Chapter 4

4 Polynomial functions

Objectives

- To revise the properties of **quadratic functions**.
- To add, subtract and multiply polynomials.
- To be able to use the technique of **equating coefficients**.
- To **divide polynomials**.
- To use the **remainder theorem**, the **factor theorem** and the **rational-root theorem** to identify the linear factors of cubic and quartic polynomials.
- To draw and use **sign diagrams**.
- To find the rules for given polynomial graphs.
- To apply **polynomial functions** to problem solving.

A polynomial function of degree 2 is called a **quadratic function**. The general rule for such a function is

$$f(x) = ax^2 + bx + c, \quad a \neq 0$$

A polynomial function of degree 3 is called a **cubic function**. The general rule for such a function is

$$f(x) = ax^3 + bx^2 + cx + d, \quad a \neq 0$$

A polynomial function of degree 4 is called a **quartic function**. The general rule for such a function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad a \neq 0$$

In this chapter we revise quadratic functions, and build on our previous study of cubic and quartic functions.

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4A Quadratic functions

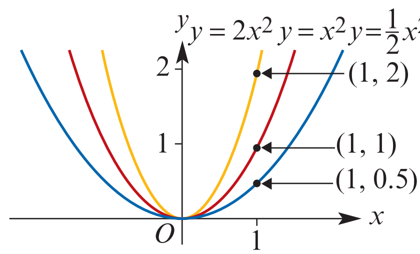
In this section, we revise material on quadratic functions covered in Mathematical Methods Units 1 & 2.

Transformations of parabolas

Dilation from the x-axis

For $a > 0$, the graph of the function $y = ax^2$ is obtained from the graph of $y = x^2$ by a dilation of factor a from the x-axis.

The graphs on the right are those of $y = x^2$, $y = 2x^2$ and $y = \frac{1}{2}x^2$, i.e. $a = 1, 2$ and $\frac{1}{2}$.

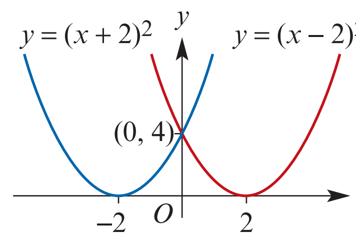


Translation parallel to the x-axis

The graphs of $y = (x + 2)^2$ and $y = (x - 2)^2$ are shown.

For $h > 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of h units in the negative direction of the x-axis.

For $h < 0$, the graph of $y = (x + h)^2$ is obtained from the graph of $y = x^2$ by a translation of $-h$ units in the positive direction of the x-axis.

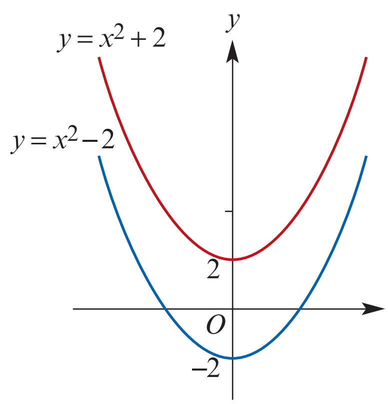


Translation parallel to the y-axis

The graphs of $y = x^2 + 2$ and $y = x^2 - 2$ are shown.

For $k > 0$, the graph of $y = x^2 + k$ is obtained from the graph of $y = x^2$ by a translation of k units in the positive direction of the y-axis.

For $k < 0$, the translation is in the negative direction of the y-axis.



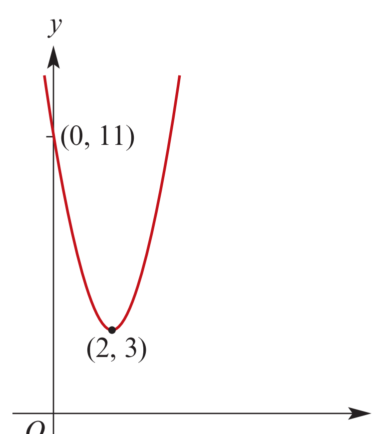
Combinations of transformations

The graph of the function

$$f(x) = 2(x - 2)^2 + 3$$

is obtained by transforming the graph of the function $f(x) = x^2$ by:

- dilation of factor 2 from the x-axis
- translation of 2 units in the positive direction of the x-axis
- translation of 3 units in the positive direction of the y-axis.



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▸ Graphing quadratics in turning point form

By applying dilations, reflections and translations to the basic parabola $y = x^2$, we can sketch the graph of any quadratic expressed in **turning point form** $y = a(x - h)^2 + k$:

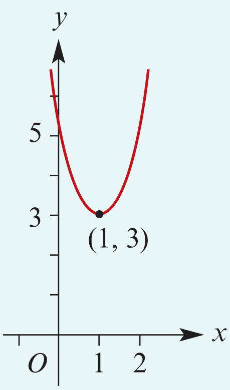
- If $a > 0$, the graph has a minimum point.
- If $a < 0$, the graph has a maximum point.
- The vertex is the point (h, k).
- The axis of symmetry is $x = h$.
- If h and k are positive, then the graph of $y = a(x - h)^2 + k$ is obtained from the graph of $y = ax^2$ by translating h units in the positive direction of the x-axis and k units in the positive direction of the y-axis.
- Similar results hold for different combinations of h and k positive and negative.

Example 1

Sketch the graph of $y = 2(x - 1)^2 + 3$.

Solution

The graph of $y = 2x^2$ is translated 1 unit in the positive direction of the x-axis and 3 units in the positive direction of the y-axis.



The vertex has coordinates $(1, 3)$.

The axis of symmetry is the line $x = 1$.

The graph will be narrower than $y = x^2$.

The range is $[3, \infty)$.

To add further detail to our graph, we can find the axis intercepts:

y-axis intercept

When $x = 0$, $y = 2(0 - 1)^2 + 3 = 5$.

x-axis intercepts

In this example, the minimum value of y is 3, and so y cannot be 0. Therefore this graph has no x-axis intercepts.

Note: Another way to see this is to let $y = 0$ and try to solve for x:

$$\begin{aligned} 0 &= 2(x - 1)^2 + 3 \\ -3 &= 2(x - 1)^2 \\ -\tfrac{3}{2} &= (x - 1)^2 \end{aligned}$$

As the square root of a negative number is not a real number, this equation has no real solutions.

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Example 2

Sketch the graph of $y = -(x+1)^2 + 4$.

Solution

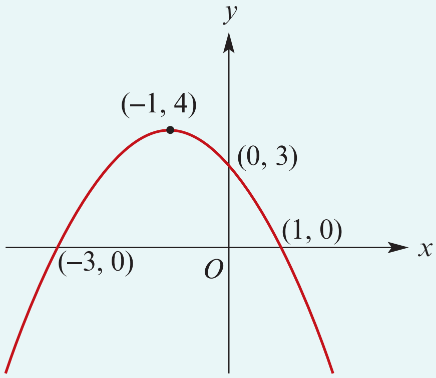
The vertex has coordinates $(-1, 4)$ and so the axis of symmetry is the line $x = -1$.

When $x = 0$, $y = -(0+1)^2 + 4 = 3$.
$\therefore$ the y-axis intercept is 3.

When $y = 0$,

$$\begin{aligned} -(x+1)^2 + 4 &= 0 \\ (x+1)^2 &= 4 \\ x + 1 &= \pm 2 \\ x &= \pm 2 - 1 \end{aligned}$$

$\therefore$ the x-axis intercepts are 1 and -3.



▶ The axis of symmetry

For a quadratic function written in polynomial form $y = ax^2 + bx + c$, the axis of symmetry of its graph has the equation $x = -\dfrac{b}{2a}$.

Therefore the x-coordinate of the turning point is $-\dfrac{b}{2a}$. Substitute this value into the quadratic polynomial to find the y-coordinate of the turning point.

Example 3

For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x-h)^2 + k$, and hence find the maximum or minimum value and the range:

a $y = x^2 - 4x + 3$ **b** $y = -2x^2 + 12x - 7$

Solution

a The x-coordinate of the turning point is 2.

When $x = 2$, $y = 4 - 8 + 3 = -1$.

The coordinates of the turning point are $(2, -1)$. Hence the equation is $y = (x-2)^2 - 1$.

The minimum value is -1 and the range is $[-1, \infty)$.

Explanation

Here $a = 1$ and $b = -4$, so the axis of symmetry is $x = -\left(\dfrac{-4}{2}\right) = 2$.

For the turning point form $y = a(x-h)^2 + k$, we have found that $a = 1$, $h = 2$ and $k = -1$.

Since $a > 0$, the parabola has a minimum.

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b The x-coordinate of the turning point is 3.

When $x = 3$, $y = -2 \times (3)^2 + 12 \times 3 - 7 = 11$.

The coordinates of the turning point are $(3, 11)$. Hence the equation is $y = -2(x-3)^2 + 11$.

The maximum value is 11 and the range is $(-\infty, 11]$.

Here $a = -2$ and $b = 12$, so the axis of symmetry is $x = -\left(\frac{12}{-4}\right) = 3$.

For the turning point form $y = a(x-h)^2 + k$, we have found that $a = -2$, $h = 3$ and $k = 11$.

Since $a < 0$, the parabola has a maximum.

Graphing quadratics in polynomial form

It is not essential to convert a quadratic to turning point form in order to sketch its graph.

For a quadratic in polynomial form, we can find the x- and y-axis intercepts and the axis of symmetry by other methods and use these details to sketch the graph.

Step 1 Find the y-axis intercept.

Step 2 Find the x-axis intercepts.

Step 3 Find the equation of the axis of symmetry.

Step 4 Find the coordinates of the turning point.

Example 4

Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of $y = x^2 + x - 12$.

Solution

Step 1 $c = -12$. Therefore the y-axis intercept is -12.

Step 2 Let $y = 0$. Then

$$0 = x^2 + x - 12$$
$$0 = (x+4)(x-3)$$
$$\therefore \quad x = -4 \quad \text{or} \quad x = 3$$

The x-axis intercepts are -4 and 3.

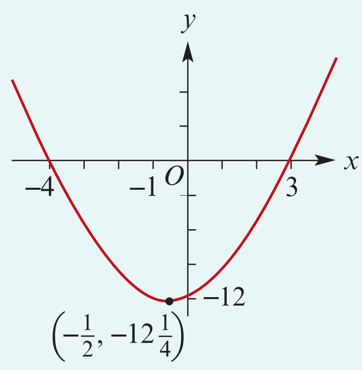
Step 3 The axis of symmetry is the line with equation

$$x = \frac{-4+3}{2} = -\frac{1}{2}$$

Step 4 When $x = -\frac{1}{2}$, $y = (-\frac{1}{2})^2 + (-\frac{1}{2}) - 12$

$$= -12\tfrac{1}{4}$$

The turning point has coordinates $(-\frac{1}{2}, -12\frac{1}{4})$.



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Completing the square

By completing the square, all quadratics in polynomial form, $y = ax^2 + bx + c$, may be transposed into turning point form, $y = a(x - h)^2 + k$. We have seen that this can be used to sketch the graphs of quadratic polynomials.

To complete the square of $x^2 + bx + c$:

- Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.

To complete the square of $ax^2 + bx + c$:

- First take out a as a factor and then complete the square inside the bracket.

Example 5

By completing the square, write the quadratic $f(x) = 2x^2 - 4x - 5$ in turning point form, and hence sketch the graph of $y = f(x)$.

Solution

$$\begin{aligned} f(x) &= 2x^2 - 4x - 5 \\ &= 2\left(x^2 - 2x - \frac{5}{2}\right) \\ &= 2\left(x^2 - 2x + 1 - 1 - \frac{5}{2}\right) && \text{add and subtract } \left(\frac{b}{2}\right)^2 \text{ to `complete the square'} \\ &= 2\left[(x^2 - 2x + 1) - \frac{7}{2}\right] \\ &= 2\left[(x - 1)^2 - \frac{7}{2}\right] \\ &= 2(x - 1)^2 - 7 \end{aligned}$$

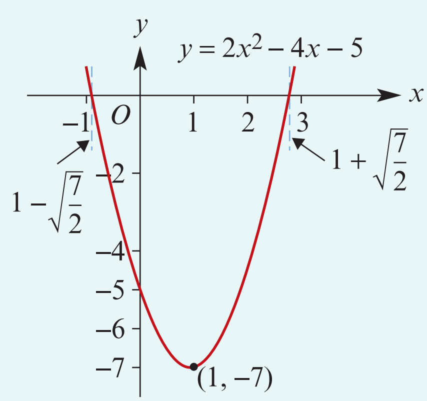
The x-axis intercepts can be determined after completing the square:

$$\begin{aligned} 2x^2 - 4x - 5 &= 0 \\ 2(x - 1)^2 - 7 &= 0 \\ (x - 1)^2 &= \frac{7}{2} \\ x - 1 &= \pm\sqrt{\frac{7}{2}} \end{aligned}$$

$$\therefore \quad x = 1 + \sqrt{\frac{7}{2}} \quad \text{or} \quad x = 1 - \sqrt{\frac{7}{2}}$$

This information can now be used to sketch the graph:

- The y-axis intercept is $c = -5$.
- The turning point is $(1, -7)$.
- The x-axis intercepts are $1 + \sqrt{\frac{7}{2}}$ and $1 - \sqrt{\frac{7}{2}}$.



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▶ The quadratic formula

The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

It should be noted that the equation of the axis of symmetry can be derived from this general formula: the axis of symmetry is the line with equation

$$x = -\frac{b}{2a}$$

Example 6

Sketch the graph of $f(x) = -3x^2 - 12x - 7$ by:

- finding the equation of the axis of symmetry
- finding the coordinates of the turning point
- using the general quadratic formula to find the x-axis intercepts.

Solution

Since $c = -7$, the y-axis intercept is -7.

Axis of symmetry
$$\begin{aligned} x &= -\frac{b}{2a} \\ &= -\left(\frac{-12}{2 \times (-3)}\right) \\ &= -2 \end{aligned}$$

Turning point

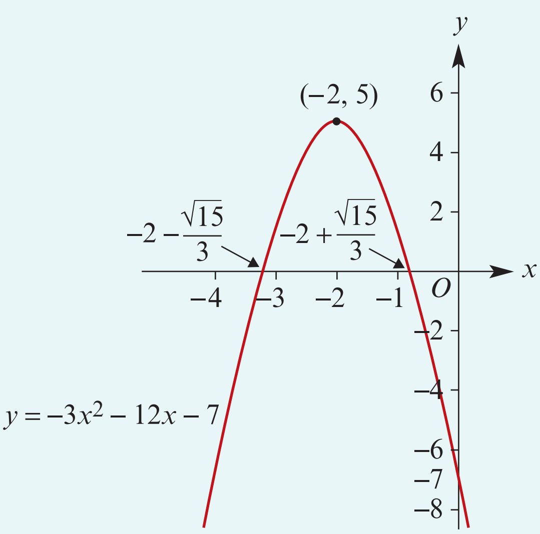
When $x = -2$, $y = -3(-2)^2 - 12(-2) - 7 = 5$.

The turning point coordinates are $(-2, 5)$.

x-axis intercepts

$$-3x^2 - 12x - 7 = 0$$

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(-3)(-7)}}{2(-3)} \\ &= \frac{12 \pm \sqrt{60}}{-6} \\ &= \frac{12 \pm 2\sqrt{15}}{-6} \\ &= -2 \pm \tfrac{1}{3}\sqrt{15} \end{aligned}$$



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▶ The discriminant

The **discriminant** Δ of a quadratic polynomial $ax^2 + bx + c$ is

$$\Delta = b^2 - 4ac$$

For the equation $ax^2 + bx + c = 0$:

- If $\Delta > 0$, there are two solutions.
- If $\Delta = 0$, there is one solution.
- If $\Delta < 0$, there are no real solutions.

For the equation $ax^2 + bx + c = 0$ where a, b and c rational numbers:

- If Δ is a perfect square and $\Delta \neq 0$, then the equation has two rational solutions.
- If $\Delta = 0$, then the equation has one rational solution.
- If Δ is not a perfect square and $\Delta > 0$, then the equation has two irrational solutions.

Example 7

Without sketching graphs, determine whether the graph of each of the following functions crosses, touches or does not intersect the x-axis:

a $f(x) = 2x^2 - 4x - 6$
b $f(x) = -4x^2 + 12x - 9$
c $f(x) = 3x^2 - 2x + 8$

Solution	*Explanation*
a $\Delta = b^2 - 4ac$	
$= (-4)^2 - 4 \times 2 \times (-6)$	Here $a = 2$, $b = -4$, $c = -6$.
$= 16 + 48$	
$= 64 > 0$	
The graph crosses the x-axis twice.	As $\Delta > 0$, there are two x-axis intercepts.
b $\Delta = b^2 - 4ac$	
$= (12)^2 - 4 \times (-4) \times (-9)$	Here $a = -4$, $b = 12$, $c = -9$.
$= 144 - 144$	
$= 0$	
The graph touches the x-axis once.	As $\Delta = 0$, there is only one x-axis intercept.
c $\Delta = b^2 - 4ac$	
$= (-2)^2 - 4 \times 3 \times 8$	Here $a = 3$, $b = -2$, $c = 8$.
$= 4 - 96$	
$= -92 < 0$	
The graph does not intersect the x-axis.	As $\Delta < 0$, there are no x-axis intercepts.

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Example 8

Find the values of m for which the equation $3x^2 - 2mx + 3 = 0$ has:

a one solution **b** no solution **c** two distinct solutions.

