Chapter 10

10 Applications of differentiation

Objectives

- To be able to find the equations of the **tangent** and the **normal** at a given point on a curve.
- To be able to find the **stationary points** on the curves of certain polynomial functions and state the nature of such points.
- To use differentiation techniques to **sketch graphs**.
- To solve **maximum and minimum problems**.
- To use the derivative of a function in **rates of change** problems.

In this chapter we continue our study of differential calculus. There are two main aspects of this chapter. One is to apply our knowledge of the derivative to sketching graphs and solving maximum and minimum problems. The other is to see that the derivative can be used to define instantaneous rate of change.

The new techniques for sketching graphs of polynomial functions are a useful addition to the skills that were introduced in Chapter 4. At that stage, rather frustratingly, we were only able to determine the coordinates of turning points of cubic and quartic functions using technology. The new techniques are also used for determining maximum or minimum values for problems set in a 'real world' context.

The use of the derivative to determine instantaneous rates of change is a very important application of calculus. One of the first areas of applied mathematics to be studied in the seventeenth century was motion in a straight line. The problems of kinematics were the motivation for Newton's work on calculus.

ISBN 978-1-107-56747-4

10A Tangents and normals

The derivative of a function is a new function that gives the measure of the gradient of the tangent at each point on the curve. Having the gradient, we can find the equation of the tangent line at a given point on the curve.

Suppose that (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent at (x_1, y_1) is given by

$$y - y_1 = f'(x_1)(x - x_1)$$

Example 1

Find the equation of the tangent to the curve $y = x^3 + \frac{1}{2}x^2$ at the point $x = 1$.

Solution

When $x = 1$, $y = \frac{3}{2}$, and so $\left(1, \frac{3}{2}\right)$ is a point on the tangent.

Since $\frac{dy}{dx} = 3x^2 + x$, the gradient of the tangent at $x = 1$ is 4.

Hence the equation of the tangent is

$$y - \frac{3}{2} = 4(x - 1)$$

i.e. $$y = 4x - \frac{5}{2}$$

The **normal** to a curve at a point on the curve is the line that passes through the point and is perpendicular to the tangent at that point.

Recall from Chapter 2 that two lines with gradients m_1 and m_2 are perpendicular if and only if $m_1 m_2 = -1$.

Thus, if a tangent has gradient m, the normal has gradient $-\frac{1}{m}$.

Example 2

Find the equation of the normal to the curve with equation $y = x^3 - 2x^2$ at the point $(1, -1)$.

Solution

The point $(1, -1)$ is on the normal.

Since $\frac{dy}{dx} = 3x^2 - 4x$, the gradient of the normal at $x = 1$ is $\frac{-1}{-1} = 1$.

Hence the equation of the normal is

$$y - (-1) = 1(x - 1)$$

i.e. $$y = x - 2$$

ISBN 978-1-107-56747-4

Example 3

Find the equation of the tangent to the curve with equation $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$ at the point on the graph where $x = 4$.

Solution

Let $y = x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$. Then $\dfrac{dy}{dx} = \dfrac{3}{2}x^{\frac{1}{2}} - 2x^{-\frac{1}{2}}$.

When $x = 4$,

$$y = 4^{\frac{3}{2}} - 4 \times 4^{\frac{1}{2}} = 0$$

and $$\frac{dy}{dx} = \frac{3}{2} \times 4^{\frac{1}{2}} - 2 \times 4^{-\frac{1}{2}} = 2$$

Hence the equation of the tangent is

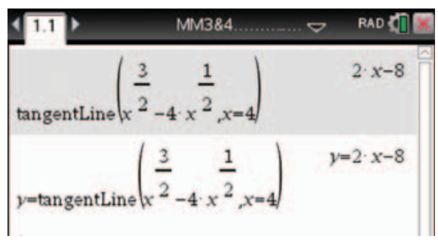
$$y - 0 = 2(x - 4)$$

i.e. $$y = 2x - 8$$

Using the TI-Nspire

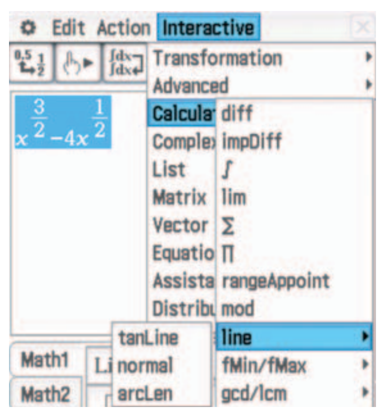
Use (menu) > **Calculus** > **Tangent Line** and complete as shown.

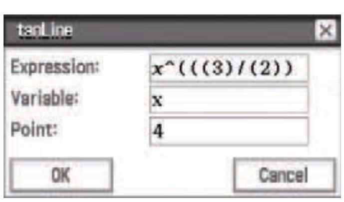
Note: The equation of the tangent can also be found in a **Graphs** application.



Using the Casio ClassPad

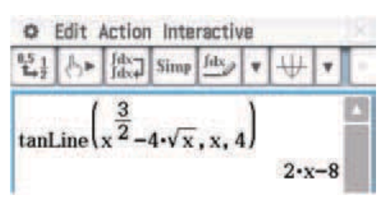
- In $\sqrt{\alpha}$ Main, enter and highlight the expression $x^{\frac{3}{2}} - 4x^{\frac{1}{2}}$.
- Go to **Interactive** > **Calculation** > **line** > **tanLine**.
- Enter the x-value 4 in the tanLine window and tap OK.





- Write your answer as an equation: $y = 2x - 8$.

Note: You can also obtain the tangent line by sketching the graph and using **Analysis** > **Sketch** > **Tangent**.



ISBN 978-1-107-56747-4

Example 4

Find the equation of the tangent to the graph of $y = \sin x$ at the point where $x = \frac{\pi}{3}$.

Solution

Let $y = \sin x$. Then $\frac{dy}{dx} = \cos x$. When $x = \frac{\pi}{3}$, $y = \frac{\sqrt{3}}{2}$ and $\frac{dy}{dx} = \frac{1}{2}$.

Therefore the equation of the tangent is

$$y - \frac{\sqrt{3}}{2} = \frac{1}{2}\left(x - \frac{\pi}{3}\right)$$

i.e. $$y = \frac{x}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{2}$$

Example 5

Find the equations of the tangent and normal to the graph of $y = -\cos x$ at the point $\left(\frac{\pi}{2}, 0\right)$.

Solution

First find the gradient of the curve at this point:

$\frac{dy}{dx} = \sin x$ and so, when $x = \frac{\pi}{2}$, $\frac{dy}{dx} = 1$.

The equation of the tangent is

$$y - 0 = 1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = x - \frac{\pi}{2}$$

The gradient of the normal is -1 and therefore the equation of the normal is

$$y - 0 = -1\left(x - \frac{\pi}{2}\right)$$

i.e. $$y = -x + \frac{\pi}{2}$$

The following example shows two situations in which we can view a graph as having a 'vertical tangent line' at a point where the derivative is not defined.

Example 6

Find the equation of the tangent to:

a $f(x) = x^{\frac{1}{3}}$ where $x = 0$ **b** $f(x) = x^{\frac{2}{3}}$ where $x = 0$.

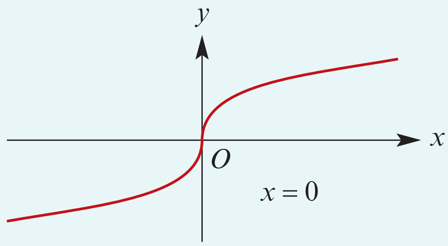
Solution

a The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0$.

The graph has a **vertical tangent** at $x = 0$.



ISBN 978-1-107-56747-4

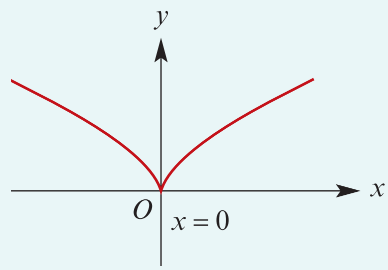
b $f(x) = x^{\frac{2}{3}}$

The derivative of f is not defined at $x = 0$.

For $x \in \mathbb{R} \setminus \{0\}$, $f'(x) = \dfrac{2}{3}x^{-\frac{1}{3}}$.

It is clear that f is continuous at $x = 0$ and that $f'(x) \to \infty$ as $x \to 0^+$ and $f'(x) \to -\infty$ as $x \to 0^-$.

There is a **cusp** at $x = 0$, and the graph of $y = f(x)$ has a **vertical tangent** at $x = 0$.



Section summary

- Equation of a tangent Suppose (x_1, y_1) is a point on the curve $y = f(x)$. Then, if f is differentiable at $x = x_1$, the equation of the tangent to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$.
- Gradient of normal If a tangent has gradient m, the normal has gradient $-\dfrac{1}{m}$.

Exercise 10A

Example 1 **1** Find the equation of the tangent to the curve $y = x^2 - 1$ at the point $(2, 3)$.

Example 2 **2** Find the equation of the normal to the curve $y = x^2 + 3x - 1$ at the point where the curve cuts the y-axis.

3 Find the equations of the normals to the curve $y = x^2 - 5x + 6$ at the points where it cuts the x-axis.

4 Find the equations of the tangent and the normal to the curve $y = (2x + 1)^9$ at the point $(0, 1)$.

5 Find the coordinates of the point on $y = x^2 - 5$ at which the curve has gradient 3. Hence find the value of c for which the line $y = 3x + c$ is tangent to $y = x^2 - 5$.

6 Find the equations of **i** the tangent and **ii** the normal at the point corresponding to the given x-value on each of the following curves:

a $y = x^2 - 2;\ x = 1$
b $y = x^2 - 3x - 1;\ x = 0$
c $y = \dfrac{1}{x};\ x = -1$
d $y = (x - 2)(x^2 + 1);\ x = -1$
e $y = \sqrt{3x + 1};\ x = 0$
f $y = \sqrt{x};\ x = 1$
g $y = x^{\frac{2}{3}} + 1;\ x = 1$
h $y = x^3 - 8x;\ x = 2$
i $y = x^3 - 3x^2 + 2;\ x = 2$
j $y = 2x^3 + x^2 - 4x + 1;\ x = 1$

Example 3 **7** Use a CAS calculator to find the equation of the tangent to the curve with equation $y = 4x^{\frac{5}{2}} - 8x^{\frac{3}{2}}$ at the point on the graph where $x = 4$.

ISBN 978-1-107-56747-4

8 Find the equation of the tangent at the point corresponding to the given x-value on each of the following curves:

a $y = \dfrac{x^2 - 1}{x^2 + 1}$; $x = 0$ **b** $y = \sqrt{3x^2 + 1}$; $x = 1$

c $y = \dfrac{1}{2x - 1}$; $x = 0$ **d** $y = \dfrac{1}{(2x - 1)^2}$; $x = 1$

Example 4, 5

9 Find the equation of the tangent to each of the following curves at the given x-value:

a $y = \sin(2x)$; $x = 0$ **b** $y = \cos(2x)$; $x = \dfrac{\pi}{2}$ **c** $y = \tan x$; $x = \dfrac{\pi}{4}$

d $y = \tan(2x)$; $x = 0$ **e** $y = \sin x + x\sin(2x)$; $x = 0$ **f** $y = x - \tan x$; $x = \dfrac{\pi}{4}$

10 For each function, find the equation of the tangent to the graph at the given value of x:

a $f(x) = e^x + e^{-x}$; $x = 0$ **b** $f(x) = \dfrac{e^x - e^{-x}}{2}$; $x = 0$

c $f(x) = x^2 e^{2x}$; $x = 1$ **d** $f(x) = e^{\sqrt{x}}$; $x = 1$

e $f(x) = xe^{x^2}$; $x = 1$ **f** $f(x) = x^2 e^{-x}$; $x = 2$

11 **a** Find the equation of the tangent and the normal to the graph of $f(x) = \log_e x$ at the point $(1, 0)$.

b Find the equation of the tangent to the graph of $f(x) = \log_e(2x)$ at the point $\left(\dfrac{1}{2}, 0\right)$.

c Find the equation of the tangent to the graph of $f(x) = \log_e(kx)$ at the point $\left(\dfrac{1}{k}, 0\right)$, where $k \in \mathbb{R}^+$.

Example 6

12 Find the equation of the tangent at the point where $y = 0$ for each of the following curves:

a $y = x^{\frac{1}{5}}$ **b** $y = x^{\frac{3}{5}}$ **c** $y = (x - 4)^{\frac{1}{3}}$

d $y = (x + 5)^{\frac{2}{3}}$ **e** $y = (2x + 1)^{\frac{1}{3}}$ **f** $y = (x + 5)^{\frac{4}{5}}$

13 The tangent to the curve with equation $y = \tan(2x)$ at the point where $x = \dfrac{\pi}{8}$ meets the y-axis at the point A. Find the distance OA, where O is the origin.

14 The tangent to the curve with equation $y = 2e^x$ at the point $(a, 2e^a)$ passes through the origin. Find the value of a.

15 The tangent to the curve with equation $y = \log_e x$ at the point $(a, \log_e a)$ passes through the origin. Find the value of a.

16 The tangent to the curve with equation $y = x^2 + 2x$ at the point $(a, a^2 + 2a)$ passes through the origin. Find the value of a.

17 The tangent to the curve with equation $y = x^3 + x$ at the point $(a, a^3 + a)$ passes through the point $(1, 1)$. Find the value of a.

ISBN 978-1-107-56747-4

10B Rates of change

The derivative was defined geometrically in the previous chapter. However, the process of differentiation may also be used to tackle many kinds of problems involving rates of change.

For the function with rule $f(x)$:

- The **average rate of change** for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The **instantaneous rate of change** of f with respect to x when $x = a$ is given by $f'(a)$.

The derivative $\dfrac{dy}{dx}$ gives the instantaneous rate of change of y with respect to x.

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

Example 7

For the function with rule $f(x) = x^2 + 2x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

Solution

a Average rate of change $= \dfrac{f(3) - f(2)}{3 - 2} = 15 - 8 = 7$

b Average rate of change

$$\begin{aligned} &= \frac{f(2+h) - f(2)}{2 + h - 2} \\ &= \frac{(2+h)^2 + 2(2+h) - 8}{h} \\ &= \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\ &= \frac{6h + h^2}{h} = 6 + h \end{aligned}$$

c The derivative is $f'(x) = 2x + 2$. When $x = 2$, the instantaneous rate of change is $f'(2) = 6$. This can also be seen from the result of part **b**.

Example 8

A balloon develops a microscopic leak and gradually decreases in volume. Its volume, V cm^3, at time t seconds is $V = 600 - 10t - \dfrac{1}{100}t^2$, $t \geq 0$.

a Find the rate of change of volume after:

i 10 seconds **ii** 20 seconds

b For how long could the model be valid?

ISBN 978-1-107-56747-4

Solution

a $\dfrac{dV}{dt} = -10 - \dfrac{t}{50}$

i When $t = 10$, $\dfrac{dV}{dt} = -10\frac{1}{5}$

i.e. the volume is decreasing at a rate of $10\frac{1}{5}$ cm^3 per second.

ii When $t = 20$, $\dfrac{dV}{dt} = -10\frac{2}{5}$

i.e. the volume is decreasing at a rate of $10\frac{2}{5}$ cm^3 per second.

b The model will not be meaningful when $V < 0$. Consider $V = 0$.

$$600 - 10t - \frac{1}{100}t^2 = 0$$

$$\therefore \quad t = 100(\sqrt{31} - 5) \quad \text{or} \quad t = -100(\sqrt{31} + 5)$$

The model may be suitable for $0 \le t \le 100(\sqrt{31} - 5)$.

Example 9

A pot of liquid is put on the stove. When the temperature of the liquid reaches 80°C, the pot is taken off the stove and placed on the kitchen bench. The temperature in the kitchen is 20°C. The temperature of the liquid, T°C, at time t minutes is given by

$$T = 20 + 60e^{-0.3t}$$

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 80$ **ii** $T = 30$

Solution

a By rearranging $T = 20 + 60e^{-0.3t}$, we see that $e^{-0.3t} = \dfrac{T - 20}{60}$.

Now $\quad T = 20 + 60e^{-0.3t}$

$$\therefore \quad \frac{dT}{dt} = -18e^{-0.3t}$$

Hence
$$\frac{dT}{dt} = -18\left(\frac{T-20}{60}\right)$$
$$= -3\left(\frac{T-20}{10}\right)$$
$$= 0.3(20 - T)$$

b **i** When $T = 80$, $\dfrac{dT}{dt} = 0.3(20 - 80)$
$= -18$

The liquid is cooling at a rate of 18°C per minute.

ii When $T = 30$, $\dfrac{dT}{dt} = 0.3(20 - 30)$
$= -3$

The liquid is cooling at a rate of 3°C per minute.

ISBN 978-1-107-56747-4

Cambridge University Press

▶ Motion in a straight line

Position, velocity and acceleration were introduced for an object moving in a straight line in Mathematical Methods Units 1 & 2.

Position (x m) is specified with respect to a reference point O on the line. Velocity (v m/s) and acceleration (a m/s^2) are given by:

velocity $v = \dfrac{dx}{dt}$ acceleration $a = \dfrac{dv}{dt}$

Example 10

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by the formula $x = t^3 - 6t^2 + 9t$. Find:

a at what times and in what positions the particle will have zero velocity

b its acceleration at those instants

c its velocity when its acceleration is zero.

Solution

Velocity $v = \dfrac{dx}{dt} = 3t^2 - 12t + 9$

a When $v = 0$,

$$3(t^2 - 4t + 3) = 0$$
$$(t - 1)(t - 3) = 0$$
$$t = 1 \text{ or } t = 3$$

i.e. the velocity is zero when $t = 1$ and $t = 3$ and where $x = 4$ and $x = 0$.

b Acceleration $a = \dfrac{dv}{dt} = 6t - 12$

When $t = 1$, $a = -6$ m/s^2

When $t = 3$, $a = 6$ m/s^2

c The acceleration is zero when $6t - 12 = 0$, i.e. when $t = 2$.

When $t = 2$, the velocity $v = 3 \times 4 - 24 + 9$

$= -3$ m/s

Section summary

For the function with rule $f(x)$:

- The average rate of change for $x \in [a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$.
- The instantaneous rate of change of f with respect to x when $x = a$ is given by $f'(a)$.

