Task 1: Simultaneous equations

1) Solve this set of simultaneous equations

$$\begin{cases} x + 2y + 3z = 5 \\ 2x + 5y + 3z = 3 \\ x + 8z = 17 \end{cases}$$

$x + 2y + 3z = 5$
-> $x = 5 - 2y - 3z$

$2x + 5y + 3z = 3$
-> $2(5 - 2y - 3z) + 5y + 3z = 3$
-> $10 - 4y - 6z + 5y + 3z = 3$
-> $10 - y - 3z = 3$

$x + 8z = 17$
-> $5 - 2y - 3z + 8z = 17$
-> $5 - 2y + 5z = 17$

$$\begin{cases} x = 5 - 2y - 3z \\ 10 + y - 3z = 3 \\ 5 - 2y + 5z = 17 \end{cases}$$

$10 + y - 3z = 3$
-> $y - 3z = -7$
$y = -7 + 3z$

$$\begin{cases} x = 5 - 2y - 3z \\ y = -7 + 3z \\ 5 - 2y + 5z = 17 \end{cases}$$

$5 - 2y + 5z = 17$
-> $5 - 2(-7 + 3z) + 5z = 17$
-> $5 + 14 - 6z + 5z = 17$
-> $19 - z = 17$
$-z = -2$
$z = 2$

$$\begin{cases} x = 5 - 2y - 3z \\ y = -7 + 3z \\ z = 2 \end{cases}$$

$y = -7 + 3(2)$
-> $y = -7 + 6$
$y = -1$

$$\begin{cases} x = 5 - 2y - 3z \\ y = -1 \\ z = 2 \end{cases}$$

$x = 5 - 2(-1) - 3(2)$
-> $x = 5 + 2 - 6$
$x = 1$

$$\begin{cases} x = 1 \\ y = -1 \\ z = 2 \end{cases}$$

Verification

$$\begin{cases} 1 + 2(-1) + 3(2) = 5 \\ 2(1) + 5(-1) + 3(2) = 3 \\ 1 + 8(2) = 17 \end{cases}$$

$$\begin{cases} 1 - 2 + 6 = 5 \\ 2 - 5 + 6 = 3 \\ 1 + 16 = 17 \end{cases}$$

$$\begin{cases} 5 = 5 \\ 3 = 3 \\ 17 = 17 \end{cases}$$

2) Solve for z and identify the plane intersection point.

$$\begin{cases} 1) : 3x + 5y - 4z = 7 \\ 2) : -3x - 2y + 4z = -1 \\ 3) : 6x + y - 8z = -4 \end{cases}$$

Solution - Plane intersection point
$x = -1$
$y = 2$
$z = 0$

The equation $-27x + 36z - 20 = 7$ fulfils the plane intersection line.

Working out

Rearranging for y
$3) : 6x + y - 8z = -4$
-> $y = -4 - 6x + 8z$

Eliminating y from the other two equations
$1) : 3x + 5y - 4z = 7$
-> $3x + 5(-4 - 6x + 8z) - 4z = 7$
-> $3x - 20 - 30x + 40z - 4z = 7$
$-27x + 36z - 20 = 7$
We use this equation for the plane intersection line.

$2): -3x - 2y + 4z = -1$
-> $-3x - 2(-4 - 6x + 8z) + 4z = -1$
-> $-3x + 8 + 12x - 16z + 4z = -1$
$8 + 9x + -12z = -1$

$$\begin{cases} 1): -27x + 36z - 20 = 7 \\ 2): 8 + 9x + -12z = -1 \\ 3): y = -4 - 6x + 8z \end{cases}$$

Rearranging for x
$2): 8 + 9x + -12z = -1$
-> $9x + -4z = -9$
-> $9x = -9 + 4z$
$x = \frac{-9+12z}{9}$

Substitute known x value into the final equation to find z
$1): -27\left(\frac{-9+12z}{9}\right) + 36z - 20 = 7$
-> $-27(\frac{1}{9}(-9 + 12z))$
-> $-3(-9 + 12z)$
-> $27 - 48z$
-> $27 - 38z + 36z - 20 = 7$
-> $7 + 2z = 7$
-> $2z = 0$
$z = 0$

