5

The derivative

Introduction

Calculus, the mathematics of change and motion, relies on the fundamental concept of the derivative. Derivatives arise in each of the sciences, and in many other fields of study. Velocity and acceleration in physics, currents in electronics, rates of reaction in chemistry, growth of populations in biology and marginal revenue in economics are all direct applications of the derivative.

This topic covers average rates of change, the definition of the derivative as an instantaneous rate of change, the gradient of a function represented as a derivative, and the simple rules needed to differentiate polynomials, exponentials, logarithms, and trigonometric functions.

After studying this topic, you should be able to:

- find the average rate of change of a function;
- understand and use the rules to find the derivatives of polynomial, trigonometric, exponential and logarithmic functions;
- find and classify stationary points;
- sketch polynomials using the derivative.

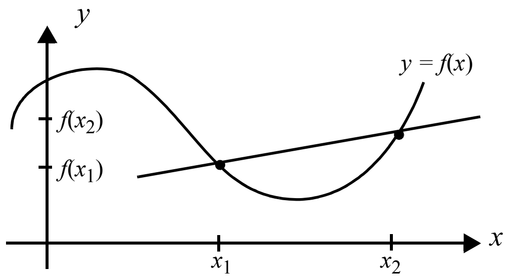
5.1 Rates of change

1. Given the function $y = f(x)$, the **average rate of change of *f* is the change in *f* divided by the change in *x***.

   In function notation, for $x_1 \le x \le x_2$,

   $$\text{average rate of change} = \frac{\text{change in } f(x)}{\text{change in } x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

   Pictorially, the average rate of change is given by the slope of the straight line joining two points on the curve, as shown in the diagram .

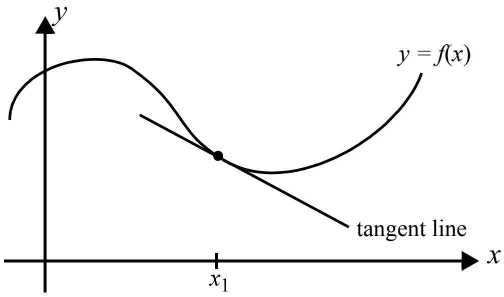
   

   For the straight line shown, the slope is given by the usual formula,

   i.e. $m = \dfrac{rise}{run} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

2. If x represents time, and $y = f(x)$ represents displacement, then the average rate of change is the average speed.

3. Given the function $y = f(x)$, the **instantaneous rate of change of *f*** when $x = x_1$ **is the slope of the tangent line to the curve** $y = f(x)$ at the point where $x = x_1$.



The formula for the instantaneous rate of change of *f* when $x = x_1$ is obtained from the average rate of change by first writing $x_2 = x_1 + h$.

The instantaneous rate of change of f is then given by

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{x_1 + h - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}$$

The smaller the distance between x_1 and x_2, i.e. the smaller the distance between *h* and 0, the closer the value of the average rate of change to the instantaneous rate of change of *f* when $x = x_1$.

So, as *h* approaches 0, $\dfrac{f(x_1 + h) - f(x_1)}{h}$ approaches the instantaneous rate of change of $f(x)$ at x_1.

Mathematically, the formula for the instantaneous rate of change of *f* when $x = x_1$ is given by

$$\text{instantaneous rate of change} = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h}$$

The process of letting *h* approach 0 is called a limit, and is called 'the limit as *h* tends to 0'.

Note: When $h = 0$, $\dfrac{f(x_1 + h) - f(x_1)}{h} = \dfrac{f(x_1) - f(x_1)}{0} = \dfrac{0}{0}$, which is undefined.

4. The formula for the instantaneous rate of change of *f* at **any** value of *x* is given by replacing x_1 by *x* in the limit used above. This gives the **gradient function** : $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.

Examples

1. The following table shows the distance covered by a swimmer at various times during a 30 second swim

Distance (m.)	0	12	23	32	42	51	60
Time (s.)	0	5	10	15	20	25	30

Find the average speed

(i) over the first 5 seconds

(ii) between 10 and 20 seconds

(iii) over the last 5 seconds

(iv) over the entire 30 seconds.

2. The temperatures at various times on a certain day are shown in the following table.

Temperature	9° C	27° C	29° C	x	21° C
Time	7 a.m.	11 a.m.	3 p.m.	7 p.m.	11 p.m

(i) Find the average rate of change of temperature between 7 a.m. and 3 p.m.

(ii) Find the average rate of change of temperature between 11 a.m. and 11 p.m.

(iii) Given that the average rate of change of temperature between 7 p.m. and 11 p.m. is $-1.5°$ C per hour, find the temperature, x, at 7 p.m.

1. The required average speed:

(i) over the first 5 seconds is $\frac{12-0}{5-0} = \frac{12}{5} = 2.4$ (m/sec);

(ii) between 10 and 20 seconds is $\frac{42-23}{20-10} = \frac{19}{10} = 1.9$ (m/sec);

(iii) over the last 5 seconds is $\frac{60-51}{30-25} = \frac{9}{5} = 1.8$ (m/sec);

(iv) over the entire 30 seconds is $\frac{60-0}{30-0} = \frac{60}{30} = 2.0$ (m/sec).

Note: The units of the rate of change here are the units for distance (m.) divided by the units for time (sec.).

2. The required rate of change of temperature:

(i) between 7 a.m. and 3 p.m. is $\frac{29-9}{15-7} = \frac{20}{8} = 2.5$ (° C per hour)

(ii) between 11 a.m. and 11 p.m.is

$$\frac{21-27}{23-11} = \frac{-6}{12} = -0.5 \text{ (° C per hour)}$$

(iii) Since the average rate of change of temperature between 7 p.m. and 11 p.m. is $-1.5°$ C per hour,

$\frac{21-x}{11-7} = -1.5$ $\quad\quad$ $\therefore \frac{21-x}{4} = -1.5$

$\therefore 21-x = -1.5 \times 4$ $\quad\quad$ $\therefore 21-x = -6$

$\therefore -x = -27$ $\quad\quad$ $\therefore x = 27$

Hence, the temperature at 7 p.m.is $27°$ C.

Problems

1. The following table shows the total distance covered by a walker at various times during a hike

Distance (km)	0	2.4	3.9	4.8	6.0	7.1	8.4
Time (hours.)	0	0.5	1	1.5	2	2.5	3

Find the average speed

(i) over the first hour

(ii) between 1.5 and 3 hours

(iii) over the last half hour

(iv) over the entire 3 hours.