ISBN 978-1-107-56747-4

Exercise 10B

Skillshet

Example 7

1 For the function with rule $f(x) = 3x^2 + 6x$, find:

a the average rate of change for $x \in [2, 3]$

b the average rate of change for the interval $[2, 2 + h]$

c the instantaneous rate of change of f with respect to x when $x = 2$.

2 Express each of the following in symbols:

a the rate of change of volume (V) with respect to time (t)

b the rate of change of surface area (S) of a sphere with respect to radius (r)

c the rate of change of volume (V) of a cube with respect to edge length (x)

d the rate of change of area (A) with respect to time (t)

e the rate of change of volume (V) of water in a glass with respect to depth of water (h)

Example 8

3 If your interest (I) in Mathematical Methods can be expressed as

$$I = \frac{4}{(t+1)^2}$$

where t is the time in days measured from the first day of Term 1, how fast is your interest waning when $t = 10$?

4 A reservoir is being emptied and the quantity of water, V m^3, remaining in the reservoir t days after it starts to empty is given by

$$V(t) = 10^3(90 - t)^3$$

a At what rate is the reservoir being emptied at time t?

b How long does it take to empty the reservoir?

c What is the volume of water in the reservoir when $t = 0$?

d After what time is the reservoir being emptied at 3×10^5 m^3/day?

e Sketch the graph of $V(t)$ against t.

f Sketch the graph of $V'(t)$ against t.

5 A coffee percolator allows 1000 mL of water to flow into a filter in 20 minutes. The volume which has flowed into the filter at time t minutes is given by

$$V(t) = \frac{1}{160}\left(5t^4 - \frac{t^5}{5}\right), \quad 0 \le t \le 20$$

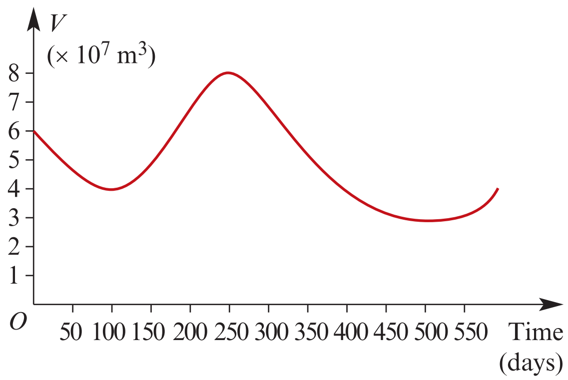
a At what rate is the water flowing into the filter at time t minutes?

b Sketch the graph of $\frac{dV}{dt}$ against t for $0 \le t \le 20$.

c When is the rate of flow greatest?

ISBN 978-1-107-56747-4

6 The graph shows the volume, V m^3, of water in a reservoir at time t days.



a At what times is the rate of flow from the reservoir 0 m^3/day?

b Find an estimate for the rate of flow at $t = 200$.

c Find the average rate of flow for the interval $[100, 250]$.

d State the times for which there is net flow into the reservoir.

7 A car tyre is inflated to a pressure of 30 units. Eight hours later it is found to have deflated to a pressure of 10 units. The pressure, P, at time t hours is given by

$$P = P_0 e^{-\lambda t}$$

a Find the values of P_0 and λ.

b At what time would the pressure be 8 units?

c Find the rate of loss of pressure at:

i time $t = 0$

ii time $t = 8$

Example 9 **8** A liquid is heated to a temperature of 90°C and then allowed to cool in a room in which the temperature is 15°C. While the liquid is cooling, its temperature, T°C, at time t minutes is given by $T = 15 + 75e^{-0.3t}$.

a Find the rate of change of temperature with respect to time in terms of T.

b Find the rate of change of temperature with respect to time when:

i $T = 90$

ii $T = 60$

iii $T = 30$

9 If $y = 3x + 2\cos x$, find $\dfrac{dy}{dx}$ and hence show that y increases as x increases.

10 The volume of water in a reservoir at time t is given by $V(t) = 3 + 2\sin\left(\dfrac{t}{4}\right)$.

a Find the volume in the reservoir at time $t = 10$.

b Find the rate of change of the volume of water in the reservoir at time $t = 10$.

ISBN 978-1-107-56747-4

Example 10 **11** A particle moves along a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = 2t^3 - 9t^2 + 12t$.

- **a** Find the velocity, v, as a function of t.
- **b** At what times and in what positions will the particle have zero velocity?
- **c** Find its acceleration at those instants.
- **d** Find its velocity when its acceleration is zero.

12 A particle moves in a straight line such that its position, x cm, relative to a point O at time t seconds is given by the equation $x = 8 + 2t - t^2$. Find:

- **a** its initial position
- **b** its initial velocity
- **c** when and where the velocity is zero
- **d** its acceleration at time t.

13 A particle is moving in a straight line such that its position, x cm, relative to a point O at time t seconds is given by $x = \sqrt{2t^2 + 2}$. Find:

- **a** the velocity as a function of t
- **b** the acceleration as a function of t
- **c** the velocity and acceleration when $t = 1$.

14 A vehicle is travelling in a straight line away from point O. Its distance from O after t seconds is $0.4e^t$ metres. Find the velocity of the vehicle when $t = 0$, $t = 1$, $t = 2$.

15 A manufacturing company has a daily output on day t of a production run given by $y = 600(1 - e^{-0.5t})$, where the first day of the production run is $t = 0$.

- **a** Sketch the graph of y against t. (Assume a continuous model.)
- **b** Find the instantaneous rate of change of output y with respect to t on the 10th day.

16 For each of the following, find $\frac{dy}{dx}$ in terms of y:

- **a** $y = e^{-2x}$
- **b** $y = Ae^{kx}$

17 The mass, m kg, of radioactive lead remaining in a sample t hours after observations began is given by $m = 2e^{-0.2t}$.

- **a** Find the mass left after 12 hours.
- **b** Find how long it takes for the mass to fall to half of its value at $t = 0$.
- **c** Find how long it takes for the mass to fall to **i** one-quarter and **ii** one-eighth of its value at $t = 0$.
- **d** Express the rate of decay as a function of m.

ISBN 978-1-107-56747-4

10C Stationary points

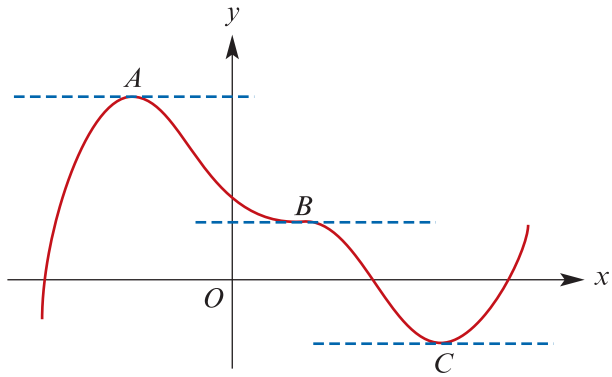
In the previous chapter, we have seen that the gradient of the tangent at a point $(a, f(a))$ on the curve with rule $y = f(x)$ is given by $f'(a)$.

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

In the graph shown, there are stationary points at A, B and C.

At such points, the tangents are parallel to the x-axis (illustrated as dashed lines).



The reason for the name *stationary point* becomes clear if we look at an application to the motion of a particle.

Example 11

A particle is moving in a straight line. Its position, x metres, relative to a point O on the line at time t seconds is given by

$$x = 9t - \frac{1}{3}t^3, \quad 0 \le t \le 4$$

Find the particle's maximum distance from O. (Here the particle is always on the right of O and so its distance from O is its position.)

Solution

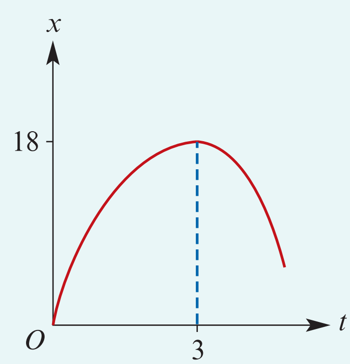
$$\frac{dx}{dt} = 9 - t^2$$

The maximum distance from O occurs when $\frac{dx}{dt} = 0$.

So $t = 3$ or $t = -3$. But $t = -3$ lies outside the domain.

At $t = 3$, $x = 18$.

Thus the stationary point is $(3, 18)$ and the maximum distance from O is 18 metres.



Note: The stationary point occurs when the rate of change of position with respect to time (the velocity) is zero. At this moment, the particle is stationary.

ISBN 978-1-107-56747-4

Cambridge University Press

Example 12

Find the stationary points of the following functions:

a $y = 9 + 12x - 2x^2$ **b** $y = 4 + 3x - x^3$ **c** $p = 2t^3 - 5t^2 - 4t + 13,\ t > 0$

Solution

a
$$y = 9 + 12x - 2x^2$$
$$\frac{dy}{dx} = 12 - 4x$$

A stationary point occurs when $\frac{dy}{dx} = 0$, i.e. when $12 - 4x = 0$.

Hence $x = 3$ and
$$y = 9 + 12 \times 3 - 2 \times 3^2 = 27$$

The stationary point is $(3, 27)$.

b
$$y = 4 + 3x - x^3$$
$$\frac{dy}{dx} = 3 - 3x^2$$
$$\frac{dy}{dx} = 0 \text{ implies } 3(1 - x^2) = 0$$
$$\therefore \quad x = \pm 1$$

The stationary points are $(1, 6)$ and $(-1, 2)$.

c
$$p = 2t^3 - 5t^2 - 4t + 13$$
$$\frac{dp}{dt} = 6t^2 - 10t - 4, \quad t > 0$$
$$\frac{dp}{dt} = 0 \text{ implies } 2(3t^2 - 5t - 2) = 0$$
$$(3t + 1)(t - 2) = 0$$
$$\therefore \quad t = -\frac{1}{3} \text{ or } t = 2$$

But $t > 0$, and so the only acceptable solution is $t = 2$. The corresponding stationary point is $(2, 1)$.

Example 13

Find the stationary points of the following functions:

a $y = \sin(2x),\ x \in [0, 2\pi]$ **b** $y = e^{2x} - x$ **c** $y = x \log_e(2x),\ x \in (0, \infty)$

Solution

a
$$y = \sin(2x)$$
$$\frac{dy}{dx} = 2\cos(2x)$$
$$\text{So } \frac{dy}{dx} = 0 \text{ implies } 2\cos(2x) = 0$$
$$\cos(2x) = 0$$
$$2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \text{ or } \frac{7\pi}{2}$$
$$\therefore \quad x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4} \text{ or } \frac{7\pi}{4}$$

The stationary points are $\left(\frac{\pi}{4}, 1\right), \left(\frac{3\pi}{4}, -1\right), \left(\frac{5\pi}{4}, 1\right)$ and $\left(\frac{7\pi}{4}, -1\right)$.

ISBN 978-1-107-56747-4

b $y = e^{2x} - x$

$$\frac{dy}{dx} = 2e^{2x} - 1$$

So $\dfrac{dy}{dx} = 0$ implies

$$\begin{aligned} 2e^{2x} - 1 &= 0 \\ e^{2x} &= \frac{1}{2} \\ \therefore \quad x &= \frac{1}{2}\log_e\left(\frac{1}{2}\right) \\ &= -\frac{1}{2}\log_e 2 \end{aligned}$$

When $x = -\dfrac{1}{2}\log_e 2$,

$$\begin{aligned} y &= e^{2\times\frac{1}{2}\log_e\left(\frac{1}{2}\right)} + \frac{1}{2}\log_e 2 \\ &= \frac{1}{2} + \frac{1}{2}\log_e 2 \end{aligned}$$

The coordinates of the stationary point are $(-\frac{1}{2}\log_e 2, \frac{1}{2} + \frac{1}{2}\log_e 2)$.

c $y = x\log_e(2x)$

$$\frac{dy}{dx} = \log_e(2x) + 1$$

So $\dfrac{dy}{dx} = 0$ implies

$$\begin{aligned} \log_e(2x) + 1 &= 0 \\ \log_e(2x) &= -1 \\ 2x &= e^{-1} \\ \therefore \quad x &= \frac{1}{2e} \end{aligned}$$

When $x = \dfrac{1}{2e}$, $y = \dfrac{1}{2e}\log_e\left(\dfrac{2}{2e}\right)$

$$= \frac{-1}{2e}$$

The coordinates of the stationary point are $\left(\dfrac{1}{2e}, \dfrac{-1}{2e}\right)$.

Example 14

The curve with equation $y = x^3 + ax^2 + bx + c$ passes through the point $(0, 5)$ and has a stationary point at $(2, 7)$. Find a, b and c.

Solution

When $x = 0$, $y = 5$. Thus $5 = c$.

$\dfrac{dy}{dx} = 3x^2 + 2ax + b$ and at $x = 2$, $\dfrac{dy}{dx} = 0$. Therefore

$$12 + 4a + b = 0 \qquad (1)$$

The point $(2, 7)$ is on the curve and so

$$\begin{aligned} 8 + 4a + 2b + 5 &= 7 \\ \therefore \quad 6 + 4a + 2b &= 0 \qquad (2) \end{aligned}$$

Subtracting (1) from (2) gives $-6 + b = 0$. Thus $b = 6$. Substitute in (1):

$$\begin{aligned} 12 + 4a + 6 &= 0 \\ 4a &= -18 \end{aligned}$$

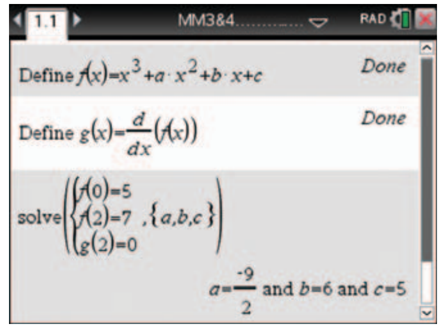
Hence $a = -\dfrac{9}{2}$, $b = 6$ and $c = 5$.

ISBN 978-1-107-56747-4

Cambridge University Press

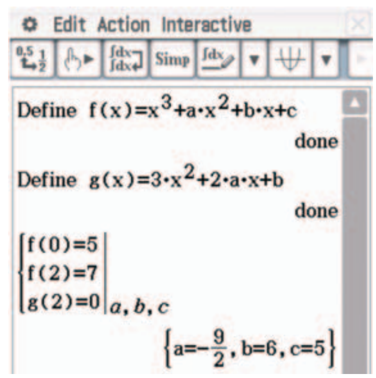
Using the TI-Nspire

- Use (menu) > **Actions** > **Define** to define $f(x) = x^3 + ax^2 + bx + c$.
- Define $g(x)$ to be the derivative ((menu) > **Calculus** > **Derivative**) of $f(x)$ as shown.
- Use the simultaneous equations solver ((menu) > **Algebra** > **Solve System of Equations** > **Solve System of Equations**) to find a, b and c given that $f(0) = 5$, $f(2) = 7$ and $g(2) = 0$.



Using the Casio ClassPad

- Use **Interactive** > **Define** to define the functions $f(x) = x^3 + ax^2 + bx + c$ and $g(x) = 3x^2 + 2ax + b$.
- In (Math1), tap ({▭) twice.
- Enter the equations and variables as shown and tap (EXE).



Section summary

- A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.
- Equivalently, a point $(a, f(a))$ on $y = f(x)$ is a stationary point if $\frac{dy}{dx} = 0$ when $x = a$.

Exercise 10C

Example 12

1 Find the stationary points for each of the following:

a $f(x) = x^3 - 12x$
b $g(x) = 2x^2 - 4x$
c $h(x) = 5x^4 - 4x^5$
d $f(t) = 8t + 5t^2 - t^3$ for $t > 0$
e $g(z) = 8z^2 - 3z^4$
f $f(x) = 5 - 2x + 3x^2$
g $h(x) = x^3 - 4x^2 - 3x + 20,\ x > 0$
h $f(x) = 3x^4 - 16x^3 + 24x^2 - 10$

Example 13

2 Find the stationary points of the following functions:

a $y = e^{2x} - 2x$
b $y = x\log_e(3x),\ x \in (0, \infty)$
c $y = \cos(2x),\ x \in [-\pi, \pi]$
d $y = xe^x$
e $y = x^2e^{-x}$
f $y = 2x\log_e x,\ x \in (0, \infty)$

ISBN 978-1-107-56747-4

3 **a** The curve with rule $f(x) = x^2 - ax + 9$ has a stationary point when $x = 3$. Find a.

b The curve with rule $h(x) = x^3 - bx^2 - 9x + 7$ has a stationary point when $x = -1$. Find b.

Example 14

4 The curve with equation $y = x^3 + bx^2 + cx + d$ passes through the point $(0, 3)$ and has a stationary point at $(1, 3)$. Find b, c and d.

5 The tangent to the curve of $y = ax^2 + bx + c$ at the point where $x = 2$ is parallel to the line $y = 4x$. There is a stationary point at $(1, -3)$. Find the values of a, b and c.

6 The graph of $y = ax^3 + bx^2 + cx + d$ touches the line $2y + 6x = 15$ at the point $A(0, 7\frac{1}{2})$ and has a stationary point at $B(3, -6)$. Find the values of a, b, c and d.

7 The curve with equation $y = ax + \dfrac{b}{2x - 1}$ has a stationary point at $(2, 7)$. Find:

a the values of a and b **b** the coordinates of the other stationary point.

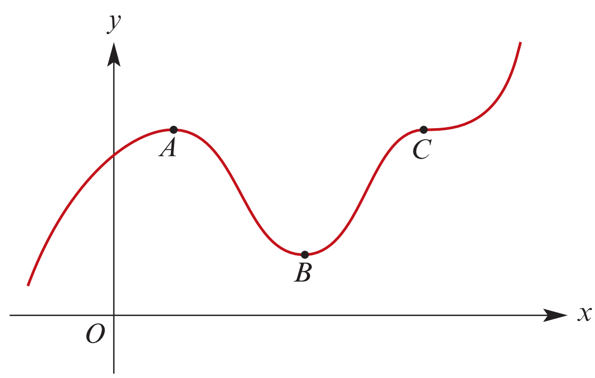
8 Find the x-coordinates, in terms of n, of the stationary points of the curve with equation $y = (2x - 1)^n(x + 2)$, where n is a natural number.

9 Find the x-coordinates of the stationary points of the curve with equation $y = (x^2 - 1)^n$ where n is an integer greater than 1.

