SIT190 - WEEK 6

Overview

- Pythagoras theorem
- Trigonometric Ratios
- Special triangles
- Unit circle
- Solving trigonometric problems

Continuing the Quest

In the next two weeks, your team will be travelling through Ancient Greece.
The artefact that you seek is some missing fragments of the Antikythera mechanism.

Image: Antikythera Mechanism by ACM Digital Library `https://dl.acm.org/cms/attachment/0baf57b5-d289-49db-a2d1-df4edc621892/f1.jpg`

Continuing the Quest

- Antikythera is ∼2000 year old computer (ca 100BCE)
- Divers found the wreck of ship off the Island of Antikythera in 1901
- Amongst a massive amount of treasure were pieces of a mechanical computer
- Astronomical device to track sun, moon and paths of planets
- View the story: `https://youtu.be/UpLcnAIpVRA`
- Reconstructed: `https://youtu.be/RLPVCJjTNgk`

Image: Antikythera Mechanism by ACM Digital Library `https://dl.acm.org/cms/attachment/0baf57b5-d289-49db-a2d1-df4edc621892/f1.jpg`

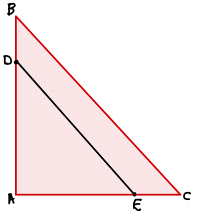
Ancient Greece

Mathematics in Ancient Greece was likely influenced by Egyptian mathematics (Thales and Pythagoras visited there) and the kingdom of Lydia provided the opportunity of trade of goods and ideas between Mesopotamia and Greek Ionian settlements.
There are no mathematical records of many achievements attributed to people such as Pythagoras and Thales that date back to their time. Much is based on later writings that attribute results to them.

Thales of Miletus ca 624-548BCE

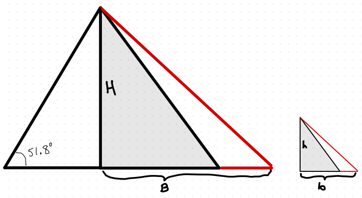
As you begin your journey in Greece, you meet an an old man name Thales:

- Tradition says that Thales predicted a solar eclipse in 585BCE
- One of Seven Wise Men
- Similar triangles: If a line is drawn parallel to a side of a triangle and intersecting the other 2 sides, then smaller and larger triangles are congruent.
  - Corresponding angles are the same
  - Ratios of side lengths are the same ($\frac{BA}{DA} = \frac{CA}{EA} = \frac{BC}{DE}$)



Thales of Miletus

Thales *"succeeded in measuring the height of the pyramids by observing the length of the shadow at the moment when a man's shadow is equal to his own height."*
(Attributed to Hieronymus (4th century BCE) by Diogenes (2nd century AD))



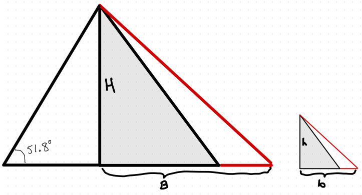
Reference: `http://www.jstor.com/stable/2690810`

Thales of Miletus

How did Thales do this (if indeed he really did)?

- The sun's rays, rod (man) and shadow make a right angled triangle.
- The sun's rays, the shadow, and the height H of the pyramid make another right angled triangle.
- Two triangles are similar
- $\frac{H}{h} = \frac{B}{b} \Rightarrow H = \frac{Bh}{b}$.

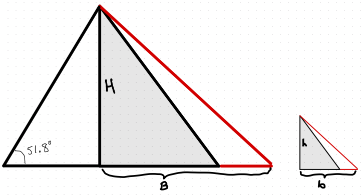
The *similar triangles*



Reference: `http://www.jstor.com/stable/2690810`

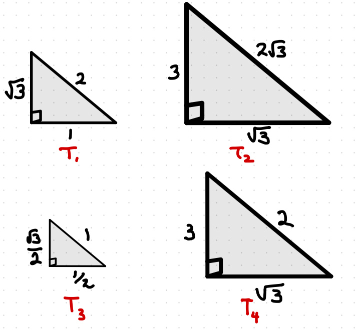
Give it a go

Suppose $B = 20m$, $h = 10m$ and $b = 2m$.
What would the height of the Pyramid be?



The First Challenge

One property of similar triangles is the ratio of the corresponding sides is the same. Can your band identify which of these triangles are similar triangles?



The First Challenge

Now consider the angles in two similar triangles.

- Create a right-angled triangle A by cutting a 2cm x 2cm square in half along the diagonal.
- Create a right-angled triangle B by cutting a 4cm x 4cm square in half along the diagonal.
- Compare the ratios of the corresponding sides to confirm that these are similar triangles.
- Compare the corresponding angles by moving the smaller triangle to the corners of the larger triangle.