2. The temperatures at various times on a certain day are shown in the following table.

Temperature	$2.8°$ C	$11.6°$ C	$14.8°$ C	x	$8.4°$ C
Time	7 a.m.	11 a.m.	3 p.m.	7 p.m.	11 p.m

(i) Find the average rate of change of temperature between 11a.m. and 3 p.m.

(ii) Find the average rate of change of temperature between 7 a.m. and 3 p.m.

(iii) Given that the average rate of change of temperature between7 p.m. and 11 p.m. is $-1.1°$ C per hour, find the temperature, x, at 7 p.m.

Answers

1. (i) 4.8 (km/h) $\quad$ (ii) 2.4 (km/h)

(iii) 2.6 (km/h) $\quad$ (iv) 2.8 (km/h).

2. (i) 0.8 (°C per hour) (ii) 1.5 (°C per hour) (iii) 12.8 (°C).

5.2 The derivative

1. Given the function $y = f(x)$, the instantaneous rate of change of f at the general point x is called **the derivative** of $y = f(x)$.

   The derivative is most commonly denoted by $\frac{dy}{dx}$, $f'(x)$, or y'.

   By definition,

   $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

   (the first-principles formula for the derivative).

2. **All derivatives can be calculated without using the first-principles formula.** Example 1 below indicates that using the formula can be quite tedious.

3. The **derivative represents the slope** (gradient) of a curve at any point on that curve.

4. The rule for finding the derivative of a power of x is sometimes called **the power rule**, i.e. if $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$, where n is a constant.

   The power rule applies for **any** constant *n,* including fractions and decimals.

   For instance, if $y = x^{7/5}$, then $\frac{dy}{dx} = \frac{7}{5}x^{7/5 - 1} = \frac{7}{5}x^{2/5}$.

   Note that, before differentiating, y must be in the form $y = x^n$, e.g.

   $y = \frac{1}{x^6}$ must be re-written as $y = x^{-6}$, before differentiating. Then

   $\frac{dy}{dx} = -6\,x^{-6-1} = -6\,x^{-7}$

5. The rule for finding the derivative of a constant multiple of a function is intuitively obvious:

   if $y = kf(x)$, then $\frac{dy}{dx} = k\frac{df}{dx}$, where k is a constant.

   For instance, if $y = 6x^{5/3}$, then $\frac{dy}{dx} = 6 \times \frac{5}{3}x^{5/3 - 1} = 10x^{2/3}$.

6. The rule for finding the derivative of a sum or difference of functions is also intuitively obvious:

   if $y = f(x) \pm g(x)$, then $\frac{dy}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$.

For instance, if $y = x^{5/3} - x^{1/4}$, then

$$\frac{dy}{dx} = \frac{5}{3}x^{5/3-1} - \frac{1}{4}x^{1/4-1} = \frac{5}{3}x^{2/3} - \frac{1}{4}x^{-3/4}.$$

7. The derivative of any constant function is 0, i.e. if $y = c$, then $\frac{dy}{dx} = 0$.

   Similarly, the derivative of any constant multiple of x is a constant, i.e. if $y = mx = mx^1$, then $\frac{dy}{dx} = m \times 1x^{1-1} = mx^0 = m$.

   More generally, for the straight line $y = mx + c$, these 2 results give $\frac{dy}{dx} = m + 0 = m$ (as expected, since m is the slope of $y = mx + c$).

8. The above rules can be used to differentiate any polynomial, e.g. for the most general quadratic $y = ax^2 + bx + c$,

$$\frac{dy}{dx} = a \times 2x^{2-1} + b + 0$$

$$\therefore \frac{dy}{dx} = 2ax + b.$$

9. An important application of rate of change is that of the **motion of an object in a straight line**. The conventional notation is as follows:

   ***s* is the displacement of the object;**

   ***t* is the time;**

   ***v* is the velocity;**

   ***a* is the acceleration.**

   Using this notation, $v = \frac{ds}{dt}$, and $a = \frac{dv}{dt}$.

   Hence, if the displacement is given, the velocity can be found by differentiation. Then, the acceleration can be found by differentiating the velocity.

Examples

1. Use the first-principles formula $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to find the derivative of the following functions

   (i) $y = f(x) = 4x - 7$ (ii) $y = f(x) = x^2$.

2. Find $\frac{dy}{dx}$ for

   (i) $y = 3\sqrt{x}$ (ii) $y = \sqrt{3x}$ (iii) $y = \frac{3}{\sqrt{x}}$

   (iv) $y = \frac{1}{3\sqrt{x}}$ (v) $y = x^4 - \frac{1}{x^4}$ (vi) $y = 4x^{1/6} + \pi^3$

(vii) $y = 4x^2 - 5x + 6$ (viii) $y = 2x^3 + 7x^2 + 9x - 1$.

3. Find the slope of the curve $y = 5x^2 + x - 3$ at:

   (i) the point $(-1, 1)$ (ii) the point $(2, 19)$.

4. It takes 12 minutes to fill an oil tank. The volume of oil V litres, after time t minutes, is given by

$$V = 24t - t^2, \quad \text{for } 0 \le t \le 12 \quad .$$

   (i) Find the instantaneous rate of change of volume when $t = 10$.

   (ii) Find the volume of oil in the tank when the instantaneous rate of change of volume is 23 litres/minute.

   (iii) Find the instantaneous rate of change of volume when the volume of oil in the tank is 44 litres.

5. The displacement s (metres) of an object at time t (seconds) is given by $s = -t^3 + 3t^2 + 6t$, for $t \ge 0$. Find

   (i) the velocity and acceleration

   (ii) the velocity after 2 seconds

   (iii) the acceleration after 7 seconds

   (iv) the velocity when the acceleration is 0

   (v) the acceleration when the velocity is -3 (metres/second).

1. (i) For $y = f(x) = 4x - 7$,

$$f(x+h) = 4(x+h) - 7 = 4x + 4h - 7.$$

$$\therefore f(x+h) - f(x) = 4x + 4h - 7 - (4x - 7)$$

$$\therefore f(x+h) - f(x) = 4x + 4h - 7 - 4x + 7 = 4h$$

$$\therefore \frac{f(x+h) - f(x)}{h} = \frac{4h}{h} = 4$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} 4 = 4.$$

i.e. $f'(x) = 4$.