10 Find the coordinates of the stationary points of the curve with equation $y = \dfrac{x}{x^2 + 1}$.

10D Types of stationary points

The graph of $y = f(x)$ shown has three stationary points A, B, C.



A Point A is called a **local maximum** point.
Notice that immediately to the left of A the gradient is positive, and immediately to right the gradient is negative.

gradient	+	0	−
shape of f	/	—	\

B Point B is called a **local minimum** point.
Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

gradient	−	0	+
shape of f	\	—	/

ISBN 978-1-107-56747-4

C Point C is called a **stationary point of inflection**. The gradient is positive immediately to the left and right of C.

gradient	+	0	+
shape of f	/	—	/

Clearly it is also possible to have stationary points of inflection such that the gradient is negative immediately to the left and right.

gradient	−	0	−
shape of f	\	—	\

Stationary points of types A and B are referred to as **turning points**.

Example 15

For the function $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 3x^3 - 4x + 1$:

a Find the stationary points and state their nature.

b Sketch the graph.

Solution

a The derivative is $f'(x) = 9x^2 - 4$.

The stationary points occur where $f'(x) = 0$:

$$9x^2 - 4 = 0$$
$$\therefore \quad x = \pm\frac{2}{3}$$

There are stationary points at $(-\frac{2}{3}, f(-\frac{2}{3}))$ and $(\frac{2}{3}, f(\frac{2}{3}))$, that is, at $(-\frac{2}{3}, 2\frac{7}{9})$ and $(\frac{2}{3}, -\frac{7}{9})$.
So $f'(x)$ is of constant sign for each of

$$\{\, x : x < -\tfrac{2}{3} \,\}, \quad \{\, x : -\tfrac{2}{3} < x < \tfrac{2}{3} \,\} \quad \text{and} \quad \{\, x : x > \tfrac{2}{3} \,\}$$

To calculate the sign of $f'(x)$ for each of these sets, simply choose a representative number in the set.

Thus $f'(-1) = 9 - 4 = 5 > 0$
$f'(0) = 0 - 4 = -4 < 0$
$f'(1) = 9 - 4 = 5 > 0$

We can now put together the table shown on the right.

x		$-\frac{2}{3}$		$\frac{2}{3}$	
$f'(x)$	+	0	−	0	+
shape of f	/	—	\	—	/

There is a local maximum at $(-\frac{2}{3}, 2\frac{7}{9})$ and a local minimum at $(\frac{2}{3}, -\frac{7}{9})$.

b To sketch the graph of this function we need to find the axis intercepts and investigate the behaviour of the graph for $x > \frac{2}{3}$ and $x < -\frac{2}{3}$.

The y-axis intercept is $f(0) = 1$.

To find the x-axis intercepts, consider $f(x) = 0$, which implies $3x^3 - 4x + 1 = 0$.
Using the factor theorem, we find that $x - 1$ is a factor of $3x^3 - 4x + 1$.
By division:

$$3x^3 - 4x + 1 = (x - 1)(3x^2 + 3x - 1)$$

ISBN 978-1-107-56747-4

Now $f(x) = (x - 1)(3x^2 + 3x - 1) = 0$ implies that $x = 1$ or $3x^2 + 3x - 1 = 0$.

We have

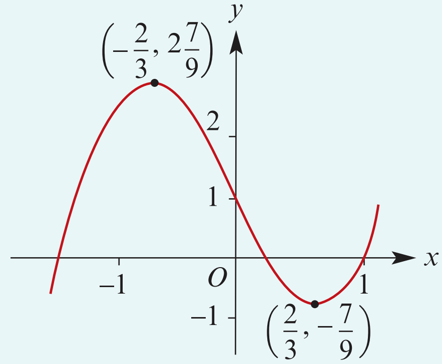
$$\begin{aligned} 3x^2 + 3x - 1 &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{1}{4} - \frac{1}{3}\right] \\ &= 3\left[\left(x + \frac{1}{2}\right)^2 - \frac{21}{36}\right] \\ &= 3\left(x + \frac{1}{2} - \frac{\sqrt{21}}{6}\right)\left(x + \frac{1}{2} + \frac{\sqrt{21}}{6}\right) \end{aligned}$$

Thus the x-axis intercepts are at

$$x = -\frac{1}{2} + \frac{\sqrt{21}}{6}, \qquad x = -\frac{1}{2} - \frac{\sqrt{21}}{6}, \qquad x = 1$$

For $x > \frac{2}{3}$, $f(x)$ becomes larger.

For $x < \frac{2}{3}$, $f(x)$ becomes smaller.



A CAS calculator can be used to plot the graph of a function and determine its key features, including:

- the value of the function at any point
- the value of its derivative at any point
- the axis intercepts
- the local maximum and local minimum points.

Example 16

Plot the graph of $y = x^3 - 19x + 20$ and determine:

a the value of y when $x = -4$

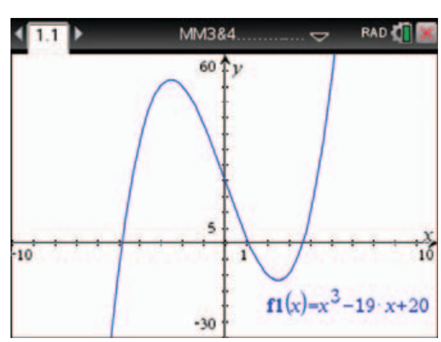
b the values of x when $y = 0$

c the value of $\frac{dy}{dx}$ when $x = -1$

d the coordinates of the local maximum.

Using the TI-Nspire

Graph $y = x^3 - 19x + 20$ in an appropriate window (menu > **Window/Zoom** > **Window Settings**).



ISBN 978-1-107-56747-4

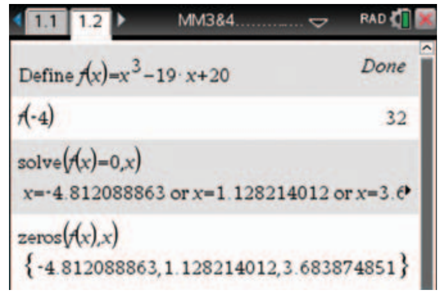
Cambridge University Press

Define $f(x) = x^3 - 19x + 20$.

a $f(-4) = 32$

b Use solve($f(x) = 0,\ x$).

Note: Alternatively, (menu) > **Algebra** > **Zeros** can be used to solve equations equal to zero as shown.

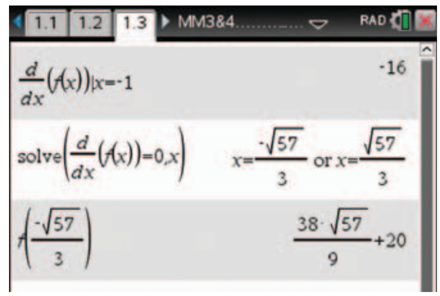


c Find the derivative of $f(x)$ at $x = -1$ as shown.

d To find the stationary points, use

$$\text{solve}\left(\frac{d}{dx}(f(x)) = 0,\ x\right)$$

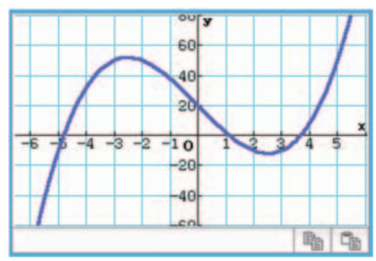
and then substitute to find the y-coordinate.



Note: Since the function was also defined in the **Graphs** application as $f1$, the name $f1$ could have been used in place of f in these calculations.

Using the Casio ClassPad

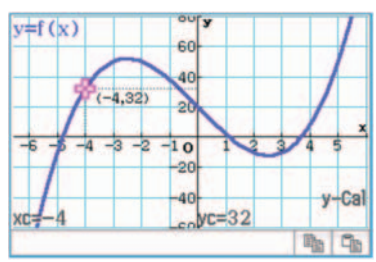
- Define $f(x) = x^3 - 19x + 20$.
- Tap [graph icon] to open the graph window.
- Drag $f(x)$ into the graph window.
- Adjust the window using [window icon].



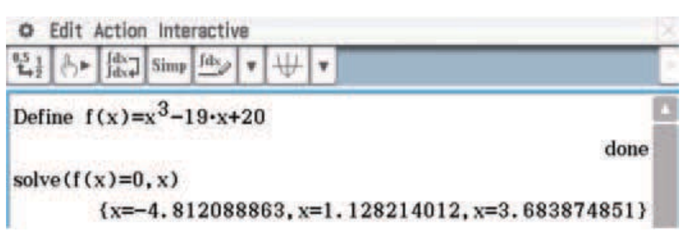
a To find $f(-4) = 32$, there are two methods:

1 In Main $\sqrt{\alpha}$, type $f(-4)$ and tap (EXE).

2 In [graph icon], go to **Analysis** > **G-Solve** > **x-Cal/y-Cal** > **y-Cal** and type -4.



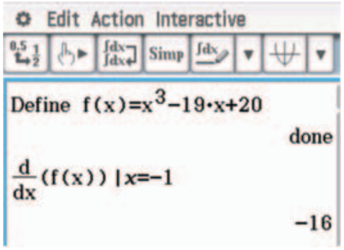
b
- In Main $\sqrt{\alpha}$, enter and highlight $f(x) = 0$.
- Go to **Interactive** > **Equation/Inequality** > **solve**.
- Rotate the screen and press ► to view all solutions.



ISBN 978-1-107-56747-4

c To find $\frac{dy}{dx}$ when $x = -1$:

- In Main $\sqrt{\alpha}$, enter and highlight $f(x)$.
- Go to **Interactive** > **Calculation** > **diff** and then tap OK.
- Select | from Math3 and type $x = -1$ as shown.
- Tap EXE.



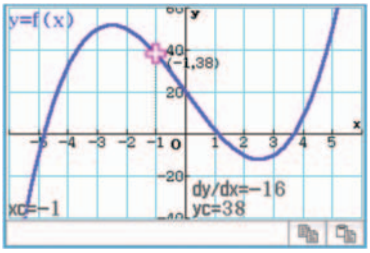
Using the graph window

To view the derivative at any point on a graph, first ensure that the Derivative/Slope setting is activated:

- Go to settings ⚙, select **Graph Format**, tick **Derivative/Slope** and tap Set.

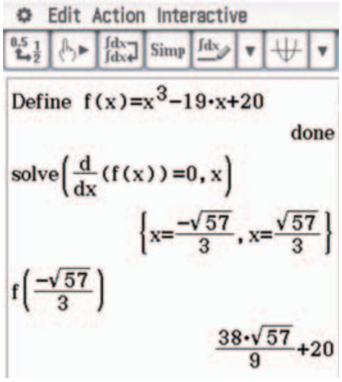
Now in ⧺:

- Go to **Analysis** > **Trace**, type -1 and tap OK.



d To find the local maximum:

- In Main $\sqrt{\alpha}$, solve $\frac{d}{dx}(f(x)) = 0$ as shown.
- Substitute to find the y-coordinate.



Alternatively, find **fMax** using an appropriate domain:

fMax(f(x), x, −6, 0)

$$\left\{\text{MaxValue}=\frac{38\cdot\sqrt{57}}{9}+20,\ x=\frac{-\sqrt{57}}{3}\right\}$$

Example 17

Sketch the graph of $f: \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x^3}$.

Solution

As $x \to -\infty$, $f(x) \to 0$.

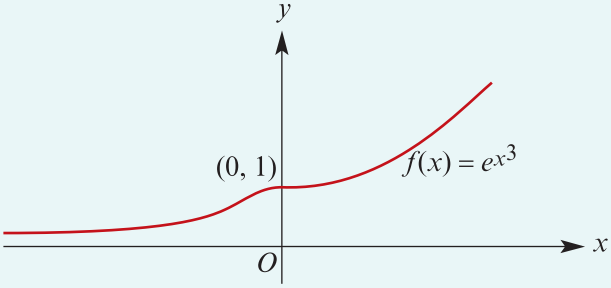
Axis intercepts

When $x = 0$, $f(x) = 1$.

Stationary points

$f'(x) = 3x^2e^{x^3}$

So $f'(x) = 0$ implies $x = 0$.



The gradient of f is always greater than or equal to 0, which means that $(0, 1)$ is a stationary point of inflection.

ISBN 978-1-107-56747-4

Example 18

For $f\colon (0, \infty) \to \mathbb{R}$, $f(x) = x \log_e x$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

Solution

a $f'(x) = x \times \dfrac{1}{x} + \log_e x$ (product rule)

$= 1 + \log_e x$

b $f(x) = x \log_e x$

Thus $f(x) = 0$ implies $x = 0$ or $\log_e x = 0$.

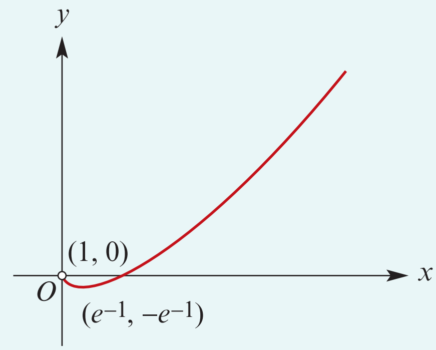
Since $x \in (0, \infty)$, the only solution is $x = 1$.

c $f'(x) = 0$ implies $1 + \log_e x = 0$.

Therefore $\log_e x = -1$ and so $x = e^{-1}$.

d When $x = e^{-1}$, $y = e^{-1} \log_e(e^{-1})$

$= e^{-1} \times (-1) = -e^{-1}$



Example 19

Find the local maximum and local minimum points of $f(x) = 2\sin x + 1 - 2\sin^2 x$, where $0 < x < 2\pi$.

Solution

Find $f'(x)$ and solve $f'(x) = 0$:

$$f(x) = 2\sin x + 1 - 2\sin^2 x$$

$$\therefore \quad f'(x) = 2\cos x - 4\sin x \cos x$$

$$= 2\cos x \cdot (1 - 2\sin x)$$

Thus $f'(x) = 0$ implies

$$\cos x = 0 \quad \text{or} \quad 1 - 2\sin x = 0$$

i.e. $\cos x = 0$ or $\sin x = \frac{1}{2}$

i.e. $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$ or $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$

We have $f\left(\dfrac{\pi}{2}\right) = 1$, $f\left(\dfrac{3\pi}{2}\right) = -3$, $f\left(\dfrac{\pi}{6}\right) = \dfrac{3}{2}$ and $f\left(\dfrac{5\pi}{6}\right) = \dfrac{3}{2}$

x		$\frac{\pi}{6}$		$\frac{\pi}{2}$		$\frac{5\pi}{6}$		$\frac{3\pi}{2}$	
$f'(x)$	+	0	−	0	+	0	−	0	+
shape of f	╱	‾	╲	_	╱	‾	╲	_	╱

Local maxima at $\left(\dfrac{\pi}{6}, \dfrac{3}{2}\right)$ and $\left(\dfrac{5\pi}{6}, \dfrac{3}{2}\right)$. Local minima at $\left(\dfrac{\pi}{2}, 1\right)$ and $\left(\dfrac{3\pi}{2}, -3\right)$.

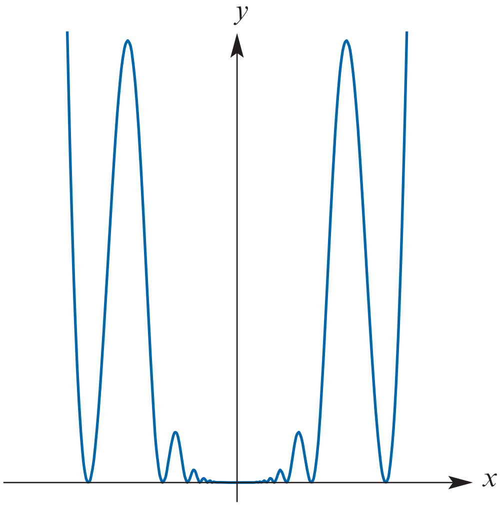
ISBN 978-1-107-56747-4

Bad behaviour? In this course, and in school courses around the world, we deal with functions that are 'conveniently behaved'. This avoids some complications.

For an example of a function which is not in this category, consider

$$f(x) = \begin{cases} x^4 \sin^2\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

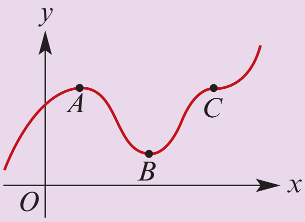
The derivative of this function is defined for all $x \in \mathbb{R}$. In any open interval around $x = 0$, the graph of this function has infinitely many stationary points, no matter how small the interval.



Section summary

A point $(a, f(a))$ on a curve $y = f(x)$ is said to be a **stationary point** if $f'(a) = 0$.

Types of stationary points



A Point A is a **local maximum**:
- $f'(x) > 0$ immediately to the left of A
- $f'(x) < 0$ immediately to the right of A.

B Point B is a **local minimum**:
- $f'(x) < 0$ immediately to the left of B
- $f'(x) > 0$ immediately to the right of B.

C Point C is a **stationary point of inflection**.

Stationary points of types A and B are called **turning points**.

Exercise 10D

Skillsheet
Example 15

1 For each of the following derivative functions, write down the values of x at which the derivative is zero and prepare a gradient table (as in Example 15) showing whether the corresponding points on the graph of $y = f(x)$ are local maxima, local minima or stationary points of inflection:

a $f'(x) = 4x^2$ **b** $f'(x) = (x - 2)(x + 5)$
c $f'(x) = (x + 1)(2x - 1)$ **d** $f'(x) = -x^2 + x + 12$
e $f'(x) = x^2 - x - 12$ **f** $f'(x) = 5x^4 - 27x^3$
g $f'(x) = (x - 1)(x - 3)$ **h** $f'(x) = -(x - 1)(x - 3)$

ISBN 978-1-107-56747-4

2 Find the stationary points on each of the following curves and state their nature:

a $y = x(x^2 - 12)$ **b** $y = x^2(3 - x)$

c $y = x^3 - 5x^2 + 3x + 2$ **d** $y = 3 - x^3$

e $y = 3x^4 + 16x^3 + 24x^2 + 3$ **f** $y = x(x^2 - 1)$

3 Sketch the graph of each of the following, finding **i** axis intercepts and **ii** stationary points:

a $y = 4x^3 - 3x^4$ **b** $y = x^3 - 6x^2$ **c** $y = 3x^2 - x^3$

d $y = x^3 + 6x^2 + 9x + 4$ **e** $y = (x^2 - 1)^5$ **f** $y = (x^2 - 1)^4$

4 **a** Find the stationary points of the graph of $y = 2x^3 + 3x^2 - 12x + 7$, stating the nature of each.

b Show that the graph passes through $(1, 0)$.

c Find the other axis intercepts.

d Sketch the graph.

