10.3HD

Task 1: Integration

1) Explain why the following functions are or are not probability density functions

a) $f(x) = \frac{7}{x}$ for $x \in [1, e^{0.5}]$

Evidence and conclusion

- Positive division gives a positive > 0 result, so property 1 is supported
- The final result is not $= 1$, dismissing property 2

As property 1 is supported but 2 is not supported, this function is not a valid probability density function.

Working out

Testing property 1

Since a positive number divided by a positive number (like all those x values found in the domain) equals a positive number, we can safely say that property 1 is supported.

For further evidence, $\frac{7}{1} = 7$ and $\frac{7}{e^{0.5}} \approx 4.25$.

Testing property 2

$\int_1^{e^{0.5}} (\frac{7}{x})$
-> $7\ln(|x|)$
$= [7\ln(|x|)]_1^{e^{0.5}}$

Upper bound (b)
$7\ln(|e^{0.5}|)$
-> 7×0.5
$= \frac{7}{2}$

Lower bound (a)
$7\ln(|1|)$
-> 7×0
$= 0$

Final results
$\frac{7}{2} - 0$
$= \frac{7}{2}$

b) $f(x) = \frac{1}{8}(16 - x^4)$ for $x \in [0, 4]$

Evidence and conclusion

- $0..2$ give x-values ≥ 0 but $3..4$ are < 0, not supporting property 1
- The final result is not $= 1$, dismissing property 2

As property 1 and 2 are not supported, this function is not a valid probability density function.

Working out

Testing property 1

Testing bounds
In order from $x = 0..\, x = 4$,
Results: $f(x) = 2$, $\frac{15}{8}, 0, -\frac{65}{8}, -30$

Testing property 2

$\int_0^4 \frac{1}{8}(16 - x^4)dx$
-> $\frac{1}{8} \times \int_0^4 (16 - x^4)dx$
-> $\frac{1}{8} \times (\frac{16}{1}x - \frac{1}{5}x^5)$
-> $\frac{1}{8} \times (16x - \frac{1}{5}x^5)$
$= \frac{1}{8} \times [16x - \frac{1}{5}x^5]_0^4$

Upper bound (b), $x = 4$
$\frac{1}{8} \times (16(4) - \frac{1}{5}(4)^5)$
-> $\frac{1}{8}(64 - \frac{1}{5}1024)$
-> $\frac{1}{8}(\frac{320}{5} - \frac{1024}{5})$
-> $\frac{1}{8}(-\frac{704}{5})$
-> $-\frac{704}{40}$
$= -\frac{88}{5}$

Lower bound (a), $x = 0$
$\frac{1}{8}(\times 16(0) - \frac{1}{5}(0)^5)$
-> $\frac{1}{8} \times 0 \times 0$
$= 0$

Final results
$-\frac{88}{5} - 0$
$= -\frac{88}{5}$

c) $f(x) = \frac{7}{3}\sin(x)$ for $x \in [0, 5\pi]$

Evidence and conclusion

- There were multiple x values that were negative, so we can not support property 1
- The final result is not $= 1$, dismissing property 2 as valid

As property 1 and 2 are not supported, this function is not a valid probability density function.

Working out

Testing property 1

$\frac{7}{3}\sin(x)$
We will test x values $0, \frac{1}{2}, \pi, \frac{3\pi}{2}, \frac{9\pi}{5} 5\pi$
Results: $f(x) = 0, \approx 1.12, 0, -\frac{7}{3}, \approx -1.37, 0$

Testing property 2

$\int_0^{5\pi}(\frac{7}{3}\sin(x))dx$
-> $-\frac{7}{3}\cos(x)$
$[-\frac{7}{3}\cos(x)]_0^{5\pi}$

Upper bound (b), $x = 5\pi$
$-\frac{7}{3}\cos(5\pi)$
-> $-\frac{7}{3} \times -1$
$= \frac{7}{3}$

Lower bound (a), $x = 0$
$-\frac{7}{3}\cos(x)$
-> $-\frac{7}{3} \times 1$
$= -\frac{7}{3}$