Reflection

- If A and B are similar triangles, then
  - the ratios between the corresponding sides are the same ,and
  - the corresponding angles are the same.
- So if we rescale a triangle by increasing the lengths of all sides by a factor k what happens to the angles?

Head in the Clouds

- Many ancient mathematicians were motivated by astrology.
- One of Aesop's fables is based on the story that Thales was so absorbed in his study of the sky that he fell into a well.
- His rescuer derided him for trying to study the heavens and ignoring what was underfoot.

Although Thales has provided you with some insights into similar triangles, you recognise that you need additional information in this quest and so continue are your way. On this part of the quest you need to gather skills in trigonometry.

Trignometry

- Triangle measurement - *trigonometry*: trigonon (triangle) and metria (measure) in Greek.
- Measuring angles
- Measuring the lengths of sides of triangles
- Trigonometric functions - relations between angles and ratio of side-lengths of triangles.

Pythagoras ca 571-497 BCE

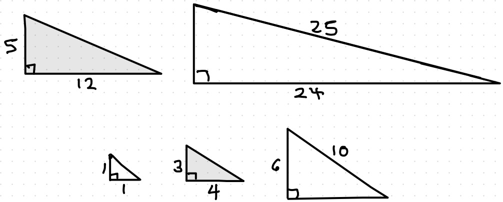
Your band recalls a well-known theorem called *Pythagoras Theorem*. This theorem states that any right angled triangle with sides of lengths a, b and c where c is the length of the hypotenuse satisfies the following formula:

$$a^2 + b^2 = c^2.$$

Although this theorem is attributed to Pythagoras, there is evidence that it was known much earlier (recall the Plimpton Tablet had Pythagorean Triples).

Pythagorean Triples

Your band seeks to infiltrate the Pythagorean secret sect as you believe that they have some information that may help you find the Antikythera fragments. Your band obtains some information that will prove useful.



Your band needs to find the lengths of the sides of these triangles. Do you notice anything different about one of these triangles?

Reflection

- Which side is the hypotenuse?
- There is some interesting information on Pythagorean Triples `https://mathworld.wolfram.com/PythagoreanTriple.html`
  - Side lengths are integers
  - Primitive Pythagorean Triples (side lengths are relatively prime)
  - Using matrices to obtain a new triple from an old triple.
- Similar triangles
- ‘Odd’ triangle

Pythagoras ca 571-497 BCE

Pythagoras was the leader of the secret sect called the Pythagoreans.

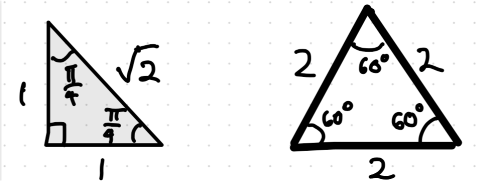
- Pythagoras Theorem is attributed to him but evident in many earlier cultures.
- Men and women were members of the sect `https://www.repository.cam.ac.uk/handle/1810/277046`
- Relationships between the frequencies of harmonious notes
- Polygonal numbers (eg Squares - $1, 4, 9, 16, \ldots$)
- Square of a number n is equal to the sum of the first n odd numbers (eg $5^2 = 1 + 3 + 5 + 7 + 9$)
- Sum of the angles in a triangle is 180 degrees
- Believed that universe could be described in terms of integers.

Pythagoras ca 571-497 BCE

One of the Pythagoreans, Hippasus discovered that $\sqrt{2}$ is not rational. This was unacceptable to the Pythagorean concept of numbers being integers or ratios of integers. Rumour says that he was drowned at sea for revealing his discovery.

Special Triangles

Your band has also uncovered the secret of irrational numbers, but unlike the Pythagorean you are not daunted by this knowledge. The $1, 1, \sqrt{2}$ triangle is one of two special triangles.



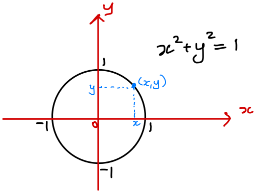
An equilateral triangle is one where all angles are 60^o and all sides have the same length. The other special triangle can be obtained by cutting the equilateral triangle with sides of length 2 exactly in half to obtain a right-angled triangle.

Reflection

- Your band has 2 special triangles - the $1, 1, \sqrt{2}$ triangle, and the one you have found by cutting the equilateral triangle in half.
  - Give the lengths of the sides of this right-angled triangle.
  - Give the angles of this right angled triangle.
- These triangles will be useful for solving trigonometric problems. But it will be useful to have the angles in both radians and degrees. Can you create a copy of each triangle with radians and another with degrees?