5 **a** Show that the polynomial $P(x) = x^3 + ax^2 + b$ has a stationary point at $x = 0$ for all a and b.

b Given that $P(x)$ has a second stationary point at $(-2, 6)$, find the values of a and b and the nature of both stationary points.

6 Sketch the graph of $f(x) = (2x - 1)^5(2x - 4)^4$.

a State the coordinates of the axis intercepts.

b State the coordinates and nature of each stationary point.

7 **a** Sketch the graphs of $f(x) = (4x^2 - 1)^6$ and $g(x) = (4x^2 - 1)^5$ on the one set of axes.

b **i** Find $\{ x : (4x^2 - 1)^6 > (4x^2 - 1)^5 \}$. **ii** Find $\{ x : f'(x) > g'(x) \}$.

8 Sketch the graph of each of the following. State the axis intercepts and the coordinates of stationary points.

a $y = x^3 + x^2 - 8x - 12$ **b** $y = 4x^3 - 18x^2 + 48x - 290$

9 For each of the following, find the coordinates of the stationary points and determine their nature:

a $f(x) = 3x^4 + 4x^3$ **b** $f(x) = x^4 + 2x^3 - 1$ **c** $f(x) = 3x^3 - 3x^2 + 12x + 9$

10 Consider the function f defined by $f(x) = \frac{1}{8}(x - 1)^3(8 - 3x) + 1$.

a Show that $f(0) = 0$ and $f(3) = 0$.

b Show that $f'(x) = \frac{3}{8}(x - 1)^2(9 - 4x)$ and specify the values of x for which $f'(x) \geq 0$.

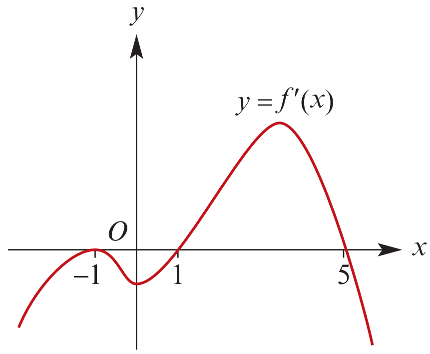
c Sketch the graph of $y = f(x)$.

11 Sketch the graph of $y = 3x^4 - 44x^3 + 144x^2$, finding the coordinates of all turning points.

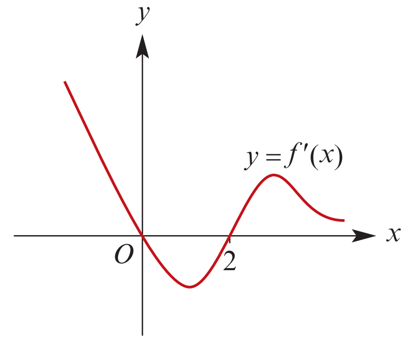
ISBN 978-1-107-56747-4

12 Each graph below shows the graph of f' for a function f. Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point.

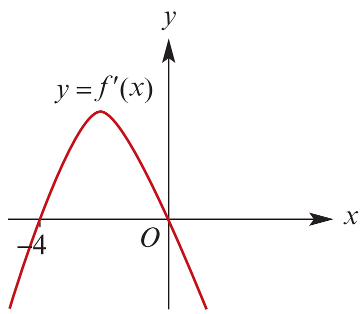
a



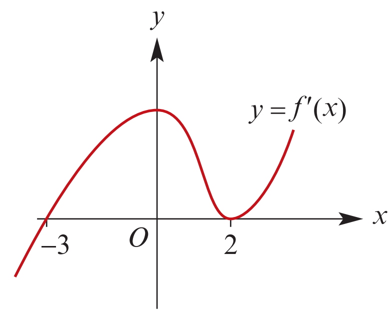
b



c



d



13 Find the coordinates of the stationary points, and state the nature of each, for the curve with equation:

a $y = x^4 - 16x^2$

b $y = x^{2m} - 16x^{2m-2}$, where m is a natural number greater than or equal to 2.

Example 17 **14** Sketch the graph of $f(x) = e^{-\frac{x^2}{2}}$.

15 Let $f(x) = x^2e^x$. Find $\{\, x : f'(x) < 0 \,\}$.

16 Find the values of x for which $100e^{-x^2+2x-5}$ increases as x increases and hence find the maximum value of $100e^{-x^2+2x-5}$.

17 Let $f(x) = e^x - 1 - x$.

a Find the minimum value of $f(x)$.

b Hence show $e^x \geq 1 + x$ for all real x.

18 For $f(x) = x + e^{-x}$:

a Find the position and nature of any stationary points.

b Find, if they exist, the equations of any asymptotes.

c Sketch the graph of $y = f(x)$.

19 The curve $y = e^x(px^2 + qx + r)$ is such that the tangents at $x = 1$ and $x = 3$ are parallel to the x-axis. The point with coordinates $(0, 9)$ is on the curve. Find p, q and r.

ISBN 978-1-107-56747-4

Cambridge University Press

20 **a** Let $y = e^{4x^2-8x}$. Find $\dfrac{dy}{dx}$.

b Find the coordinates of the stationary point on the curve of $y = e^{4x^2-8x}$ and state its nature.

c Sketch the graph of $y = e^{4x^2-8x}$.

d Find the equation of the normal to the curve of $y = e^{4x^2-8x}$ at the point where $x = 2$.

21 On the same set of axes, sketch the graphs of $y = \log_e x$ and $y = \log_e(5x)$, and use them to explain why $\dfrac{d}{dx}\big(\log_e x\big) = \dfrac{d}{dx}\big(\log_e(5x)\big)$.

Example 18 **22** For the function $f\colon (0,\infty) \to \mathbb{R}$, $f(x) = x^2 \log_e x$:

a Find $f'(x)$.

b Solve the equation $f(x) = 0$.

c Solve the equation $f'(x) = 0$.

d Sketch the graph of $y = f(x)$.

23 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = x^3 - 3x^2 - 9x + 11$. Sketch the graph of:

a $y = f(x)$ **b** $y = 2f(x)$ **c** $y = f(x+2)$ **d** $y = f(x-2)$ **e** $y = -f(x)$

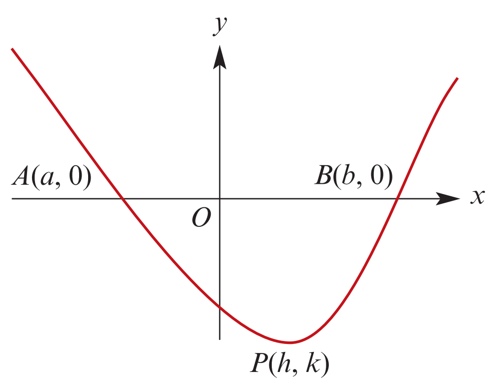
24 Let $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = 2 + 3x - x^3$. Sketch the graph of:

a $y = f(x)$ **b** $y = -2f(x)$ **c** $y = 2f(x-1)$ **d** $y = f(x) - 3$ **e** $y = 3f(x+1)$

25 The graph shown opposite has equation $y = f(x)$. Suppose a dilation of factor p from the x-axis followed by a translation of ℓ units in the positive direction of the x-axis is applied to the graph. For the graph of the image, state:

a the axis intercepts

b the coordinates of the turning point.



Example 19 **26** Find the values of x for which the graph of $y = f(x)$ has a stationary point and state the nature of each stationary point. Consider $0 \le x \le 2\pi$ only.

a $f(x) = 2\cos x - (2\cos^2 x - 1)$

b $f(x) = 2\cos x + 2\sin x \cos x$

c $f(x) = 2\sin x - (2\cos^2 x - 1)$

d $f(x) = 2\sin x + 2\sin x \cos x$

27 The graph of a quartic function passes through the points with coordinates $(1, 21)$, $(2, 96)$, $(5, 645)$, $(6, 816)$ and $(7, 861)$.

a Find the rule of the quartic and plot the graph. Determine the turning points and axis intercepts.

b Plot the graph of the derivative on the same screen.

c Find the value of the function when $x = 10$.

d For what value(s) of x is the value of the function 500?

ISBN 978-1-107-56747-4

Cambridge University Press

10E Absolute maximum and minimum values

Local maximum and minimum values were discussed in the previous section. These are often not the actual maximum and minimum values of the function.

For a function defined on an interval:

- the actual maximum value of the function is called the **absolute maximum**
- the actual minimum value of the function is called the **absolute minimum**.

The corresponding points on the graph of the function are not necessarily stationary points.

More precisely, for a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the absolute maximum value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the absolute minimum value of the function.

Example 20

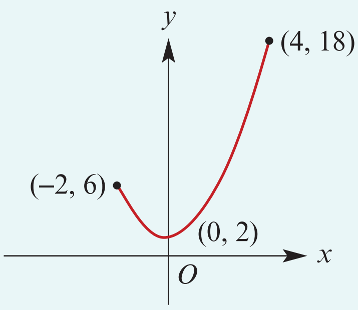
Let $f\colon [-2, 4] \to \mathbb{R}$, $f(x) = x^2 + 2$. Find the absolute maximum value and the absolute minimum value of the function.

Solution

The maximum value is 18 and occurs when $x = 4$.

The minimum value is 2 and occurs when $x = 0$.

(Note that the absolute minimum occurs at a stationary point of the graph. The absolute maximum occurs at an endpoint, not at a stationary point.)



Example 21

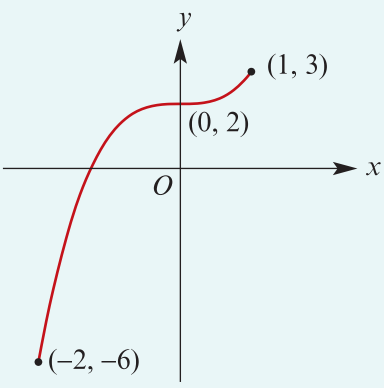
Let $f\colon [-2, 1] \to \mathbb{R}$, $f(x) = x^3 + 2$. Find the maximum and minimum values of the function.

Solution

The maximum value is 3 and occurs when $x = 1$.

The minimum value is -6 and occurs when $x = -2$.

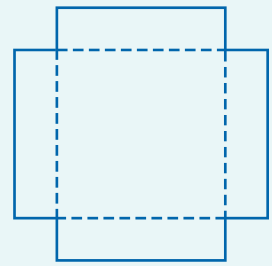
(Note that the absolute maximum and minimum values do not occur at stationary points.)



ISBN 978-1-107-56747-4

Example 22

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



a Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.

b Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.

c Sketch the graph of V against x for a suitable domain.

d If the height of the box must be less than 0.3 m, i.e. $x \le 0.3$, what will be the maximum volume of the box?

Solution

a The box has length and width $2 - 2x$ metres, and has height x metres. Thus

$$\begin{aligned} V &= (2-2x)^2 x \\ &= (4 - 8x + 4x^2)x \\ &= 4x^3 - 8x^2 + 4x \end{aligned}$$

b Let $V(x) = 4x^3 - 8x^2 + 4x$. A local maximum will occur when $V'(x) = 0$.
We have $V'(x) = 12x^2 - 16x + 4$, and so $V'(x) = 0$ implies that

$$\begin{aligned} 12x^2 - 16x + 4 &= 0 \\ 3x^2 - 4x + 1 &= 0 \\ (3x-1)(x-1) &= 0 \\ \therefore \quad x = \frac{1}{3} &\text{ or } x = 1 \end{aligned}$$

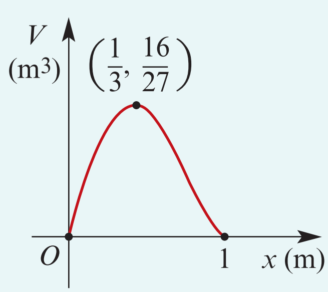
But, when $x = 1$, the length of the box is $2 - 2x = 0$. Therefore the only value to be considered is $x = \frac{1}{3}$. We show the entire chart for completeness.

x		$\frac{1}{3}$		1
$V'(x)$	+	0	−	0
shape of V	/	—	\	—

The maximum occurs when $x = \frac{1}{3}$.

$$\begin{aligned} \therefore \text{ Maximum volume} &= \left(2 - 2 \times \frac{1}{3}\right)^2 \times \frac{1}{3} \\ &= \frac{16}{27} \text{ m}^3 \end{aligned}$$

c



d The local maximum of $V(x)$ defined on $[0, 1]$ is at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

But $\frac{1}{3}$ is not in the interval $[0, 0.3]$.

Since $V'(x) > 0$ for all $x \in [0, 0.3]$, the maximum volume for this situation occurs when $x = 0.3$ and is 0.588 m^3.

ISBN 978-1-107-56747-4

Section summary

For a continuous function f defined on an interval $[a, b]$:

- if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
- if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

Exercise 10E

Skillsheet
Example 20

1 Let $f\colon [-3, 3] \to \mathbb{R}$, $f(x) = 2 - 8x^2$. Find the absolute maximum value and the absolute minimum value of the function.

Example 21

2 Let $f\colon [-3, 2] \to \mathbb{R}$, $f(x) = x^3 + 2x + 3$. Find the absolute maximum value and the absolute minimum value of the function for its domain.

3 Let $f\colon [-1.5, 2.5] \to \mathbb{R}$, $f(x) = 2x^3 - 6x^2$. Find the absolute maximum and absolute minimum values of the function.

4 Let $f\colon [-2, 6] \to \mathbb{R}$, $f(x) = 2x^4 - 8x^2$. Find the absolute maximum and absolute minimum values of the function.

Example 22

5 A rectangular block is such that the sides of its base are of length x cm and $3x$ cm. The sum of the lengths of all its edges is 20 cm.

a Show that the volume, V cm^3, of the block is given by $V = 15x^2 - 12x^3$.

b Find $\dfrac{dV}{dx}$.

c Find the coordinates of the local maximum of the graph of V against x for $x \in [0, 1.25]$.

d If $x \in [0, 0.8]$, find the absolute maximum value of V and the value of x for which this occurs.

e If $x \in [0, 1]$, find the absolute maximum value of V and the value of x for which this occurs.

6 Variables x, y and z are such that $x + y = 30$ and $z = xy$.

a If $x \in [2, 5]$, find the possible values of y.

b Find the absolute maximum and absolute minimum values of z.

7 Consider the function $f\colon [2, 3] \to \mathbb{R}$, $f(x) = \dfrac{1}{x-1} + \dfrac{1}{4-x}$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

ISBN 978-1-107-56747-4

8 A piece of string 10 metres long is cut into two pieces to form two squares.

a If one piece of string has length x metres, show that the combined area of the two squares is given by $A = \frac{1}{8}(x^2 - 10x + 50)$.

b Find $\frac{dA}{dx}$.

c Find the value of x that makes A a minimum.

d If two squares are formed but $x \in [0, 1]$, find the maximum possible combined area of the two squares.

9 Find the absolute maximum and minimum values of the function $g\colon [2.1, 8] \to \mathbb{R}$, $g(x) = x + \frac{1}{x-2}$.

10 Consider the function $f\colon [0, 3] \to \mathbb{R}$, $f(x) = \frac{1}{x+1} + \frac{1}{4-x}$.

a Find $f'(x)$.

b Find the coordinates of the stationary point of the graph of $y = f(x)$.

c Find the absolute maximum and absolute minimum of the function.

11 For the function $f\colon \left[-\frac{\pi}{2}, \frac{\pi}{8}\right] \to \mathbb{R}$, $f(x) = \sin(2x)$, state the absolute maximum and minimum values of the function.

12 For the function $f\colon \left[0, \frac{\pi}{8}\right] \to \mathbb{R}$, $f(x) = \cos(2x)$, state the absolute maximum and minimum values of the function.

13 For the function $f\colon [-1, 8] \to \mathbb{R}$, $f(x) = 2 - x^{\frac{2}{3}}$, sketch the graph and state the absolute maximum and minimum values of the function.

14 For the function $f\colon [-1, 2] \to \mathbb{R}$, $f(x) = 2e^x + e^{-x}$, sketch the graph and state the absolute maximum and minimum values of the function.

15 For the function $f\colon [-2, 2] \to \mathbb{R}$, $f(x) = 2e^{(x-1)^2}$, sketch the graph and state the absolute maximum and minimum values of the function.

16 For the function $f\colon [6, 10] \to \mathbb{R}$, $f(x) = (x-5)\log_e\left(\frac{x-5}{10}\right)$, sketch the graph and state the absolute maximum and minimum values of the function.

10F Maximum and minimum problems

Many practical problem require that some quantity (for example, cost of manufacture or fuel consumption) be **minimised**, that is, be made as small as possible. Other problems require that some quantity (for example, profit on sales or attendance at a concert) be **maximised**, that is, be made as large as possible. We can use differential calculus to solve many of these problems.

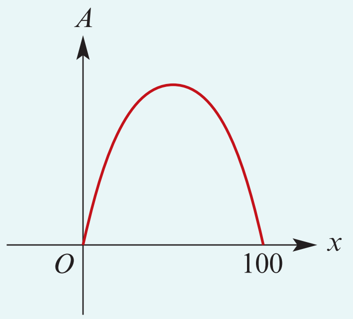
ISBN 978-1-107-56747-4

Example 23

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution

Let the length of the rectangle be x metres. Then the width is $100 - x$ metres and the area is A m^2, where



$$\begin{aligned} A &= x(100 - x) \\ &= 100x - x^2 \end{aligned}$$

The maximum value of A occurs when $\frac{dA}{dx} = 0$.

$$\frac{dA}{dx} = 100 - 2x$$

$$\therefore \quad \frac{dA}{dx} = 0 \text{ implies } x = 50$$

From the gradient chart, the maximum area occurs when $x = 50$.

x		50	
$\frac{dA}{dx}$	+	0	−
shape of A	/	—	\

The pen with maximum area has dimensions 50 m by 50 m, and so has area 2500 m^2.

Example 24

Two variables x and y are such that $x^4y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value and show that this value of z is a minimum.