Note: this agrees with the fact that the slope of the straight line r $y = f(x) = 4x - 7$ is 4 (constant slope for all values of x).

(ii) For $y = f(x) = x^2$,

$$f(x+h) = (x+h)^2 = x^2 + 2xh + h^2.$$

$$\therefore f(x+h) - f(x) = x^2 + 2xh + h^2 - x^2$$

$$\therefore f(x+h) - f(x) = 2xh + h^2$$

$$\therefore \frac{f(x+h)-f(x)}{h} = \frac{2xh+h^2}{h} = \frac{h(2x+h)}{h} = 2x+h$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}(2x+h) = 2x+0 = 2x$$

i.e. $f'(x) = 2x$.

Note that the limit is obtained by cancelling as far as is possible, then replacing h by 0.

2. (i) $y = 3\sqrt{x} \qquad \therefore y = 3x^{1/2}$

$$\therefore \frac{dy}{dx} = 3\times\frac{1}{2}x^{1/2-1} = \frac{3}{2}x^{-1/2}$$

(ii) $y = \sqrt{3x} \qquad \therefore y = \sqrt{3}\sqrt{x} = \sqrt{3}x^{1/2}$

$$\therefore \frac{dy}{dx} = \sqrt{3}\times\frac{1}{2}x^{1/2-1} = \frac{\sqrt{3}}{2}x^{-1/2}$$

(iii) $y = \dfrac{3}{\sqrt{x}} \qquad \therefore y = \dfrac{3}{x^{1/2}} = 3x^{-1/2}$

$$\therefore \frac{dy}{dx} = 3\times\frac{-1}{2}x^{-1/2-1} = \frac{-3}{2}x^{-3/2}$$

(iv) $y = \dfrac{1}{3\sqrt{x}} \qquad \therefore y = \dfrac{1}{3x^{1/2}} = \dfrac{1}{3}x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{3}\times\left(\frac{-1}{2}\right)x^{-1/2-1} = \frac{-1}{6}x^{-3/2}$$

(v) $y = x^4 - \dfrac{1}{x^4} \qquad \therefore y = x^4 - x^{-4}$

$$\therefore \frac{dy}{dx} = 4x^{4-1} - (-4)x^{-4-1} = 4x^3 + 4x^{-5}$$

(vi) $y = 4x^{1/6} + \pi^3$

$$\therefore \frac{dy}{dx} = \left(4\times\frac{1}{6}\right)x^{1/6-1} + 0 = \frac{2}{3}x^{-5/6} \quad \text{(Note: } \pi^3 \text{ is a constant)}$$

(vii) $y = 4x^2 - 5x + 6$

$$\therefore \frac{dy}{dx} = 4\times 2x^{2-1} - 5 + 0 = 8x - 5$$

(viii) $y = 2x^3 + 7x^2 + 9x - 1$

$$\therefore \frac{dy}{dx} = 2\times 3x^{3-1} + 7\times 2x^{2-1} + 9 + 0 = 6x^2 + 14x + 9.$$

3. For $y = 5x^2 + x - 3$, $\qquad \dfrac{dy}{dx} = 5 \times 2x^{2-1} + 1 + 0 = 10x + 1$

Hence, the slope at a point is $10x + 1$.

(i) at the point $(-1, 1)$, i.e. when $x = -1$,

$\dfrac{dy}{dx} = (10)(-1) + 1 = -10 + 1 = -9$

(ii) at the point $(2, 19)$, i.e. when $x = 2$,

$\dfrac{dy}{dx} = 10 \times 2 + 1 = 20 + 1 = 21$.

Note that when $x = -1$, $y = 5x^2 + x - 3 = 5 - 1 - 3 = 1$, i.e. the point $(-1, 1)$ **is** a point on the curve.

Similarly, when $x = 2$, $y = 5 \times 4 + 2 - 3 = 20 + 2 - 3 = 19$, i.e. the point $(-2, 19)$ **is also** a point on the curve.

4. Given $V = 24t - t^2$, for $0 \le t \le 12$, the instantaneous rate of change of volume is given by $\dfrac{dV}{dt} = 24 - 2t$

(i) When $t = 10$, $\dfrac{dV}{dt} = 24 - 20 = 4$ (litres per minute).

(ii) The question is to find V when $\dfrac{dV}{dt} = 23$, i.e. when

$24 - 2t = 23 \qquad \therefore -2t = 23 - 24 = -1$

$\therefore t = \dfrac{-1}{-2} = \dfrac{1}{2}$.

When $t = \dfrac{1}{2}$, $V = 24t - t^2 = 24 \times \dfrac{1}{2} - \left(\dfrac{1}{2}\right)^2 = 12 - \dfrac{1}{4} = \dfrac{47}{4}$,

i.e. $V = \dfrac{47}{4} = 11.75$ (litres).

(iii) The question is to find $\dfrac{dV}{dt}$ when $V = 44$, i.e. when

$24t - t^2 = 44 \qquad \therefore 0 = t^2 - 24t + 44 \qquad$ (a quadratic)

Factorising gives $t^2 - 24t + 44 = (t - 2)(t - 22) = 0$,
with solutions $t = 2$ and $t = 22$.

Since $0 \le t \le 12$, the only permissible solution is $t = 2$,

i.e. 2 (minutes).

When $t = 2$, $\dfrac{dV}{dt} = 24 - 4 = 20$ (litres per minute).

5. Given $s = -t^3 + 3t^2 + 6t$, for $t \geq 0$,

(i) the velocity is given by $v = \dfrac{ds}{dt} = -3t^2 + 6t + 6$ (m/sec), and

the acceleration by $a = \dfrac{dv}{dt} = -6t + 6$ (m/sec/sec).

(ii) When $t = 2$, $v = -3 \times 4 + 6 \times 2 + 6 = -12 + 12 + 6 = 6$, i.e. the velocity is 6 (m/sec).

(iii) When $t = 7$, $a = -6 \times 7 + 6 = -42 + 6 = -36$, i.e. the acceleration is -36 (m/sec/sec).

(iv) When the acceleration is 0, $-6t + 6 = 0$

$6t = 6 \qquad \therefore t = 1$.

When $t = 1$, $v = -3 + 6 + 6 = 9$ (m/sec).

(v) When the velocity is -3, $-3t^2 + 6t + 6 = -3$

$\therefore -3t^2 + 6t + 9 = 0$. Dividing by -3 gives

$t^2 - 2t - 3 = 0$ Factorising gives

$t^2 - 2t - 3 = (t-3)(t+1) = 0$, which has solutions

$t = 3$ and $t = -1$.