Pythagoreans

Your band manages to infiltrate the sect who are impressed that you have the knowledge of two irrational numbers that arise from triangles.



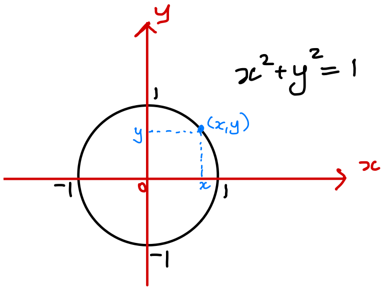
You discover that there is a link between triangles and circles ...

Hipparchus 190-120BCE

- First known table of chords called 'Of Lines Inside a Circle' (Hipparchus 140 BCE).
- Tables have not survived, but claim is that there were 12 books of tables of chords.
- Introduced the Babylonian measurement of angles in degrees.
- The chord of an angle is the line connecting two points on the circle separated by that angle.
- Tabulated chord(θ)$= 2\sin(\frac{\theta}{2})$ for every 7.5 degrees.

Circles and Triangles

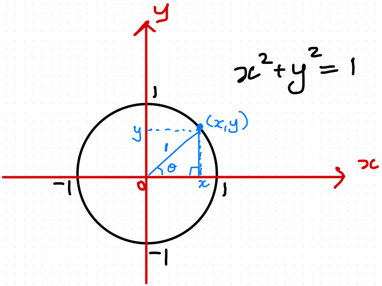
- The relation $x^2 + y^2 = 1$ gives a circle with centre at the origin and radius 1.
- It is not a function as some x values have more than one y value. For example, when $x = \frac{1}{2}$, $y = \pm\frac{\sqrt{3}}{2}$.
- We consider a point (x, y) on this circle.



Circles and Triangles

We can create a right-angled triangle as follows:

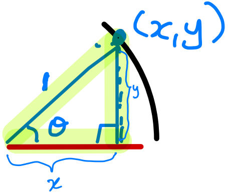
- The line connecting this point and the origin has length 1 (the radius of this circle).
- The angle between this line and the x axis we label θ.
- A second line connects our point (x, y) with the point $(x, 0)$. The length of this line is:
- A third line connects the origin $(0, 0)$ with the point $(x, 0)$. The length of this line is:



Circles and Triangles

The trigonometric functions relate the angle to the ratio between two sides of the triangle.

Function	Sides	
sine	opposite, hypotenuse	$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} =$
cosine	adjacent, hypotenuse	$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} =$
tangent	opposite, adjacent	$\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}} =$ $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$



Reflection

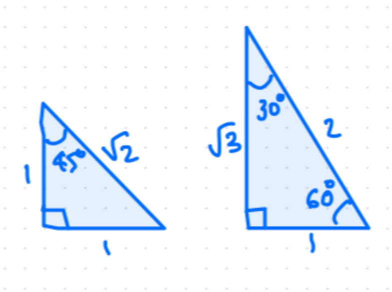
We use the mnemonic SOHCAHTOA.

- We have found functions that relate angles to the ratio between two sides of a right-angled triangle.
- What values would you need to find the length of the side of a right-angled triangle using one of these functions?
  - One of the triangle's angles that is not the right-angle and the length of a side?
  - The length of two sides of the triangle?
- What values would you need to find one of the triangle's angles that is not the right-angle using one of these functions?
  - One of the triangle's angles that is not the right-angle and the length of a side?
  - The length of two sides of the triangle?

Special Triangles

Using the special triangles, complete this table:

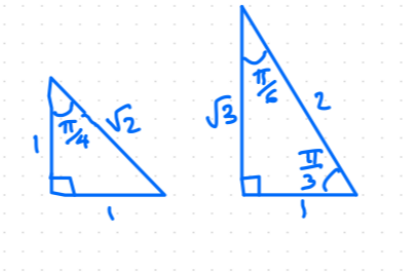
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
30^o			
45^o			
60^o			



Special Triangles

Is this table any harder to fill?