Solution

Obtain y in terms of x from the equation $x^4y = 8$:

$$y = 8x^{-4}$$

Substitute in the equation $z = x + y$:

$$z = x + 8x^{-4} \qquad (1)$$

Now z is expressed in terms of one variable, x. Differentiate with respect to x:

$$\frac{dz}{dx} = 1 - 32x^{-5}$$

A stationary point occurs where $\frac{dz}{dx} = 0$:

$$\begin{aligned} 1 - 32x^{-5} &= 0 \\ 32x^{-5} &= 1 \\ x^5 &= 32 \\ \therefore \quad x &= 2 \end{aligned}$$

ISBN 978-1-107-56747-4

There is a stationary point at $x = 2$. The corresponding value of y is $8 \times 2^{-4} = \frac{1}{2}$.

Now substitute in equation (1) to find z:

$$z = 2 + \frac{8}{16} = 2\tfrac{1}{2}$$

Determine the nature of the stationary point using a gradient chart.

x		2	
$\dfrac{dz}{dx}$	$-$	0	$+$
shape of z	$\searrow$	—	$\nearrow$

The minimum value of z is $2\frac{1}{2}$ and occurs when $x = 2$ and $y = \frac{1}{2}$.

Example 25

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution

Let the radius of the circular end of the tin be r cm, let the height of the tin be h cm and let the volume of the tin be V cm^3.

Obtain equations for the surface area and the volume.

Surface area: $\quad 100 = 2\pi r^2 + 2\pi r h \quad (1)$

Volume: $\quad V = \pi r^2 h \quad (2)$

The process we follow now is very similar to Example 23. Obtain h in terms of r from equation (1):

$$h = \frac{1}{2\pi r}(100 - 2\pi r^2)$$

Substitute in equation (2):

$$V = \pi r^2 \times \frac{1}{2\pi r}(100 - 2\pi r^2)$$

$$\therefore \quad V = 50r - \pi r^3 \qquad (3)$$

A stationary point of the graph of $V = 50r - \pi r^3$ occurs when $\dfrac{dV}{dr} = 0$.

$$\frac{dV}{dr} = 0 \text{ implies } 50 - 3\pi r^2 = 0$$

$$\therefore \quad r = \pm\sqrt{\frac{50}{3\pi}} \approx \pm 2.3$$

But $r = -2.3$ does not fit the practical situation.

Substitute $r = 2.3$ in equation (3) to find V:

$$V \approx 76.78$$

So there is a stationary point at $(2.3, 76.8)$.

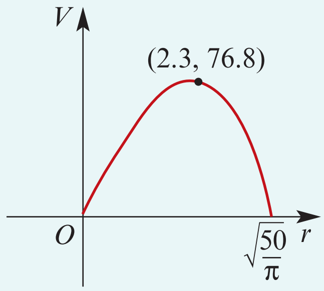
ISBN 978-1-107-56747-4

Use a gradient chart to determine the nature of this stationary point.

r		2.3	
$\dfrac{dV}{dr}$	$+$	0	$-$
shape of V	/	—	\

The maximum volume is 76.78 cm^3 correct to two decimal places.

It can be observed that the volume is given by a function f with rule $f(r) = 50r - \pi r^3$ and domain $\left[0, \sqrt{\frac{50}{\pi}}\right]$, giving the graph on the right.



If the greatest radius the canister can have is 2 cm, then the function f has domain $[0, 2]$. It has been seen that $f'(r) > 0$ for all $r \in [0, 2]$. The maximum value occurs when $r = 2$. The maximum volume in this case is $f(2) = 100 - 8\pi \approx 74.87$ cm^3.

In some situations the variables may not be continuous. For instance, one of them may only take integer values. In such cases it is not strictly valid to use techniques of differentiation to solve the problem. However, in some problems we may model the non-continuous case with a continuous function so that the techniques of differential calculus may be used. Examples 26 and 27 illustrate this.

Example 26

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution

Let x denote the monthly fee. Then the number of subscribers is $1000 + 100\left(\dfrac{5-x}{0.1}\right)$.

(Note that we are treating a discrete situation with a continuous function.)

Let R denote the revenue. Then

$$\begin{aligned} R &= x(1000 + 1000(5 - x)) \\ &= 1000(6x - x^2) \\ \therefore \quad \frac{dR}{dx} &= 1000(6 - 2x) \end{aligned}$$

Thus $\dfrac{dR}{dx} = 0$ implies $6 - 2x = 0$ and hence $x = 3$.

The gradient chart is shown.

x		3	
$\dfrac{dR}{dx}$	$+$	0	$-$
shape of R	/	—	\

For maximum revenue, the monthly fee should be \$3 and this gives a total revenue of \$9000.

ISBN 978-1-107-56747-4

Example 27

A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x, of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution

Number of production runs per year $= \dfrac{10\,000}{x}$

Set-up costs for these production runs $= 40\left(\dfrac{10\,000}{x}\right)$

Let C be the total set-up and carrying costs. Then

$$\begin{aligned} C &= x^{\frac{3}{2}} + \frac{400\,000}{x} \\ &= x^{\frac{3}{2}} + 400\,000x^{-1}, \quad x > 0 \end{aligned}$$

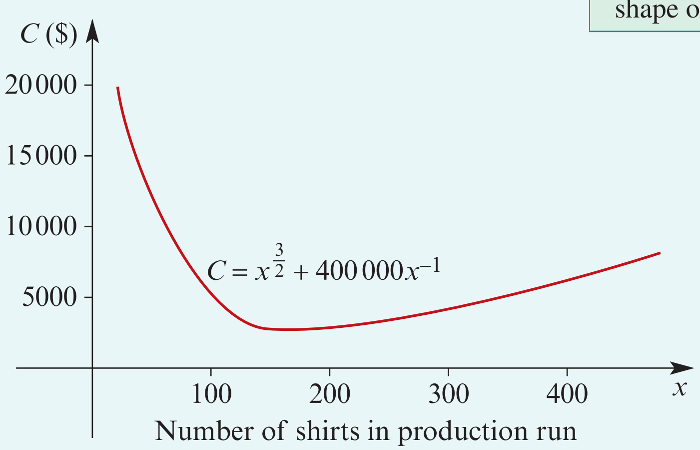
$$\therefore \quad \frac{dC}{dx} = \frac{3}{2}x^{\frac{1}{2}} - \frac{400\,000}{x^2}$$

Thus $\dfrac{dC}{dx} = 0$ implies

$$\begin{aligned} \frac{3}{2}x^{\frac{1}{2}} &= \frac{400\,000}{x^2} \\ x^{\frac{5}{2}} &= \frac{400\,000 \times 2}{3} \\ \therefore \quad x &\approx 148.04 \end{aligned}$$

Each production run should be 148 shirts.

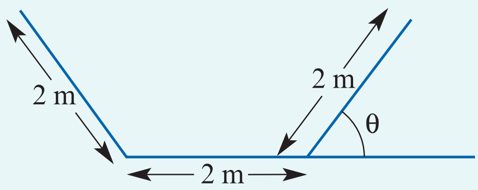
x		148.04	
$\dfrac{dC}{dx}$	−	0	+
shape of C	\	—	/



ISBN 978-1-107-56747-4

Example 28

The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution

Let A m^2 be the area of the trapezium. Then

$$A = \frac{1}{2} \times 2\sin\theta \times (2 + 2 + 4 \times \cos\theta)$$
$$= \sin\theta \cdot (4 + 4\cos\theta)$$

and $\quad A'(\theta) = \cos\theta \cdot (4 + 4\cos\theta) - 4\sin^2\theta$

$$= 4\cos\theta + 4\cos^2\theta - 4(1 - \cos^2\theta)$$
$$= 4\cos\theta + 8\cos^2\theta - 4$$

The maximum will occur when $A'(\theta) = 0$:

$$8\cos^2\theta + 4\cos\theta - 4 = 0$$
$$2\cos^2\theta + \cos\theta - 1 = 0$$
$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$
$$\therefore \quad \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

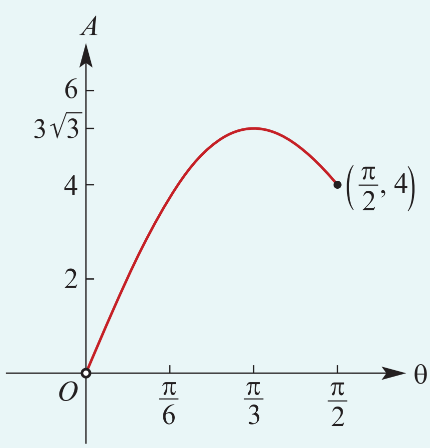
The practical restriction on θ is that $0 < \theta \le \frac{\pi}{2}$.

Therefore the only possible solution is $\theta = \frac{\pi}{3}$, and a gradient chart confirms that $\frac{\pi}{3}$ gives a maximum.

θ		$\frac{\pi}{3}$	
$A'(\theta)$	+	0	−
shape of A	/	—	\

When $\theta = \frac{\pi}{3}$, $A = \frac{\sqrt{3}}{2}(4 + 2) = 3\sqrt{3}$,

i.e. the maximum cross-sectional area is $3\sqrt{3}$ m^2.

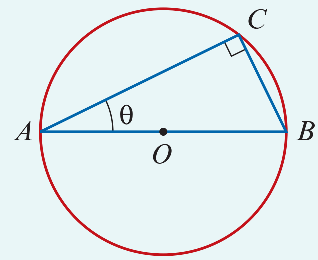


ISBN 978-1-107-56747-4

Cambridge University Press

Example 29

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



a If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

b Find the value of θ for which $\frac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of T $(0° < \theta° < 90°)$.

Solution

a Time taken $= \dfrac{\text{distance travelled}}{\text{speed}}$

Therefore the swim takes $\dfrac{4\cos\theta}{3}$ hours and the walk takes $\dfrac{4\theta}{4}$ hours.

Thus the total time taken is given by $T = \frac{1}{3}(4\cos\theta + 3\theta)$.

b $\dfrac{dT}{d\theta} = \dfrac{1}{3}(-4\sin\theta + 3)$

The stationary point occurs where $\dfrac{dT}{d\theta} = 0$, and $\frac{1}{3}(-4\sin\theta + 3) = 0$ implies $\sin\theta = \dfrac{3}{4}$.

Therefore $\theta = 48.59°$ to two decimal places.

From the gradient chart, the value of T is a maximum when $\theta = 48.59°$.

θ		$48.59°$	
$\dfrac{dT}{d\theta}$	$+$	0	$-$
shape of T	⁄	—	＼

Notes:

- The maximum time taken is 1.73 hours.
- If the man swims straight across the lake, it takes $1\frac{1}{3}$ hours.
- If he walks around all the way around the edge, it takes approximately 1.57 hours.

Example 30

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution

$$N(t) = 36te^{-0.1t}$$

$$\therefore \quad N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$$

$$= e^{-0.1t}(36 - 3.6t)$$

Thus $N'(t) = 0$ implies $t = 10$.

The maximum population is $N(10) = 360e^{-1} \approx 132$.

ISBN 978-1-107-56747-4

Cambridge University Press

Maximum rates of increase and decrease

We know that when we take the derivative of a function we obtain a new function, the derivative, which gives the instantaneous rate of change. We can apply the same technique to the new function to find the maximum rate of increase or decrease.

Remember:

- If $\dfrac{dy}{dx} > 0$, then y is increasing as x increases.
- If $\dfrac{dy}{dx} < 0$, then y is decreasing as x increases.

We illustrate this technique by revisiting Example 30.

Example 31

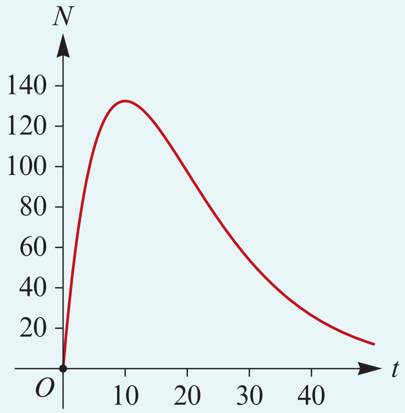
Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

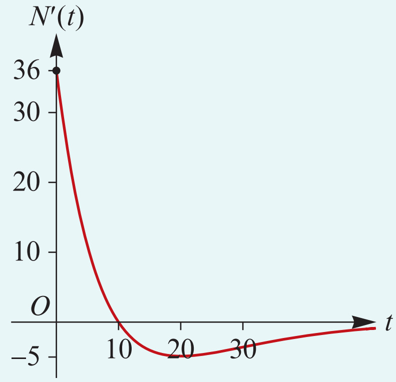
b Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution

a $N(t) = 36te^{-0.1t}$



$N'(t) = 36e^{-0.1t} - 3.6te^{-0.1t}$



b Let $R(t) = N'(t) = e^{-0.1t}(36 - 3.6t)$ be the rate of change of the population.

From the graph, the maximum value of $R(t)$ occurs at $t = 0$. Thus the maximum rate of increase of the population is $R(0) = 36$ bacteria per unit of time.

We now calculate

$$\begin{aligned} R'(t) &= -7.2e^{-0.1t} + 0.36te^{-0.1t} \\ &= e^{-0.1t}(-7.2 + 0.36t) \end{aligned}$$

Thus $R'(t) = 0$ implies $t = 20$.

The minimum value of $R(t)$ occurs at $t = 20$. Since $R(20) = -36e^{-2} \approx -4.9$, the maximum rate of decrease of the population is 4.9 bacteria per unit of time.

ISBN 978-1-107-56747-4

▸ The second derivative and points of inflection*

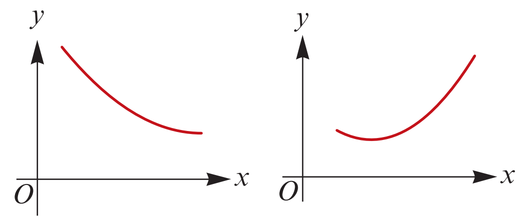
In Example 31, we used the derivative of the derivative, called the **second derivative**, to find the maximum rate of change. The second derivative can also be used in graph sketching.

For a function f with $y = f(x)$, the second derivative of f is denoted by f'' or by $\dfrac{d^2y}{dx^2}$.

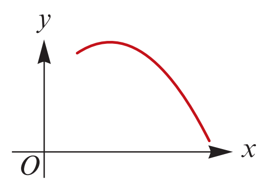
Concave up and concave down

Let f be a function defined on an interval (a, b), and assume that both $f'(x)$ and $f''(x)$ exist for all $x \in (a, b)$.

If $f''(x) > 0$ for all $x \in (a, b)$, then the gradient of the curve $y = f(x)$ is increasing in the interval (a, b). The curve is **concave up**.



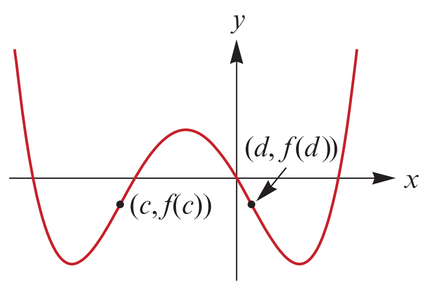
If $f''(x) < 0$ for all $x \in (a, b)$, then the gradient of the curve $y = f(x)$ is decreasing in the interval (a, b). The curve is **concave down**.



Inflection points

A point where a curve changes from concave up to concave down or from concave down to concave up is called a **point of inflection**.

In the graph on the right, there are points of inflection at $x = c$ and $x = d$.



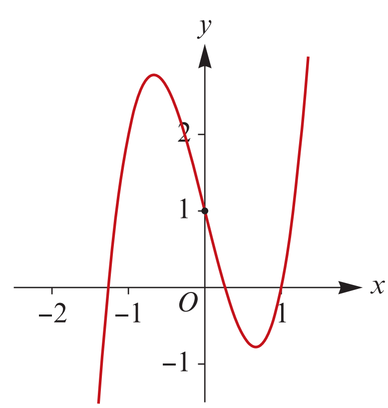
At a point of inflection of a twice differentiable function f, we must have $f''(x) = 0$. However, this condition does not necessarily guarantee a point of inflection. At a point of inflection, there must also be a change of concavity.

We return to the graph of Example 15. The function is $f(x) = 3x^3 - 4x + 1$. Here $f'(x) = 9x^2 - 4$ and $f''(x) = 18x$. We can observe:

- $f''(0) = 0$
- $f''(x) > 0$ for $x > 0$
- $f''(x) < 0$ for $x < 0$

Hence the curve is concave down to the left of 0, and the curve is concave up to the right of 0.

You may like to use this technique when you are sketching graphs in the future.



* This material is not included in the study design of Mathematical Methods Units 3 & 4.

ISBN 978-1-107-56747-4

Section summary

Here are some steps for solving maximum and minimum problems:

- Where possible, draw a diagram to illustrate the problem. Label the diagram and designate your variables and constants. Note the values that the variables can take.
- Write an expression for the quantity that is going to be maximised or minimised. Form an equation for this quantity in terms of a single independent variable. This may require some algebraic manipulation.
- If $y = f(x)$ is the quantity to be maximised or minimised, find the values of x for which $f'(x) = 0$.
- Test each point for which $f'(x) = 0$ to determine whether it is a local maximum, a local minimum or neither.
- If the function $y = f(x)$ is defined on an interval, such as $[a, b]$ or $[0, \infty)$, check the values of the function at the endpoints.

Exercise 10F

Example 23

1 Find the maximum area of a rectangular field that can be enclosed by 100 m of fencing.

Example 24

2 Find two positive numbers that sum to 4 and such that the sum of the cube of the first and the square of the second is as small as possible.

3 For $x + y = 100$, prove that the product $P = xy$ is a maximum when $x = y$ and find the maximum value of P.

4 A farmer has 4 km of fencing wire and wishes to fence a rectangular piece of land through which flows a straight river, which is to be utilised as one side of the enclosure. How can this be done to enclose as much land as possible?

5 Two positive quantities p and q vary in such a way that $p^3q = 9$. Another quantity z is defined by $z = 16p + 3q$. Find values of p and q that make z a minimum.

Example 25

6 A cuboid has a total surface area of 150 cm^2 with a square base of side length x cm.

a Show that the height, h cm, of the cuboid is given by $h = \dfrac{75 - x^2}{2x}$.

b Express the volume of the cuboid in terms of x.

c Hence determine its maximum volume as x varies.