Since $t \geq 0$, the only permissible solution is $t = 3$.

When $t = 3$, $a = -6 \times 3 + 6 = -18 + 6 = -12$ (m/sec/sec).

Problems

1. Find $\dfrac{dy}{dx}$ for

(i) $y = 2\sqrt{x}$ (ii) $y = \sqrt{2x}$ (iii) $y = \dfrac{2}{\sqrt{x}}$

(iv) $y = \dfrac{1}{2\sqrt{x}}$ (v) $y = x^3 - \dfrac{1}{x^2}$ (vi) $y = 6x^{2/3} + \sqrt{3}$

(vii) $y = 7x^2 - 9x + 8$ (viii) $y = 3x^3 - 7x^2 + 10x - 1$.

2. Find the slope of the curve $y = 7x^2 - 4x - 6$ at

(i) the point $(-1, 5)$ (ii) the point $(2, 14)$.

3. It takes 10 minutes to fill a water tank. The volume of water V litres, after time t minutes, is given by

$$V = 20t - t^2, \quad \text{for } 0 \leq t \leq 10.$$

(i) Find the instantaneous rate of change of volume when $t = 9$.

(ii) Find the volume of water in the tank when the instantaneous rate of change of volume is 14 litres/minute.

(iii) Find the instantaneous rate of change of volume when the volume of water in the tank is 84 litres.

4. The displacement s (metres) of an object at time t (seconds) is given by $s = -2t^3 + 6t^2 + 18t$, for $t \geq 0$. Find

(i) the velocity and acceleration

(ii) the velocity after 2 seconds

(iii) the acceleration after 3 seconds

(iv) the velocity when the acceleration is 0

(v) the acceleration when the velocity is -30 (metres/second).

Answers

1. (i) $y = 2\sqrt{x} \quad \therefore y = 2x^{1/2}$

$$\therefore \frac{dy}{dx} = 2 \times \frac{1}{2}x^{1/2-1} = x^{-1/2}$$

(ii) $y = \sqrt{2x} \quad \therefore y = \sqrt{2}\sqrt{x} = \sqrt{2}x^{1/2}$

$$\therefore \frac{dy}{dx} = \sqrt{2} \times \frac{1}{2}x^{1/2-1} = \frac{\sqrt{2}}{2}x^{-1/2} \; (= \frac{1}{\sqrt{2}}x^{1/2})$$

(iii) $y = \dfrac{2}{\sqrt{x}} \quad \therefore y = \dfrac{2}{x^{1/2}} = 2x^{-1/2}$

$$\therefore \frac{dy}{dx} = 2 \times \frac{-1}{2}x^{-1/2-1} = -x^{-3/2}$$

(iv) $y = \dfrac{1}{2\sqrt{x}} \quad \therefore y = \dfrac{1}{2x^{1/2}} = \dfrac{1}{2}x^{-1/2}$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \left(\frac{-1}{2}\right)x^{-1/2-1} = \frac{-1}{4}x^{-3/2}$$

(v) $y = x^3 - \dfrac{1}{x^2} \quad \therefore y = x^3 - x^{-2}$

$$\therefore \frac{dy}{dx} = 3x^{3-1} - (-2)x^{-2-1} = 3x^2 + 2x^{-3}$$

(vi) $y = 6x^{2/3} + \sqrt{3}$

$$\therefore \frac{dy}{dx} = \left(6 \times \frac{2}{3}\right)x^{2/3-1} + 0 = 4x^{-1/3}$$

(vii) $y = 7x^2 - 9x + 8$

$$\therefore \frac{dy}{dx} = 2 \times 7x^{2-1} - 9 + 0 = 14x - 9$$

(viii) $y = 3x^3 - 7x^2 + 10x - 1$

$$\therefore \frac{dy}{dx} = 3 \times 3x^{3-1} - 7 \times 2x^{2-1} + 10 - 0 = 9x^2 - 14x + 10 .$$

2. As $y = 7x^2 - 4x - 6$, $\frac{dy}{dx} = 14x - 4$

(i) At the point $(-1, 5)$, $\frac{dy}{dx} = -14 - 4 = -18$.

(ii) At the point $(2, 14)$, $\frac{dy}{dx} = 28 - 4 = 24$.

3. As $V = 20t - t^2$, $\frac{dV}{dt} = 20 - 2t$

(i) When $t = 9$, $\frac{dV}{dt} = 20 - 18 = 2$ (litres per minute).

(ii) When $\frac{dV}{dt} = 14$, i.e. when $20 - 2t = 14$

$\therefore -2t = 14 - 20 = -6$ $\qquad \therefore t = 3$.

When $t = 3$, $V = 20t - t^2 = 20 \times 3 - 3^2 = 60 - 9 = 51$ (litres)

(iii) When $V = 84$, i.e. when

$20t - t^2 = 84$ $\qquad \therefore 0 = t^2 - 20t + 84$

As $t^2 - 20t + 84 = (t-6)(t-14) = 0$, the solutions are $t = 6$ and $t = 14$.

Since $0 \le t \le 10$, the only permissible solution is $t = 6$, i.e. 6 (minutes).

When $t = 6$, $\frac{dV}{dt} = 20 - 12 = 8$ (litres per minute).

4. As $s = -2t^3 + 6t^2 + 18t$,

(i) the velocity $v = \frac{ds}{dt} = -6t^2 + 12t + 18$ (m/sec), and

the acceleration $a = \frac{dv}{dt} = -12t + 12$ (m/sec/sec).

(ii) When $t = 2$, the velocity is $v = -6 \times 4 + 12 \times 2 + 18 = -24 + 24 + 18 = 18$ (m/sec).

(iii) When $t = 3$, the acceleration is $a = -12 \times 3 + 12 = -36 + 12 = -24$ (m/sec/sec).

(iv) When $a = 0$, $-12t + 12 = 0$

$12t = 12 \qquad \therefore t = 1$.

When $t = 1$, $v = -6 + 12 + 18 = 24$ (m/sec).

(v) When $v = -30$, $-6t^2 + 12t + 18 = -30$, i.e.

$$-6t^2 + 12t + 48 = 0$$

$\therefore t^2 - 2t - 8 = 0 \qquad \therefore (t-4)(t+2) = 0$

$t = 4$ and $t = -2$.