Solution

For the quadratic $3x^2 - 2mx + 3$, the discriminant is $\Delta = 4m^2 - 36$.

a For one solution:

$$\Delta = 0$$

i.e. $4m^2 - 36 = 0$

$m^2 = 9$

$\therefore \quad m = \pm 3$

b For no solution:

$$\Delta < 0$$

i.e. $4m^2 - 36 < 0$

From the graph, this is equivalent to

$-3 < m < 3$

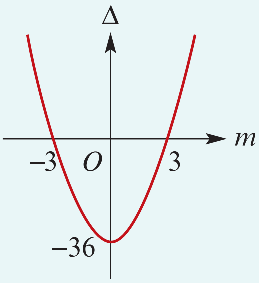
c For two distinct solutions:

$$\Delta > 0$$

i.e. $4m^2 - 36 > 0$

From the graph it can be seen that

$m > 3$ or $m < -3$



Section summary

- The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- The axis of symmetry of the graph of $y = ax^2 + bx + c$ has equation $x = -\frac{b}{2a}$.
- By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into the turning point form $y = a(x - h)^2 + k$.
- To complete the square of $x^2 + bx + c$:
  - Take half the coefficient of x (that is, $\frac{b}{2}$) and add and subtract its square $\frac{b^2}{4}$.
- To complete the square of $ax^2 + bx + c$:
  - First take out a as a factor and then complete the square inside the bracket.
- The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

From the formula it can be seen that:

- If $b^2 - 4ac > 0$, there are two solutions.
- If $b^2 - 4ac = 0$, there is one solution.
- If $b^2 - 4ac < 0$, there are no real solutions.

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Exercise 4A

Example 1, 2

1 Sketch the graphs of the following functions:

a $f(x) = 2(x - 1)^2$
b $f(x) = 2(x - 1)^2 - 2$
c $f(x) = -2(x - 1)^2$
d $f(x) = 4 - 2(x + 1)^2$
e $f(x) = 4 + 2(x + \frac{1}{2})^2$
f $f(x) = 2(x + 1)^2 - 1$
g $f(x) = 3(x - 2)^2 - 4$
h $f(x) = (x + 1)^2 - 1$
i $f(x) = 5x^2 - 1$
j $f(x) = 2(x + 1)^2 - 4$

Example 3

2 For each of the following quadratic functions, use the axis of symmetry to find the turning point of the graph, express the function in the form $y = a(x - h)^2 + k$, and hence find the maximum or minimum value and the range:

a $f(x) = x^2 + 3x - 2$
b $f(x) = x^2 - 6x + 8$
c $f(x) = 2x^2 + 8x - 6$
d $f(x) = 4x^2 + 8x - 7$
e $f(x) = 2x^2 - 5x$
f $f(x) = 7 - 2x - 3x^2$
g $f(x) = -2x^2 + 9x + 11$

Example 4

3 Find the x- and y-axis intercepts and the turning point, and hence sketch the graph of each of the following:

a $y = -x^2 + 2x$
b $y = x^2 - 6x + 8$
c $y = -x^2 - 5x - 6$
d $y = -2x^2 + 8x - 6$
e $y = 4x^2 - 12x + 9$
f $y = 6x^2 + 3x - 18$

Example 5

4 Sketch the graph of each of the following by first completing the square:

a $y = x^2 + 2x - 6$
b $y = x^2 - 4x - 10$
c $y = -x^2 - 5x - 3$
d $y = -2x^2 + 8x - 10$
e $y = x^2 - 7x + 3$

Example 6

5 Sketch the graph of $f(x) = 3x^2 - 2x - 1$ by first finding the equation of the axis of symmetry, then finding the coordinates of the vertex, and finally using the quadratic formula to calculate the x-axis intercepts.

6 Sketch the graph of $f(x) = -3x^2 - 2x + 2$ by first finding the equation of the axis of symmetry, then finding the coordinates of the vertex, and finally using the quadratic formula to calculate the x-axis intercepts.

7 Sketch the graphs of the following functions, clearly labelling the axis intercepts and turning points:

a $f(x) = x^2 + 3x - 2$
b $f(x) = 2x^2 + 4x - 7$
c $f(x) = 5x^2 - 10x - 1$
d $f(x) = -2x^2 + 4x - 1$
e $y = 2.5x^2 + 3x + 0.3$
f $y = -0.6x^2 - 1.3x - 0.1$

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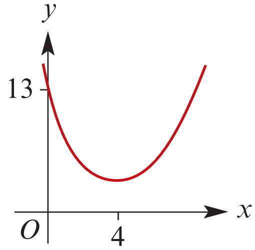
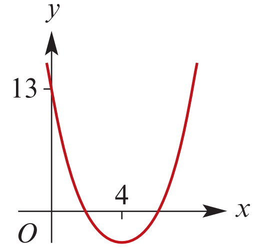
8 **a** Which of the graphs shown could represent $y = (x - 4)^2 - 3$?

b Which graph could represent $y = 3 - (x - 4)^2$?

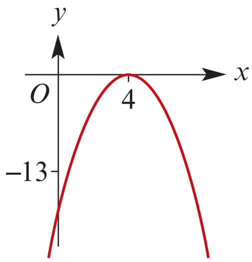
A



B



C



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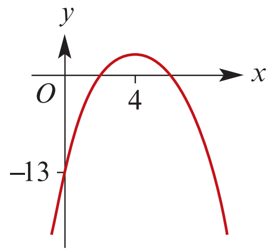


9 Match each of the following functions with the appropriate graph below:

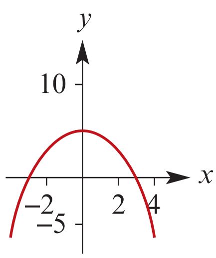
a $y = \frac{1}{3}(x + 4)(8 - x)$

b $y = x^2 - \frac{x}{2} + 1$

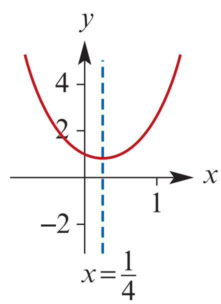
c $y = -10 + 2(x - 1)^2$

d $y = \frac{1}{2}(9 - x^2)$

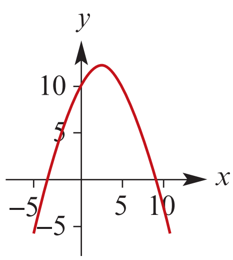
A



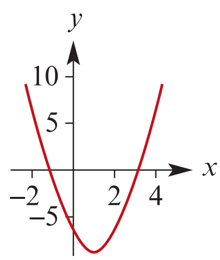
B



C



D



Example 7 **10** Without sketching the graphs of the following functions, determine whether they cross, touch or do not intersect the x-axis:

a $f(x) = x^2 - 5x + 2$ **b** $f(x) = -4x^2 + 2x - 1$ **c** $f(x) = x^2 - 6x + 9$

d $f(x) = 8 - 3x - 2x^2$ **e** $f(x) = 3x^2 + 2x + 5$ **f** $f(x) = -x^2 - x - 1$

Example 8 **11** For which values of m does the equation $mx^2 - 2mx + 3 = 0$ have:

a two solutions for x **b** one solution for x?

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12 Find the value of m for which $(4m + 1)x^2 - 6mx + 4$ is a perfect square.

13 Find the values of a for which the equation $(a - 3)x^2 + 2ax + (a + 2) = 0$ has no solutions for x.

14 Prove that the equation $x^2 + (a + 1)x + (a - 2) = 0$ always has two distinct solutions.

15 Show that the equation $(k + 1)x^2 - 2x - k = 0$ has a solution for all values of k.

16 For which values of k does the equation $kx^2 - 2kx = 5$ have:

a two solutions for x **b** one solution for x?

17 For which values of k does the equation $(k - 3)x^2 + 2kx + (k + 2) = 0$ have:

a two solutions for x **b** one solution for x?

18 Show that the equation $ax^2 - (a + b)x + b = 0$ has a solution for all values of a and b.

4B Determining the rule for a parabola

In this section we revise methods for finding the rule of a quadratic function from information about its graph. The following three forms are useful. You will see others in the worked examples.

1 $y = a(x - e)(x - f)$ This can be used if two x-axis intercepts and the coordinates of one other point are known.

2 $y = a(x - h)^2 + k$ This can be used if the coordinates of the turning point and one other point are known.

3 $y = ax^2 + bx + c$ This can be used if the coordinates of three points on the parabola are known.

Example 9

A parabola has x-axis intercepts -3 and 4 and it passes through the point $(1, 24)$. Find the rule for this parabola.

Solution

$y = a(x + 3)(x - 4)$

When $x = 1$, $y = 24$. Thus

$$24 = a(1 + 3)(1 - 4)$$
$$24 = -12a$$
$$\therefore \quad a = -2$$

The rule is $y = -2(x + 3)(x - 4)$.

Explanation

Two x-axis intercepts are given. Therefore use the form $y = a(x - e)(x - f)$.

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Example 10

The coordinates of the turning point of a parabola are $(2, 6)$ and the parabola passes through the point $(3, 3)$. Find the rule for this parabola.

Solution

$y = a(x - 2)^2 + 6$

When $x = 3$, $y = 3$. Thus

$$3 = a(3 - 2)^2 + 6$$
$$3 = a + 6$$
$$\therefore \quad a = -3$$

The rule is $y = -3(x - 2)^2 + 6$.

Explanation

The coordinates of the turning point and one other point on the parabola are given. Therefore use $y = a(x - h)^2 + k$.

Example 11

A parabola passes through the points $(1, 4)$, $(0, 5)$ and $(-1, 10)$. Find the rule for this parabola.

Solution

$y = ax^2 + bx + c$

When $x = 1$, $y = 4$.
When $x = 0$, $y = 5$.
When $x = -1$, $y = 10$.

Therefore

$$4 = a + b + c \qquad (1)$$
$$5 = c \qquad (2)$$
$$10 = a - b + c \qquad (3)$$

Substitute from equation (2) into equations (1) and (3):

$$-1 = a + b \qquad (1')$$
$$5 = a - b \qquad (3')$$

Add (1′) and (3′):

$$4 = 2a$$
$$\therefore \quad a = 2$$

Substitute into equation (1′):

$$-1 = 2 + b$$
$$\therefore \quad b = -3$$

The rule is $y = 2x^2 - 3x + 5$.

Explanation

The coordinates of three points on the parabola are given. Therefore we substitute values into the general polynomial form $y = ax^2 + bx + c$ to obtain three equations in three unknowns.

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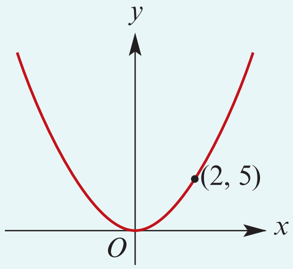
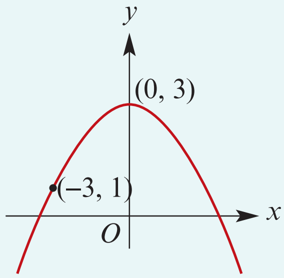
Example 12

Find the equation of each of the following parabolas:

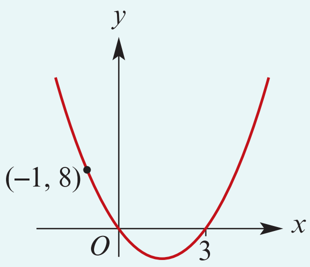
a



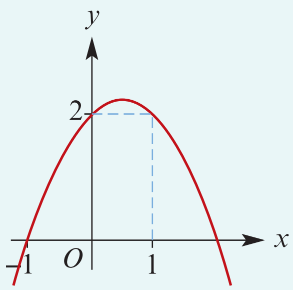
b



c



d



Solution

a This is of the form $y = ax^2$ (since the graph has its vertex at the origin).
As the point $(2, 5)$ is on the parabola,

$$5 = a(2)^2$$

$$\therefore \quad a = \frac{5}{4}$$

The rule is $y = \frac{5}{4}x^2$.

b This is of the form $y = ax^2 + c$ (since the graph is symmetric about the y-axis).

For $(0, 3)$: $\quad 3 = a(0)^2 + c$

$$\therefore \quad c = 3$$

For $(-3, 1)$: $\quad 1 = a(-3)^2 + 3$

$$1 = 9a + 3$$

$$\therefore \quad a = -\frac{2}{9}$$

The rule is $y = -\frac{2}{9}x^2 + 3$.

c This is of the form $y = ax(x - 3)$.
As the point $(-1, 8)$ is on the parabola,

$$8 = -a(-1 - 3)$$

$$8 = 4a$$

$$\therefore \quad a = 2$$

The rule is $y = 2x(x - 3)$.

d This is of the form $y = ax^2 + bx + c$.
The y-axis intercept is 2 and so $c = 2$.
As $(-1, 0)$ and $(1, 2)$ are on the parabola,

$$0 = a - b + 2 \qquad (1)$$

$$2 = a + b + 2 \qquad (2)$$

Add equations (1) and (2):

$$2 = 2a + 4$$

$$2a = -2$$

$$\therefore \quad a = -1$$

Substitute $a = -1$ in (1) to obtain $b = 1$.
The rule is $y = -x^2 + x + 2$.

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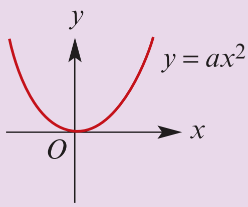
Section summary

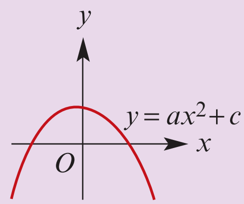
To find a quadratic rule to fit given points, first choose the best form of quadratic expression to work with. Then substitute in the coordinates of the known points to determine the unknown parameters. Some possible forms are given here:

i



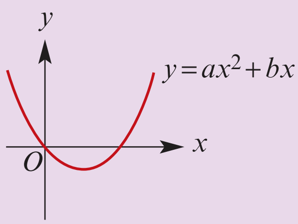
One point is needed to determine a.

ii



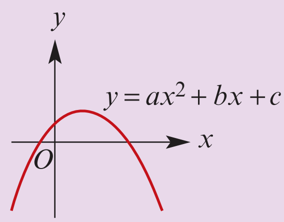
Two points are needed to determine a and c.

iii



Two points are needed to determine a and b.

iv



Three points are needed to determine a, b and c.

Exercise 4B

Example 9
Skillsheet

1 A parabola has x-axis intercepts -3 and -2 and it passes through the point $(1, -24)$. Find the rule for this parabola.

2 A parabola has x-axis intercepts -3 and $-\frac{3}{2}$ and it passes through the point $(1, 20)$. Find the rule for this parabola.

Example 10

3 The coordinates of the turning point of a parabola are $(-2, 4)$ and the parabola passes through the point $(4, 58)$. Find the rule for this parabola.

4 The coordinates of the turning point of a parabola are $(-2, -3)$ and the parabola passes through the point $(-3, -5)$. Find the rule for this parabola.

Example 11

5 A parabola passes through the points $(1, 19)$, $(0, 18)$ and $(-1, 7)$. Find the rule for this parabola.

6 A parabola passes through the points $(2, -14)$, $(0, 10)$ and $(-4, 10)$. Find the rule for this parabola.

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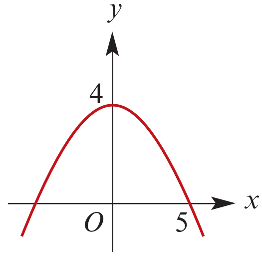
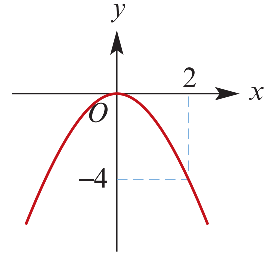
Example 12

7 Determine the equation of each of the following parabolas:

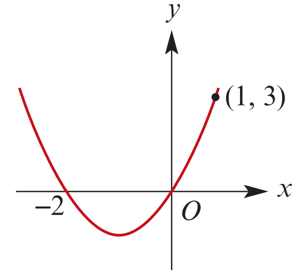
a



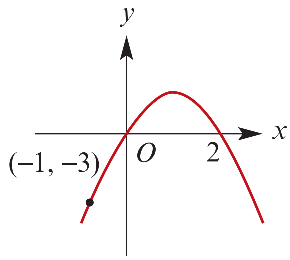
b



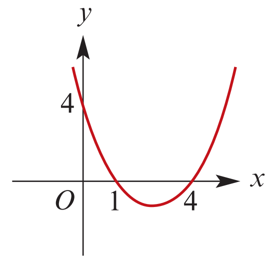
c



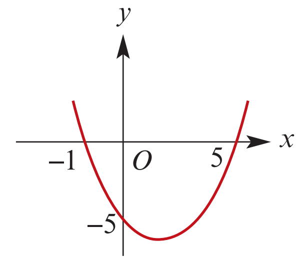
d



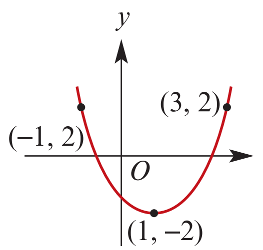
e



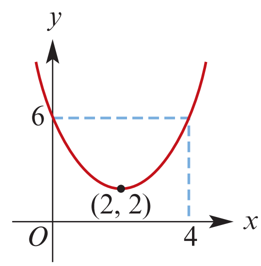
f



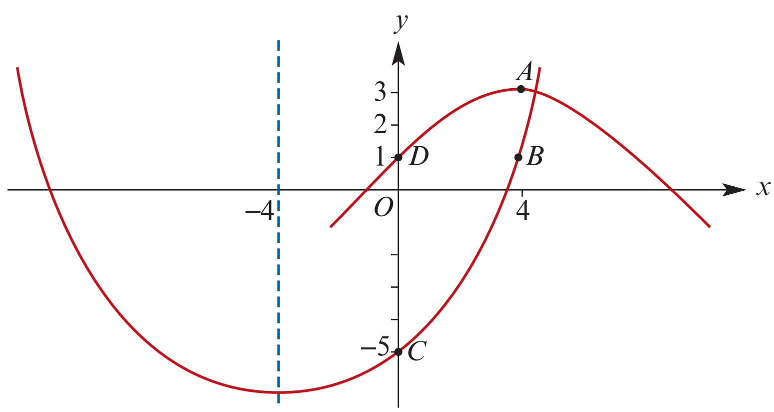
g



h



8 Find quadratic expressions for the two curves in the diagram, given that the coefficient of x in each case is 1. The marked points are $A(4, 3)$, $B(4, 1)$, $C(0, -5)$ and $D(0, 1)$.