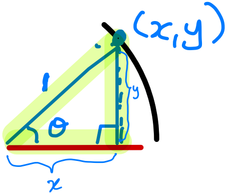
θ	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$\frac{\pi}{6}$			
$\frac{\pi}{4}$			
$\frac{\pi}{3}$			



Circles and Triangles

Any point (x, y) on the circle $x^2 + y^2 = 1$ can be represented as

$$(\cos(\theta), \sin(\theta)).$$



Applying Pythagoras Theorem to this triangle, and noting that $x = \sin(\theta)$ and $y = \cos(\theta)$ we obtain

$$\sin^2(\theta) + \cos^2(\theta) = 1.$$

Reflection

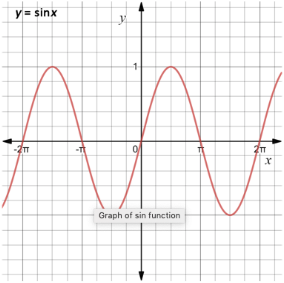
Note: that $\sin^2(\theta) + \cos^2(\theta) = 1$ means that we can have positive and negative evaluations. Consider the following cases:

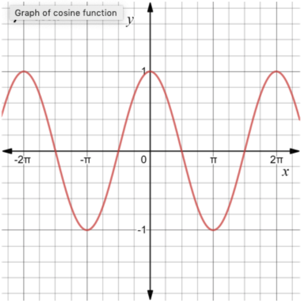
θ	$x = \cos(\theta)$	$y = \sin(\theta)$	$\sin^2(\theta) + \cos^2(\theta)$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	
$\frac{3\pi}{4}$	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	
$\frac{5\pi}{4}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	
$\frac{7\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	

We need to know which quadrant the angle is in to find the sign.

Graphs of trigonometric functions

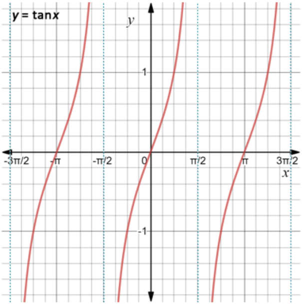
- For $\sin(\theta)$ and $\cos(\theta)$ the behaviour repeats for each interval of 2π
- Remember the triangle had hypotenuse of length 1, so the largest absolute value of these functions is 1.
- The other two sides (x and y) have lengths in $[-1, 1]$
- Can your band identify the places where the graphs below lie below the x-axis?





Graphs of trigonometric functions

- The function $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
- The domain of this function excludes places where $\cos(\theta) = ?$
- The domain is $\mathbb{R} \setminus \{\frac{(2n+1)\pi}{2} : n \in \mathbb{Z}\}$.
- There is no absolute value of this function, so its range is $(-\infty, \infty)$
- Can your band identify the places where this graph lies below the x-axis?



CAST

Can you identify the sign of these functions in these intervals:

	Quadrant 1 $(0, \frac{\pi}{2})$	Quadrant 2 $(\frac{\pi}{2}, \pi)$	Quadrant 3 $(\pi, \frac{3\pi}{2})$	Quadrant 4 $(\frac{3\pi}{2}, 2\pi)$
$\sin(\theta)$	$+$	$+$	$-$	$-$
$\cos(\theta)$				
$\tan(\theta)$				

Reflection

CAST and $\sin^2(\theta) + \cos^2(\theta) = 1$ are a powerful combination.

- If we have $\sin(\theta)$ we can find (up to sign) $\cos(\theta)$ using $\sin^2(\theta) + \cos^2(\theta) = 1$, and
- If we have $\cos(\theta)$ we can find (up to sign) $\sin(\theta)$ using $\sin^2(\theta) + \cos^2(\theta) = 1$.
- Then CAST can be used to give us the sign

An aside

We can find other interesting and useful properties of these functions when we observe the graphs. For example:

1. $\cos(\theta) = cos(-\theta)$
2. $\sin(\theta) = -\sin(-\theta)$

Solving Trigonometric expressions

The Pythagorean sect had a strict rule of secrecy, but your band are resourceful and can build on their knowledge to solve many problems.

Here are some tools that you have:

1. Pythagoras theorem: $a^2 + b^2 = c^2$ where c is the length of the hypotenuse.
2. Sum of angles in a triangle: The angles in a triangle sum to 180^o and so a right angled triangle has two angles that sum to 90^o and the right angle that is 90^o.
3. Special triangles: The $1, 1, \sqrt{2}$ and $1, 2, \sqrt{3}$ triangles.
4. Similar triangles: Scaling does not change the angles.
5. SOHCAHTOA: Evaluation of trigonometric functions with respect to sides of triangle.
6. CAST: Sign of the trigonometric function based on quadrant.
7. $\sin^2(\theta) + \cos^2(\theta) = 1$

Solving Trigonometric expressions

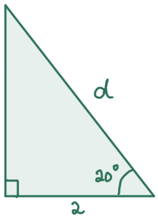
The Pythagoreans agree to trade magi knowledge, if you can demonstrate that your band has the power to solve a range of problems. Can your band identify which of the tools you could use for the following:

1. To find $\cos(\theta)$ when we know $\sin(\theta)$.
2. To find $\cos(\frac{\pi}{4})$
3. To find the length of a side of a triangle when you know the lengths of the other 2 sides.
4. To find the angle in a triangle when you know the other two angles.
5. To find $\sin(30^o)$
6. To find the angles of a triangle with sides $4, 4, 4\sqrt{2}$
7. To find the angle in a triangle when the side opposite it has length 4, and the side adjacent to it has length 5.
8. To find the sign of $\cos(\theta)$ when θ is in the 3rd quadrant.