Since $t \geq 0$, the only permissible solution is $t = 4$.

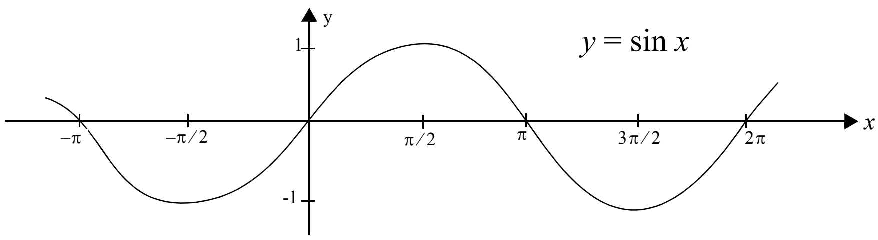
When $t = 4$, $a = -12 \times 4 + 12 = -48 + 12 = -36$ (m/sec/sec).

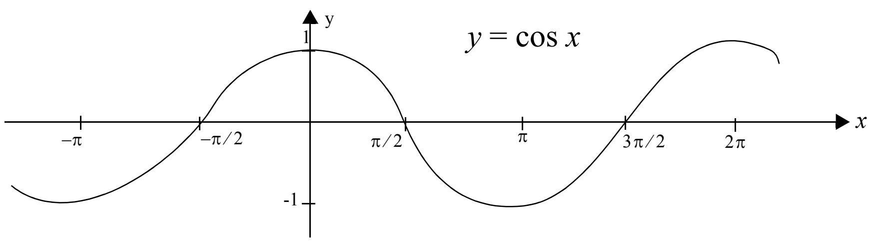
5.3 Trigonometric, exponential and logarithmic functions

1. Given the graph of a function $y = f(x)$, the graph of its derivative $\dfrac{dy}{dx}$ can be drawn (roughly). Although this is not a rigorous method of finding the derivative, it does give an illustration of the possible prescription for $\dfrac{dy}{dx}$.

2. The graphs of the two trigonometric functions $y = \sin x$ and $y = \cos x$ are reproduced below. From the graphs, two results for derivatives follow: If $y = \sin x$, $\qquad \dfrac{dy}{dx} = \cos x$.

   If $y = \cos x$, $\qquad \dfrac{dy}{dx} = -\sin x$.

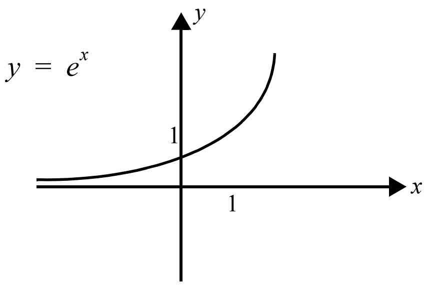


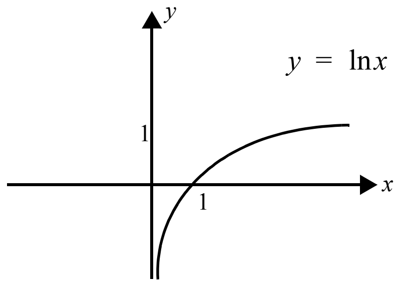


3. The graphs of the two functions $y = e^x$ and $y = \ln x$ are reproduced below. From the graphs, two results for derivatives follow:

$$\text{If } y = e^x, \quad \frac{dy}{dx} = e^x;$$

$$\text{If } y = \ln x, \quad \frac{dy}{dx} = \frac{1}{x}.$$





4. The results for the derivatives of trigonometric, exponential and logarithmic functions can be generalised as follows:
For any constant k,

$$\text{if } y = \sin kx, \qquad \frac{dy}{dx} = k\cos kx;$$

$$\text{if } y = \cos kx, \qquad \frac{dy}{dx} = -k\sin kx;$$

$$\text{if } y = e^{kx}, \qquad \frac{dy}{dx} = ke^{kx};$$

$$\text{if } y = \ln x, \qquad \frac{dy}{dx} = \frac{1}{x}.$$

Note that the angle (kx)appearing in a trigonometric function is preserved after differentiation. Similarly, the power (kx) appearing in an exponential function is also preserved.

5. In summary, the 5 major results for the derivatives of the commonly used functions are shown in the table below.

Function: y	**Derivative:** $\frac{dy}{dx}$
x^n	nx^{n-1}
$\sin kx$	$k\cos kx$
$\cos kx$	$-k\sin kx$
e^{kx}	ke^{kx}
$\ln x$	$\frac{1}{x}$

Examples

1. Find $\frac{dy}{dx}$ for

(i) $y = \sin(3x)$ (ii) $y = \cos(5x)$

(iii) $y = 4e^{2x}$ (iv) $y = 8\sin\left(\frac{x}{2}\right)$

(v) $y = 4\ln x + 7e^{-2x}$ (vi) $y = \ln(5x)$

(vii) $y = \ln(x^4)$ (viii) $y = e^{3x} + 3e^x - \frac{4}{e^{2x}}$

(ix) $y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{x}{2}\right)$ (x) $y = x^3 + 3e^x$.

1. (i) $y = \sin(3x)$

$$\therefore \frac{dy}{dx} = 3\cos(3x) \qquad (k = 3)$$

(ii) $y = \cos(5x)$

$$\therefore \frac{dy}{dx} = -5\sin(5x) \qquad (k = 5)$$

(iii) $y = 4e^{2x}$

$$\therefore \frac{dy}{dx} = 4 \times 2e^{2x} = 8e^{2x} \qquad (k = 2)$$

(iv) $y = 8\sin\left(\frac{x}{2}\right)$ (Note that $\frac{x}{2} = \frac{1}{2}x$)

$$\therefore \frac{dy}{dx} = 8 \times \frac{1}{2}\cos\left(\frac{x}{2}\right) = 4\cos\left(\frac{x}{2}\right) \qquad (k = \tfrac{1}{2})$$

(v) $y = 4\ln x + 7e^{-2x}$

$$\therefore \frac{dy}{dx} = 4 \times \frac{1}{x} + 7 \times (-2e^{-2x}) \qquad (k = -2)$$

$$\therefore \frac{dy}{dx} = \frac{4}{x} - 14e^{-2x}$$

(vi) $y = \ln(5x)$ Using the log rules, $y = \ln 5 + \ln x$.