9 The graph of the quadratic function $f(x) = A(x + b)^2 + B$ has a vertex at $(-2, 4)$ and passes through the point $(0, 8)$. Find the values of A, b and B.

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4C The language of polynomials

- A **polynomial function** is a function that can be written in the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
- The number 0 is called the **zero polynomial**.
- The **leading term**, $a_n x^n$, of a polynomial is the term of highest index among those terms with a non-zero coefficient.
- The **degree of a polynomial** is the index n of the leading term.
- A **monic polynomial** is a polynomial whose leading term has coefficient 1.
- The **constant term** is the term of index 0. (This is the term not involving x.)

Example 13

Let $P(x) = x^4 - 3x^3 - 2$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$

Solution

a $P(1) = 1^4 - 3 \times 1^3 - 2$
$= 1 - 3 - 2$
$= -4$

b $P(-1) = (-1)^4 - 3 \times (-1)^3 - 2$
$= 1 + 3 - 2$
$= 2$

c $P(2) = 2^4 - 3 \times 2^3 - 2$
$= 16 - 24 - 2$
$= -10$

d $P(-2) = (-2)^4 - 3 \times (-2)^3 - 2$
$= 16 + 24 - 2$
$= 38$

Example 14

a Let $P(x) = 2x^4 - x^3 + 2cx + 6$. If $P(1) = 21$, find the value of c.

b Let $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$. If $Q(-1) = Q(2) = 0$, find the values of a and b.

Solution

a $P(x) = 2x^4 - x^3 + 2cx + 6$ and $P(1) = 21$.

$$P(1) = 2(1)^4 - (1)^3 + 2c + 6$$
$$= 2 - 1 + 2c + 6$$
$$= 7 + 2c$$

Since $P(1) = 21$,

$$7 + 2c = 21$$
$$\therefore \quad c = 7$$

Explanation

We will substitute $x = 1$ into $P(x)$ to form an equation and solve.

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b $Q(x) = 2x^6 - x^3 + ax^2 + bx + 20$ and $Q(-1) = Q(2) = 0$.

First find $Q(-1)$ and $Q(2)$ in terms of a and b.

$$\begin{aligned} Q(-1) &= 2(-1)^6 - (-1)^3 + a(-1)^2 - b + 20 \\ &= 2 + 1 + a - b + 20 \\ &= 23 + a - b \end{aligned}$$

$$\begin{aligned} Q(2) &= 2(2)^6 - (2)^3 + a(2)^2 + 2b + 20 \\ &= 128 - 8 + 4a + 2b + 20 \\ &= 140 + 4a + 2b \end{aligned}$$

Since $Q(-1) = Q(2) = 0$, this gives

Form simultaneous equations in a and b by putting $Q(-1) = 0$ and $Q(2) = 0$.

$$23 + a - b = 0 \qquad (1)$$
$$140 + 4a + 2b = 0 \qquad (2)$$

Divide (2) by 2:

$$70 + 2a + b = 0 \qquad (3)$$

Add (1) and (3):

$$93 + 3a = 0$$
$$\therefore \quad a = -31$$

Substitute in (1) to obtain $b = -8$.

► The arithmetic of polynomials

The operations of addition, subtraction and multiplication for polynomials are naturally defined, as shown in the following examples.

Let $P(x) = x^3 + 3x^2 + 2$ and $Q(x) = 2x^2 + 4$. Then

$$\begin{aligned} P(x) + Q(x) &= (x^3 + 3x^2 + 2) + (2x^2 + 4) \\ &= x^3 + 5x^2 + 6 \end{aligned}$$

$$\begin{aligned} P(x) - Q(x) &= (x^3 + 3x^2 + 2) - (2x^2 + 4) \\ &= x^3 + x^2 - 2 \end{aligned}$$

$$\begin{aligned} P(x)Q(x) &= (x^3 + 3x^2 + 2)(2x^2 + 4) \\ &= (x^3 + 3x^2 + 2) \times 2x^2 + (x^3 + 3x^2 + 2) \times 4 \\ &= 2x^5 + 6x^4 + 4x^2 + 4x^3 + 12x^2 + 8 \\ &= 2x^5 + 6x^4 + 4x^3 + 16x^2 + 8 \end{aligned}$$

The sum, difference and product of two polynomials is a polynomial.

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Example 15

Let $P(x) = x^3 - 6x + 3$ and $Q(x) = x^2 - 3x + 1$. Find:

a $P(x) + Q(x)$ **b** $P(x) - Q(x)$ **c** $P(x)Q(x)$

Solution

a $P(x) + Q(x)$

$$\begin{aligned} &= x^3 - 6x + 3 + x^2 - 3x + 1 \\ &= x^3 + x^2 - 6x - 3x + 3 + 1 \\ &= x^3 + x^2 - 9x + 4 \end{aligned}$$

b $P(x) - Q(x)$

$$\begin{aligned} &= x^3 - 6x + 3 - (x^2 - 3x + 1) \\ &= x^3 - 6x + 3 - x^2 + 3x - 1 \\ &= x^3 - x^2 - 6x + 3x + 3 - 1 \\ &= x^3 - x^2 - 3x + 2 \end{aligned}$$

c
$$\begin{aligned} P(x)Q(x) &= (x^3 - 6x + 3)(x^2 - 3x + 1) \\ &= x^3(x^2 - 3x + 1) - 6x(x^2 - 3x + 1) + 3(x^2 - 3x + 1) \\ &= x^5 - 3x^4 + x^3 - 6x^3 + 18x^2 - 6x + 3x^2 - 9x + 3 \\ &= x^5 - 3x^4 + (x^3 - 6x^3) + (18x^2 + 3x^2) - (6x + 9x) + 3 \\ &= x^5 - 3x^4 - 5x^3 + 21x^2 - 15x + 3 \end{aligned}$$

We use the notation $\deg(f)$ to denote the degree of a polynomial f. For $f, g \neq 0$, we have

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$
$$\deg(f \times g) = \deg(f) + \deg(g)$$

Equating coefficients

Two polynomials P and Q are equal only if their corresponding coefficients are equal. For two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, they are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

For example, if

$$P(x) = 4x^3 + 5x^2 - x + 3 \quad \text{and} \quad Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$$

then $P(x) = Q(x)$ if and only if $b_3 = 4$, $b_2 = 5$, $b_1 = -1$ and $b_0 = 3$.

Example 16

The polynomial $P(x) = x^3 + 3x^2 + 2x + 1$ can be written in the form $(x - 2)(x^2 + bx + c) + r$ where b, c and r are real numbers. Find the values of b, c and r.

Solution

Expand the required form:

$$\begin{aligned} (x - 2)(x^2 + bx + c) + r &= x(x^2 + bx + c) - 2(x^2 + bx + c) + r \\ &= x^3 + bx^2 + cx - 2x^2 - 2bx - 2c + r \\ &= x^3 + (b - 2)x^2 + (c - 2b)x - 2c + r \end{aligned}$$

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If $x^3 + 3x^2 + 2x + 1 = x^3 + (b-2)x^2 + (c-2b)x - 2c + r$ for all real numbers x, then by equating coefficients:

coefficient of x^2	$3 = b - 2$	$\therefore\ b = 5$
coefficient of x	$2 = c - 2b$	$\therefore\ c = 2b + 2 = 12$
constant term	$1 = -2c + r$	$\therefore\ r = 2c + 1 = 25$

Hence $b = 5$, $c = 12$ and $r = 25$.

This means $P(x) = (x-2)(x^2 + 5x + 12) + 25$.

Example 17

a If $x^3 + 3x^2 + 3x + 8 = a(x+1)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b Show that $x^3 + 6x^2 + 6x + 8$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

Solution

a Expand the right-hand side of the equation:

$$\begin{aligned} a(x+1)^3 + b &= a(x^3 + 3x^2 + 3x + 1) + b \\ &= ax^3 + 3ax^2 + 3ax + a + b \end{aligned}$$

If $x^3 + 3x^2 + 3x + 8 = ax^3 + 3ax^2 + 3ax + a + b$ for all $x \in \mathbb{R}$, then by equating coefficients:

coefficient of x^3	$1 = a$
coefficient of x^2	$3 = 3a$
coefficient of x	$3 = 3a$
constant term	$8 = a + b$

Hence $a = 1$ and $b = 7$.

b Expand the proposed form:

$$\begin{aligned} a(x+c)^3 + b &= a(x^3 + 3cx^2 + 3c^2x + c^3) + b \\ &= ax^3 + 3cax^2 + 3c^2ax + c^3a + b \end{aligned}$$

Suppose $x^3 + 6x^2 + 6x + 8 = ax^3 + 3cax^2 + 3c^2ax + c^3a + b$ for all $x \in \mathbb{R}$. Then

coefficient of x^3	$1 = a$	(1)
coefficient of x^2	$6 = 3ca$	(2)
coefficient of x	$6 = 3c^2a$	(3)
constant term	$8 = c^3a + b$	(4)

From (1), we have $a = 1$. So from (2), we have $c = 2$.
But substituting $a = 1$ and $c = 2$ into (3) gives $6 = 12$, which is a contradiction.

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Section summary

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \ldots, a_n$ are real numbers with $a_n \neq 0$.
  The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

Exercise 4C

Example 13

1 Let $P(x) = x^3 - 2x^2 + 3x + 1$. Find:

a $P(1)$ **b** $P(-1)$ **c** $P(2)$ **d** $P(-2)$ **e** $P(\frac{1}{2})$ **f** $P(-\frac{1}{2})$

2 Let $P(x) = x^3 + 3x^2 - 4x + 6$. Find:

a $P(0)$ **b** $P(1)$ **c** $P(2)$ **d** $P(-1)$ **e** $P(a)$ **f** $P(2a)$

Example 14

3 **a** Let $P(x) = x^3 + 3x^2 - ax - 30$. If $P(2) = 0$, find the value of a.

b Let $P(x) = x^3 + ax^2 + 5x - 14$. If $P(3) = 68$, find the value of a.

c Let $P(x) = x^4 - x^3 - 2x + c$. If $P(1) = 6$, find the value of c.

d Let $P(x) = 2x^6 - 5x^3 + ax^2 + bx + 12$. If $P(-1) = P(2) = 0$, find a and b.

e Let $P(x) = x^5 - 2x^4 + ax^3 + bx^2 + 12x - 36$. If $P(3) = P(1) = 0$, find a and b.

Example 15

4 Let $f(x) = 2x^3 - x^2 + 3x$, $g(x) = 2 - x$ and $h(x) = x^2 + 2x$. Simplify each of the following:

a $f(x) + g(x)$ **b** $f(x) + h(x)$ **c** $f(x) - g(x)$

d $3f(x)$ **e** $f(x)\,g(x)$ **f** $g(x)\,h(x)$

g $f(x) + g(x) + h(x)$ **h** $f(x)\,h(x)$

5 Expand each of the following products and collect like terms:

a $(x - 2)(x^2 - 3x + 4)$ **b** $(x - 5)(x^2 - 2x + 3)$ **c** $(x + 1)(2x^2 - 3x - 4)$

d $(x + 2)(x^2 + bx + c)$ **e** $(2x - 1)(x^2 - 4x - 3)$

Example 16

6 It is known that $x^3 - x^2 - 6x - 4 = (x + 1)(x^2 + bx + c)$ for all values of x, for suitable values of b and c.

a Expand $(x + 1)(x^2 + bx + c)$ and collect like terms.

b Find b and c by equating coefficients.

c Hence write $x^3 - x^2 - 6x - 4$ as a product of three linear factors.

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Example 17

7 **a** If $2x^3 - 18x^2 + 54x - 49 = a(x-3)^3 + b$ for all $x \in \mathbb{R}$, find the values of a and b.

b If $-2x^3 + 18x^2 - 54x + 52 = a(x+c)^3 + b$ for all $x \in \mathbb{R}$, find the values of a, b and c.

c Show that $x^3 - 5x^2 - 2x + 24$ cannot be written in the form $a(x+c)^3 + b$ for real numbers a, b and c.

8 Find the values of A and B such that $A(x+3) + B(x+2) = 4x + 9$ for all real numbers x.

9 Find the values of A, B and C in each of the following:

a $x^2 - 4x + 10 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

b $4x^2 - 12x + 14 = A(x+B)^2 + C$ for all $x \in \mathbb{R}$

c $x^3 - 9x^2 + 27x - 22 = A(x+B)^3 + C$ for all $x \in \mathbb{R}$.

4D Division and factorisation of polynomials

The division of polynomials was introduced in Mathematical Methods Units 1 & 2.

When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

$$P(x) = D(x)Q(x) + R(x)$$

and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.

Here $P(x)$ is the **dividend** and $D(x)$ is the **divisor**.

The following example illustrates the process of dividing.

Example 18

Divide $x^3 + x^2 - 14x - 24$ by $x + 2$.

Solution

$$\begin{array}{r|l}
 & x^2 - x - 12 \\
\hline
x+2 & x^3 + x^2 - 14x - 24 \\
 & x^3 + 2x^2 \\
\cline{2-2}
 & -x^2 - 14x - 24 \\
 & -x^2 - 2x \\
\cline{2-2}
 & -12x - 24 \\
 & -12x - 24 \\
\cline{2-2}
 & 0
\end{array}$$

Explanation

- Divide x, from $x + 2$, into the leading term x^3 to get x^2.
- Multiply x^2 by $x + 2$ to give $x^3 + 2x^2$.
- Subtract from $x^3 + x^2 - 14x - 24$, leaving $-x^2 - 14x - 24$.
- Now divide x, from $x + 2$, into $-x^2$ to get $-x$.
- Multiply $-x$ by $x + 2$ to give $-x^2 - 2x$.
- Subtract from $-x^2 - 14x - 24$, leaving $-12x - 24$.
- Divide x into $-12x$ to get -12.
- Multiply -12 by $x + 2$ to give $-12x - 24$.
- Subtract from $-12x - 24$, leaving remainder of 0.

In this example we see that $x + 2$ is a factor of $x^3 + x^2 - 14x - 24$, as the remainder is zero. Thus $(x^3 + x^2 - 14x - 24) \div (x + 2) = x^2 - x - 12$ with zero remainder.

$$\therefore \quad \frac{x^3 + x^2 - 14x - 24}{x + 2} = x^2 - x - 12$$

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Example 19

Divide $3x^4 - 9x^2 + 27x - 8$ by $x - 2$.

Solution

$$
\begin{array}{r@{}l}
 & 3x^3 + 6x^2 + 3x + 33 \\
\cline{2-2}
x - 2 & \big)\, 3x^4 + 0x^3 - 9x^2 + 27x - 8 \\
 & 3x^4 - 6x^3 \\
\cline{2-2}
 & \qquad 6x^3 - 9x^2 + 27x - 8 \\
 & \qquad 6x^3 - 12x^2 \\
\cline{2-2}
 & \qquad\qquad 3x^2 + 27x - 8 \\
 & \qquad\qquad 3x^2 - 6x \\
\cline{2-2}
 & \qquad\qquad\qquad 33x - 8 \\
 & \qquad\qquad\qquad 33x - 66 \\
\cline{2-2}
 & \qquad\qquad\qquad\qquad 58
\end{array}
$$

Therefore

$$3x^4 - 9x^2 + 27x - 8 = (x - 2)(3x^3 + 6x^2 + 3x + 33) + 58$$

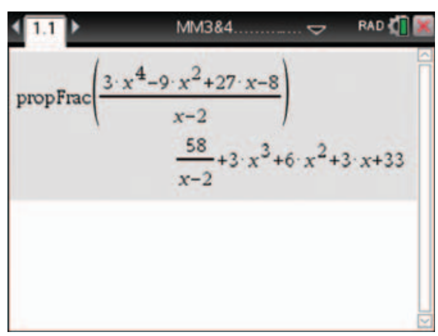
or, equivalently,

$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2} = 3x^3 + 6x^2 + 3x + 33 + \frac{58}{x - 2}$$

In this example, the dividend is $3x^4 - 9x^2 + 27x - 8$, the divisor is $x - 2$, and the remainder is 58.

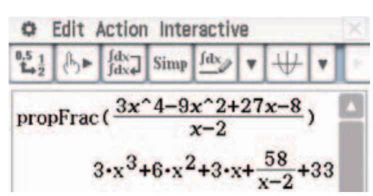
Using the TI-Nspire

Use **propFrac** from (menu) > **Algebra** > **Fraction Tools** > **Proper Fraction** as shown.