Use known z value to find x
$x = \frac{-9+4z}{9}$
-> $x = \frac{-9+4(0)}{9}$
-> $\frac{-9}{9}$
$x = -1$

Use known x/z value to find y
$y = -4 - 6x + 8z$
-> $y = -4 - 6(-1) + 8(0)$
-> $y = -4 + 6 + 0$
$y = 2$

Task 2: Matrices

1/2) Find M, a 2×2 rotation matrix to rotate $\begin{bmatrix} x \\ y \end{bmatrix}$ around the origin

$M \times \begin{bmatrix} x \\ y \end{bmatrix}$ should rotate this by θ° around the origin counter-clockwise.

$$M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

(Math Easy Solutions; 2023)

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ) \\ \sin(180^\circ) & \cos(180^\circ) \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -0 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} -1 * x + 0 * x \\ 0 * y + -1 * y \end{bmatrix} = \begin{bmatrix} -x \\ -y \end{bmatrix}$$

The coordinates have their signs flipped, in this case $(x, y) \to (-x, -y)$.

Math Easy Solutions; (6 March 2023) [Rotating a Vector with the Rotation Matrix] [video]; Math Easy Solutions, YouTube; accessed 9 April 2024

Task 3: Functions

1) Compare these functions

$y = t^2$

- Domain: Any real number: $-\infty < t < \infty$
- Range: Any real, positive number: $y \geq 0$
  $y = (t+1)^2$
- Domain: Any real number: $-\infty < t < \infty$
- Range: Any real, positive number: $y \geq 0$
  $y = t^2 + 1$
- Domain: Any real number: $-\infty < t < \infty$
- Range: Any real, positive number: $y \geq 1$

a) $y = (t+1)^2$ is $y = t^2$ shifted left (towards negative) in the x-axis

b) $t^2 + 1$ is $y = t^2$ shifted upwards (towards positives) in the y-axis

c) The function obtained by shifting $y = t^2$ one unit down and one unit right is $y = (t^2 - 1) - 1$

d) The function obtained by shifting $y = t^2$ up h units and k units right is $y = (t^2 - k) + h$

2) Compare the functions

$y = \ln(t)$
$y = \ln(t+1)$
$y = \ln(t) + 1$

In $y = \ln(t+1)$, the graph is shifted left by 1 unit, so instead of $\ln(t)$ which collides with $(1, \text{undefined})$, this equation collides with $(0, 0)$
In $y = \ln(t) + 1$, the graph is moved upwards by 1 unit

Working out - $y = \ln(t+1)$
y-intercept:
-> $y = \ln(0+1) = 0$
x-intercept:
-> $0 = \ln(t+1)$
-> $e^0 = t+1$
-> $1 = t+1$
-> $t = 0$
$(0, 0)$

Working out - $y = \ln(t)$
y-intercept:
-> $y = \ln(0)$
$y = \text{undefined}$
x-intercept:
-> $0 = \ln(t)$
-> $e^0 = t$
$t = 1$

$(1, \text{undefined})$

3) Solve the following equations for x, giving all solutions in the given domain

a) $2\sin\left(2x + \frac{\pi}{4}\right) = \sqrt{2}$, domain is $x \in [-\pi, \pi]$

What we're trying to find is the x coordinates where $y = \sqrt{2}$

Solution
The x values for this function are $\pi, -\pi, 0, \frac{\pi}{4}, \frac{-3\pi}{4}$ radians given the domain $x \in [-\pi, \pi]$