$$\therefore \frac{dy}{dx} = 0 + \frac{1}{x} = \frac{1}{x}$$ (Note that $\ln 5$ is a constant)

(vii) $y = \ln(x^4)$ Using the log rules, $y = 4\ln x$.

$$\therefore \frac{dy}{dx} = 4 \times \frac{1}{x} = \frac{4}{x}.$$

(viii) $y = e^{3x} + 3e^{x} - \frac{4}{e^{2x}}$ $\therefore y = e^{3x} + 3e^{x} - 4e^{-2x}$

$$\therefore \frac{dy}{dx} = 3e^{3x} + 3e^{x} - 4 \times (-2e^{-2x}) = 3e^{3x} + 3e^{x} + 8e^{-2x}$$

($k = 3$, $k = 1$, and $k = -2$, respectively)

(ix) $y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{x}{2}\right)$ (Note that $\frac{x}{2} = \frac{1}{2}x$)

$$y = \frac{1}{4}\cos(8x) - 3\sin\left(\frac{1}{2}x\right)$$

$$\therefore \frac{dy}{dx} = \frac{1}{4} \times (-8)\sin(8x) - 3 \times \frac{1}{2}\cos\left(\frac{1}{2}x\right)$$

($k = 8$, and $k = \frac{1}{2}$, respectively)

$$\therefore \frac{dy}{dx} = -2\sin(8x) - \frac{3}{2}\cos\left(\frac{x}{2}\right)$$

(x) $y = x^3 + 3e^{x}$

$$\therefore \frac{dy}{dx} = 3x^2 + 3e^{x} \qquad (k = 1)$$

Problems

1. Find $\dfrac{dy}{dx}$ for:

(i) $y = \sin(6x)$ (ii) $y = \cos(2x)$

(iii) $y = 5e^{3x}$ (iv) $y = 6\sin\left(\dfrac{x}{3}\right)$

(v) $y = 5\ln x - 3e^{-4x}$ (vi) $y = \ln(2x)$

(vii) $y = \ln(x^2)$ (viii) $y = 2e^{5x} + 3e^{4x} - \dfrac{7}{e^x}$

(ix) $y = \dfrac{2}{3}\cos(9x) - 5\sin\left(\dfrac{x}{4}\right)$ (x) $y = x^4 + 3e^{4x}$.

Answers

1. (i) $y = \sin(6x)$ $\therefore \dfrac{dy}{dx} = 6\cos(6x)$ $(k = 6)$

(ii) $y = \cos(2x)$ $\therefore \dfrac{dy}{dx} = -2\sin(2x)$ $(k = 2)$

(iii) $y = 5e^{3x}$ $\therefore \dfrac{dy}{dx} = 15e^{3x}$ $(k = 3)$

(iv) $y = 6\sin\left(\dfrac{x}{3}\right)$ $\therefore \dfrac{dy}{dx} = 2\cos\left(\dfrac{x}{3}\right)$ $(k = \dfrac{1}{3})$

(v) $y = 5\ln x - 3e^{-4x}$ $\therefore \dfrac{dy}{dx} = \dfrac{5}{x} + 12e^{-4x}$ $(k = -4)$

(vi) $y = \ln(2x)$ $\therefore y = \ln 2 + \ln x$ $\therefore \dfrac{dy}{dx} = \dfrac{1}{x}$

(vii) $y = \ln(x^2)$ $\therefore y = 2\ln x$ $\therefore \dfrac{dy}{dx} = \dfrac{2}{x}$

(viii) $y = 2e^{5x} + 3e^{4x} - \dfrac{7}{e^x}$ $\therefore y = 2e^{5x} + 3e^{4x} - 7e^{-x}$

$\therefore \dfrac{dy}{dx} = 10e^{5x} + 12e^{4x} + 7e^{-x}$

$(k = 5$, $k = 4$, and $k = -1$, respectively)

(ix) $y = \dfrac{2}{3}\cos(9x) - 5\sin\left(\dfrac{x}{4}\right)$ $\therefore \dfrac{dy}{dx} = -6\sin(9x) - \dfrac{5}{4}\cos\left(\dfrac{x}{4}\right)$

$(k = 9$, and $k = \dfrac{1}{4}$, respectively)

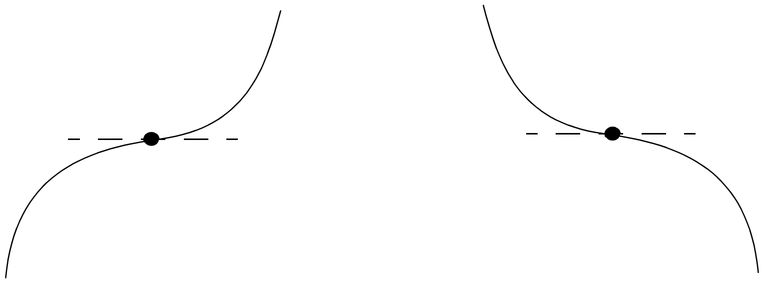
(x) $y = x^4 + 3e^{4x}$ $\therefore \dfrac{dy}{dx} = 4x^3 + 12e^{4x}$ $(k = 4)$

5.4 Maxima and minima

1. The points on a curve $y = f(x)$ at which the slope (gradient) is 0 are called **stationary points (or critical points)**. In mathematical terms, the definition of a stationary point is as follows:

   For the function $y = f(x)$, if $\frac{dy}{dx} = 0$ when $x = c$, then $x = c$ is a stationary point.

2. A stationary point can be either a **local maximum**, a **local minimum**, or a **horizontal point of inflection**. The two types of horizontal point of inflection are reproduced below.



   Note: 'horizontal' means the slope is 0, and 'inflection' means 'kink'.

3. To find all stationary points for a given function $y = f(x)$:

   I Find $\frac{dy}{dx}$;

   II Solve the equation $\frac{dy}{dx} = 0$ to obtain the x co-ordinates of each stationary point;

   III Substitute these x co-ordinates into $y = f(x)$ to find the corresponding y co-ordinates of each stationary point.

   For instance, if $y = x^2 - 6x$, then $\frac{dy}{dx} = 2x - 6$.

   Hence, for stationary points: $2x - 6 = 0$, $\quad \therefore 2x = 6$, so $x = 3$.

   Now, when $x = 3$, $y = 3^2 - 6 \times 3 = 9 - 18 = -9$,

   i.e. $(3, -9)$ is the (only) stationary point.