Using the Casio ClassPad

- Enter and highlight
$$\frac{3x^4 - 9x^2 + 27x - 8}{x - 2}$$
- Select **Interactive** > **Transformation** > **propFrac**.



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A second method for division, called **equating coefficients**, can be seen in the explanation column of the next example.

Example 20

Divide $3x^3 + 2x^2 - x - 2$ by $2x + 1$.

Solution

$$
\begin{array}{r|l}
 & \frac{3}{2}x^2 + \frac{1}{4}x - \frac{5}{8} \\
\hline
2x + 1 & 3x^3 + 2x^2 - x - 2 \\
 & 3x^3 + \frac{3}{2}x^2 \\
\cline{2-2}
 & \qquad \frac{1}{2}x^2 - x - 2 \\
 & \qquad \frac{1}{2}x^2 + \frac{1}{4}x \\
\cline{2-2}
 & \qquad\qquad -\frac{5}{4}x - 2 \\
 & \qquad\qquad -\frac{5}{4}x - \frac{5}{8} \\
\cline{2-2}
 & \qquad\qquad\qquad -1\frac{3}{8}
\end{array}
$$

Explanation

We show the alternative method here.

First write the identity

$3x^3 + 2x^2 - x - 2 = (2x + 1)(ax^2 + bx + c) + r$

Equate coefficients of x^3:

$3 = 2a$. Therefore $a = \frac{3}{2}$.

Equate coefficients of x^2:

$2 = a + 2b$. Therefore $b = \frac{1}{2}(2 - \frac{3}{2}) = \frac{1}{4}$.

Equate coefficients of x:

$-1 = 2c + b$. Therefore $c = \frac{1}{2}(-1 - \frac{1}{4}) = -\frac{5}{8}$.

Equate constant terms:

$-2 = c + r$. Therefore $r = -2 + \frac{5}{8} = -\frac{11}{8}$.

A third method, called **synthetic division**, is described in the Interactive Textbook.

Dividing by a non-linear polynomial

We give one example of dividing by a non-linear polynomial. The technique is exactly the same as when dividing by a linear polynomial.

Example 21

Divide $3x^3 - 2x^2 + 3x - 4$ by $x^2 - 1$.

Solution

$$
\begin{array}{r|l}
 & 3x - 2 \\
\hline
x^2 + 0x - 1 & 3x^3 - 2x^2 + 3x - 4 \\
 & 3x^3 + 0x^2 - 3x \\
\cline{2-2}
 & \qquad -2x^2 + 6x - 4 \\
 & \qquad -2x^2 + 0x + 2 \\
\cline{2-2}
 & \qquad\qquad\quad 6x - 6
\end{array}
$$

Therefore

$$3x^3 - 2x^2 + 3x - 4 = (x^2 - 1)(3x - 2) + 6x - 6$$

or, equivalently,

$$\frac{3x^3 - 2x^2 + 3x - 4}{x^2 - 1} = 3x - 2 + \frac{6x - 6}{x^2 - 1}$$

Explanation

We write $x^2 - 1$ as $x^2 + 0x - 1$.

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► The remainder theorem and the factor theorem

The following two results are recalled from Mathematical Methods Units 1 & 2.

The remainder theorem

Suppose that, when the polynomial $P(x)$ is divided by $x - \alpha$, the quotient is $Q(x)$ and the remainder is R. Then

$$P(x) = (x - \alpha)Q(x) + R$$

Now, as the two expressions are equal for all values of x, they are equal for $x = \alpha$.

$$\therefore \quad P(\alpha) = (\alpha - \alpha)Q(\alpha) + R \qquad \therefore \quad R = P(\alpha)$$

i.e. the remainder when $P(x)$ is divided by $x - \alpha$ is equal to $P(\alpha)$. We therefore have

$$P(x) = (x - \alpha)Q(x) + P(\alpha)$$

More generally:

Remainder theorem

When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.

Example 22

Find the remainder when $P(x) = 3x^3 + 2x^2 + x + 1$ is divided by $2x + 1$.

Solution

By the remainder theorem, the remainder is

$$P\left(-\frac{1}{2}\right) = 3\left(-\frac{1}{2}\right)^3 + 2\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1$$

$$= -\frac{3}{8} + \frac{2}{4} - \frac{1}{2} + 1 = \frac{5}{8}$$

The factor theorem

Now, in order for $x - \alpha$ to be a factor of the polynomial $P(x)$, the remainder must be zero. We state this result as the **factor theorem**.

Factor theorem

For a polynomial $P(x)$:

- If $P(\alpha) = 0$, then $x - \alpha$ is a factor of $P(x)$.
- Conversely, if $x - \alpha$ is a factor of $P(x)$, then $P(\alpha) = 0$.

More generally:

- If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
- Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.

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Example 23

Given that $x + 1$ and $x - 2$ are factors of $6x^4 - x^3 + ax^2 - 6x + b$, find the values of a and b.

Solution

Let $P(x) = 6x^4 - x^3 + ax^2 - 6x + b$.

By the factor theorem, we have $P(-1) = 0$ and $P(2) = 0$. Hence

$$6 + 1 + a + 6 + b = 0 \qquad (1)$$

$$96 - 8 + 4a - 12 + b = 0 \qquad (2)$$

Rearranging gives:

$$a + b = -13 \qquad (1')$$

$$4a + b = -76 \qquad (2')$$

Subtract (1′) from (2′):

$$3a = -63$$

Therefore $a = -21$ and, from (1′), $b = 8$.

Example 24

Show that $x + 1$ is a factor of $x^3 - 4x^2 + x + 6$ and hence find the other linear factors.

Solution

Let $\quad P(x) = x^3 - 4x^2 + x + 6$

Then $\quad P(-1) = (-1)^3 - 4(-1)^2 + (-1) + 6$

$\qquad\qquad = 0$

Thus $x + 1$ is a factor (by the factor theorem).

Divide by $x + 1$ to find the other factor:

$$
\begin{array}{r l}
 & x^2 - 5x + 6 \\
\cline{2-2}
x + 1 \big) & x^3 - 4x^2 + x + 6 \\
 & x^3 + x^2 \\
\cline{2-2}
 & \quad -5x^2 + x + 6 \\
 & \quad -5x^2 - 5x \\
\cline{2-2}
 & \qquad\quad 6x + 6 \\
 & \qquad\quad 6x + 6 \\
\cline{2-2}
 & \qquad\qquad\quad 0
\end{array}
$$

$$\begin{aligned} \therefore\ x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) \\ &= (x + 1)(x - 3)(x - 2) \end{aligned}$$

The linear factors of $x^3 - 4x^2 + x + 6$ are $(x + 1)$, $(x - 3)$ and $(x - 2)$.

Explanation

We can use the factor theorem to find one factor, and then divide to find the other two linear factors.

Here is an alternative method:

Once we have found that $x + 1$ is a factor, we know that we can write

$$x^3 - 4x^2 + x + 6 = (x + 1)(x^2 + bx + c)$$

By equating constant terms, we have $6 = 1 \times c$. Hence $c = 6$.

By equating coefficients of x^2, we have $-4 = 1 + b$. Hence $b = -5$.

$$\therefore \quad x^3 - 4x^2 + x + 6 = (x + 1)(x^2 - 5x + 6)$$

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Sums and differences of cubes

If $P(x) = x^3 - a^3$, then $x - a$ is a factor and so by division:

$$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

If a is replaced by $-a$, then

$$x^3 - (-a)^3 = (x - (-a))(x^2 + (-a)x + (-a)^2)$$

This gives:

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$$

Example 25

Factorise:

a $8x^3 + 64$

b $125a^3 - b^3$

Solution

a $$\begin{aligned} 8x^3 + 64 &= (2x)^3 + (4)^3 \\ &= (2x + 4)(4x^2 - 8x + 16) \end{aligned}$$

b $$\begin{aligned} 125a^3 - b^3 &= (5a)^3 - b^3 \\ &= (5a - b)(25a^2 + 5ab + b^2) \end{aligned}$$

The rational-root theorem

Consider the cubic polynomial

$$P(x) = 2x^3 - x^2 - x - 3$$

If the equation $P(x) = 0$ has a solution α that is an integer, then α divides the constant term -3. We can easily show that $P(1) \neq 0$, $P(-1) \neq 0$, $P(3) \neq 0$ and $P(-3) \neq 0$. Hence the equation $P(x) = 0$ has no solution that is an integer.

Does it have a rational solution, that is, a fraction for a solution?

The **rational-root theorem** helps us with this. It says that if α and β have highest common factor 1 (i.e. α and β are relatively prime) and $\beta x + \alpha$ is a factor of $2x^3 - x^2 - x - 3$, then β divides 2 and α divides -3.

Therefore, if $-\dfrac{\alpha}{\beta}$ is a solution of the equation $P(x) = 0$ (where α and β are relatively prime), then β must divide 2 and α must divide -3. So the only value of β that needs to be considered is 2, and $\alpha = \pm 3$ or $\alpha = \pm 1$.

We can test these through the factor theorem. That is, check $P\left(\pm\dfrac{1}{2}\right)$ and $P\left(\pm\dfrac{3}{2}\right)$. We find

$$\begin{aligned} P\left(\frac{3}{2}\right) &= 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right) - 3 \\ &= 2 \times \frac{27}{8} - \frac{9}{4} - \frac{3}{2} - 3 \\ &= 0 \end{aligned}$$

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We have found that $2x - 3$ is a factor of $P(x) = 2x^3 - x^2 - x - 3$.

Dividing through we find that

$$2x^3 - x^2 - x - 3 = (2x - 3)(x^2 + x + 1)$$

We can show that $x^2 + x + 1$ has no linear factors by showing that the discriminant of this quadratic is negative.

Example 26

Use the rational-root theorem to help factorise $P(x) = 3x^3 + 8x^2 + 2x - 5$.

Solution

$$P(1) = 8 \neq 0, \qquad P(-1) = -2 \neq 0,$$

$$P(5) = 580 \neq 0, \qquad P(-5) = -190 \neq 0,$$

$$P\left(-\frac{5}{3}\right) = 0$$

Therefore $3x + 5$ is a factor.

Dividing gives

$$3x^3 + 8x^2 + 2x - 5 = (3x + 5)(x^2 + x - 1)$$

We complete the square for $x^2 + x - 1$ to factorise:

$$\begin{aligned} x^2 + x - 1 &= x^2 + x + \frac{1}{4} - \frac{1}{4} - 1 \\ &= \left(x + \frac{1}{2}\right)^2 - \frac{5}{4} \\ &= \left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right) \end{aligned}$$

Hence

$$P(x) = (3x + 5)\left(x + \frac{1}{2} + \frac{\sqrt{5}}{2}\right)\left(x + \frac{1}{2} - \frac{\sqrt{5}}{2}\right)$$

Explanation

The only possible integer solutions are ± 5 or ± 1. So there are no integer solutions. We now use the rational-root theorem.

If $-\dfrac{\alpha}{\beta}$ is a solution, the only value of β that needs to be considered is 3 and $\alpha = \pm 5$ or $\alpha = \pm 1$.

Here is the complete statement of the theorem:

Rational-root theorem

Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime).

If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.

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Solving polynomial equations

The factor theorem may be used in the solution of equations.

Example 27

Factorise $P(x) = x^3 - 4x^2 - 11x + 30$ and hence solve the equation $x^3 - 4x^2 - 11x + 30 = 0$.

Solution

$$P(1) = 1 - 4 - 11 + 30 \neq 0$$
$$P(-1) = -1 - 4 + 11 + 30 \neq 0$$
$$P(2) = 8 - 16 - 22 + 30 = 0$$

Therefore $x - 2$ is a factor.

Dividing $x^3 - 4x^2 - 11x + 30$ by $x - 2$ gives

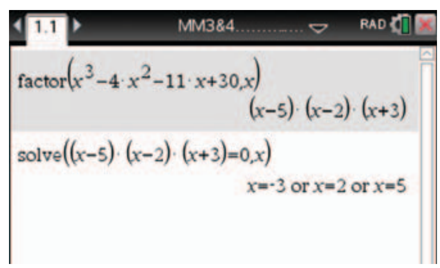
$$\begin{aligned} P(x) &= (x-2)(x^2 - 2x - 15) \\ &= (x-2)(x-5)(x+3) \end{aligned}$$

Now we see that $P(x) = 0$ if and only if

$$x - 2 = 0 \quad \text{or} \quad x - 5 = 0 \quad \text{or} \quad x + 3 = 0$$
$$\therefore \quad x = 2 \quad \text{or} \quad x = 5 \quad \text{or} \quad x = -3$$

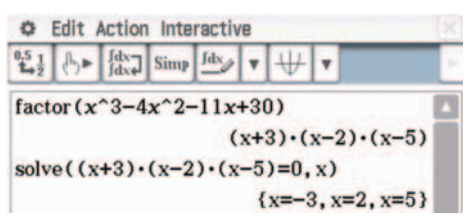
Using the TI-Nspire

Use **factor** (menu > **Algebra** > **Factor**) and **solve** (menu > **Algebra** > **Solve**) as shown.



Using the Casio ClassPad

- Enter and highlight $x^3 - 4x^2 - 11x + 30$.
- Select **Interactive** > **Transformation** > **factor**.
- Copy and paste the answer to the next entry line.
- Select **Interactive** > **Equation/Inequality** > **solve** and ensure the variable is x.



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Section summary

- Division of polynomials When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
- Two methods for dividing polynomials are long division and equating coefficients.
- Remainder theorem When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- Factor theorem
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.
- Rational-root theorem Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Difference of two cubes: $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
- Sum of two cubes: $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Exercise 4D

Skillsheet
Example 18

1 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 14x + 24, \ x + 4$

b $2x^3 + x^2 - 25x + 12, \ x - 3$

Example 19

2 For each of the following, divide the first polynomial by the second:

a $x^3 - x^2 - 15x + 25, \ x + 3$

b $2x^3 - 4x + 12, \ x - 3$

Example 20

3 For each of the following, divide the first polynomial by the second:

a $2x^3 - 2x^2 - 15x + 25, \ 2x + 3$

b $4x^3 + 6x^2 - 4x + 12, \ 2x - 3$

4 For each of the following, divide the first expression by the second:

a $2x^3 - 7x^2 + 15x - 3, \ x - 3$

b $5x^5 + 13x^4 - 2x^2 - 6, \ x + 1$

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Example 21 **5** For each of the following, divide the first expression by the second:

a $x^4 - 9x^3 + 25x^2 - 8x - 2,\ x^2 - 2$

b $x^4 + x^3 + x^2 - x - 2,\ x^2 - 1$

Example 22 **6** **a** Find the remainder when $x^3 + 3x - 2$ is divided by $x + 2$.

b Find the value of a for which $(1 - 2a)x^2 + 5ax + (a - 1)(a - 8)$ is divisible by $x - 2$ but not by $x - 1$.

7 Given that $f(x) = 6x^3 + 5x^2 - 17x - 6$:

a Find the remainder when $f(x)$ is divided by $x - 2$.

b Find the remainder when $f(x)$ is divided by $x + 2$.

c Factorise $f(x)$ completely.

8 **a** Prove that the expression $x^3 + (k - 1)x^2 + (k - 9)x - 7$ is divisible by $x + 1$ for all values of k.

b Find the value of k for which the expression has a remainder of 12 when divided by $x - 2$.

Example 23 **9** The polynomial $f(x) = 2x^3 + ax^2 - bx + 3$ has a factor $x + 3$. When $f(x)$ is divided by $x - 2$, the remainder is 15.

a Calculate the values of a and b.

b Find the other two linear factors of $f(x)$.

10 The expression $4x^3 + ax^2 - 5x + b$ leaves remainders of -8 and 10 when divided by $2x - 3$ and $x - 3$ respectively. Calculate the values of a and b.

11 Find the remainder when $(x + 1)^4$ is divided by $x - 2$.

12 Let $P(x) = x^5 - 3x^4 + 2x^3 - 2x^2 + 3x + 1$.

a Show that neither $x - 1$ nor $x + 1$ is a factor of $P(x)$.

b Given that $P(x)$ can be written in the form $(x^2 - 1)Q(x) + ax + b$, where $Q(x)$ is a polynomial and a and b are constants, hence or otherwise, find the remainder when $P(x)$ is divided by $x^2 - 1$.

Example 24 **13** Show that $x + 1$ is a factor of $2x^3 - 5x^2 - 4x + 3$ and find the other linear factors.

14 **a** Show that both $x - \sqrt{3}$ and $x + \sqrt{3}$ are factors of $x^4 + x^3 - x^2 - 3x - 6$.

b Hence write down one quadratic factor of $x^4 + x^3 - x^2 - 3x - 6$, and find a second quadratic factor.

