6 The triangle ABC is isosceles. The vertices are $A(5, 0)$, $B(13, 0)$ and $C(9, 10)$.

a Find the coordinates of the midpoints M and N of AC and BC respectively.

b Find the equation of the lines:

i AC

ii BC

iii MN

c Find the equations of the lines perpendicular to AC and BC, passing through the points M and N respectively, and find the coordinates of their intersection point.

2E Matrices

This section provides a brief introduction to matrices. In Chapter 3, we will see that the transformations we consider in this course can be determined through matrix arithmetic. We will consider the inverse of a 2×2 matrix only in the context of transformations; this is done in Section 3J. Additional information and exercises on matrices are available in the Interactive Textbook.

▶ Matrix notation

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The following are examples of matrices:

$$\begin{bmatrix} -3 & 4 \\ 5 & 6 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ 7 \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2} & \pi & 3 \\ 0 & 0 & 1 \\ \sqrt{2} & 0 & \pi \end{bmatrix} \qquad \begin{bmatrix} 5 \end{bmatrix}$$

The size, or **dimension**, of a matrix is described by specifying the number of **rows** (horizontal lines) and **columns** (vertical lines).

The dimensions of the above matrices are, in order:

$$2 \times 2, \qquad 2 \times 1, \qquad 3 \times 3, \qquad 1 \times 1$$

The first number represents the number of rows, and the second the number of columns.

In this book we are only interested in 2×2 matrices and 2×1 matrices.

If $\mathbf{A}$ is a matrix, then a_{ij} will be used to denote the entry that occurs in row i and column j of $\mathbf{A}$. Thus a 2×2 matrix may be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A general 2×1 matrix may be written as

$$\mathbf{B} = \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix}$$

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A matrix is, then, a way of recording a set of numbers, arranged in a particular way. As in Cartesian coordinates, the order of the numbers is significant. Although the matrices

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

have the same numbers and the same number of entries, they are different matrices (just as (2, 1) and (1, 2) are the coordinates of different points).

Two matrices **A** and **B** are **equal**, and we can write **A** = **B**, when:

- they have the same number of rows and the same number of columns, and
- they have the same number or entry at corresponding positions.

Addition, subtraction and multiplication by a scalar

Addition is defined for two matrices only when they have the same dimension. In this case, the sum of the two matrices is found by adding corresponding entries.

For example,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 4 & 3 \end{bmatrix}$$

and $$\begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} \\ a_{21} + b_{21} \end{bmatrix}$$

Subtraction is defined in a similar way: If two matrices have the same dimension, then their difference is found by subtracting corresponding entries.

Example 7

Find:

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Solution

a $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix}$

b $\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

It is useful to define **multiplication of a matrix by a real number**. If **A** is an $m \times n$ matrix and k is a real number (also called a **scalar**), then $k\mathbf{A}$ is an $m \times n$ matrix whose entries are k times the corresponding entries of **A**. Thus

$$3\begin{bmatrix} 2 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -6 \\ 0 & 3 \end{bmatrix}$$

These definitions have the helpful consequence that, if a matrix is added to itself, the result is twice the matrix, i.e. $\mathbf{A} + \mathbf{A} = 2\mathbf{A}$. Similarly, the sum of n matrices each equal to **A** is $n\mathbf{A}$ (where n is a natural number).

The $m \times n$ matrix with all entries equal to zero is called the **zero matrix**.

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Example 8

If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix}$, find the matrix $\mathbf{X}$ such that $2\mathbf{A} + \mathbf{X} = \mathbf{B}$.

Solution

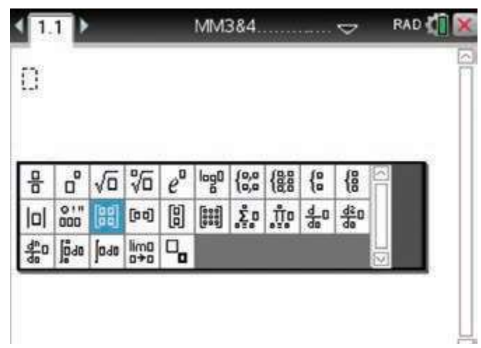
If $2\mathbf{A} + \mathbf{X} = \mathbf{B}$, then $\mathbf{X} = \mathbf{B} - 2\mathbf{A}$. Therefore

$$\begin{aligned} \mathbf{X} &= \begin{bmatrix} 0 & -4 \\ -2 & 8 \end{bmatrix} - 2\begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 - 2 \times 3 & -4 - 2 \times 2 \\ -2 - 2 \times (-1) & 8 - 2 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} -6 & -8 \\ 0 & 6 \end{bmatrix} \end{aligned}$$