4. A stationary point can be classified as a local maximum, a local minimum, or a horizontal point of inflection using the following **First Derivative Test**.

   The stationary point at $x = c$ is:

   (a) a local maximum if, for $\begin{cases} x < c, & f'(x) > 0 \\ x > c, & f'(x) < 0 \end{cases}$;

(b) a local minimum if, for $\begin{cases} x < c, & f'(x) < 0 \\ x > c, & f'(x) > 0 \end{cases}$;

(c) a horizontal point of inflection otherwise.

Using the sign pattern notation for $f\ '(x)$ (not $f(x)$)

a local maximum has the pattern +, 0, -

a local minimum has the pattern -, 0, +

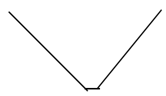
a horizontal point of inflection has the pattern -, 0, - or +, 0, +

For instance, the function $y = x^2 - 6x$ has $\frac{dy}{dx} = 2x - 6$. There is only one stationary point, at $(3, -9)$ (see note 3 above). This stationary point can be classified by investigating the sign of $\frac{dy}{dx}$ (i.e. $f\ '(x)$) on either side of $x = 3$.

Choosing, say, $x = 2$, $\frac{dy}{dx} = 4 - 6 = -2 < 0$.

Choosing, say, $x = 4$, $\frac{dy}{dx} = 8 - 6 = 2 > 0$.

So, the sign pattern of $f\ '(x)$ near $x = 3$ is -, 0, +, and the point $(3, -9)$ is a local minimum. Alternatively, from the above analysis of the sign of $f\ '(x)$, the following (rough) diagram can be drawn.



This also shows that $(3, -9)$ is a local minimum.

Examples

1. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, a local minimum, or a horizontal point of inflection.

   (i) $y = -3x^2 + 12x + 2$

   (ii) $y = x^3 - 6x^2 + 12x + 9$

   (iii) $y = 3x^3 - 9x^2 + 1$

1. (i) $y = -3x^2 + 12x + 2$

$$\therefore \frac{dy}{dx} = -6x + 12.$$

For stationary points, $-6x + 12 = 0$ $\therefore -6x = -12$

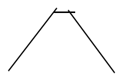
$$\therefore x = \frac{-12}{-6} = 2.$$

When $x = 2$, $y = -3 \times 4 + 12 \times 2 + 2 = -12 + 24 + 2 = 14$.

So, the one stationary point is $(2, 14)$.

Now, for $x < 2$, e.g. $x = 0$, $\frac{dy}{dx} = 0 + 12 = 12 > 0$.

Also, for $x > 2$, e.g. $x = 3$, $\frac{dy}{dx} = -18 + 12 = -6 < 0$.

So, near $x = 2$, the curve looks like: 

$\therefore (2, 14)$ is a local maximum.

(ii) $y = x^3 - 6x^2 + 12x + 9$

$$\therefore \frac{dy}{dx} = 3x^2 - 12x + 12$$

For stationary points, $3x^2 - 12x + 12 = 0$. Dividing by 3,

$x^2 - 4x + 4 = 0$. Factorising gives

$$x^2 - 4x + 4 = (x-2)(x-2) = 0$$

$$\therefore x = 2$$

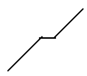
When $x = 2$, $y = 8 - 6 \times 4 + 12 \times 2 + 9 = 8 - 24 + 24 + 9 = 17$

So, the one stationary point is $(2, 17)$.

Now, for $x < 2$, e.g. $x = 0$, $\frac{dy}{dx} = 0 - 0 + 12 = 12 > 0$.

Also, for $x > 2$, e.g. $x = 3$,

$$\frac{dy}{dx} = 3 \times 9 - 12 \times 3 + 12 = 27 - 36 + 12 = 3 > 0.$$

So, near $x = 2$, the curve looks like: 

$\therefore (2, 17)$ is a horizontal point of inflection.

Note: as $\frac{dy}{dx} = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2$

(a perfect square), the slope cannot be negative, so $x = 2$ has to be a horizontal point of inflection.

(iii) $y = 3x^3 - 9x^2 + 1$

$$\therefore \frac{dy}{dx} = 9x^2 - 18x$$

For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

$x^2 - 2x = 0$. Factorising gives

$x^2 - 2x = x(x-2) = 0$

$\therefore x = 0$, and $x = 2$ (2 stationary points)

When $x = 0$, $y = 0 - 0 + 1 = 1$

So, one stationary point is $(0, 1)$.

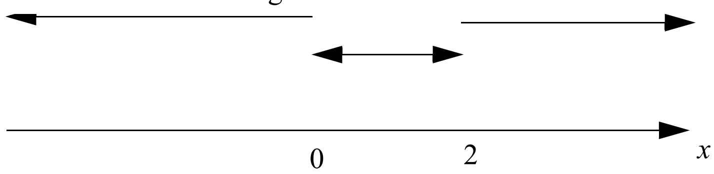
When $x = 2$, $y = 3 \times 8 - 9 \times 4 + 1 = 24 - 36 + 1 = -11$

So, the other stationary point is $(2, -11)$.

It is necessary to find the sign of $f'(x)$ in 3 separate intervals, i.e.

$x < 0$; $0 < x < 2$; $x > 2$,

as shown in the diagram below:



So, for $x < 0$, e.g. $x = -1$,

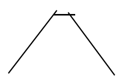
$$\frac{dy}{dx} = 9 \times 1 - 18 \times (-1) = 9 + 18 = 27 > 0$$

For $0 < x < 2$, e.g. $x = 1$,

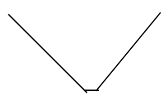
$$\frac{dy}{dx} = 9 \times 1 - 18 \times 1 = 9 - 18 = -9 < 0$$

Also, for $x > 2$, e.g. $x = 3$,

$$\frac{dy}{dx} = 9 \times 9 - 18 \times 3 = 81 - 54 = 27 > 0.$$

So, near $x = 0$, the curve looks like: 

$\therefore (0, 1)$ is a local maximum.

Also, near $x = 2$, the curve looks like: 

$\therefore (2, -11)$ is a local minimum.

Problems

1. For each of the following functions find the x and y co-ordinates of the stationary points. Then classify each of the points as either a local maximum, a local minimum, or a horizontal point of inflection.