Example 25 **15** Factorise each of the following:

a $8a^3 + 27b^3$ **b** $64 - a^3$

c $125x^3 + 64y^3$ **d** $(a - b)^3 + (a + b)^3$

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Example 26 **16** Use the rational-root theorem to help factorise each of the following:

a $12x^3 + 20x^2 - x - 6$

b $4x^3 - 2x^2 + 6x - 3$

17 Use the rational-root theorem to help factorise each of the following:

a $4x^3 + 3x - 18$

b $8x^3 - 12x^2 - 2x + 3$

Example 27 **18** Solve each of the following equations for x:

a $(2 - x)(x + 4)(x - 2)(x - 3) = 0$

b $x^3(2 - x) = 0$

c $(2x - 1)^3(2 - x) = 0$

d $(x + 2)^3(x - 2)^2 = 0$

e $x^4 - 4x^2 = 0$

f $x^4 - 9x^2 = 0$

g $12x^4 + 11x^3 - 26x^2 + x + 2 = 0$

h $x^4 + 2x^3 - 3x^2 - 4x + 4 = 0$

i $6x^4 - 5x^3 - 20x^2 + 25x - 6 = 0$

19 Find the x-axis intercepts and y-axis intercept of the graph of each of the following:

a $y = x^3 - x^2 - 2x$

b $y = x^3 - 2x^2 - 5x + 6$

c $y = x^3 - 4x^2 + x + 6$

d $y = 2x^3 - 5x^2 + x + 2$

e $y = x^3 + 2x^2 - x - 2$

f $y = 3x^3 - 4x^2 - 13x - 6$

g $y = 5x^3 + 12x^2 - 36x - 16$

h $y = 6x^3 - 5x^2 - 2x + 1$

i $y = 2x^3 - 3x^2 - 29x - 30$

20 The expressions $px^4 - 5x + q$ and $x^4 - 2x^3 - px^2 - qx - 8$ have a common factor $x - 2$. Find the values of p and q.

21 Find the remainder when $f(x) = x^4 - x^3 + 5x^2 + 4x - 36$ is divided by $x + 1$.

22 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $x^3 - 11x^2 - 125x + 1287$

b $x^3 - 9x^2 - 121x + 1089$

c $2x^3 - 9x^2 - 242x + 1089$

d $4x^3 - 367x + 1287$

23 Factorise each of the following:

a $x^4 - x^3 - 43x^2 + x + 42$

b $x^4 + 4x^3 - 27x - 108$

24 Factorise each of the following polynomials, using a calculator to help find at least one linear factor:

a $2x^4 - 25x^3 + 57x^2 + 9x + 405$

b $x^4 + 13x^3 + 40x^2 + 81x + 405$

c $x^4 + 3x^3 - 4x^2 + 3x - 135$

d $x^4 + 4x^3 - 35x^2 - 78x + 360$

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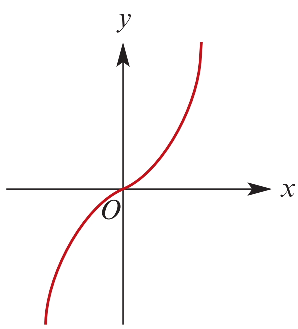
4E The general cubic function

Not all cubic functions can be written in the form $f(x) = a(x - h)^3 + k$. In this section we consider the general cubic function. The form of a general cubic function is

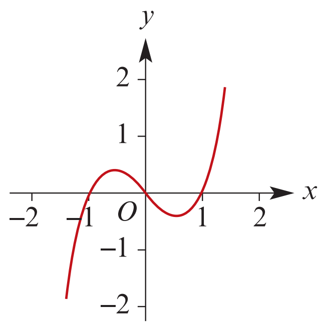
$$f(x) = ax^3 + bx^2 + cx + d, \quad \text{where } a \neq 0$$

It is impossible to fully investigate cubic functions without the use of calculus. Cubic functions will be revisited in Chapter 10.

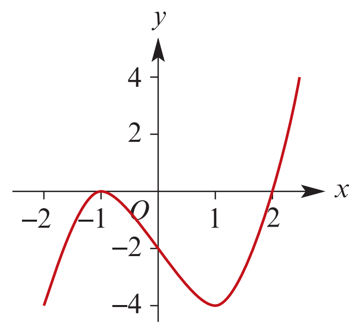
The 'shapes' of cubic graphs vary. Below is a gallery of cubic graphs, demonstrating the variety of 'shapes' that are possible.



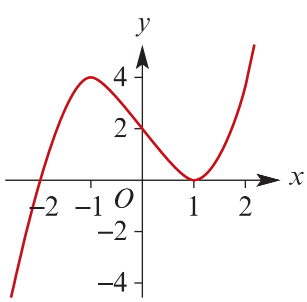
$f(x) = x^3 + x$



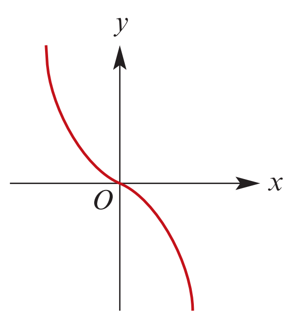
$f(x) = x^3 - x$



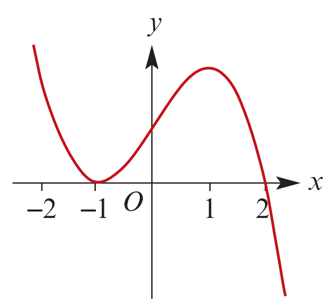
$f(x) = x^3 - 3x - 2$



$f(x) = x^3 - 3x + 2$



$f(x) = -x^3 - x$



$f(x) = -x^3 + 3x + 2$

Notes:

- A cubic graph can have one, two or three x-axis intercepts.
- Not all cubic graphs have a stationary point. For example, the graph of $f(x) = x^3 + x$ shown above has no points of zero gradient.
- The turning points do not occur symmetrically between consecutive x-axis intercepts as they do for quadratics. Differential calculus must be used to determine them.
- If a cubic graph has a turning point on the x-axis, this corresponds to a **repeated factor**. For example, the graph of $f(x) = x^3 - 3x - 2$ shown above has a turning point at $(-1, 0)$. The factorisation is $f(x) = (x + 1)^2(x - 2)$.

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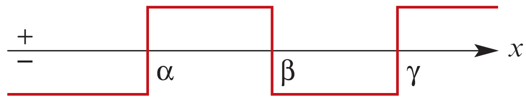
Sign diagrams

A **sign diagram** is a number-line diagram that shows when an expression is positive or negative. For a cubic function with rule $f(x) = (x - \alpha)(x - \beta)(x - \gamma)$, where $\alpha < \beta < \gamma$, the sign diagram is as shown.



Example 28

Draw a sign diagram for the cubic function $f(x) = x^3 - 4x^2 - 11x + 30$.

Solution

From Example 27, we have

$$f(x) = (x + 3)(x - 2)(x - 5)$$

Therefore $f(-3) = f(2) = f(5) = 0$. We note that

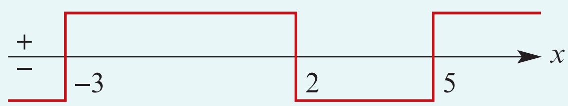
$f(x) > 0$ for $x > 5$

$f(x) < 0$ for $2 < x < 5$

$f(x) > 0$ for $-3 < x < 2$

$f(x) < 0$ for $x < -3$

Hence the sign diagram may be drawn as shown.



Example 29

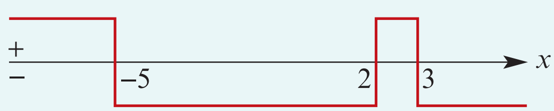
For the cubic function with rule $f(x) = -x^3 + 19x - 30$:

a Sketch the graph of $y = f(x)$ using a calculator to find the coordinates of the turning points, correct to two decimal places.

b Sketch the graph of $y = \frac{1}{2}f(x - 1)$.

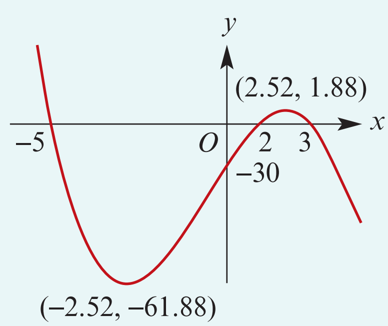
Solution

a
$$\begin{aligned} f(x) &= -x^3 + 19x - 30 \\ &= (3 - x)(x - 2)(x + 5) \\ &= -(x + 5)(x - 2)(x - 3) \end{aligned}$$



The x-axis intercepts are at $x = -5$, $x = 2$ and $x = 3$ and the y-axis intercept is at $y = -30$.

The turning points can be found using a CAS calculator. The method is described following this example.



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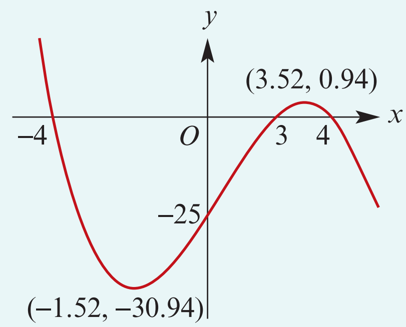
b The rule for the transformation is

$$(x, y) \rightarrow \left(x + 1, \frac{1}{2}y\right)$$

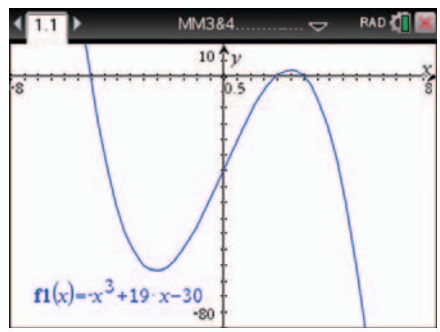
This is a dilation of factor $\frac{1}{2}$ from the x-axis followed by a translation 1 unit to the right.
Transformations of the turning points:

$$(2.52, 1.88) \rightarrow (3.52, 0.94)$$
$$(-2.52, -61.88) \rightarrow (-1.52, -30.94)$$

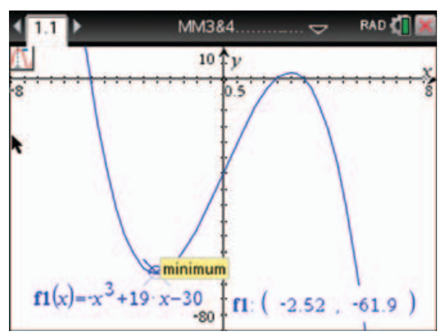


Using the TI-Nspire

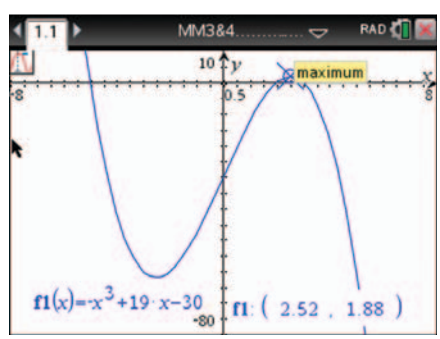
- Enter the function in a **Graphs** page.
- Use (menu) > **Window/Zoom** > **Window Settings** to set an appropriate window.
- Use either (menu) > **Trace** > **Graph Trace** or (menu) > **Analyze Graph** > **Maximum or Minimum** to display the approximate (decimal) coordinates of key points on the graph.



- In **Graph Trace**, the tracing point (×) can be moved either by using the arrow keys (◄ ►) or by typing a specific x-value then (enter). When the tracing point reaches a local minimum, it displays 'minimum'.
- Pressing (enter) will paste the coordinates to the point on the graph.
- Press (esc) to exit the command.



- Here **Graph Trace** has been used to find the turning points of the cubic function.
- If you use **Analyze Graph** instead, select the lower bound by moving to the left of the key point and clicking ( ) and then select the upper bound by moving to the right (►) of the key point and clicking.



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Using the Casio ClassPad

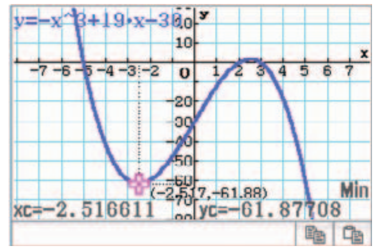
a ■ In the $\sqrt[\text{Main}]{\alpha}$ screen, define the function f.

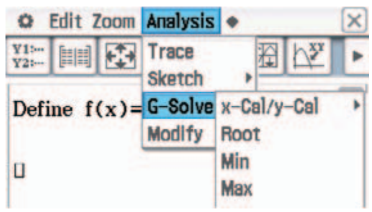
■ Tap [graph icon] to open the graph window.

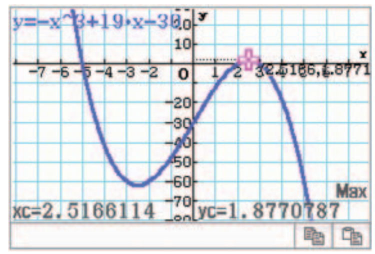
■ Highlight the function and drag into the graph window.

■ To find the local minimum, select **Analysis** > **G-Solve** > **Min**.

■ To find the local maximum, select **Analysis** > **G-Solve** > **Max**.



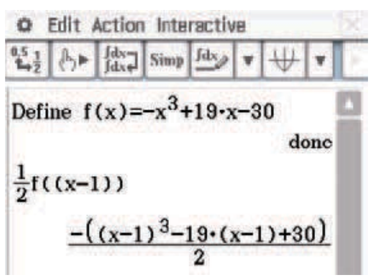


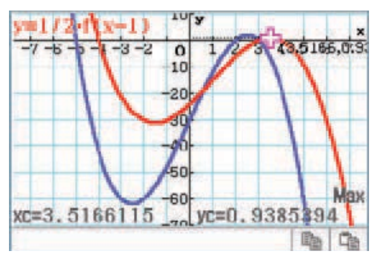


b ■ Enter the transformed function as $\frac{1}{2}f(x-1)$.

■ Highlight the transformed function and drag into the graph window.

■ The coordinates of the turning points can be found as above.





Section summary

■ The graph of a cubic function can have one, two or three x-axis intercepts.

■ The graph of a cubic function can have zero, one or two stationary points.

■ To sketch a cubic in factorised form $y = a(x-\alpha)(x-\beta)(x-\gamma)$:

- Find the y-axis intercept.
- Find the x-axis intercepts.
- Prepare a sign diagram.
- Consider the y-values as x increases to the right of all x-axis intercepts.
- Consider the y-values as x decreases to the left of all x-axis intercepts.

■ If there is a repeated factor to the power 2, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 4E

Example 28 **1** Draw a sign diagram for each of the following expressions:

a $(3 - x)(x - 1)(x - 6)$ **b** $(3 + x)(x - 1)(x + 6)$
c $(x - 5)(x + 1)(2x - 6)$ **d** $(4 - x)(5 - x)(1 - 2x)$
e $(x - 5)^2(x - 4)$ **f** $(x - 5)^2(4 - x)$

2 First factorise and then draw a sign diagram for each of the following expressions:

a $x^3 - 4x^2 + x + 6$ **b** $4x^3 + 3x^2 - 16x - 12$
c $x^3 - 7x^2 + 4x + 12$ **d** $2x^3 + 3x^2 - 11x - 6$

Example 29 **3** **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 - 2x^2 + 1$.
b On the same screen, plot the graphs of:
i $y = f(x - 2)$ **ii** $y = f(x + 2)$ **iii** $y = 3f(x)$

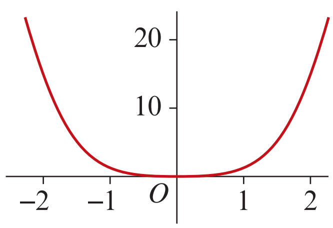
4 **a** Use a calculator to plot the graph of $y = f(x)$ where $f(x) = x^3 + x^2 - 4x + 2$.
b On the same screen, plot the graphs of:
i $y = f(2x)$ **ii** $y = f\left(\frac{x}{2}\right)$ **iii** $y = 2f(x)$

4F Polynomials of higher degree

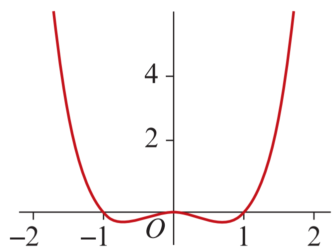
The general form for a quartic function is

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e, \quad \text{where } a \neq 0$$

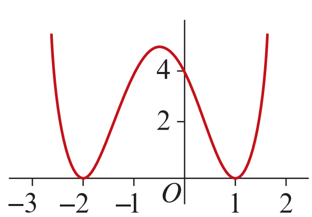
A gallery of quartic functions is shown below.



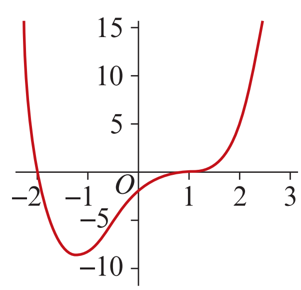
$f(x) = x^4$



$f(x) = x^4 - x^2$



$f(x) = (x - 1)^2(x + 2)^2$



$f(x) = (x - 1)^3(x + 2)$

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The techniques that have been developed for cubic functions may now be applied to quartic functions and to polynomial functions of higher degree in general.

For a polynomial $P(x)$ of degree n, there are at most n solutions to the equation $P(x) = 0$. Therefore the graph of $y = P(x)$ has at most n x-axis intercepts.

The graph of a polynomial of even degree may have no x-axis intercepts: for example, $P(x) = x^2 + 1$. But the graph of a polynomial of odd degree must have at least one x-axis intercept.