Example 26, 27

7 A manufacturer finds that the daily profit, \$$P$, from selling n articles is given by $P = 100n - 0.4n^2 - 160$.

a **i** Find the value of n which maximises the daily profit.

ii Find the maximum daily profit.

b Sketch the graph of P against n. (Use a continuous graph.)

c State the allowable values of n for a profit to be made.

d Find the value of n which maximises the profit per article.

ISBN 978-1-107-56747-4

8 The number of salmon swimming upstream in a river to spawn is approximated by $s(x) = -x^3 + 3x^2 + 360x + 5000$ with x representing the temperature of the water in degrees (°C). (This function is valid only if $6 \le x \le 20$.) Find the water temperature that produces the maximum number of salmon swimming upstream.

9 The number of mosquitos, $M(x)$ in millions, in a certain area depends on the average daily rainfall, x mm, during September and is approximated by

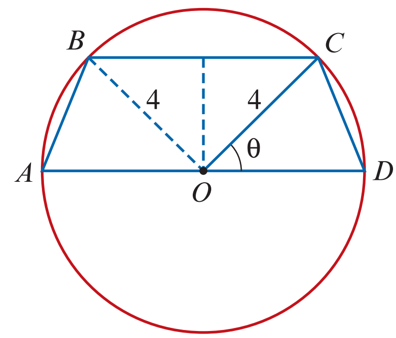
$$M(x) = \frac{1}{30}(50 - 32x + 14x^2 - x^3) \qquad \text{for } 0 \le x \le 10$$

Find the rainfall that will produce the maximum and the minimum number of mosquitos.

Example 28 **10** $ABCD$ is a trapezium with $AB = CD$. The vertices are on a circle with centre O and radius 4 units. The line segment AD is a diameter of the circle.

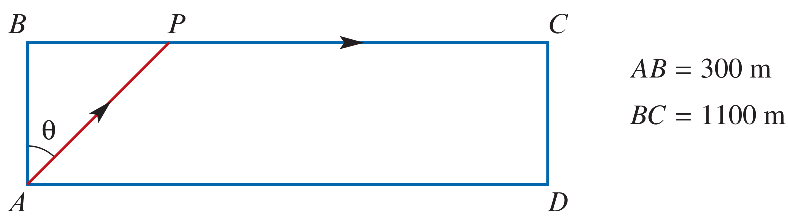
a Find BC in terms of θ.

b Find the area of the trapezium in terms of θ and hence find the maximum area.



11 Find the point on the parabola $y = x^2$ that is closest to the point $(3, 0)$.

Example 29 **12** The figure shows a rectangular field in which $AB = 300$ m and $BC = 1100$ m.



a An athlete runs across the field from A to P at 4 m/s. Find the time taken to run from A to P in terms of θ.

b The athlete, on reaching P, immediately runs to C at 5 m/s. Find the time taken to run from P to C in terms of θ.

c Use the results from **a** and **b** to show that the total time taken, T seconds, is given by $T = 220 + \dfrac{75 - 60\sin\theta}{\cos\theta}$.

d Find $\dfrac{dT}{d\theta}$.

e Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and show that this is the value of θ for which T is a minimum.

f Find the minimum value of T and find the distance of point P from B that will minimise the athlete's running time.

ISBN 978-1-107-56747-4

Example 30 **13** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Example 31 **14** The number $N(t)$ of insects in a population at time t is given by $N(t) = 50te^{-0.1t}$.

a Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

b Find the maximum rates of increase and decrease of the population and the times at which these occur.

15 Water is being poured into a flask. The volume, V mL, of water in the flask at time t seconds is given by

$$V(t) = \frac{3}{4}\left(10t^2 - \frac{t^3}{3}\right), \quad 0 \le t \le 20$$

a Find the volume of water in the flask when:

i $t = 0$ **ii** $t = 20$

b Find $V'(t)$, the rate of flow of water into the flask.

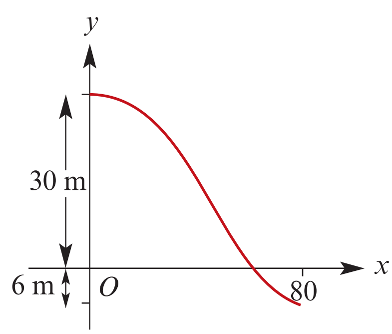
c Sketch the graph of $V(t)$ against t for $0 \le t \le 20$.

d Sketch the graph of $V'(t)$ against t for $0 \le t \le 20$.

e At what time is the flow greatest and what is the flow at this time?

16 A section of a roller coaster can be described by the rule

$$y = 18\cos\left(\frac{\pi x}{80}\right) + 12, \quad 0 \le x \le 80$$



a Find the gradient function, $\frac{dy}{dx}$.

b Sketch the graph of $\frac{dy}{dx}$ against x.

c State the coordinates of the point on the track for which the magnitude of the gradient is maximum.

17 The depth, $D(t)$ metres, of water at the entrance to a harbour at t hours after midnight on a particular day is given by

$$D(t) = 10 + 3\sin\left(\frac{\pi t}{6}\right), \quad 0 \le t \le 24$$

a Sketch the graph of $y = D(t)$ for $0 \le t \le 24$.

b Find the values of t for which $D(t) \ge 8.5$.

c Find the rate at which the depth is changing when:

i $t = 3$ **ii** $t = 6$ **iii** $t = 12$

d **i** At what times is the depth increasing most rapidly?

ii At what times is the depth decreasing most rapidly?

ISBN 978-1-107-56747-4

10G Families of functions

Example 32

Consider the family of functions with rules of the form $f(x) = (x - a)^2(x - b)$, where a and b are positive constants with $b > a$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(a, 0)$ is always a local maximum.

d Find the values of a and b if the stationary points occur where $x = 3$ and $x = 4$.

Solution

a Use a CAS calculator to find that $f'(x) = (x - a)(3x - a - 2b)$.

b The coordinates of the stationary points are $(a, 0)$ and $\left(\dfrac{a + 2b}{3}, \dfrac{4(a - b)^3}{27}\right)$.

c If $x < a$, then $f'(x) > 0$, and if $a < x < \dfrac{a + 2b}{3}$, then $f'(x) < 0$.

Therefore the stationary point at $(a, 0)$ is a local maximum.

d Since $a < b$, we must have $a = 3$ and $\dfrac{a + 2b}{3} = 4$. Therefore $b = \dfrac{9}{2}$.

Example 33

The graph of $y = x^3 - 3x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^3 - 3x^2$.

b Find the coordinates of the turning points of its image.

Solution

a The turning points have coordinates $(0, 0)$ and $(2, -4)$.

b The turning points of the image are (a, b) and $(2 + a, -4 + b)$.

Example 34

A cubic function with rule $f(x) = ax^3 + bx^2 + cx$ has a stationary point at $(1, 6)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a stationary point at $x = 2$.

Solution

a Since $f(1) = 6$, we obtain

$$a + b + c = 6 \qquad (1)$$

Since $f'(x) = 3ax^2 + 2bx + c$ and $f'(1) = 0$, we obtain

$$3a + 2b + c = 0 \qquad (2)$$

The solution of equations (1) and (2) is $a = c - 12$ and $b = 18 - 2c$.

ISBN 978-1-107-56747-4

b The rule is

$$f(x) = (c - 12)x^3 + (18 - 2c)x^2 + cx$$
$$\therefore \quad f'(x) = 3(c - 12)x^2 + 2(18 - 2c)x + c$$

If $f'(2) = 0$, then

$$12(c - 12) + 4(18 - 2c) + c = 0$$
$$5c - 72 = 0$$
$$\therefore \quad c = \frac{72}{5}$$

Exercise 10G

Example 32

1 Consider the family of functions with rules $f(x) = (x - 1)^2(x - b)$, where $b > 1$.

a Find the derivative of $f(x)$ with respect to x.

b Find the coordinates of the stationary points of the graph of $y = f(x)$.

c Show that the stationary point at $(1, 0)$ is always a local maximum.

d Find the value of b if the stationary points occur where $x = 1$ and $x = 4$.

Example 33

2 The graph of the function $y = x^4 - 4x^2$ is translated by a units in the positive direction of the x-axis and b units in the positive direction of the y-axis (where a and b are positive constants).

a Find the coordinates of the turning points of the graph of $y = x^4 - 4x^2$.

b Find the coordinates of the turning points of its image.

Example 34

3 A cubic function f has rule $f(x) = ax^3 + bx^2 + cx$. The graph has a stationary point at $(1, 10)$.

a Find a and b in terms of c.

b Find the value of c for which the graph has a stationary point at $x = 3$.

4 Consider the function $f\colon [0, \infty) \to \mathbb{R}$ defined by $f(x) = x^2 - ax^3$, where a is a real number with $a > 0$.

a Determine the intervals on which f is a strictly decreasing function and the intervals on which f is a strictly increasing function.

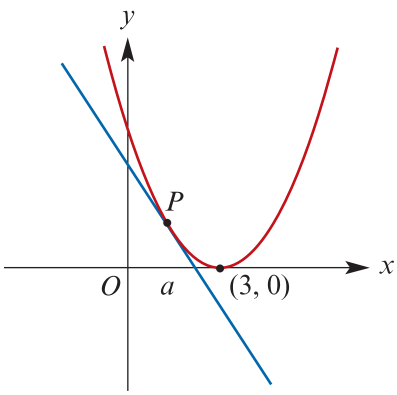
b Find the equation of the tangent to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

c Find the equation of the normal to the graph of f at the point $\left(\frac{1}{a}, 0\right)$.

d What is the range of f?

ISBN 978-1-107-56747-4

5 A line with equation $y = mx + c$ is a tangent to the curve $y = (x - 3)^2$ at a point $P(a, y)$ where $0 < a < 3$.



a **i** Find the gradient of the curve at $x = a$ for $0 < a < 3$.

ii Hence express m in terms of a.

b State the coordinates of the point P, expressing your answer in terms of a.

c Find the equation of the tangent where $x = a$.

d Find the x-axis intercept of the tangent.

6 **a** The graph of $f(x) = x^4$ is translated to the graph of $y = f(x + h)$. Find the possible values of h if $f(1 + h) = 16$.

b The graph of $f(x) = x^3$ is transformed to the graph of $y = f(ax)$. Find the possible value of a if the graph of $y = f(ax)$ passes through the point with coordinates $(1, 8)$.

c The quartic function with equation $y = ax^4 - bx^3$ has a turning point with coordinates $(1, 16)$. Find the values of a and b.

7 Consider the cubic function with rule $f(x) = (x - a)^2(x - 1)$ where $a > 1$.

a Find the coordinates of the turning points of the graph of $y = f(x)$.

b State the nature of each of the turning points.

c Find the equation of the tangent to the curve at the point where:

i $x = 1$ **ii** $x = a$ **iii** $x = \dfrac{a+1}{2}$

8 Consider the quartic function with rule $f(x) = (x - 1)^2(x - b)^2$ where $b > 1$.

a Find the derivative of f.

b Find the coordinates of the turning points of f.

c Find the value of b such that the graph of $y = f(x)$ has a turning point at $(2, 1)$.

9 A cubic function has rule $y = ax^3 + bx^2 + cx + d$. It passes through the points $(1, 6)$ and $(10, 8)$ and has turning points where $x = -1$ and $x = 1$. Find the values of a, b, c and d.

10 A quartic function f has rule $f(x) = ax^4 + bx^3 + cx^2 + dx$. The graph has a stationary point at $(1, 1)$ and passes through the point $(-1, 4)$.

a Find a, b and c in terms of d.

b Find the value of d for which the graph has a stationary point at $x = 4$.

ISBN 978-1-107-56747-4

Review

Chapter summary

Spreadsheet AS Nrich

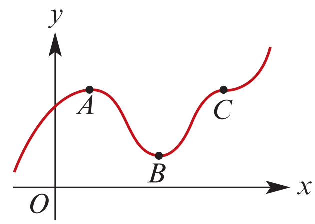
- **Tangents and normals**

  Let (x_1, y_1) be a point on the curve $y = f(x)$. If f is differentiable at $x = x_1$, then
  - the equation of the **tangent** to the curve at (x_1, y_1) is given by $y - y_1 = f'(x_1)(x - x_1)$
  - the equation of the **normal** to the curve at (x_1, y_1) is given by $y - y_1 = \dfrac{-1}{f'(x_1)}(x - x_1)$.

- **Stationary points**

  A point with coordinates $(a, f(a))$ on a curve $y = f(x)$ is a **stationary point** if $f'(a) = 0$.

  The graph shown has three stationary points: A, B and C.

  A Point A is a **local maximum** point. Notice that immediately to the left of A the gradient is positive, and immediately to the right the gradient is negative.

  B Point B is a **local minimum** point. Notice that immediately to the left of B the gradient is negative, and immediately to the right the gradient is positive.

  C Point C is a **stationary point of inflection**.

  Stationary points of types A and B are referred to as **turning points**.

- **Maximum and minimum values**

  For a continuous function f defined on an interval $[a, b]$:
  - if M is a value of the function such that $f(x) \leq M$ for all $x \in [a, b]$, then M is the **absolute maximum** value of the function
  - if N is a value of the function such that $f(x) \geq N$ for all $x \in [a, b]$, then N is the **absolute minimum** value of the function.

- **Motion in a straight line**

  For an object moving in a straight line with position x at time t:

  $$\text{velocity } v = \frac{dx}{dt} \qquad \text{acceleration } a = \frac{dv}{dt}$$

Technology-free questions

1 **a** Find the equation of the tangent to the curve $y = x^3 - 8x^2 + 15x$ at the point with coordinates $(4, -4)$.

b Find the coordinates of the point where the tangent meets the curve again.

2 Find the equation of the tangent to the curve $y = 3x^2$ at the point where $x = a$. If this tangent meets the y-axis at P, find the y-coordinate of P in terms of a.

ISBN 978-1-107-56747-4

Review

3 **a** Find the equation of the tangent to the curve with equation $y = x^3 - 7x^2 + 14x - 8$ at the point where $x = 1$.

b Find the x-coordinate of a second point on this curve at which the tangent is parallel to the tangent at $x = 1$.

4 Use the formula $A = \pi r^2$ for the area of a circle to find:

a the average rate at which the area of a circle changes with respect to the radius as the radius increases from $r = 2$ to $r = 3$

b the instantaneous rate at which the area changes with respect to r when $r = 3$.

5 For each of the following, find the stationary points of the graph and state their nature:

a $f(x) = 4x^3 - 3x^4$

b $g(x) = x^3 - 3x - 2$

c $h(x) = x^3 - 9x + 1$

6 Sketch the graph of $y = x^3 - 6x^2 + 9x$.

7 The derivative of the function $y = f(x)$ is $\dfrac{dy}{dx} = (x-1)^2(x-2)$. Find the x-coordinate and state the nature of each stationary point.

8 Find the equation of the tangent to the curve $y = x^3 - 3x^2 - 9x + 11$ at $x = 2$.

9 Let $f\colon \mathbb{R} \to \mathbb{R}$ where $f(x) = 3 + 6x^2 - 2x^3$. Determine the values of x for which the graph of $y = f(x)$ has a positive gradient.

10 For what value(s) of x do the graphs of $y = x^3$ and $y = x^3 + x^2 + x - 2$ have the same gradient?

11 For the function with rule $f(x) = (x-1)^{\frac{4}{5}}$:

a State the values for which the function is differentiable, and find the rule for f'.

b Find the equations of the tangents at the points $(2, 1)$ and $(0, 1)$.

c Find the coordinates of the point of intersection of the two tangents.

12 A spherical bubble, initially of radius 1 cm, expands steadily, its radius increases by 1 cm/s and it bursts after 5 seconds.

a Find the rate of increase of volume with respect to the radius when the radius is 4 cm.

b Find the rate of increase of volume with respect to time when the radius is 4 cm.

13 A vehicle is travelling in a straight line away from a point O. Its distance from O after t seconds is $0.25e^t$ metres. Find the velocity of the vehicle at $t = 0$, $t = 1$, $t = 2$, $t = 4$.

ISBN 978-1-107-56747-4

Review

14 The temperature, θ°C, of material inside a nuclear power station at time t seconds after a reaction begins is given by $\theta = \frac{1}{4}e^{100t}$.

a Find the rate of increase of temperature at time t.

b Find the rate of increase of temperature when $t = \frac{1}{20}$.

15 Find the equation of the tangent to $y = e^x$ at $(1, e)$.

16 The diameter of a tree (D cm) t years after 1 January 2010 is given by $D = 50e^{kt}$.

a Prove that $\frac{dD}{dt} = cD$ for some constant c.

b If $k = 0.2$, find the rate of increase of D when $D = 100$.

17 Find the minimum value of $e^{3x} + e^{-3x}$.

18 **a** Find the equation of the tangent to $y = \log_e x$ at the point $(e, 1)$.

b Find the equation of the tangent to $y = 2\sin\left(\frac{x}{2}\right)$ at the point $\left(\frac{\pi}{2}, \sqrt{2}\right)$.

c Find the equation of the tangent to $y = \cos x$ at the point $\left(\frac{3\pi}{2}, 0\right)$.

d Find the equation of the tangent to $y = \log_e(x^2)$ at the point $(-\sqrt{e}, 1)$.