Final results
$\frac{7}{3} - -\frac{7}{3}$
$= \frac{14}{3}$

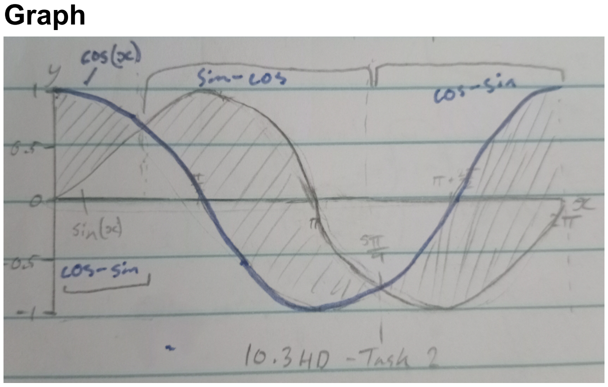
Task 2: $\sin$ and $\cos$ graph

Find the total area between $y = \sin(x)$ and $y = \cos(x)$ given the domain of $x \in [0, 2\pi]$

Answers

$\text{total area} = 4\sqrt{2}$ given $y = \sin(x)$, $y = \cos(x)$, and the domain $x \in [0, 2\pi]$

Graph



Working out

As the functions intersect, the area is not found directly by $[f(x)-g(x)]_0^{2\pi}$. We need to split it up into further intervals.

According to the graph, there is a clear point of intersection at every $\frac{\pi}{4}+n\pi$ point (n starts at 0). We will use these intersection points and graph to distinguish the interval points for the integral.

$\int_0^{2\pi}(\sin(x)-\cos(x))dx$
-> $[-\cos(x)-\sin(x)]_0^{2\pi}$

$\int_0^{2\pi}(\cos(x)-\sin(x))dx$
-> $[\sin(x)+\cos(x)]_0^{2\pi}$

The intervals we will use are therefore going to be:

- 0 to $\frac{\pi}{4}$
  - $[\sin(x)+\cos(x)]_{\frac{\pi}{4}}^{0}$
- $\frac{\pi}{4}$ to $\frac{5\pi}{4}$
  - $[-\cos(x)-\sin(x)]_{\frac{5\pi}{4}}^{\frac{\pi}{4}}$
- $\frac{5\pi}{4}$ to 2π
  - $[\sin(x)+\cos(x)]_{2\pi}^{\frac{5\pi}{4}}$

$$[\sin(x)+\cos(x)]_0^{\frac{\pi}{4}}$$

> $\sin(0)+\cos(0)$
> $0+1$
> $=1$

> $\sin(\frac{\pi}{4})+\cos(\frac{\pi}{4})$
> $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$
> $\frac{2}{\sqrt{2}}$
> $=\sqrt{2}$

$=\sqrt{2}-1$

$$[-\cos(x)-\sin(x)]_{\frac{\pi}{4}}^{\frac{5\pi}{4}}$$

> $-\cos\left(\frac{\pi}{4}\right)-\sin(\frac{\pi}{4})$
> $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}$
> $-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}$
> $\frac{-\sqrt{2}-\sqrt{2}}{2}$
> $=-\sqrt{2}$

> $-\cos\left(\frac{5\pi}{4}\right)-\sin(\frac{5\pi}{4})$
> $\frac{\sqrt{2}}{2}--\frac{\sqrt{2}}{2}$
> $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}$
> $=\sqrt{2}$

$\sqrt{2} - -\sqrt{2}$
-> $2\sqrt{2}$

$$\left[\sin(x) + \cos(x)\right]_{\frac{5\pi}{4}}^{2\pi}$$

> $\sin\left(\frac{5\pi}{4}\right) + \cos(\frac{5\pi}{4})$
> $\frac{-\sqrt{2}}{2} + \frac{-\sqrt{2}}{2}$
> $= -\sqrt{2}$

$\sin(2\pi) + \cos(2\pi)$

> $0 + 1$
> $= 1$

$1 - -\sqrt{2}$
$= \sqrt{2} + 1$

$(\sqrt{2} - 1) + (2\sqrt{2}) + (\sqrt{2} + 1)$
-> $2\sqrt{2} + 2\sqrt{2}$
$= 4\sqrt{2}$