Example 30

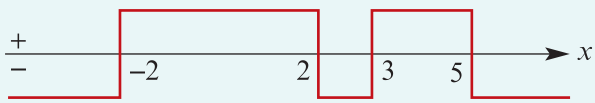
Draw a sign diagram for each quartic expression:

a $(2 - x)(x + 2)(x - 3)(x - 5)$

b $x^4 + x^2 - 2$

Solution

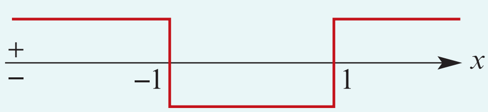
a



b Let $P(x) = x^4 + x^2 - 2$.
Then $P(1) = 1 + 1 - 2 = 0$.
Thus $x - 1$ is a factor.

$$
\begin{array}{r|l}
 & x^3 + x^2 + 2x + 2 \\
\hline
x - 1 & x^4 + 0x^3 + x^2 + 0x - 2 \\
 & x^4 - x^3 \\
\hline
 & \quad x^3 + x^2 + 0x - 2 \\
 & \quad x^3 - x^2 \\
\hline
 & \qquad 2x^2 + 0x - 2 \\
 & \qquad 2x^2 - 2x \\
\hline
 & \qquad\quad 2x - 2 \\
 & \qquad\quad 2x - 2 \\
\hline
 & \qquad\qquad 0
\end{array}
$$

$$
\begin{aligned}
\therefore \quad P(x) &= (x - 1)(x^3 + x^2 + 2x + 2) \\
&= (x - 1)[x^2(x + 1) + 2(x + 1)] \\
&= (x - 1)(x + 1)(x^2 + 2)
\end{aligned}
$$



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Example 31

For $p(x) = x^4 - 2x^2 + 1$, find the coordinates of the points where the graph of $y = p(x)$ intersects the x- and y-axes, and hence sketch the graph.

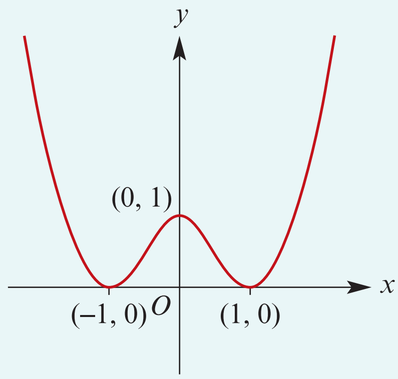
Solution

Note that

$$\begin{aligned} p(x) &= (x^2)^2 - 2(x^2) + 1 \\ &= (x^2 - 1)^2 \\ &= [(x-1)(x+1)]^2 \\ &= (x-1)^2(x+1)^2 \end{aligned}$$

Therefore the x-axis intercepts are 1 and -1.

When $x = 0$, $y = 1$. So the y-axis intercept is 1.



Explanation

Alternatively, we can factorise $p(x)$ by using the factor theorem and division.

Note that

$$p(1) = 1 - 2 + 1 = 0$$

Therefore $x - 1$ is a factor.

$$\begin{aligned} p(x) &= (x-1)(x^3 + x^2 - x - 1) \\ &= (x-1)[x^2(x+1) - (x+1)] \\ &= (x-1)(x+1)(x^2-1) \\ &= (x-1)^2(x+1)^2 \end{aligned}$$

Section summary

- The graph of a quartic function can have zero, one, two, three or four x-axis intercepts.
- The graph of a quartic function can have one, two or three stationary points.
- To sketch a quartic in factorised form $y = a(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$:
  - Find the y-axis intercept.
  - Find the x-axis intercepts.
  - Prepare a sign diagram.
  - Consider the y-values as x increases to the right of all x-axis intercepts.
  - Consider the y-values as x decreases to the left of all x-axis intercepts.
- If there is a repeated factor to an even power, then the y-values have the same sign immediately to the left and right of the corresponding x-axis intercept.

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Exercise 4F

Example 30 **1** Draw a sign diagram for each quartic expression:

a $(3 - x)(x + 4)(x - 5)(x - 1)$ **b** $x^4 - 2x^3 - 3x^2 + 4x + 4$

Example 31 **2** For $h(x) = 81x^4 - 72x^2 + 16$, find the coordinates of the points where the graph of $y = h(x)$ intersects the x- and y-axes, and hence sketch the graph.

Hint: First express $h(x)$ as the square of a quadratic expression.

3 **a** Use a calculator to plot the graph of $y = f(x)$, where $f(x) = x^4 - 2x^3 + x + 1$.

b On the same screen, plot the graphs of:

i $y = f(x - 2)$ **ii** $y = f(2x)$ **iii** $y = f\left(\dfrac{x}{2}\right)$

4 The graph of $y = 9x^2 - x^4$ is as shown. Sketch the graph of each of the following by applying suitable transformations:

a $y = 9(x - 1)^2 - (x - 1)^4$

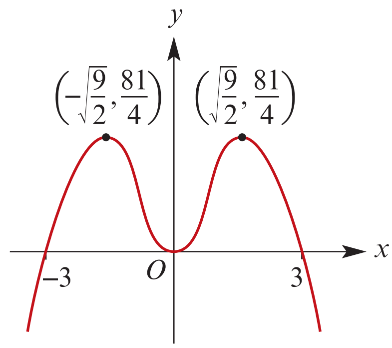
b $y = 18x^2 - 2x^4$

c $y = 18(x + 1)^2 - 2(x + 1)^4$

d $y = 9x^2 - x^4 - \dfrac{81}{4}$

e $y = 9x^2 - x^4 + 1$

(Do not find the x-axis intercepts for part **e**.)



5 Sketch the graph of $f(x) = x^6 - x^2$. (Use a calculator to find the stationary points.)

6 Sketch the graph of $f(x) = x^5 - x^3$. (Use a calculator to find the stationary points.)

4G Determining the rule for the graph of a polynomial

A straight line is determined by any two points on the line. More generally, the graph of a polynomial function of degree n is completely determined by any $n + 1$ points on the curve.

For example, for a cubic function with rule $y = f(x)$, if it is known that $f(a_1) = b_1$, $f(a_2) = b_2$, $f(a_3) = b_3$ and $f(a_4) = b_4$, then the rule can be determined.

Finding the rule for a parabola has been discussed in Section 4B.

The method for finding the rule from a graph of a cubic function will depend on what information is given in the graph.

If the cubic function has rule of the form $f(x) = a(x - h)^3 + k$ and the point of inflection (h, k) is given, then one other point needs to be known in order to find the value of a.

For those that are not of this form, the information given may be some or all of the x-axis intercepts as well as the coordinates of other points including possibly the y-axis intercept.

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Example 32

a A cubic function has rule of the form $y = a(x-2)^3 + 2$. The point $(3, 10)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+2)(x-4)$. The point $(5, 16)$ is on the graph of the function. Find the value of a.

c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(1, 16)$ and $(2, 30)$ are on the graph of the function. Find the values of a and b.

Solution

Explanation

In each of these problems, we substitute the given values to find the unknowns.

a $y = a(x-2)^3 + 2$

When $x = 3$, $y = 10$. Solve for a:

$$10 = a(3-2)^3 + 2$$
$$8 = a \times 1^3$$
$$\therefore \quad a = 8$$

The coordinates of the point of inflection of a graph which is a translation of $y = ax^3$ are known and the coordinates of one further point are known.

b $y = a(x-1)(x+2)(x-4)$

When $x = 5$, $y = 16$ and so

$$16 = a(5-1)(5+2)(5-4)$$
$$16 = 28a$$
$$\therefore \quad a = \frac{4}{7}$$

Three x-axis intercepts are known and the coordinates of a fourth point are known.

c $f(x) = ax^3 + bx$

We know $f(1) = 16$ and $f(2) = 30$:

$$16 = a + b \qquad (1)$$
$$30 = a(2)^3 + 2b \qquad (2)$$

Multiply (1) by 2 and subtract from (2):

$$-2 = 6a$$
$$\therefore \quad a = -\frac{1}{3}$$

Substitute in (1):

$$16 = -\frac{1}{3} + b$$
$$\therefore \quad b = \frac{49}{3}$$

Form simultaneous equations in a and b.

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Example 33

For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, -18)$, $(0, -5)$, $(1, -4)$ and $(2, -9)$ lie on the graph. Find the values of a, b, c and d.

Solution

The following equations can be formed:

$$-a + b - c + d = -18 \qquad (1)$$
$$d = -5 \qquad (2)$$
$$a + b + c + d = -4 \qquad (3)$$
$$8a + 4b + 2c + d = -9 \qquad (4)$$

Adding (1) and (3) gives

$$2b + 2d = -22$$

Since $d = -5$, we obtain $b = -6$.

There are now only two unknowns.

Equations (3) and (4) become:

$$a + c = 7 \qquad (3')$$
$$8a + 2c = 20 \qquad (4')$$

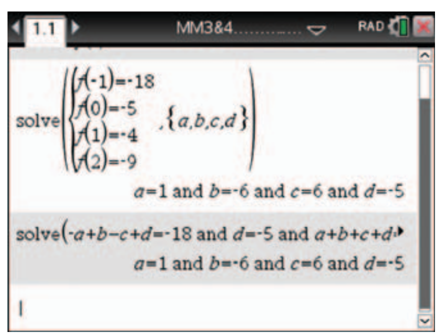
Multiply (3′) by 2 and subtract from (4′) to obtain

$$6a = 6$$

Thus $a = 1$ and $c = 6$.

Using the TI-Nspire

- Define $f(x) = ax^3 + bx^2 + cx + d$.
- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) to solve for a, b, c, d given that $f(-1) = -18$, $f(0) = -5$, $f(1) = -4$ and $f(2) = -9$.



- Alternatively, enter: solve($-a + b - c + d = -18$ and $d = -5$ and $a + b + c + d = -4$ and $8a + 4b + 2c + d = -9$, $\{a, b, c, d\}$). The word ‘and’ can be typed directly or found in the catalog (📖 1 A).

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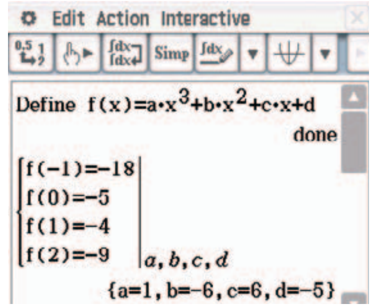
Using the Casio ClassPad

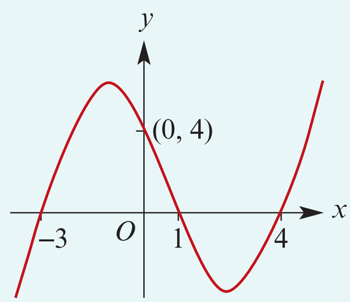
- Open the main screen and define the function $f(x) = ax^3 + bx^2 + cx + d$ using the Var keyboard.
- Tap the simultaneous equations icon three times.
- Enter $f(-1) = -18$, $f(0) = -5$, $f(1) = -4$ and $f(2) = -9$ as the simultaneous equations to be solved, with variables a, b, c, d. Tap EXE.



Note: The function name f must be selected from the abc keyboard.

Example 34

The graph shown is that of a cubic function. Find the rule for this cubic function.



Solution

From the graph, the function is of the form

$$y = a(x-4)(x-1)(x+3)$$

The point $(0, 4)$ is on the graph. Hence

$$4 = a(-4)(-1)3$$

$$\therefore \quad a = \frac{1}{3}$$

The rule is $y = \frac{1}{3}(x-4)(x-1)(x+3)$.

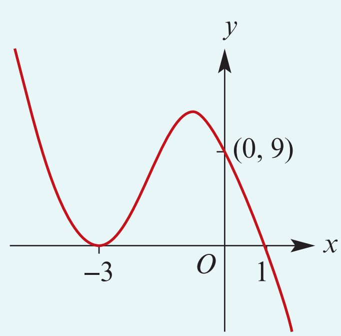
Explanation

The x-axis intercepts are -3, 1 and 4.

So $x+3$, $x-1$ and $x-4$ are linear factors.

Example 35

The graph shown is that of a cubic function. Find the rule for this cubic function.



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Solution

From the graph, the function is of the form

$$y = k(x - 1)(x + 3)^2$$

The point $(0, 9)$ is on the graph. Hence

$$9 = k(-1)(9)$$

$$\therefore \quad k = -1$$

The rule is $y = -(x - 1)(x + 3)^2$.

Explanation

The graph touches the x-axis at $x = -3$. Therefore $x + 3$ is a repeated factor.

Example 36

The graph of a cubic function passes through the points $(0, 1)$, $(1, 4)$, $(2, 17)$ and $(-1, 2)$. Find the rule for this cubic function.

Solution

The cubic function will have a rule of the form

$$y = ax^3 + bx^2 + cx + d$$

The values of a, b, c and d have to be determined.

As the point $(0, 1)$ is on the graph, we have $d = 1$.

By using the points $(1, 4)$, $(2, 17)$ and $(-1, 2)$, three simultaneous equations are produced:

$$4 = a + b + c + 1$$
$$17 = 8a + 4b + 2c + 1$$
$$2 = -a + b - c + 1$$

These become:

$$3 = a + b + c \qquad (1)$$
$$16 = 8a + 4b + 2c \qquad (2)$$
$$1 = -a + b - c \qquad (3)$$

Add (1) and (3):

$$4 = 2b$$
$$\therefore \quad b = 2$$

Substitute in (1) and (2):

$$1 = a + c \qquad (4)$$
$$8 = 8a + 2c \qquad (5)$$

Multiply (4) by 2 and subtract from (5):

$$6 = 6a$$
$$\therefore \quad a = 1$$

From (4), we now have $c = 0$. Hence the rule is $y = x^3 + 2x^2 + 1$.

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Exercise 4G

Skillsheet

Example 32

1 **a** A cubic function has rule of the form $y = a(x-5)^3 - 2$. The point $(4, 0)$ is on the graph of the function. Find the value of a.

b A cubic function has rule of the form $y = a(x-1)(x+1)(x+2)$. The point $(3, 120)$ is on the graph of the function. Find the value of a.

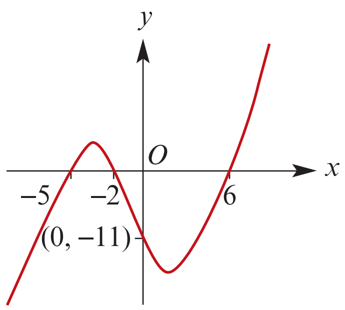
c A cubic function has rule of the form $f(x) = ax^3 + bx$. The points $(2, -20)$ and $(-1, 20)$ are on the graph of the function. Find the values of a and b.

Example 33

2 For the cubic function with rule $f(x) = ax^3 + bx^2 + cx + d$, it is known that the points with coordinates $(-1, 14)$, $(0, 5)$, $(1, 0)$ and $(2, -19)$ lie on the graph of the cubic. Find the values of a, b, c and d.

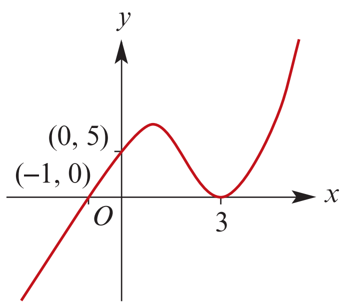
Example 34

3 Determine the rule for the cubic function with the graph shown below.



Example 35

4 Determine the rule for the cubic function with the graph shown below.



Example 36

5 Find the rule for the cubic function that passes through the following points:

a $(0, 1)$, $(1, 3)$, $(-1, -1)$ and $(2, 11)$

b $(0, 1)$, $(1, 1)$, $(-1, 1)$ and $(2, 7)$

c $(0, -2)$, $(1, 0)$, $(-1, -6)$ and $(2, 12)$

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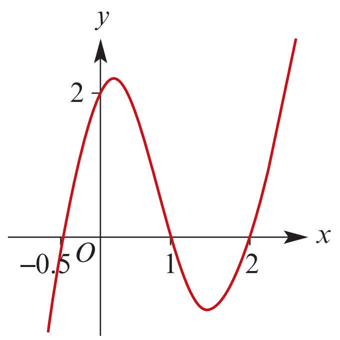
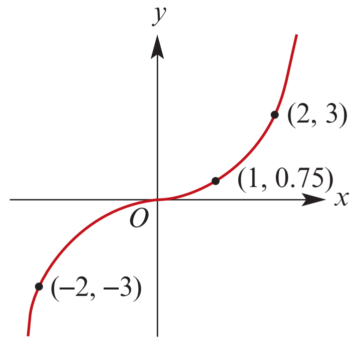
6 Find expressions which define the following cubic curves:

a

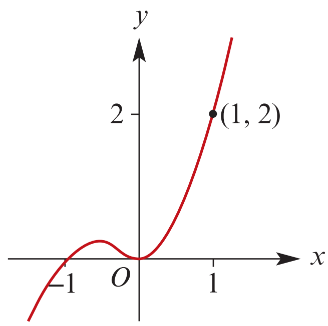


b

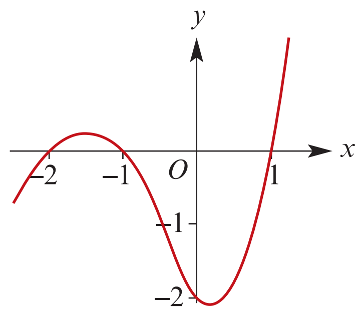


Note that (0, 0) is not a point of zero gradient.