Multiple-choice questions

1 The line with equation $y = 4x + c$ is a tangent to the curve with equation $y = x^2 - x - 5$. The value of c is

A $-\frac{45}{4}$ **B** $-1 + 2\sqrt{2}$ **C** 2 **D** $\frac{5}{2}$ **E** $-\frac{2}{5}$

2 The equation of the tangent to the curve with equation $y = x^4$ at the point where $x = 1$ is

A $y = -4x - 3$ **B** $y = \frac{1}{4}x - 3$ **C** $y = -4x$

D $y = \frac{1}{4}x + \frac{5}{4}$ **E** $y = 4x - 3$

3 For a polynomial function with rule $f(x)$, the derivative satisfies $f'(a) = f'(b) = 0$, $f'(x) > 0$ for $x \in (a, b)$, $f'(x) < 0$ for $x < a$ and $f'(x) > 0$ for $x > b$. The nature of the stationary points of the graph of $y = f(x)$ is

A local maximum at $(a, f(a))$ and local minimum at $(b, f(b))$

B local minimum at $(a, f(a))$ and local maximum at $(b, f(b))$

C stationary point of inflection at $(a, f(a))$ and local minimum at $(b, f(b))$

D stationary point of inflection at $(a, f(a))$ and local maximum at $(b, f(b))$

E local minimum at $(a, f(a))$ and stationary point of inflection at $(b, f(b))$

ISBN 978-1-107-56747-4

4 The graph of a polynomial function with rule $y = f(x)$ has a local maximum at the point with coordinates $(a, f(a))$. The graph also has a local minimum at the origin, but no other stationary points. The graph of the function with rule $y = -2f\left(\frac{x}{2}\right) + k$, where k is a positive real number, has

A a local maximum at the point with coordinates $(2a, -2f(a) + k)$

B a local minimum at the point with coordinates $\left(\frac{a}{2}, 2f(a) + k\right)$

C a local maximum at the point with coordinates $\left(\frac{a}{2}, -2f(a) + k\right)$

D a local maximum at the point with coordinates $(2a, -2f(a) - k)$

E a local minimum at the point with coordinates $(2a, -2f(a) + k)$

5 For $f(x) = x^3 - x^2 - 1$, the values of x for which the graph of $y = f(x)$ has stationary points are

A $\frac{2}{3}$ only **B** 0 and $\frac{2}{3}$ **C** 0 and $-\frac{2}{3}$ **D** $-\frac{1}{3}$ and 1 **E** $\frac{1}{3}$ and -1

6 A function f is differentiable for all values of x in $[0, 6]$, and the graph with equation $y = f(x)$ has a local minimum point at $(2, 4)$. The equation of the tangent at the point with coordinates $(2, 4)$ is

A $y = 2x$ **B** $x = 2$ **C** $y = 4$ **D** $2x - 4y = 0$ **E** $4x - 2y = 0$

7 The volume, V cm^3, of a solid is given by the formula $V = -10x(2x^2 - 6)$ where x cm is a particular measurement. The value of x for which the volume is a maximum is

A 0 **B** 1 **C** $\sqrt{2}$ **D** $\sqrt{3}$ **E** 2

8 The equation of the normal to the curve with equation $y = x^2$ at the point where $x = a$ is

A $y = \frac{-1}{2a}x + 2 + a^2$ **B** $y = \frac{-1}{2a}x + \frac{1}{2} + a^2$ **C** $y = 2ax - a^2$

D $y = 2ax + 3a^2$ **E** $y = \frac{1}{2a}x + 2 + a^2$

9 For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - ex$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(1, \frac{1}{e}\right)$ **B** $(1, e)$ **C** $(0, 1)$ **D** $(1, 0)$ **E** $(e, 1)$

10 The equation of the tangent to $y = e^{ax}$ at the point $\left(\frac{1}{a}, e\right)$ is

A $y = e^{ax-1} + 1$ **B** $y = ae^{ax}x$ **C** $y = 1 - ae^{ax}$ **D** $y = \frac{e^2 x}{a}$ **E** $y = aex$

11 Under certain conditions, the number of bacteria, N, in a sample increases with time, t hours, according to the rule $N = 4000e^{0.2t}$. The rate, to the nearest whole number of bacteria per hour, that the bacteria are growing 3 hours from the start is

A 1458 **B** 7288 **C** 16 068 **D** 80 342 **E** 109 731

ISBN 978-1-107-56747-4

12 The gradient of the tangent to the curve $y = x^2 \cos(5x)$ at the point where $x = \pi$ is

A $5\pi^2$ **B** $-5\pi^2$ **C** 5π **D** -5π **E** -2π

13 The equation of the tangent to the curve with equation $y = e^{-x} - 1$ at the point where the curve crosses the y-axis is

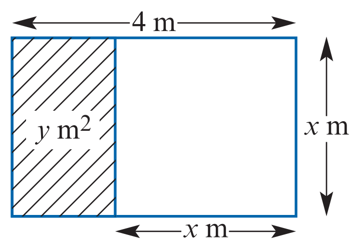
A $y = x$ **B** $y = -x$ **C** $y = \frac{1}{2}x$ **D** $y = -\frac{1}{2}x$ **E** $y = -2x$

14 For $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{ax} - \frac{ax}{e}$, the coordinates of the turning point of the graph of $y = f(x)$ are

A $\left(-\frac{1}{a}, 0\right)$ **B** $\left(\frac{1}{a}, \frac{1}{e}\right)$ **C** $\left(-\frac{1}{a}, \frac{2}{e}\right)$ **D** $\left(-1, \frac{1}{e}\right)$ **E** $(1, 0)$

Extended-response questions

1 The diagram shows a rectangle with sides 4 m and x m and a square with side x m. The area of the shaded region is y m^2.



a Find an expression for y in terms of x.

b Find the set of possible values for x.

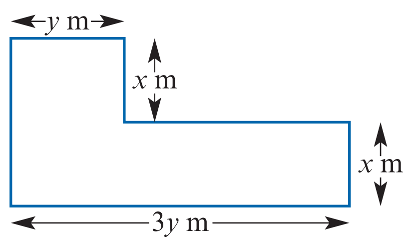
c Find the maximum value of y and the corresponding value of x.

d Explain briefly why this value of y is a maximum.

e Sketch the graph of y against x.

f State the set of possible values for y.

2 A flower bed is to be L-shaped, as shown in the figure, and its perimeter is 48 m.



a Write down an expression for the area, A m^2, in terms of y and x.

b Find y in terms of x.

c Write down an expression for A in terms of x.

d Find the values of x and y that give the maximum area.

e Find the maximum area.

3 It costs $(12 + 0.008x)$ dollars per kilometre to operate a truck at x kilometres per hour. In addition it costs \$14.40 per hour to pay the driver.

a What is the total cost per kilometre if the truck is driven at:

i 40 km/h **ii** 64 km/h?

b Write an expression for C, the total cost per kilometre, in terms of x.

c Sketch the graph of C against x for $0 < x < 120$.

d At what speed should the truck be driven to minimise the total cost per kilometre?

ISBN 978-1-107-56747-4

Review

4 A box is to be made from a 10 cm by 16 cm sheet of metal by cutting equal squares out of the corners and bending up the flaps to form the box. Let the lengths of the sides of the squares be x cm and let the volume of the box formed be V cm^3.

a Show that $V = 4(x^3 - 13x^2 + 40x)$.

b State the set of x-values for which the expression for V in terms of x is valid.

c Find the values of x such that $\frac{dV}{dx} = 0$.

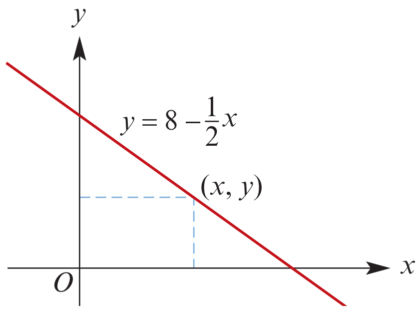
d Find the dimensions of the box if the volume is to be a maximum.

e Find the maximum volume of the box.

f Sketch the graph of V against x for the domain established in **b**.

5 A rectangle has one vertex at the origin, another on the positive x-axis, another on the positive y-axis and a fourth on the line $y = 8 - \frac{x}{2}$.

What is the greatest area the rectangle can have?



6 At a factory the time, T seconds, spent in producing a certain size metal component is related to its weight, w kg, by $T = k + 2w^2$, where k is a constant.

a If a 5 kg component takes 75 seconds to produce, find k.

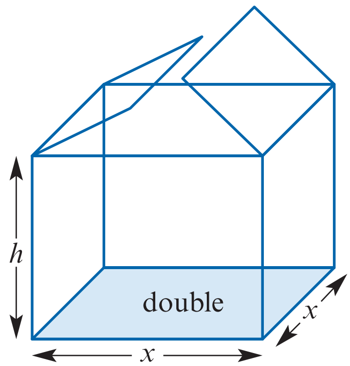
b Sketch the graph of T against w.

c Write down an expression for the average time A (in seconds per kilogram).

d **i** Find the weight that yields the minimum average machining time.

ii State the minimum average machining time.

7 A manufacturer produces cardboard boxes that have a square base. The top of each box consists of a double flap that opens as shown. The bottom of the box has a double layer of cardboard for strength. Each box must have a volume of 12 cubic metres.



a Show that the area of cardboard required is given by $C = 3x^2 + 4xh$.

b Express C as a function of x only.

c Sketch the graph of C against x for $x > 0$.

d **i** What dimensions of the box will minimise the amount of cardboard used?

ii What is the minimum area of cardboard used?

8 An open tank is to be constructed with a square base and vertical sides to contain 500 m^3 of water. What must be the area of sheet metal used in its construction if this area is to be a minimum?

ISBN 978-1-107-56747-4

Review

9 A piece of wire of length 1 m is bent into the shape of a sector of a circle of radius a cm and sector angle θ. Let the area of the sector be A cm^2.

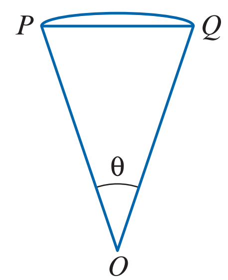
a Find A in terms of a and θ.

b Find A in terms of θ.

c Find the value of θ for which A is a maximum.

d Find the maximum area of the sector.

10 A piece of wire of fixed length, L cm, is bent to form the boundary $OPQO$ of a sector of a circle. The circle has centre O and radius r cm. The angle of the sector is θ radians.



a Show that the area, A cm^2, of the sector is given by

$$A = \frac{1}{2}rL - r^2$$

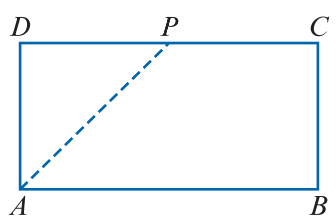
b **i** Find a relationship between r and L for which $\frac{dA}{dr} = 0$.

ii Find the corresponding value of θ.

iii Determine the nature of the stationary point found in **i**.

c Show that, for the value of θ found in **b ii**, the area of the triangle OPQ is approximately 45.5% of the area of sector OPQ.

11 A Queensland resort has a large swimming pool as illustrated, with $AB = 75$ m and $AD = 30$ m. A boy can swim at 1 m/s and run at $1\frac{2}{3}$ m/s. He starts at A, swims to a point P on DC, and runs from P to C. He takes 2 seconds to pull himself out of the pool.
Let $DP = x$ m and the total time taken be T s.



a Show that $T = \sqrt{x^2 + 900} + \frac{3}{5}(75 - x) + 2$.

b Find $\frac{dT}{dx}$.

c **i** Find the value of x for which the time taken is a minimum.

ii Find the minimum time.

d Find the time taken if the boy runs from A to D and then from D to C.

12 **a** Find the equation of the tangent to the curve $y = e^x$ at the point $(1, e)$.

b Find the equation of the tangent to the curve $y = e^{2x}$ at the point $\left(\frac{1}{2}, e\right)$.

c Find the equation of the tangent to the curve $y = e^{kx}$ at the point $\left(\frac{1}{k}, e\right)$.

d Show that $y = xke$ is the only tangent to the curve $y = e^{kx}$ which passes through the origin.

e Hence determine for what values of k the equation $e^{kx} = x$ has:

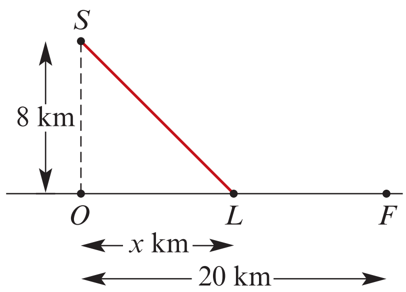
i a unique real solution

ii no real solution.

ISBN 978-1-107-56747-4

Review

13 The point S is 8 km offshore from the point O, which is located on the straight shore of a lake, as shown in the diagram. The point F is on the shore, 20 km from O. Contestants race from the start, S, to the finish, F, by rowing in a straight line to some point, L, on the shore and then running along the shore to F. A certain contestant rows at 5 km per hour and runs at 15 km per hour.



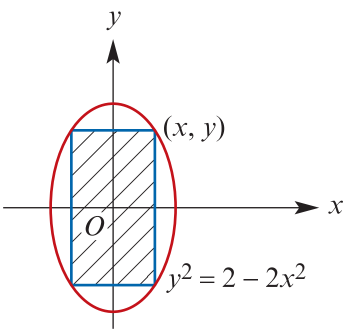
a Show that, if the distance OL is x km, the time taken by this contestant to complete the course is (in hours):

$$T(x) = \frac{\sqrt{64 + x^2}}{5} + \frac{20 - x}{15}$$

b Show that the time taken by this contestant to complete the course has its minimum value when $x = 2\sqrt{2}$. Find this time.

14 At noon the captain of a ship sees two fishing boats approaching. One of them is 10 km due east and travelling west at 8 km/h. The other is 6 km due north and travelling south at 6 km/h. At what time will the fishing boats be closest together and how far apart will they be?

15 A rectangular beam is to be cut from a non-circular tree trunk whose cross-sectional outline can be represented by the equation $y^2 = 2 - 2x^2$.



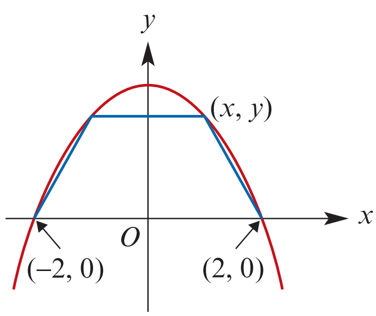
a Show that the area of the cross-section of the beam is given by $A = 4x\sqrt{2 - 2x^2}$ where x is the half-width of the beam.

b State the possible values for x.

c Find the value of x for which the cross-sectional area of the beam is a maximum and find the corresponding value of y.

d Find the maximum cross-sectional area of the beam.

16 An isosceles trapezium is inscribed in the parabola $y = 4 - x^2$ as illustrated.



a Show that the area of the trapezium is

$$\frac{1}{2}(4 - x^2)(2x + 4)$$

b Show that the trapezium has its greatest area when $x = \frac{2}{3}$.

c Repeat with the parabola $y = a^2 - x^2$:

i Show that the area, A, of the trapezium is given by $(a^2 - x^2)(a + x)$.

ii Use the product rule to find $\frac{dA}{dx}$.

iii Show that a maximum occurs when $x = \frac{a}{3}$.

ISBN 978-1-107-56747-4

17 Assume that the number of bacteria present in a culture at time t is given by $N(t)$ where $N(t) = 24te^{-0.2t}$. At what time will the population be at a maximum? Find the maximum population.

18 It is believed that, for some time after planting in ideal conditions, the area covered by a particular species of ground-cover plant has a rate of increase of y cm^2/week, given by $y = -t^3 + bt^2 + ct$ where t is the number of weeks after planting.

t	1	2
y	10	24

a Find b and c using the table of observations on the right.

b Assume that the model is accurate for the first 8 weeks after planting. When during this period is:

i the area covered by the plant a maximum

ii the rate of increase in area a maximum?

c According to the model, if the plant covered 100 cm^2 when planted, what area will it cover after 4 weeks?

d Discuss the implications for the future growth of the plant if the model remains accurate for longer than the first 4 weeks.

19 Let $f(x) = x^3 - 3x^2 + 6x - 10$.

a Find the coordinates of the point on the graph of f for which $f'(x) = 3$.

b Express $f'(x)$ in the form $a(x + p)^2 + q$.

c Hence show that the gradient of f is greater than 3 for all points on the curve of f other than the point found in **a**.

20 A curve with equation of the form $y = ax^3 + bx^2 + cx + d$ has zero gradient at the point $\left(\frac{1}{3}, \frac{4}{27}\right)$ and also touches, but does not cross, the x-axis at the point $(1, 0)$.

a Find a, b, c and d.

b Find the values of x for which the curve has a negative gradient.

c Sketch the curve.

21 The volume of water, V m^3, in a reservoir when the depth indicator shows y metres is given by the formula

$$V = \frac{\pi}{3}[(y + 630)^3 - 630^3]$$

a Find the volume of water in the reservoir when $y = 40$.

b Find the rate of change of volume with respect to depth, y.

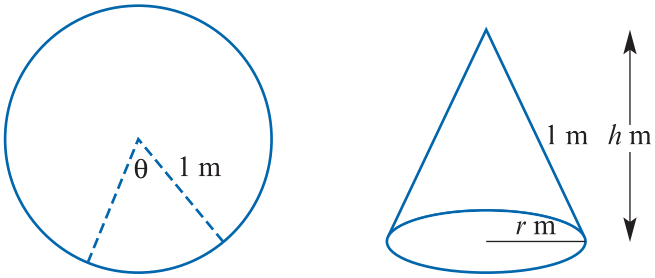
c Sketch the graph of V against y for $0 \le y \le 60$.

d If $y = 60$ m is the maximum depth of the reservoir, find the capacity (m^3) of the reservoir.

e If $\dfrac{dV}{dt} = 20\,000 - 0.005\pi(y + 630)^2$, where t is the time in days from 1 January, sketch the graph of $\dfrac{dV}{dt}$ against y for $0 \le y \le 60$.

ISBN 978-1-107-56747-4

22 A cone is made by cutting out a sector with central angle θ from a circular piece of cardboard of radius 1 m and joining the two cut edges to form a cone of slant height 1 m as shown in the following diagrams.



The volume of a cone is given by the formula $V = \frac{1}{3}\pi r^2 h$.

a **i** Find r in terms of θ.

ii Find h in terms of θ.

iii Show that $V = \frac{1}{3}\pi\left(\frac{2\pi - \theta}{2\pi}\right)^2\sqrt{1 - \left(\frac{2\pi - \theta}{2\pi}\right)^2}$.

b Find the value of V when $\theta = \frac{\pi}{4}$.

c Find the value(s) of θ for which the volume of the cone is 0.3 m^3.

d **i** Use a calculator to determine the value of θ that maximises the volume of the cone.

ii Find the maximum volume.

e Determine the maximum volume using calculus.

23 **a** For the function with rule $f(x) = x^3 + ax^2 + bx$, plot the graph of each of the following using a calculator. (Give axis intercepts, coordinates of stationary points and the nature of stationary points.)

i $a = 1,\ b = 1$

ii $a = -1,\ b = -1$

iii $a = 1,\ b = -1$

iv $a = -1,\ b = 1$

b **i** Find $f'(x)$.

ii Solve the equation $f'(x) = 0$ for x, giving your answer in terms of a and b.

c **i** Show that the graph of $y = f(x)$ has exactly one stationary point if $a^2 - 3b = 0$.

ii If $b = 3$, find the corresponding value(s) of a which satisfy $a^2 - 3b = 0$. Find the coordinates of the stationary points and state the nature of each.

iii Using a calculator, plot the graph(s) of $y = f(x)$ for these values of a and b.

iv Plot the graphs of the corresponding derivative functions on the same set of axes.

d State the relationship between a and b if no stationary points exist for the graph of $y = f(x)$.

