Task 1: Matrices - Inverses

\1)

$$\begin{bmatrix} 8 & -16 \\ -2 & 4 \end{bmatrix}$$

Determinant:
$\frac{1}{ad-bc}$
-> $8 \times 4 - -16 \times -2$
-> $32 - 32 = 0$
Since $\frac{1}{0} = \text{undefined}$, this matrix does not have an inverse

\2)

$$\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix}$$

Determinant:
-> $5 \times 4 - 2 \times 7$
-> $20 - 14 = 6$
-> $\frac{1}{6}$
The determinant is non-zero, so the matrix does have an inverse

Inverse:

$$\begin{bmatrix} 5 & 2 \\ 7 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & -2 \\ -7 & 5 \end{bmatrix} \times \frac{1}{6} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{7}{6} & \frac{5}{6} \end{bmatrix}$$

Task 2: Simultaneous equations

Sakae bought 4 pens, 2 notepads, and paid $13
Pritika bough 8 pens, 1 notepads, and paid $8

What was the cost/pen and cost/notebook?

$$\begin{cases} 4p + 2n = 13 \\ 8p + 1n = 8 \end{cases}$$

$$\begin{bmatrix} 4 & 2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} p \\ n \end{bmatrix} = \begin{bmatrix} 13 \\ 8 \end{bmatrix}$$

Determinant: $ad - bc$
$4 \times 1 - 2 \times 8 = -12$
There therefore is a valid inverse matrix. We will times the flipped matrix using $\frac{1}{-12}$.

$$\begin{bmatrix} 4 & 2 \\ 8 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 \\ -8 & 4 \end{bmatrix}$$

$$-\frac{1}{12} \times \begin{bmatrix} 1 & -2 \\ -8 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{2}{12} \\ \frac{8}{12} & -\frac{4}{12} \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$$

Now that we have the inverse matrix, we can multiply both sides of the equation by it, removing the non-inverted matrix and finding the solution for the $\begin{bmatrix} p \\ n \end{bmatrix}$ matrix.

Full equation:

$$\begin{bmatrix} -\frac{1}{12} & \frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 8 & 1 \end{bmatrix} \begin{bmatrix} p \\ n \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} & \frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 13 \\ 8 \end{bmatrix}$$

Solving $A \times A^{-1}$

$$\begin{bmatrix} -\frac{1}{12} \times 4 + \frac{1}{6} \times 8 & -\frac{1}{12} \times 2 + \frac{1}{6} \times 1 \\ \frac{2}{3} \times 4 + -\frac{1}{3} \times 8 & \frac{2}{3} \times 2 + -\frac{1}{3} \times 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p \\ n \end{bmatrix} = \begin{bmatrix} 1 \times p + 0 \times n \\ 0 \times p + 1 \times n \end{bmatrix} = \begin{bmatrix} p \\ n \end{bmatrix}$$

Solving $A^{-1} \times \begin{bmatrix} 13 \\ 8 \end{bmatrix}$

$$\begin{bmatrix} -\frac{1}{12} & \frac{1}{6} \\ \frac{2}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 13 \\ 8 \end{bmatrix} = \begin{bmatrix} -\frac{1}{12} \times 13 + \frac{1}{6} \times 8 \\ \frac{2}{3} \times 13 + -\frac{1}{3} \times 8 \end{bmatrix} = \begin{bmatrix} -\frac{13}{12} + \frac{8}{6} \\ \frac{26}{3} + -\frac{8}{3} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{13}{12} + \frac{8}{6} \\ \frac{26}{3} + -\frac{8}{3} \end{bmatrix} = \begin{bmatrix} -\frac{13}{12} + \frac{16}{12} = \frac{3}{12} \\ \frac{18}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ 6 \end{bmatrix}$$

The matrix method is significantly quicker to do only if you have a calculator to quickly multiply matrices and find the inverse. Without a calculator, the elimination method seems to be best for me. Doing them manually and typing formulas for each answer was excruciating, without typing the working out and with a calculator, it is lightning fast. Matrices do not come intuitively to me, but it is ultimately very satisfying to see the answer 'appear' of nowhere.