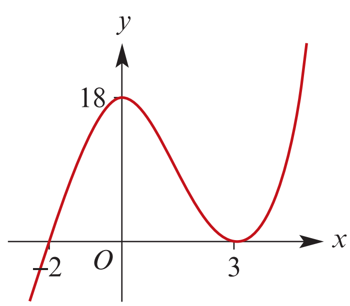
c



d



e



7 Find the rule of the cubic function for which the graph passes through the points with coordinates:

a $(0, 135), (1, 156), (2, 115), (3, 0)$

b $(-2, -203), (0, 13), (1, 25), (2, -11)$

8 Find the rule of the quartic function for which the graph passes through the points with coordinates:

a $(-1, 43), (0, 40), (2, 70), (6, 1618), (10, 670)$

b $(-3, 119), (-2, 32), (-1, 9), (0, 8), (1, 11)$

c $(-3, 6), (-1, 2), (1, 2), (3, 66), (6, 1227)$

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4H Solution of literal equations and systems of equations

Literal equations

We solved linear literal equations in Section 2B. We now look at non-linear equations. They certainly can be solved with a CAS calculator, but full setting out is shown here.

Example 37

Solve each of the following literal equations for x:

a $x^2 + kx + k = 0$ **b** $x^3 - 3ax^2 + 2a^2x = 0$ **c** $x(x^2 - a) = 0$, where $a > 0$

Solution

a The quadratic formula gives

$$x = \frac{-k \pm \sqrt{k^2 - 4k}}{2}$$

A real solution exists only for $k^2 - 4k \geq 0$, that is, for $k \geq 4$ or $k \leq 0$.

b

$$x^3 - 3ax^2 + 2a^2x = 0$$
$$x(x^2 - 3ax + 2a^2) = 0$$
$$x(x - a)(x - 2a) = 0$$

Hence $x = 0$ or $x = a$ or $x = 2a$.

c $x(x^2 - a) = 0$ implies $x = 0$ or $x = \sqrt{a}$ or $x = -\sqrt{a}$.

In the next example, we use the following two facts about power functions:

- If n is an odd natural number, then $b^n = a$ is equivalent to $b = a^{\frac{1}{n}}$.
- If n is an even natural number, then $b^n = a$ is equivalent to $b = \pm a^{\frac{1}{n}}$, where $a \geq 0$.

Note that care must be taken with even powers: for example, $x^2 = 2$ is equivalent to $x = \pm\sqrt{2}$.

Example 38

Solve each of the following equations for x:

a $ax^3 - b = c$ **b** $a(x + b)^3 = c$ **c** $x^4 = c$, where $c > 0$

d $ax^{\frac{1}{5}} = b$ **e** $x^5 - c = d$

Solution

a

$$ax^3 - b = c$$
$$ax^3 = b + c$$
$$x^3 = \frac{b + c}{a}$$
$$\therefore \quad x = \left(\frac{b + c}{a}\right)^{\frac{1}{3}}$$

b

$$a(x + b)^3 = c$$
$$(x + b)^3 = \frac{c}{a}$$
$$x + b = \left(\frac{c}{a}\right)^{\frac{1}{3}}$$
$$\therefore \quad x = \left(\frac{c}{a}\right)^{\frac{1}{3}} - b$$

c

$$x^4 = c$$
$$\therefore \quad x = \sqrt[4]{c} \text{ or } x = -\sqrt[4]{c}$$

d

$$ax^{\frac{1}{5}} = b$$
$$x^{\frac{1}{5}} = \frac{b}{a}$$
$$\therefore \quad x = \left(\frac{b}{a}\right)^5$$

e

$$x^5 - c = d$$
$$x^5 = c + d$$
$$\therefore \quad x = (c + d)^{\frac{1}{5}}$$

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▶ Simultaneous equations

We now look at methods for finding the coordinates of the points of intersection of different graphs.

Example 39

Find the coordinates of the points of intersection of the parabola with equation $y = x^2 - 2x - 2$ and the straight line with equation $y = x + 4$.

Solution

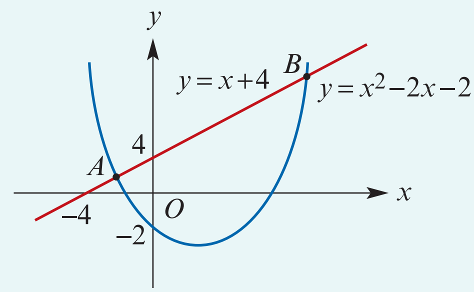
Equate the two expressions for y:

$$x^2 - 2x - 2 = x + 4$$

$$x^2 - 3x - 6 = 0$$

$$\therefore \quad x = \frac{3 \pm \sqrt{9 - 4 \times (-6)}}{2}$$

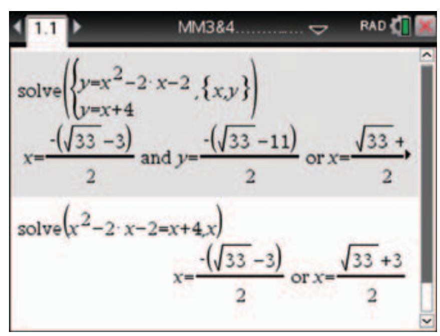
$$= \frac{3 \pm \sqrt{33}}{2}$$



The points of intersection are $A\left(\dfrac{3 - \sqrt{33}}{2}, \dfrac{11 - \sqrt{33}}{2}\right)$ and $B\left(\dfrac{3 + \sqrt{33}}{2}, \dfrac{11 + \sqrt{33}}{2}\right)$.

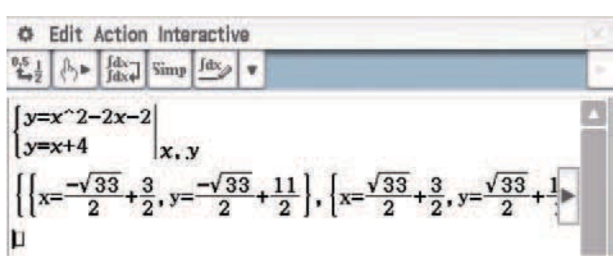
Using the TI-Nspire

- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) and complete as shown.
- Use the up arrow (▲) to move up to the answer and then use the right arrow (►) to display the remaining part of the answer.
- Alternatively, equate the two expressions for y and solve for x as shown.



Using the Casio ClassPad

- Select the simultaneous equations template.
- Enter the equations $y = x^2 - 2x - 2$ and $y = x + 4$. Set the variables as x, y.



- Tap Rotate to rotate the screen, and tap the right-arrow button (►) to view the solutions.

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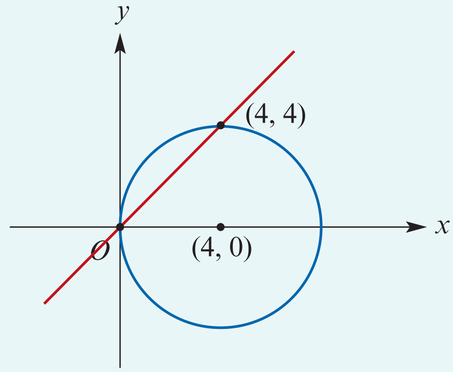
Example 40

Find the points of intersection of the circle with equation $(x-4)^2 + y^2 = 16$ and the line with equation $x - y = 0$.

Solution

Rearrange $x - y = 0$ to make y the subject.
Substitute $y = x$ into the equation of the circle:

$$\begin{aligned}
(x-4)^2 + x^2 &= 16 \\
x^2 - 8x + 16 + x^2 &= 16 \\
2x^2 - 8x &= 0 \\
2x(x-4) &= 0 \\
\therefore \quad x = 0 \text{ or } x &= 4
\end{aligned}$$



The points of intersection are $(0, 0)$ and $(4, 4)$.

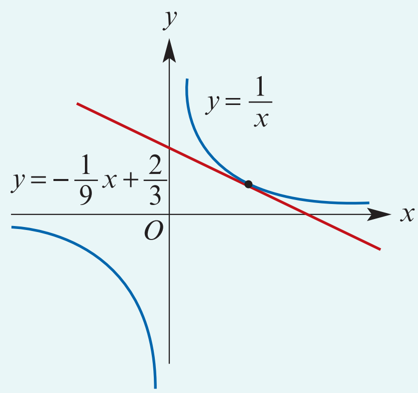
Example 41

Find the point of contact of the line with equation $\frac{1}{9}x + y = \frac{2}{3}$ and the curve with equation $xy = 1$.

Solution

Rewrite the equations as $y = -\frac{1}{9}x + \frac{2}{3}$ and $y = \frac{1}{x}$.
Equate the expressions for y:

$$\begin{aligned}
-\frac{1}{9}x + \frac{2}{3} &= \frac{1}{x} \\
-x^2 + 6x &= 9 \\
x^2 - 6x + 9 &= 0 \\
(x-3)^2 &= 0 \\
\therefore \quad x &= 3
\end{aligned}$$



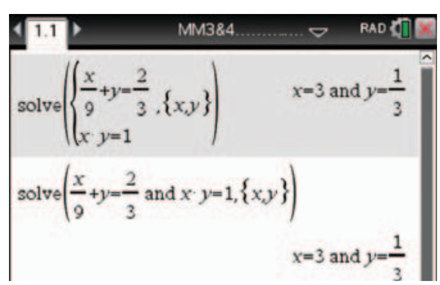
The point of intersection is $(3, \frac{1}{3})$.

Using the TI-Nspire

Two methods are shown:

- Use the simultaneous equations template (menu > **Algebra** > **Solve System of Equations** > **Solve System of Equations**).
- Alternatively, use menu > **Algebra** > **Solve**.

Note: The multiplication sign is required between x and y.



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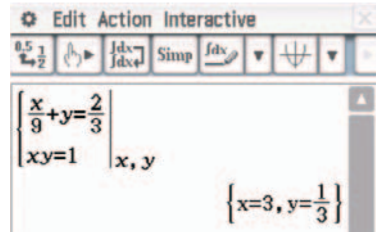
Using the Casio ClassPad

- Select the simultaneous equations template $\boxed{\{\square}$.
- Enter the equations $\frac{x}{9} + y = \frac{2}{3}$ and $xy = 1$ and set the variables as x, y.

Note: Tap $\boxed{\frac{\square}{\square}}$ to enter the fractions.



Exercise 4H

Example 37

1 Solve each of the following literal equations for x:

a $kx^2 + x + k = 0$ **b** $x^3 - 7ax^2 + 12a^2x = 0$ **c** $x(x^3 - a) = 0$

d $x^2 - kx + k = 0$ **e** $x^3 - ax = 0$ **f** $x^4 - a^4 = 0$

g $(x - a)^5(x - b) = 0$ **h** $(a - x)^4(a - x^3)(x^2 - a) = 0$

Example 38

2 Solve each of the following equations for x:

a $ax^3 + b = 2c$ **b** $ax^2 - b = c$, where $a, b, c > 0$

c $a - bx^2 = c$, where $a > c$ and $b > 0$ **d** $x^{\frac{1}{3}} = a$

e $x^{\frac{1}{n}} + c = a$, where $n \in \mathbb{N}$ and $a > c$ **f** $a(x - 2b)^3 = c$

g $ax^{\frac{1}{3}} = b$ **h** $x^3 - c = d$

Example 39

3 Find the coordinates of the points of intersection for each of the following:

a $y = x^2$, $y = x$ **b** $y - 2x^2 = 0$, $y - x = 0$ **c** $y = x^2 - x$, $y = 2x + 1$

Example 40

4 Find the coordinates of the points of intersection for each of the following:

a $x^2 + y^2 = 178$, $x + y = 16$ **b** $x^2 + y^2 = 125$, $x + y = 15$ **c** $x^2 + y^2 = 185$, $x - y = 3$

d $x^2 + y^2 = 97$, $x + y = 13$ **e** $x^2 + y^2 = 106$, $x - y = 4$

Example 41

5 Find the coordinates of the points of intersection for each of the following:

a $x + y = 28$, $xy = 187$ **b** $x + y = 51$, $xy = 518$ **c** $x - y = 5$, $xy = 126$

6 Find the coordinates of the points of intersection of the straight line with equation $y = 2x$ and the circle with equation $(x - 5)^2 + y^2 = 25$.

7 Find the coordinates of the points of intersection of the curves with equations $y = \frac{1}{x - 2} + 3$ and $y = x$.

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8 Find the coordinates of the points of intersection of the line with equation $\frac{y}{4} - \frac{x}{5} = 1$ and the circle with equation $x^2 + 4x + y^2 = 12$.

9 Find the coordinates of the points of intersection of the curve $y = \frac{1}{x+2} - 3$ and the line $y = -x$.

10 Find the coordinates of the point where the line with equation $4y = 9x + 4$ touches the parabola with equation $y^2 = 9x$.

11 Find the coordinates of the point where the line with equation $y = 2x + 3\sqrt{5}$ touches the circle $x^2 + y^2 = 9$.

12 Find the coordinates of the point where the straight line with equation $y = \frac{1}{4}x + 1$ touches the curve with equation $y = -\frac{1}{x}$.

13 Find the coordinates of the points of intersection of the curve with equation $y = \frac{2}{x-2}$ and the line $y = x - 1$.

14 Solve the simultaneous equations:

a $5x - 4y = 7$ and $xy = 6$

b $2x + 3y = 37$ and $xy = 45$

c $5x - 3y = 18$ and $xy = 24$

15 What is the condition for $x^2 + ax + b$ to be divisible by $x + c$?

16 Solve the simultaneous equations $y = x + 2$ and $y = \frac{160}{x}$.

17 Find the equations of the lines that pass through the point $(1, 7)$ and touch the parabola $y = -3x^2 + 5x + 2$.

Hint: Form a quadratic equation and consider when the discriminant Δ is zero.

18 Find the values of m for which the line $y = mx - 8$ intersects the parabola $y = x^2 - 5x + m$ twice.

19 The line $y = x + c$ meets the hyperbola $y = \frac{9}{2-x}$ once. Find the possible values of c.

20 **a** Solve the simultaneous equations $y = mx$ and $y = \frac{1}{x} + 5$ for x in terms of m.

b Find the value of m for which the graphs of $y = mx$ and $y = \frac{1}{x} + 5$ touch, and give the coordinates of this point.

c For which values of m do the graphs not meet?

21 Show that, if the line with equation $y = kx + b$ touches the curve $y = x^2 + x + 4$, then $k^2 - 2k + 4b - 15 = 0$. Hence find the equations of such lines that also pass through the point $(0, 3)$.

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Review

Chapter summary

Spreadsheet AS Nrich

Quadratic polynomials

- Turning point form
  - By completing the square, all quadratic functions in polynomial form $y = ax^2 + bx + c$ may be transposed into turning point form $y = a(x - h)^2 + k$.
  - The graph of $y = a(x - h)^2 + k$ is a parabola congruent to the graph of $y = ax^2$. The vertex (or turning point) is the point (h, k). The axis of symmetry is $x = h$.
- Axis of symmetry

  The axis of symmetry of the graph of the quadratic function $y = ax^2 + bx + c$ is the line with equation $x = -\dfrac{b}{2a}$.
- Quadratic formula

  The solutions of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by the **quadratic formula**

  $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  From the formula it can be seen that:
  - If $b^2 - 4ac > 0$, there are two solutions.
  - If $b^2 - 4ac = 0$, there is one solution.
  - If $b^2 - 4ac < 0$, there are no real solutions.

  The quantity $\Delta = b^2 - 4ac$ is called the **discriminant** of the quadratic $ax^2 + bx + c$.

Polynomials in general

- A **polynomial function** is a function that can be written in the form

  $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

  where $n \in \mathbb{N} \cup \{0\}$ and the coefficients $a_0, \dots, a_n$ are real numbers with $a_n \neq 0$.
  The **leading term** is $a_n x^n$ (the term of highest index) and the **constant term** is a_0 (the term not involving x).
- The **degree of a polynomial** is the index n of the leading term.
  - Polynomials of degree 1 are called **linear** functions.
  - Polynomials of degree 2 are called **quadratic** functions.
  - Polynomials of degree 3 are called **cubic** functions.
  - Polynomials of degree 4 are called **quartic** functions.
- The sum, difference and product of two polynomials is a polynomial. Division does not always lead to another polynomial.
- Two polynomials P and Q are equal only if their corresponding coefficients are equal. Two cubic polynomials, $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, are equal if and only if $a_3 = b_3$, $a_2 = b_2$, $a_1 = b_1$ and $a_0 = b_0$.

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Review

- Division of polynomials

  When we divide the polynomial $P(x)$ by the polynomial $D(x)$ we obtain two polynomials, $Q(x)$ the **quotient** and $R(x)$ the **remainder**, such that

  $$P(x) = D(x)Q(x) + R(x)$$

  and either $R(x) = 0$ or $R(x)$ has degree less than $D(x)$.
  Two methods for dividing polynomials are **long division** and **equating coefficients**.
- Remainder theorem

  When $P(x)$ is divided by $\beta x + \alpha$, the remainder is $P\left(-\frac{\alpha}{\beta}\right)$.
- Factor theorem
  - If $\beta x + \alpha$ is a factor of $P(x)$, then $P\left(-\frac{\alpha}{\beta}\right) = 0$.
  - Conversely, if $P\left(-\frac{\alpha}{\beta}\right) = 0$, then $\beta x + \alpha$ is a factor of $P(x)$.
- A cubic polynomial can be factorised by using the factor theorem to find the first linear factor and then using polynomial division or the method of equating coefficients to complete the factorisation.
- Rational-root theorem

  Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree n with all the coefficients a_i integers. Let α and β be integers such that the highest common factor of α and β is 1 (i.e. α and β are relatively prime). If $\beta x + \alpha$ is a factor of $P(x)$, then β divides a_n and α divides a_0.
- Difference and sum of two cubes
  - $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$
  - $x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

Technology-free questions

1 Sketch the graph of each of the following quadratic functions. Clearly indicate coordinates of the vertex and the axis intercepts.

a $h(x) = 3(x - 1)^2 + 2$ **b** $h(x) = (x - 1)^2 - 9$ **c** $f(x) = x^2 - x + 6$
d $f(x) = x^2 - x - 6$ **e** $f(x) = 2x^2 - x + 5$ **f** $h(x) = 2x^2 - x - 1$

2 The points with coordinates (1, 1) and (2, 5) lie on a parabola with equation of the form $y = ax^2 + b$. Find the values of a and b.