24 For what value of x is $\frac{\log_e x}{x}$ a maximum? That is, when is the ratio of the logarithm of a number to the number a maximum?

ISBN 978-1-107-56747-4

Review

25 Consider the function with rule $f(x) = 6x^4 - x^3 + ax^2 - 6x + 8$.

a **i** If $x + 1$ is a factor of $f(x)$, find the value of a.

ii Using a calculator, plot the graph of $y = f(x)$ for this value of a.

b Let $g(x) = 6x^4 - x^3 + 21x^2 - 6x + 8$.

i Plot the graph of $y = g(x)$.

ii Find the minimum value of $g(x)$ and the value of x for which this occurs.

iii Find $g'(x)$.

iv Using a calculator, solve the equation $g'(x) = 0$ for x.

v Find $g'(0)$ and $g'(10)$.

vi Find the derivative of $g'(x)$.

vii Show that the graph of $y = g'(x)$ has no stationary points and thus deduce that $g'(x) = 0$ has only one solution.

26 For the quartic function f with rule $f(x) = (x - a)^2(x - b)^2$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = 2(x - a)(x - b)[2x - (b + a)]$.

b **i** Solve the equation $f'(x) = 0$ for x. **ii** Solve the equation $f(x) = 0$ for x.

c Hence find the coordinates of the stationary points of the graph of $y = f(x)$.

d Plot the graph of $y = f(x)$ on a calculator for several values of a and b.

e **i** If $a = b$, then $f(x) = (x - a)^4$. Sketch the graph of $y = f(x)$.

ii If $a = -b$, find the coordinates of the stationary points.

iii Plot the graph of $y = f(x)$ for several values of a, given that $a = -b$.

27 For the quartic function f with rule $f(x) = (x - a)^3(x - b)$, where $a > 0$ and $b > 0$:

a Show that $f'(x) = (x - a)^2[4x - (3b + a)]$.

b **i** Solve the equation $f'(x) = 0$. **ii** Solve the equation $f(x) = 0$.

c Find the coordinates of the stationary points of the graph of $y = f(x)$ and state the nature of the stationary points.

d Using a calculator, plot the graph of $y = f(x)$ for several values of a and b.

e If $a = -b$, state the coordinates of the stationary points in terms of a.

f **i** State the relationship between b and a if there is a local minimum for $x = 0$.

ii Illustrate this for $b = 1$ and $a = -3$ on a calculator.

g Show that, if there is a turning point for $x = \dfrac{a + b}{2}$, then $b = a$ and $f(x) = (x - a)^4$.

28 A psychologist hypothesised that the ability of a mouse to memorise during the first 6 months of its life can be modelled by the function f given by $f\colon (0, 6) \to \mathbb{R}$, $f(x) = x \log_e x + 1$, i.e. the ability to memorise at age x months is $f(x)$.

a Find $f'(x)$.

b Find the value of x for which $f'(x) = 0$ and hence find when the mouse's ability to memorise is a minimum.

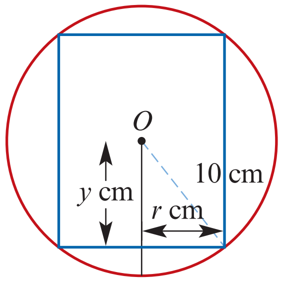
c Sketch the graph of f.

d When is the mouse's ability to memorise a maximum in this period?

ISBN 978-1-107-56747-4

Cambridge University Press

29 A cylinder is to be cut from a sphere. The cross-section through the centre of the sphere is as shown. The radius of the sphere is 10 cm. Let r cm be the radius of the cylinder.



a **i** Find y in terms of r and hence the height, h cm, of the cylinder.

ii The volume of a cylinder is given by $V = \pi r^2 h$. Find V in terms of r.

b **i** Plot the graph of V against r using a calculator.

ii Find the maximum volume of the cylinder and the corresponding values of r and h. (Use a calculator.)

iii Find the two possible values of r if the volume is 2000 cm^3.

c **i** Find $\dfrac{dV}{dr}$.

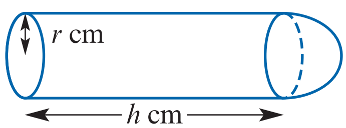
ii Hence find the exact value of the maximum volume and the volume of r for which this occurs.

d **i** Plot the graph of the derivative function $\dfrac{dV}{dr}$ against r, using a calculator.

ii From the calculator, find the values of r for which $\dfrac{dV}{dr}$ is positive.

iii From the calculator, find the values of r for which $\dfrac{dV}{dr}$ is increasing.

30 A wooden peg consists of a cylinder of length h cm and a hemispherical cap of radius r cm, and so the volume, V cm^3, of the peg is given by $V = \pi r^2 h + \dfrac{2}{3}\pi r^3$. If the surface area of the peg is 100π cm:



a Find h in terms of r.

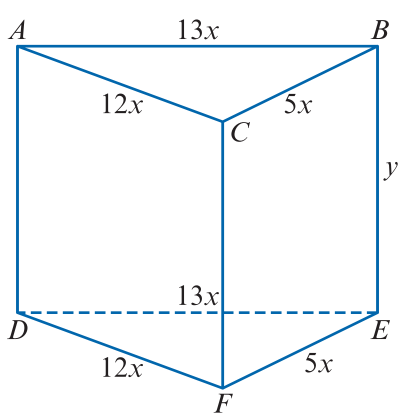
b Find V as a function of r.

c Find the possible values of r (i.e. find the domain of the function defined in **b**).

d Find $\dfrac{dV}{dr}$.

e Sketch the graph of V against r.

31 A triangular prism has dimensions as shown in the diagram. All lengths are in centimetres. The volume of the prism is 3000 cm^3.



a **i** Find y in terms of x.

ii Find the surface area of the prism, S cm^2, in terms of x.

b **i** Find $\dfrac{dS}{dx}$.

ii Find the minimum surface area, correct to two decimal places.

c Given that x is increasing at 0.5 cm/s find the rate at which the surface area is increasing when $x = 10$.

ISBN 978-1-107-56747-4

32 The kangaroo population in a certain confined region is given by $f(x) = \dfrac{100\,000}{1 + 100e^{-0.3x}}$, where x is the time in years.

a Find $f'(x)$.

b Find the rate of growth of the kangaroo population when:

i $x = 0$ **ii** $x = 4$

33 Consider the function $f : \{x : x < a\} \to \mathbb{R}$, $f(x) = 8\log_e(6 - 0.2x)$ where a is the largest value for which f is defined.

a What is the value of a?

b Find the exact values for the coordinates of the points where the graph of $y = f(x)$ crosses each axis.

c Find the gradient of the tangent to the graph of $y = f(x)$ at the point where $x = 20$.

d Find the rule of the inverse function f^{-1}.

e State the domain of the inverse function f^{-1}.

f Sketch the graph of $y = f(x)$.

34 **a** Using a calculator, plot the graphs of $f(x) = \sin x$ and $g(x) = e^{\sin x}$ on the one screen.

b Find $g'(x)$ and hence find the coordinates of the stationary points of $y = g(x)$ for $x \in [0, 2\pi]$.

c Give the range of g. **d** State the period of g.

35 **a** Show that the tangent to the graph of $y = e^x$ for $x = 0$ has equation $y = x + 1$.

b Plot the graphs of $y = e^x$ and $y = x + 1$ on a calculator.

c Let $f(x) = e^x$ and $g(x) = x + 1$. Use a calculator to investigate functions of the form

$$h(x) = af(x - b) + c \quad \text{and} \quad k(x) = ag(x - b) + c$$

Comment on your observations.

d Use the chain rule and properties of transformations to prove that, if the tangent to the curve $y = f(x)$ at the point (x_1, y_1) has equation $y = mx + c$, then the tangent to the curve $y = af(bx)$ at the point $\left(\dfrac{x_1}{b}, y_1 a\right)$ has equation $y = a(mbx + c)$.

36 A certain chemical starts to dissolve in water at time $t = 0$. It is known that, if x is the number of grams not dissolved after t hours, then

$$x = \frac{60}{5e^{\lambda t} - 3}, \quad \text{where } \lambda = \frac{1}{2}\log_e\left(\frac{6}{5}\right)$$

a Find the amount of chemical present when:

i $t = 0$ **ii** $t = 5$

b Find $\dfrac{dx}{dt}$ in terms of t.

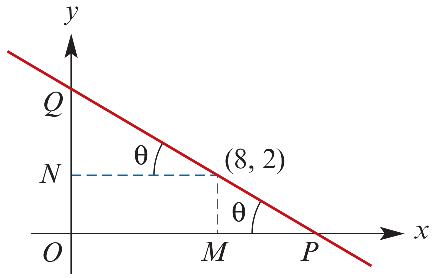
c **i** Show that $\dfrac{dx}{dt} = -\lambda x - \dfrac{\lambda x^2}{20}$.

ii Sketch the graph of $\dfrac{dx}{dt}$ against x for $x \geq 0$.

iii Write a short explanation of your result.

ISBN 978-1-107-56747-4

37 A straight line is drawn through the point $(8, 2)$ to intersect the positive y-axis at Q and the positive x-axis at P. (In this problem we will determine the minimum value of $OP + OQ$.)



a Show that the derivative of $\dfrac{1}{\tan\theta}$ is $-\operatorname{cosec}^2\theta$.

b Find MP in terms of θ.

c Find NQ in terms of θ.

d Hence find $OP + OQ$ in terms of θ. Denote $OP + OQ$ by x.

e Find $\dfrac{dx}{d\theta}$.

f Find the minimum value of x and the value of θ for which this occurs.

38 Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x - e^{-x}$.

a Find $f'(x)$.

b Find $\{ x : f(x) = 0 \}$.

c Show that $f'(x) > 0$ for all x.

d Sketch the graph of f.

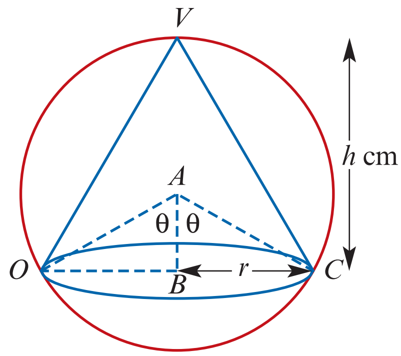
39 **a** Find all values of x for which $(\log_e x)^2 = 2\log_e x$.

b Find the gradient of each of the curves $y = 2\log_e x$ and $y = (\log_e x)^2$ at the point $(1, 0)$.

c Use these results to sketch, on one set of axes, the graphs of $y = 2\log_e x$ and $y = (\log_e x)^2$.

d Find $\{ x : 2\log_e x > (\log_e x)^2 \}$.

40 A cone is inscribed inside a sphere as illustrated. The radius of the sphere is a cm, and the magnitude of $\angle OAB$ = magnitude of $\angle CAB = \theta$. The height of the cone is h cm and the radius of the cone is r cm.



a Find h, the height of the cone, in terms of a and θ.

b Find r, the radius of the cone, in terms of a and θ.

The volume, V cm^3, of the cone is given by $V = \dfrac{1}{3}\pi r^2 h$.

c Use the results from **a** and **b** to show that

$$V = \frac{1}{3}\pi a^3 \sin^2\theta \cdot (1 + \cos\theta)$$

d Find $\dfrac{dV}{d\theta}$ (a is a constant) and hence find the value of θ for which the volume is a maximum.

e Find the maximum volume of the cone in terms of a.

ISBN 978-1-107-56747-4

41 Some bacteria are introduced into a supply of fresh milk. After t hours there are y grams of bacteria present, where

$$y = \frac{Ae^{bt}}{1 + Ae^{bt}} \qquad (1)$$

and A and b are positive constants.

a Show that $0 < y < 1$ for all values of t.

b Find $\frac{dy}{dt}$ in terms of t.

c From equation (1), show that $Ae^{bt} = \frac{y}{1-y}$.

d **i** Show that $\frac{dy}{dt} = by(1-y)$.

ii Hence, or otherwise, show that the maximum value of $\frac{dy}{dt}$ occurs when $y = 0.5$.

e If $A = 0.01$ and $b = 0.7$, find when, to the nearest hour, the bacteria will be increasing at the fastest rate.

42 Let $f\colon \mathbb{R}^+ \to \mathbb{R}$, $f(x) = \frac{e^x}{x}$.

a Find $f'(x)$.

b Find $\{ x : f'(x) = 0 \}$.

c Find the coordinates of the one stationary point and state its nature.

d **i** Find $\frac{f'(x)}{f(x)}$. **ii** Find $\lim_{x\to\infty} \frac{f'(x)}{f(x)}$ and comment.

e Sketch the graph of f.

f Over a period of years, the number of birds (n) in an island colony decreased and increased with time (t years) according to the approximate formula

$$n = \frac{ae^{kt}}{t}$$

where t is measured from 1900 and a and k are constant. If during this period the population was the same in 1965 as it was in 1930, when was it least?

43 A culture contains 1000 bacteria and 5 hours later the number has increased to 10 000. The number, N, of bacteria present at any time, t hours, is given by $N = Ae^{kt}$.

a Find the values of A and k.

b Find the rate of growth at time t.

c Show that, at time t, the rate of growth is proportional to the number of bacteria present.

d Find this rate of growth when:

i $t = 4$ **ii** $t = 50$

44 The populations of two ant colonies, A and B, are increasing according to the rules:

A population $= 2 \times 10^4 e^{0.03t}$

B population $= 10^4 e^{0.05t}$

After how many years will their populations:

a be equal **b** be increasing at the same rate?

ISBN 978-1-107-56747-4

Cambridge University Press

45 A particle on the end of a spring, which is hanging vertically, is oscillating such that its height, h metres, above the floor after t seconds is given by

$$y = 0.5 + 0.2\sin(3\pi t), \quad t \geq 0$$

a Find the greatest height above the floor and the time at which this height is first reached.

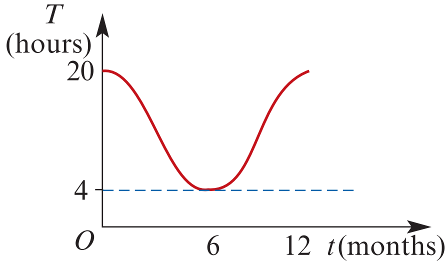
b Find the period of oscillation.

c Find the speed of the particle when $t = \frac{1}{3}, \frac{2}{3}, \frac{1}{6}$.

46 The length of night on Seal Island varies between 20 hours in midwinter and 4 hours in midsummer. The relationship between T, the number of hours of night, and t, the number of months past the longest night in 2010, is given by

$$T(t) = p + q\cos(\pi r t)$$

where p, q and r are constants.
Assume that the year consists of 12 months of equal length.
The graph of T against t is illustrated.



a Find the value of:

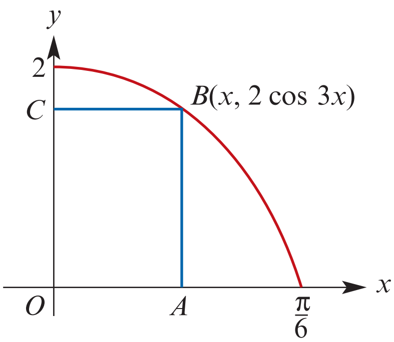
i r **ii** p and q

b Find $T'(3)$ and $T'(9)$ and find the rate of change of hours of night with respect to the number of months.

c Find the average rate of change of hours of night from $t = 0$ to $t = 6$.

d After how many months is the rate of change of hours of night a maximum?

47 A section of the graph of $y = 2\cos(3x)$ is shown in the diagram.



a Show that the area, A, of the rectangle $OABC$ in terms of x is $2x\cos(3x)$.

b **i** Find $\frac{dA}{dx}$.

ii Find $\frac{dA}{dx}$ when $x = 0$ and $x = \frac{\pi}{6}$.

c **i** On a calculator, plot the graph of $A = 2x\cos(3x)$ for $x \in \left[0, \frac{\pi}{6}\right]$.

ii Find the two values of x for which the area of the rectangle is 0.2 square units.

iii Find the maximum area of the rectangle and the value of x for which this occurs.

d **i** Show that $\frac{dA}{dx} = 0$ is equivalent to $\tan(3x) = \frac{1}{3x}$.

ii Using a calculator, plot the graphs of $y = \tan(3x)$ and $y = \frac{1}{3x}$ for $x \in \left(0, \frac{\pi}{6}\right)$ and find the coordinates of the point of intersection.

 ISBN 978-1-107-56747-4 Cambridge University Press

48 **a** A population of insects grows according to the model

$$N(t) = 1000 - t + 2e^{\frac{t}{20}} \quad \text{for } t \geq 0$$

where t is the number of days after 1 January 2000.

i Find the rate of growth of the population as a function of t.

ii Find the minimum population size and value of t for which this occurs.

iii Find $N(0)$.

iv Find $N(100)$.

v Sketch the graph of N against t for $0 \leq t \leq 100$.

b It is found that the population of another species is given by

$$N_2(t) = 1000 - t^{\frac{1}{2}} + 2e^{\frac{t^{\frac{1}{2}}}{20}}$$

i Find $N_2(0)$.

ii Find $N_2(100)$.

iii Plot the graph of $y = N_2(t)$ for $t \in [0, 5000]$ on a calculator.

iv Solve the equation $N_2'(t) = 0$ and hence give the minimum population of this species of insects.

c A third model is

$$N_3(t) = 1000 - t^{\frac{3}{2}} + 2e^{\frac{t}{20}}$$

Use a calculator to:

i plot a graph for $0 \leq t \leq 200$

ii find the minimum population and the time at which this occurs.

d **i** For N_3, find $N_3'(t)$.

ii Show that $N_3'(t) = 0$ is equivalent to $t = 20 \log_e(15\sqrt{t})$.

49 **a** Consider the curve with equation $y = (2x^2 - 5x)e^{ax}$. If the curve passes through the point with coordinates $(3, 10)$, find the value of a.

b **i** For the curve with equation $y = (2x^2 - 5x)e^{ax}$, find the x-axis intercepts.

ii Use calculus to find the x-values for which there is a turning point, in terms of a.

ISBN 978-1-107-56747-4