Task 3: Quadratics

Given $y = -x^2 - 3x - 9$:

Find the x and y-intercepts
-> $y = -0^2 - 3(0) - 9$
-> $= 0 - 0 - 9$
-> $y = -9$

-> $0 = -x^2 - 3x - 9$
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
-> $\frac{--3 \pm \sqrt{-3^2 - 4(-1)(-9)}}{2(-1)} = \frac{3 \pm \sqrt{-27}}{-2}$
-> $x^+ = \frac{3 + \sqrt{-27}}{-2}$
-> $x^- = \frac{3 - \sqrt{-27}}{-2}$
There is no real x-intercept. It therefore must be below the x-axis.

Alternate wrong x-intercept answer - could I have it explained why this is wrong?
-> $0 = -x^2 - 3x - 9$
-> $9 = -x^2 - 3x$
-> $9 + 3x = -x^2$
-> $12 = -\frac{x^2}{x}$
-> $12 = -x$
-> $x = -12$

Stationary point
Formula given $y = ax^2 + bx + c$
We want to reach the turning point formula format, $a(x-h)^2 + k$

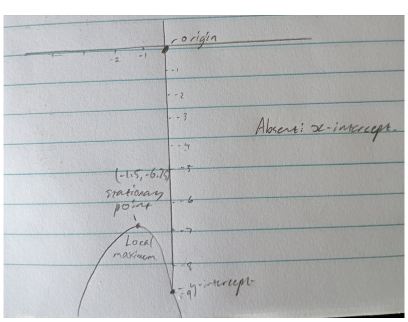
$-x^2 - 3x - 9$
-> $+\frac{b}{2}^2$ and $-\frac{b}{2}^2$ to both sides
-> $\pm\frac{9}{4}$
-> $-x^2 - 3x + \frac{9}{4} - 9 - \frac{9}{4}$
-> $(-x^2 - 3x + \frac{3}{2}^2) - (9 - \frac{9}{4})$
-> $(x+m)(x+n)$
-> $m \times n = \frac{3}{2}^2 : [(\frac{3}{2}, 1), (\frac{1}{2}, 3)]$
-> $m + n = \frac{3}{2} + \frac{3}{2} = \frac{6}{2} = 3$
-> $y = -(x + \frac{3}{2})^2 - \frac{27}{4}$

From this, we know that we can translate the graph $-\frac{27}{4}$ down the y coordinate and by $\frac{3}{2}$ across the x coordinate. And remembering the translating by a positive number on the x axis moves left instead of the intuitive right, this reaches the vertex of $(-\frac{3}{2}, -\frac{27}{4})$, or $(-1.5, -6.75)$

Verification
Using the vertex formula, $x = \frac{-\mathrm{b}}{2\,\mathrm{a}} \longrightarrow \frac{--3}{2\,(-1)} \longrightarrow -\frac{3}{2}$
At this point, the vertex is equal to $(\frac{-3}{2}, y)$

We substitute x into the found turning point formula to find the y coordinate like so:
-> $y = -(\frac{-3}{2} + \frac{3}{2})^2 - \frac{27}{4}$
-> $-(0)^2 - \frac{27}{4}$
-> $y = \frac{-27}{4} = -6.75$
The vertex/stationary point is still $(-1.5, -6.75)$

Graph
The graph displays the vertex as a local (and global) maximum, being the highest point of the parabola



Task 4: Cubic

Find the x and y-intercepts of the cubic equation $y = (x-3)(x^2-14)$

$0 = x - 3$
-> $x = 3$
$0 = x^2 - 14$
-> $x = \pm\sqrt{14}$
The x intercepts are therefore:
$(3, 0), (\sqrt{14}, 0), (-\sqrt{14}, 0)$

y-intercept
$y = (0-3)(0^2-14)$
-> -3×-14
-> $y = 42$

Verification