Pythagoreans

The Pythagoreans are impressed and so agree to trade knowledge with you. They guide you to a boat and the final test is to descend the ladder into the boat at the correct angle and for the correct distance, or face a similar fate to Hippasus.

1. The angle θ satisfies the following equation $\sin(\theta) = \frac{-\sqrt{3}}{2}$ and $\cos(\theta) = \frac{1}{2}$.
2. The distance d is the length of the side of the triangle below (approximate this to 2 decimal places).



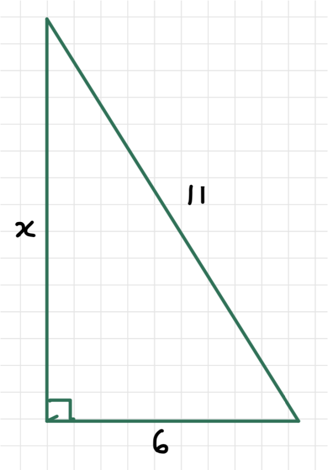
Reflection

You board the boat safely and your band is taken to the next part of your journey. On the way, we will apply our trigonometry skills and test our tools.

As we have many new tools and concepts, we will use the remainder of this week's workshop to look at some examples and how to identify the best tool for each task.

Some Examples

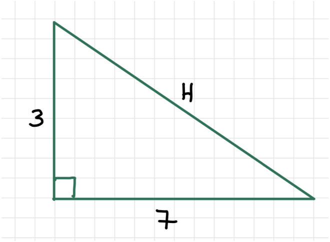
Find the length of the side x in the triangle below.



Which rule did you use?

Some Examples

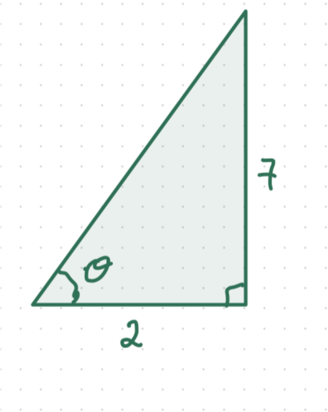
Find the length of the side x in the triangle below.



Which rule did you use?

Some Examples

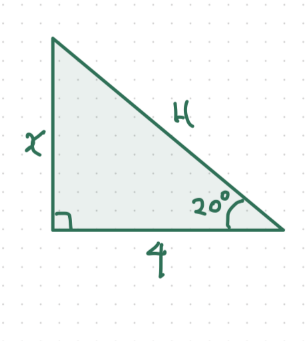
Find the angle of θ in the triangle below.



Which rule did you use? Can you find the other angles?

Some Examples

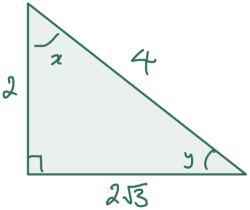
Find the length of the sides x and H in the triangle below.



Which rule did you use?

Some Examples

Can you find the angles in the triangle below.



Which rule did you use?

Some Examples

Suppose $\cos(\theta) = \frac{\sqrt{3}}{2}$ and θ is in the fourth quadrant. Find $\sin(\theta)$.

Some Examples

Suppose $\sin(\theta) = \frac{1}{\sqrt{2}}$ and θ is in the second quadrant. Find $\cos(\theta)$.

Some Examples

Suppose $\cos(\theta) = \frac{1}{2}$ and θ is in the first quadrant. Find $\sin(\theta)$ and $\tan(\theta)$.

Some Examples

Suppose $\sin(\theta) = \frac{1}{4}$ and θ is in the second quadrant. Find $\cos(\theta)$ and $\tan(\theta)$. (You can approximate to 2 decimal places.)

Some Examples

Approximate the following expressions to 2 decimal places:

- $\cos(\frac{\pi}{5})$
- $\cos(\frac{-\pi}{5})$
- $\sin(\frac{\pi}{5})$
- $\sin(\frac{-\pi}{5})$