m into the hole.
- Each $\frac{1}{4}$ of a day its tail grows $\frac{11}{4}$m.
- How long until the ever-growing snake is fully underground?

Let l be the length of the snake above the ground.
When $t = 0$ days, we have $l = 80$ metres.
The snake descends $\frac{15}{2} \times \frac{14}{5} = 21$ metres each day and grows $\frac{11}{4} \times \frac{4}{1} = 11$ metres each day, which means

$$l = 80 - 21t + 11t = 80 - 10t$$

1. What is the rate of change of the length of the snake above the ground with respect to time?

Journey to Kerala

Your band plans to travel to the Kerala School of Astronomy and Mathematics (1300 to 1600 BCE), as you suspect you may find some pages of the missing notebook there. You need to travel from Ponnani on the coast to Kerala. You have two options:

- A 54 km boat ride along the Bharathappuzha River. The distance travelled along the river from the beach in t hours is given by $s_b = 8t^3 + 4t^2 + 12t$.
- A 49km car trip. The distance travelled along the road from the beach in t hours is given by $s_c = 20t^2 - 7t$

1. How long would it take to go by (a) boat and (b) car?
2. Find the velocity and acceleration of the boat at time t.

Journey to Kerala

Your band selects the fastest form of transportation and arrives safely in Kerala. On the way you have gathered some new tools for differentiating complex functions.

- Product rule
- Quotient rule
- Chain rule
- Combinations of rules

For the remainder of the workshop we will look at some applications.

Applications

A rocket is fired from a platform 10 metre from the ground. The distance of the rocket from the ground at time t seconds is

$$s = -10t^2 + 99t + 10 \text{ for } t \in [0, 10].$$

1. Find the velocity and acceleration of the rocket.
2. What was the maximum distance that the rocket travelled from the ground? and at what time did this occur?
3. How do you know that this is the maximum and not the minimum distance?
4. . What is the velocity when $t = 5$?

Applications

It takes 15 mintues to fill a water tank. The volume V litres after t minutes is given by

$$V = 30t - t^2 \text{ where } t \in [0, 15].$$

1. What is the volume at $t = 0$ and at $t = 15$?
2. Find the instantaneous rate of change of volume when $t = 8$.
3. Find the instantaneous rate of change of volume when $t = 10$.
4. Is the instantaneous rate of change increasing or decreasing? How does this relate to $\frac{d^2V}{dt^2}$?
5. Find the volume of water in the tank when the instantaneous rate of change of volume is 10.

The displacement s metres of an object at time t seconds is given by

$$s = -8t^3 + 24t^2 + 32t \text{ for } t \geq 0.$$

1. Find the velocity and acceleration.
2. Find velocity when $t = 2$
3. Find the acceleration at $t = 3$
4. Find the velocity when acceleration is 0.
5. Find the acceleration and displacement when velocity is 50m/s.

Applications

Suppose you have 300 metres of fencing and you want to fence the largest rectangular area.

1. Find the width and length of the largest rectangular area you can fence with your fencing.
2. Suppose there is a long fence at the end of your property. Fencing the area adjacent to this fence means that you only need to fence three sides of the rectangle. What is the largest area you can fence now?
3. You wonder if you can fence a larger area if you have a triangular shaped area. Now you only have to fence two sides (you can assume the triangle is a right-angled triangle and that you use the existing fence for the hypotenuse).

Summary of Derivatvies

Function	Derivative
$y = x^n$	$\frac{dy}{dx} = nx^{n-1}$
$y = e^{kx}$	$\frac{dy}{dx} = ke^{kx}$
$y = \ln(x)$	$\frac{dy}{dx} = \frac{1}{x}$
$y = \sin(kx)$	$\frac{dy}{dx} = k\cos(kx)$
$y = \cos(kx)$	$\frac{dy}{dx} = -k\sin(kx)$

Summary of Derivatvies

Function	Derivative
$f(x) = x^n$	$f'(x) = nx^{n-1}$
$f(x) = e^{kx}$	$f'(x) = ke^{kx}$
$f(x) = \ln(x)$	$f'(x) = \frac{1}{x}$
$f(x) = \sin(kx)$	$f'(x) = k\cos(kx)$
$f(x) = \cos(kx)$	$f'(x) = -k\sin(kx)$