Using the TI-Nspire

The matrix template

- The simplest way to enter a 2×2 matrix is using the 2×2 matrix template as shown. (Access the templates using either or ctrl menu > **Math Templates**.)
- Notice that there is also a template for entering $m \times n$ matrices.



- Define the matrix $\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix}$ as shown.

  The assignment symbol := is accessed using ctrl. Use the touchpad arrows to move between the entries of the matrix.
- Define the matrix $\mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$ similarly.

Note: All variables will be changed to lower case.
Alternatively, you can store (ctrl var) the matrices if preferred.

Entering matrices directly

- To enter matrix **A** without using the template, enter the matrix row by row as [[3, 6][6, 7]].

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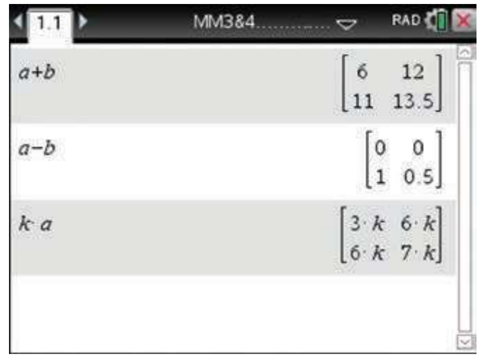
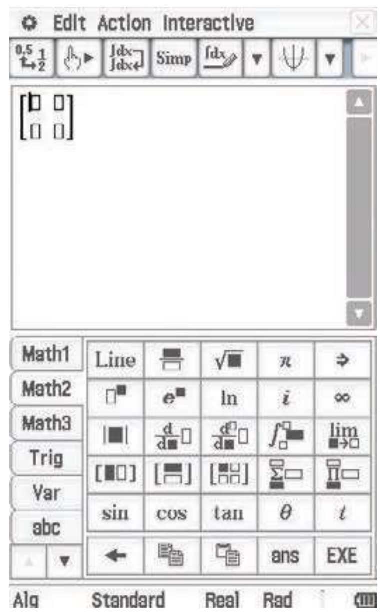
Addition, subtraction and multiplication by a scalar

- Once **A** and **B** are defined as above, the matrices **A** + **B**, **A** − **B** and k**A** can easily be determined.



Using the Casio ClassPad

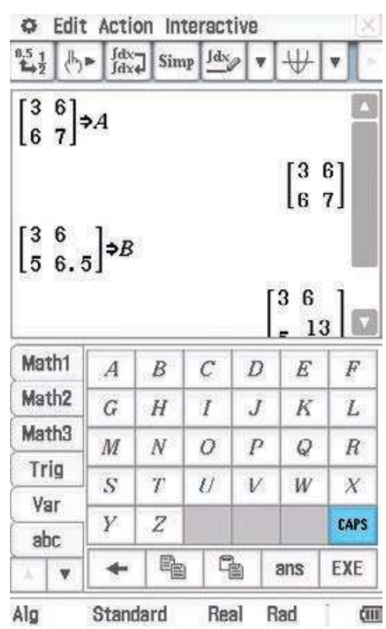
- Matrices are accessed through the Math2 keyboard.
- Select and tap on each of the entry boxes to enter the matrix values.

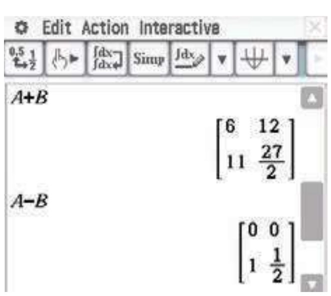


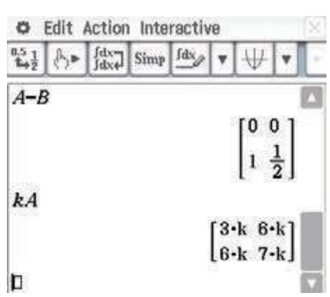
Notes:

- To expand the 2 × 2 matrix to a 3 × 3 matrix, tap on the button twice.
- To increase the number of rows, tap on the button. To increase the number of columns, tap on the button.