   (i) $y = 5x^2 - 20x + 9$

   (ii) $y = 2x^3 - 9x^2 + 12x + 1$

   (iii) $y = x^3 + 3x^2 + 3x + 1$

Answers

1. (i) $y = 5x^2 - 20x + 9 \quad \therefore \frac{dy}{dx} = 10x - 20 = 10(x-2)$

   The only stationary point is $(2, -11)$, which is a local minimum.

   (ii) $y = 2x^3 - 9x^2 + 12x + 1 \quad \therefore \frac{dy}{dx} = 6x^2 - 18x + 12$

   $$\therefore \frac{dy}{dx} = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

   There are two stationary points; $(1, 6)$ is a local maximum and $(2, 5)$ is a local minimum.

   (iii) $y = x^3 + 3x^2 + 3x + 1 \quad \therefore \frac{dy}{dx} = 3x^2 + 6x + 3$

$$\therefore \frac{dy}{dx} = 3(x^2 + 2x + 1) = 3(x+1)^2$$

The only stationary point is $(-1, 0)$, which is a horizontal point of inflection.

5.5 Graph sketching

1. The graphs of straight lines ($y = mx + c$), and parabolas ($y = ax^2 + bx + c$) can be sketched using simple methods. For polynomials of degree greater than 2, stationary points can be used as a valuable aid in sketching.
2. **To sketch a polynomial of degree greater than 2**, e.g. the cubic $y = ax^3 + bx^2 + cx + d$, the following three criteria should be examined:

   I Find the x and y-intercepts;

   II Find and classify the stationary points;

   III Sketch, labelling all intercepts, and stationary points.

Examples

1. For $y = 3x^3 - 9x^2$
   (i) find the x and y-intercepts
   (ii) find and classify the stationary points
   (iii) sketch, labelling all intercepts, and stationary points.

1. $y = 3x^3 - 9x^2$

   (i) When $x = 0$, $y = 0 - 0 = 0$.

   When $y = 0$, $3x^3 - 9x^2 = 0$

   Factorising gives $3x^3 - 9x^2 = 3x^2(x-3) = 0$,

   with solutions $x = 0$ and $x = 3$.

   So, the intercepts are $(0, 0)$, and $(3, 0)$.

   (ii) Now, $\dfrac{dy}{dx} = 9x^2 - 18x$

   For stationary points, $9x^2 - 18x = 0$. Dividing by 9,

   $x^2 - 2x = 0$. Factorising gives

   $x^2 - 2x = x(x-2) = 0$

   $\therefore x = 0$, and $x = 2$ (2 stationary points)

   When $x = 0$, $y = 0$ (an intercept, from (i))

   So, one stationary point is $(0, 0)$.

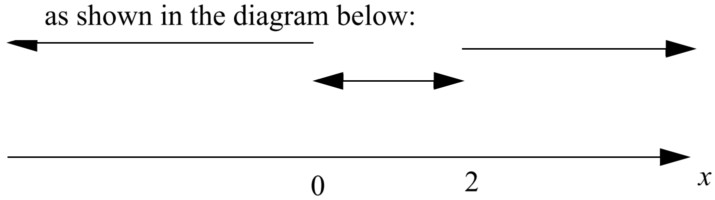
   When $x = 2$, $y = 3 \times 8 - 9 \times 4 = 24 - 36 = -12$

   So, the other stationary point is $(2, -12)$.

   It is necessary to find the sign of $f\,'(x)$ in 3 separate intervals, i.e.

   $x < 0$; $0 < x < 2$; $x > 2$,

   as shown in the diagram below:



   So, for $x < 0$, e.g. $x = -1$,

   $$\frac{dy}{dx} = 9 \times 1 - 18 \times (-1) = 9 + 18 = 27 > 0$$

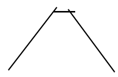
   For $0 < x < 2$, e.g. $x = 1$,

$$\frac{dy}{dx} = 9 \times 1 - 18 \times 1 = 9 - 18 = -9 < 0$$

Also, for $x > 2$, e.g. $x = 3$,

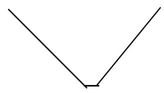
$$\frac{dy}{dx} = 9 \times 9 - 18 \times 3 = 81 - 54 = 27 > 0 .$$

So, near $x = 0$, the curve looks like:



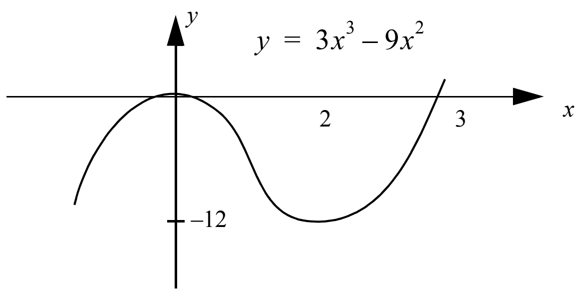
$\therefore (0, 0)$ is a local maximum.

Also, near $x = 2$, the curve looks like:



$\therefore (2, -12)$ is a local minimum.

(iii) The curve is sketched below.



Problems

1. For $y = (x-1)^3$
   (i) find the x and y-intercepts
   (ii) find and classify the stationary points
   (iii) sketch, labelling all intercepts, and stationary points.

$$y = (x-1)^3$$

Answers

1. $y = (x-1)^3$

(i) When $x = 0$, $y = (-1)^3 = -1$.

When $y = 0$, $(x-1)^3 = 0$

with one solution $x = 1$.

So, the intercepts are $(0, -1)$, and $(1, 0)$.

(ii) Now, $y = (x-1)^3 = (x-1)(x-1)^2 = (x-1)(x^2-2x+1)$

$\therefore y = x(x^2-2x+1)-(x^2-2x+1)$

$\therefore y = x^3-2x^2+x-x^2+2x-1$

$\therefore y = x^3-3x^2+3x-1$

$\therefore \dfrac{dy}{dx} = 3x^2-6x+3$

For stationary points, $3x^2-6x+3 = 0$. Dividing by 3,

$x^2-2x+1 = 0$. Factorising gives

$x^2-2x+1 = (x-1)(x-1) = 0$

$\therefore x = 1$. (1 stationary point)

When $x = 1$, $y = 0$ (an intercept, from (i))

So, the only stationary point is $(1, 0)$.

Now, since $\dfrac{dy}{dx} = 3x^2-6x+3 = 3(x^2-2x+1) = 3(x-1)^2$ can never be negative, $(1, 0)$ is a horizontal point of inflection.

The curve has positive slope (except when $x = 1$, where the slope is zero).

(iii) The curve is sketched below.