Finding the equations
$2\sin(2x + \frac{\pi}{4}) = \sqrt{2}$
-> $\sin(2x + \frac{\pi}{4}) = \frac{\pm\sqrt{2}}{2}$
-> $2x + \frac{\pi}{4} = \arccos(\frac{\pm\sqrt{2}}{2})$
-> $2x + \frac{\pi}{4} = \arccos(\frac{+\sqrt{2}}{2})$
-> $\arccos(\frac{+\sqrt{2}}{2}) = 0.7854 = \frac{\pi}{4}$
-> Add $2\pi n$ to equation for $\sin / \cos$ waves as they're periodic
$2x + \frac{\pi}{4} = \frac{\pi}{4} + 2\pi n$
-> $2x + \frac{\pi}{4} = \arccos(\frac{-\sqrt{2}}{2})$
-> $\arccos\left(\frac{-\sqrt{2}}{2}\right) = 2.356 = \frac{3\pi}{4}$
-> Add $2\pi n$ for $\sin / \cos$ waves
$2x + \frac{\pi}{4} = \frac{3\pi}{4} + 2\pi n$

Finding x values: $2x+\frac{\pi}{4}=\frac{\pi}{4}+2\pi n$
$2x+\frac{\pi}{4}=\frac{\pi}{4}+2\pi n$
-> $2x=2\pi n$
-> $x=\frac{2\pi n}{2}$
$x=\pi n$

$n=-2..2$
-> $-2\pi,-\pi,0,\pi,2\pi$
$n=-2$ and $n=2$ are out of the domain.
-> $-\pi,0,\pi$

Finding x values: $2x+\frac{\pi}{4}=\frac{3\pi}{4}+2\pi n$
$2x+\frac{\pi}{4}=\frac{3\pi}{4}+2\pi n$
-> $2x=\frac{\pi}{2}+2\pi n$
-> $x=\frac{\frac{\pi}{2}+2\pi n}{2}$
$x=\frac{\pi}{4}+\pi n$

$n=-2..2$
-> $-\frac{7\pi}{4},-\frac{3\pi}{4},\frac{\pi}{4},\frac{5\pi}{4},\frac{9\pi}{4}$
$n=-2\to-5.49$, $n=1\to3.93$, and $n=2\to7.07$ are out of the domain.
-> $-\frac{3\pi}{4},\frac{\pi}{4}$

b) $\sqrt{2}\cos(x)+1=0$, domain is $x\in[0,2\pi]$

$\sqrt{2}\cos(x)+1=0$

Solutions
The x values for this function are $-2.35619, 3.92699$ radians given the domain $x\in[0\pi,2\pi]$

-2.35619
-> 3.92699

Working out
-> $\sqrt{2}\cos(x)=-1$
-> $\cos(x)=\frac{-1}{\sqrt{2}}$
-> $x=\arccos(-\frac{1}{\sqrt{2}})$
-> $x=\arccos(-\frac{1}{\sqrt{2}})$
-> 2.35619
-> Add $2\pi n$ to equation for $\sin/\cos$ waves as they're periodic
$x=2.35619+2\pi n$
-> $x=\arccos(-\frac{1}{\sqrt{2}})$
-> $x=\arccos\left(-\frac{1}{\sqrt{2}}\right)=2.35619$
-> Apply the inverse $\cos$ mirror solution $-\arccos(\theta)$
-> $-(2.35619)=-2.35619$
-> Add $2\pi n$ to equation for $\sin/\cos$ waves as they're periodic
$x=-2.35619+2\pi n$

Finding x values: $x = 2.35619 + 2\pi n$

$x = 2.35619 + 2\pi n$

$n = -2..2$

-> $x = -10.2102, -3.927, 2.35619, 8.63938, 14.9226$

$n = -2, -1, 1, 2$ are out of the domain (max$= 2\pi = 6.28$).

-> 2.35619

Finding x values: $x = -2.35619 + 2\pi n$

$x = -2.35619 + 2\pi n$

$n = -2..2$

-> $x = -14.9226, -8.63938, -2.35619, 3.92699, 10.2102$

$n = -2, -1, 0, 2$ are out of the domain (max$= 2\pi = 6.28$)

-> 3.92699

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4) Summarise what you've learned about functions in this task

I have solidified my understanding of translating equations/functions, positive, internal values shifting x negatively, and positive, external values shifting y positively. I learnt a lot about how inverse trigonometry functions work and how they have two solutions, one the principle and the other the mirror which are both needed to find all valid x values within a given domain.