3 Solve the equation $3x^2 - 2x - 10 = 0$ by using the quadratic formula.

4 Sketch the graph of each of the following. State the coordinates of the point of zero gradient and the axis intercepts.

a $f(x) = 2(x - 1)^3 - 16$ **b** $g(x) = -(x + 1)^3 + 8$ **c** $h(x) = -(x + 2)^3 - 1$
d $f(x) = (x + 3)^3 - 1$ **e** $f(x) = 1 - (2x - 1)^3$

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5 Express each of the following in turning point form:

a $x^2 + 4x$ **b** $3x^2 + 6x$ **c** $x^2 - 4x + 6$

d $2x^2 - 6x - 4$ **e** $2x^2 - 7x - 4$ **f** $-x^2 + 3x - 4$

6 Draw a sign diagram for each of the following:

a $y = (x + 2)(2 - x)(x + 1)$ **b** $y = (x - 3)(x + 1)(x - 1)$

c $y = x^3 + 7x^2 + 14x + 8$ **d** $y = 3x^3 + 10x^2 + x - 6$

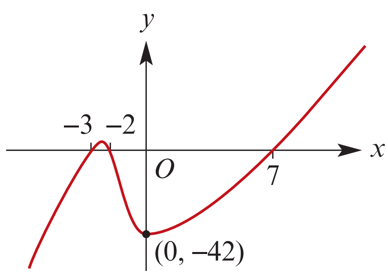
7 Without actually dividing, find the remainder when the first polynomial is divided by the second:

a $x^3 + 3x^2 - 4x + 2, \ x + 1$

b $x^3 - 3x^2 - x + 6, \ x - 2$

c $2x^3 + 3x^2 - 3x - 2, \ x + 2$

8 Determine the rule for the cubic function shown in the graph.



9 Factorise each of the following:

a $x^3 + 2x^2 - 5x - 6$ **b** $x^3 - 3x^2 - x + 3$

c $x^4 - x^3 - 7x^2 + x + 6$ **d** $x^3 + 2x^2 - 4x + 1$

10 Find the quotient and remainder when $x^2 + 4$ is divided by $x^2 - 2x + 2$.

11 Find the value of a for which $x - 2$ is a factor of $3x^3 + ax^2 + x - 2$.

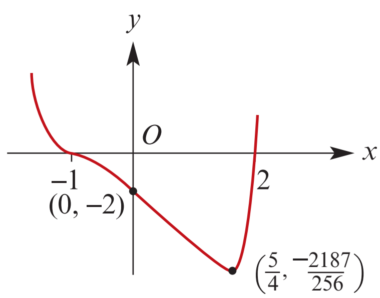
12 The graph of $f(x) = (x + 1)^3(x - 2)$ is shown. Sketch the graph of:

a $y = f(x - 1)$

b $y = f(x + 1)$

c $y = f(2x)$

d $y = f(x) + 2$



13 For what value of k is $2x^2 - kx + 8$ a perfect square?

14 Find the coordinates of the points of intersection of the graph of $y = 2x + 3$ with the graph of $y = x^2 + 3x - 9$.

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15 Find constants a, b and c such that $3x^2 - 5x + 1 = a(x + b)^2 + c$ holds for all values of x.

16 Expand $(3 + 4x)^3$.

17 Given that $x^3 - 2x^2 + 5 = ax(x - 1)^2 + b(x - 1) + c$ for all real numbers x, find the values of a, b and c.

18 Find the values of p for which the equation $4x^2 - 2px + p + 3 = 0$ has no real solutions.

19 Find the rule for the cubic function, the graph of which passes through the points $(1, 1)$, $(2, 4)$, $(3, 9)$ and $(0, 6)$.

Multiple-choice questions

1 By completing the square, the expression $5x^2 - 10x - 2$ can be written in turning point form $a(x - h)^2 + k$ as

A $(5x + 1)^2 + 5$ **B** $(5x - 1)^2 - 5$ **C** $5(x - 1)^2 - 5$
D $5(x + 1)^2 - 2$ **E** $5(x - 1)^2 - 7$

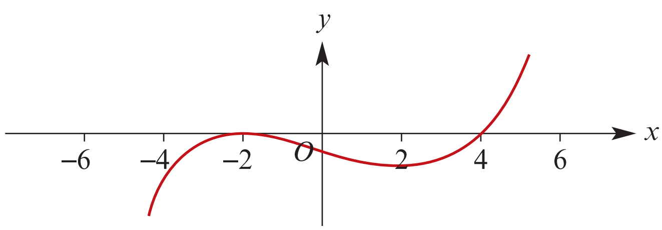
2 For which value(s) of m does the equation $mx^2 + 6x - 3 = 0$ have two real solutions?

A $m = -3$ **B** $m = 3$ **C** $m = 0$ **D** $m > -3$ **E** $m < -3$

3 $x^3 + 27$ is equal to

A $(x + 3)^3$ **B** $(x - 3)^3$ **C** $(x + 3)(x^2 - 6x + 9)$
D $(x - 3)(x^2 + 3x + 9)$ **E** $(x + 3)(x^2 - 3x + 9)$

4 The equation of the graph shown on the right is



A $y = x(x - 2)(x + 4)$
B $y = x(x + 2)(x - 4)$
C $y = (x + 2)^2(x - 4)$
D $y = (x + 2)(x - 4)^2$
E $y = (x + 2)^2(x - 4)^2$

5 If $x - 1$ is a factor of $x^3 + 3x^2 - 2ax + 1$, then the value of a is

A 2 **B** 5 **C** $\frac{2}{5}$ **D** $-\frac{2}{5}$ **E** $\frac{5}{2}$

6 $6x^2 - 8xy - 8y^2$ is equal to

A $(3x + 2y)(2x - 4y)$ **B** $(3x - 2y)(6x + 4y)$ **C** $(6x - 4y)(x + 2y)$
D $(3x - 2y)(2x + 4y)$ **E** $(6x + y)(x - 8y)$

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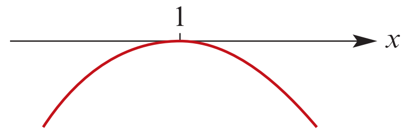
7 The diagram shows a part of the graph of a cubic polynomial function f, near the point $(1, 0)$. Which of the following could be the rule for f?

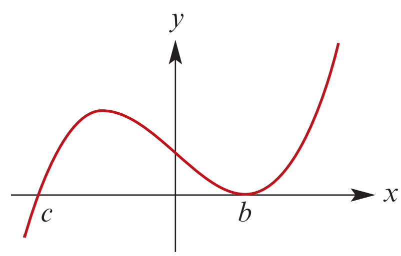


A $f(x) = x^2(x-1)$ **B** $f(x) = (x-1)^3$ **C** $f(x) = -x(x-1)^2$
D $f(x) = x(x-1)^2$ **E** $f(x) = -x(x+1)^2$

8 The coordinates of the turning point of the graph of the function $p(x) = 3((x-2)^2 + 4)$ are

A $(-2, 12)$ **B** $(-2, 4)$ **C** $(2, -12)$ **D** $(2, 4)$ **E** $(2, 12)$

9 The diagram shows part of the graph of a polynomial function. A possible equation for the graph is



A $y = (x+c)(x-b)^2$
B $y = (x-b)(x-c)^2$
C $y = (x-c)(b-x)^2$
D $y = -(x-c)(b-x)^2$
E $y = (x+b)^2(x-c)$

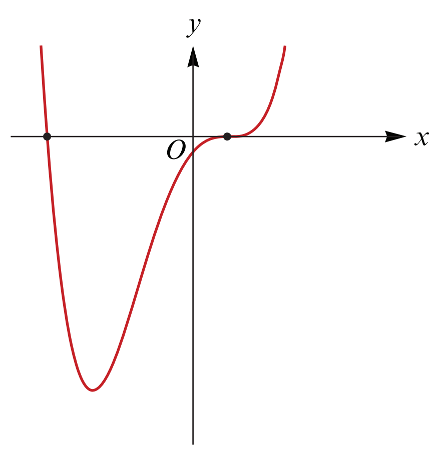
10 The number of solutions of the equation $(x^2 + a)(x-b)(x+c) = 0$, where $a, b, c \in \mathbb{R}^+$, is

A 0 **B** 1 **C** 2 **D** 3 **E** 4

11 The graph of $y = kx - 3$ meets the graph of $y = -x^2 + 2x - 12$ at two distinct points for

A $k \in [-4, 8]$ **B** $k \in \{-4, -8\}$ **C** $k \in (-\infty, -4) \cup (8, \infty)$
D $k \in (-4, 8)$ **E** $k \in (-\infty, -8) \cup (4, \infty)$

12 The function f is a quartic polynomial. Its graph is shown on the right. It has x-axis intercepts at $(a, 0)$ and $(b, 0)$, where $a > 0$ and $b < 0$. A possible rule for this function is



A $f(x) = (x-a)^2(x+b)^2$
B $f(x) = (x-a)^3(x-b)$
C $f(x) = (x-a)(x-b)^2$
D $f(x) = (x+a)^2(x-b)^2$
E $f(x) = (x-b)^3(x-a)$

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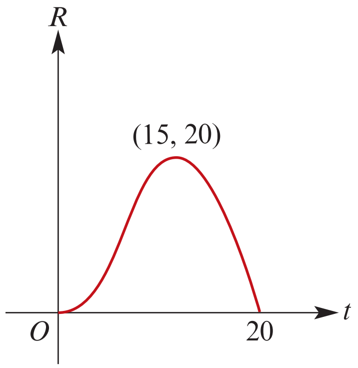
Review

Extended-response questions

1 The rate of flow of water, R mL/min, into a vessel is described by the quartic expression

$$R = kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

where t minutes is the time elapsed from the beginning of the flow. The graph is shown.



a Find the value of k.

b Find the rate of flow when $t = 10$.

c The flow is adjusted so that the new expression for the flow is

$$R_{\text{new}} = 2kt^3(20 - t), \quad \text{for } 0 \le t \le 20$$

i Sketch the graph of R_{new} against t for $0 \le t \le 20$.

ii Find the rate of flow when $t = 10$.

d Water is allowed to run from the vessel and it is found that the rate of flow from the vessel is given by

$$R_{\text{out}} = -k(t - 20)^3(40 - t), \quad \text{for } 20 \le t \le 40$$

i Sketch the graph of R_{out} against t for $20 \le t \le 40$.

ii Find the rate of flow when $t = 30$.

Hints: The graph of R_{new} against t is given by a dilation of factor 2 from the x-axis. The graph of R_{out} against t is given by the translation with rule $(t, R) \to (t + 20, R)$ followed by a reflection in the t-axis.

2 A large gas container is being deflated. The volume V (in m^3) at time t hours is given by

$$V = 4(9 - t)^3, \quad \text{for } 0 \le t \le 9$$

a Find the volume when:

i $t = 0$ **ii** $t = 9$

b Sketch the graph of V against t for $0 \le t \le 9$.

c At what time is the volume 512 m^3?

3 A hemispherical bowl of radius 6 cm contains water. The volume of water in the hemispherical bowl, where the depth of the water is x cm, is given by

$$V = \frac{1}{3}\pi x^2(18 - x) \text{ cm}^3$$

a Find the volume of water when:

i $x = 2$ **ii** $x = 3$ **iii** $x = 4$

b Find the volume when the hemispherical bowl is full.

c Sketch the graph of V against x.

d Find the depth of water when the volume is equal to $\dfrac{325\pi}{3}$ cm^3.

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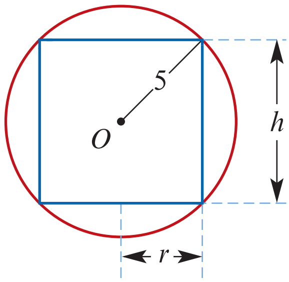
4 A metal worker is required to cut a circular cylinder from a solid sphere of radius 5 cm. A cross-section of the sphere and the cylinder is shown in the diagram.



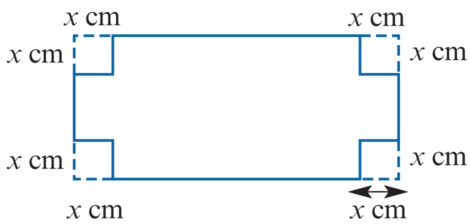
a Express r in terms of h, where r cm is the radius of the cylinder and h cm is the height of the cylinder. Hence show that the volume, V cm^3, of the cylinder is given by $V = \frac{1}{4}\pi h(100 - h^2)$.

b Sketch the graph of V against h for $0 < h < 10$.
Hint: The coordinates of the maximum point are approximately $(5.77, 302.3)$.

c Find the volume of the cylinder if $h = 6$.

d Find the height and radius of the cylinder if the volume of the cylinder is 48π cm^3.

5 An open tank is to be made from a sheet of metal 84 cm by 40 cm by cutting congruent squares of side length x cm from each of the corners.



a Find the volume, V cm^3, of the box in terms of x.

b State the maximal domain for V when it is considered as a function of x.

c Plot the graph of V against x using a calculator.

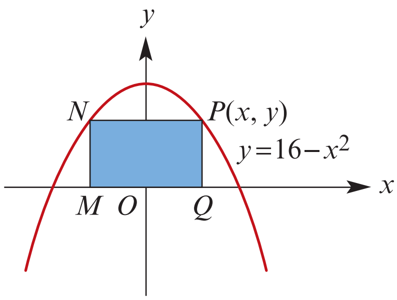
d Find the volume of the tank when:

i $x = 2$ **ii** $x = 6$ **iii** $x = 8$ **iv** $x = 10$

e Find the value(s) of x, correct to two decimal places, for which the capacity of the tank is 10 litres.

f Find, correct to two decimal places, the maximum capacity of the tank in cubic centimetres.

6 A rectangle is defined by vertices N and $P(x, y)$ on the curve with equation $y = 16 - x^2$ and vertices M and Q on the x-axis.



a **i** Find the area, A, of the rectangle in terms of x.

ii State the implied domain for the function defined by the rule given in part **i**.

b **i** Find the value of A when $x = 3$.

ii Find the value of x, correct to two decimal places, when $A = 25$.

c A cuboid has volume V given by the rule $V = xA$.

i Find V in terms of x.

ii Find the value of x, correct to two decimal places, such that $V = 100$.

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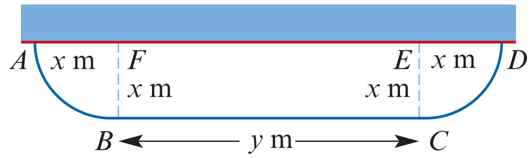
Review

7 The plan of a garden adjoining a wall is shown. The rectangle $BCEF$ is of length y m and width x m. The borders of the two end sections are quarter circles of radius x m and centres at E and F.



A fence is erected along the curves AB and CD and the straight line BC.

a Find the area, A m^2, of the garden in terms of x and y.

b If the length of the fence is 100 m, find:

  i y in terms of x

  ii A in terms of x

  iii the maximal domain of the function with the rule obtained in part **ii**.

c Find, correct to two decimal places, the value(s) of x if the area of the garden is to be 1000 m^2.

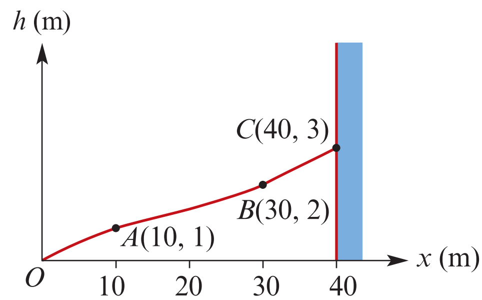
d It is decided to build the garden up to a height of $\frac{x}{50}$ metres. If the length of the fence is 100 m, find correct to two decimal places:

  i the volume, V m^3, of soil needed in terms of x

  ii the volume of soil needed for a garden of area 1000 m^2

  iii the value(s) of x for which 500 m^3 of soil is required.

8 A mound of earth is piled up against a wall. The cross-section is as shown. The coordinates of several points on the surface are given.



a Find the rule of the cubic function for which the graph passes through the points O, A, B and C.

b For what value of x is the height of the mound 1.5 metres?

c The coefficient of x^3 for the function is 'small'. Consider the quadratic formed when the x^3 term is deleted. Compare the graph of the resulting quadratic function with the graph of the cubic function.

d The mound moves and the curve describing the cross-section now passes through the points $O(0, 0)$, $A(10, 0.3)$, $B(30, 2.7)$ and $D(40, 2.8)$. Find the rule of the cubic function for which the graph passes through these points.

e Let $y = f(x)$ be the function obtained in part **a**.

  i Sketch the graph of the piecewise-defined function

$$g(x) = \begin{cases} f(x) & \text{for } 0 \le x \le 40 \\ f(80 - x) & \text{for } 40 < x \le 80 \end{cases}$$

  ii Comment on the appearance of the graph of $y = g(x)$.

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