- Matrices can be stored as a variable for later use in operations by selecting the store button ⇒ located in Math1 followed by the variable name (usually a capital letter).
- Once **A** and **B** are defined as shown, the matrices **A** + **B**, **A** − **B** and k**A** can be found. (Use the Var keyboard to enter the variable names.)







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Multiplication of matrices

Multiplication of a matrix by a real number has been discussed in the previous subsection. The definition for multiplication of matrices is less natural. The procedure for multiplying two 2×2 matrices is shown first.

Let $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}$.

$$\begin{aligned} \text{Then} \quad \mathbf{AB} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \times 5 + 3 \times 6 & 1 \times 1 + 3 \times 3 \\ 4 \times 5 + 2 \times 6 & 4 \times 1 + 2 \times 3 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 10 \\ 32 & 10 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \text{and} \quad \mathbf{BA} &= \begin{bmatrix} 5 & 1 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 5 \times 1 + 1 \times 4 & 5 \times 3 + 1 \times 2 \\ 6 \times 1 + 3 \times 4 & 6 \times 3 + 3 \times 2 \end{bmatrix} \\ &= \begin{bmatrix} 9 & 17 \\ 18 & 24 \end{bmatrix} \end{aligned}$$

Note that $\mathbf{AB} \neq \mathbf{BA}$.

If $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times r$ matrix, then the product $\mathbf{AB}$ is the $m \times r$ matrix whose entries are determined as follows:

> To find the entry in row i and column j of $\mathbf{AB}$, single out row i in matrix $\mathbf{A}$ and column j in matrix $\mathbf{B}$. Multiply the corresponding entries from the row and column and then add up the resulting products.

Note: The product $\mathbf{AB}$ is defined only if the number of columns of $\mathbf{A}$ is the same as the number of rows of $\mathbf{B}$.

Example 9

For $\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, find $\mathbf{AB}$.

Solution

$\mathbf{A}$ is a 2×2 matrix and $\mathbf{B}$ is a 2×1 matrix. Therefore $\mathbf{AB}$ is defined and will be a 2×1 matrix.

$$\mathbf{AB} = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \times 5 + 4 \times 3 \\ 3 \times 5 + 6 \times 3 \end{bmatrix} = \begin{bmatrix} 22 \\ 33 \end{bmatrix}$$

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Using the TI-Nspire

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

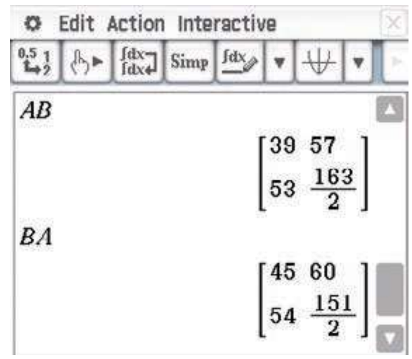
The products **AB** and **BA** are shown.

Using the Casio ClassPad

Multiplication of

$$\mathbf{A} = \begin{bmatrix} 3 & 6 \\ 6 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & 6 \\ 5 & 6.5 \end{bmatrix}$$

The products **AB** and **BA** are shown.



A matrix with the same number of rows and columns is called a **square matrix**.

For 2×2 matrices, the **identity matrix** is

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix has the property that $\mathbf{AI} = \mathbf{A} = \mathbf{IA}$, for any 2×2 matrix **A**.

In general, for the family of $n \times n$ matrices, the multiplicative identity **I** is the matrix that has ones in the 'top left' to 'bottom right' diagonal and has zeroes in all other positions.

Section summary

- A **matrix** is a rectangular array of numbers.
- Two matrices **A** and **B** are equal when:
  - they have the same number of rows and the same number of columns, and
  - they have the same number or entry at corresponding positions.
- The size or **dimension** of a matrix is described by specifying the number of rows (m) and the number of columns (n). The dimension is written $m \times n$.
- Addition is defined for two matrices only when they have the same dimension. The sum is found by adding corresponding entries.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

  Subtraction is defined in a similar way.

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