SIT190 - WEEK 5

Overview

- Index Laws
- Logarithms
- The natural logarithm and the number e
- Solving and simplifying equations

Continuing the Quest

Your band reads the instructions on the small cuneiform tablet. It contains directions to the ancient city of Larsa and a list of supplies that you should take with you on the way. Unfortunately, the numbers do not make much sense:

- $1,0$ litres of water
- 10 goats
- $1,40$ kilograms of flour
- 14 kilograms of dried dates
- $20;30$ kilograms of barley
- 4 bales of hay for the goats.

Indices

Before you attempt to understand these numbers, we will revise some properties of indices.

$a^n = a \times a \times \ldots \times a.$	eg $10^3 = 10 \times 10 \times 10$
$a^{-n} = \frac{1}{a^n}.$	eg $4^{-3} = \frac{1}{4^3} = \frac{1}{4 \times 4 \times 4}$
$a^1 = a.$	eg $7^1 = 7$
$a^0 = 1.$	eg $4^0 = 1$
$a^{\frac{1}{n}} = \sqrt[n]{a}.$	eg $4^{\frac{1}{2}} = \sqrt{4}$, eg $4^{\frac{1}{3}} = \sqrt[3]{4}$
$a^n a^m = a^{n+m}$	eg $3^2 3^4 = 3^6$
$(a^n)^m = a^{nm}$	eg $(3^2)^4 = 3^8$

Sexagesimal

The Ancient Babylonians used a base 60, or sexagesimal, number system. We use a base 10, or decimal, number system. A quick reminder about our decimal number system.

- There are ten digits.
- A number is represented by a string of digits eg. 1305.2
- The value of each digit is obtained from it's place value. eg. $1345.2 = 1 \times 10^3 + 3 \times 10^2 + 0 \times 10^1 + 5 \times 10^0 + 2 \times 10^{-1}$
- Some fractions can be represented exactly in decimal notations. eg $\frac{1}{2}$ and $\frac{1}{5}$.

Sexagesimal

In comparison, the sexagesimal system used by the Ancient Babylonians:

- There are 59 digits - we will represent these by the numbers 1 to 59.
- There was no digit representing zero. A symbol was used as an internal space.
- A number is represented by a string of digits eg. 13,4,5.2 (We have separated the sexagesimal digits by ',' for clarity.)
- This value of each digit is obtained from it's place value. eg. $13, 4, 5.2 = 13 \times 60^2 + 4 \times 60^1 + 5 \times 60^0 + 2 \times 60^{-1}$
- Some fractions can be represented exactly in sexagesimal notation.
  eg $30 \times 60^{-1} \frac{1}{2}$, $20 \times 60^{-1} = \frac{1}{3}$, $15 \times 60^{-1} = \frac{1}{4}$, $12 \times 60^{-1} = \frac{1}{5}$ and $10 \times 60^{-1} = \frac{1}{6}$.

Sexagesimal and Decimal

As an aside, consider evaluating the polynomial $f(x) = x^3 + 3x^2 + 4x + 5$ when $x = 10$ and when $x = 60$.

- $f(10)$ gives you the value of the decimal number 1345, and
- $f(60)$ gives you the sexagesimal number 1,3,4,5.

Supplies for the journey

You realise that each of the sexagesimal digits has been separated by a "," and ";" is the equivalent of the decimal point. For example, $2,3;3$ is the number $2 \times 60^1 + 3 \times 60^0 + 3 \times 60^{-1} = 120 + 3 + \frac{3}{60} = 123.05$ in decimal. Convert the list below from sexagesimal to decimal.

- $1,0$ litres of water
- 10 goats
- $1,40$ kilograms of flour
- 14 kilograms of dried dates
- $20;30$ kilograms of barley
- 4 bales of hay for the goats.

The band to find the correct quantities for the list of supplies first will be given 10 points.

Base 60

There are still remnants of the Babylonian sexagesimal system in our lives today:

- 60 minutes=1 hour.
- 60 seconds = 1 minute.
- 360 degrees in a circle.
- Distance that can be walked in a day is 12 miles = 12 hours?

Yale Tablet

The Yale Table, an Old Babylonian Tablet (ca 1800-1600BCE) contains the following sexagesimal number:

$$1;24,51,10 = 1 \times 60^0 + 24 \times 60^{-1} + 51 \times 60^{-2} + 10 \times 60^{-3}$$

which provides a numerical estimate to six decimal (or 3 sexagesmal) places of the number _______.

Approximating irrational numbers

The following method is called the *Bablyonian Method*.
Maybe this is what they used to find $n^{\frac{1}{2}}$.

1. Initial guess x
2. Update x to $\frac{1}{2}(\frac{n}{x} + x)$
3. Repeat Step 2 until 'close enough'

For example, to find $3^{\frac{1}{2}}$:

Guess $x = \frac{3}{2} = 1.5$

Update x to $\frac{1}{2}(\frac{3}{\frac{3}{2}} + \frac{3}{2}) = \frac{1}{2}(2 + \frac{3}{2}) = \frac{7}{4} = 1.75$

Update x to $\frac{1}{2}(\frac{3}{\frac{7}{4}} + \frac{7}{4}) = \frac{1}{2}(\frac{12}{7} + \frac{7}{4}) = \frac{97}{56} \approx 1.732142857$

Update x to $\frac{1}{2}(\frac{3}{\frac{97}{56}} + \frac{97}{56}) = \frac{1}{2}(\frac{168}{97} + \frac{97}{56}) = \frac{18817}{10864} \approx 1.73205081$

Reflection

While ordering supplies and approximating square roots, your band has been working with powers.

- What is 60^1?
  - In general, what is a^1?
- What is 60^0?
  - In general, what is a^0?
- What happens when the power is negative?
  - What is 10^{-2}?
  - What is a^{-n}?
- What happens when the power is a fraction?
  - What is $4^{\frac{1}{2}}$?
  - What is $8^{\frac{1}{3}}$?

Distance to Larsa

Your band reads the instructions in the cunieform tablet. The distance you must travel is encoded in an expression in the tablet. But first you must identify which expression should be used.

Simplify the following expressions so that they are expressed with all positive indices and in the simplest form:

1. $\frac{x^2-x^{-1}}{x^4-x^{-2}}$
2. $\frac{x(x^{\frac{3}{2}})^2-x}{(x^3)^2-1}$
3. $\frac{(x^{\frac{3}{2}}-1)(x^{\frac{3}{2}}+1)}{x^5-x^{-1}}$
4. $\frac{x^3}{x^3x^2+x^4x^{-2}}$
5. $\frac{x}{(x^{\frac{1}{3}})^9+x^{\frac{2}{3}}x^{\frac{-4}{6}}}$

Distance to Larsa

1. $\frac{x^2 - x^{-1}}{x^4 - x^{-2}}$
2. $\frac{x(x^{\frac{3}{2}})^2 - x}{(x^3)^2 - 1}$
3. $\frac{(x^{\frac{3}{2}} - 1)(x^{\frac{3}{2}} + 1)}{x^5 - x^{-1}}$
4. $\frac{x^3}{x^3 x^2 + x^4 x^{-2}}$
5. $\frac{x}{(x^{\frac{1}{3}})^9 + x^{\frac{2}{3}} x^{\frac{-4}{6}}}$

Can you identify the expression of interest?
Evaluate this expression when $x = -\frac{11}{10}$ and round to the nearest integer to find the distance.

Reflection

Your band has identified the expression, and can head to Larsa. It has also had practice working with indices.

- How do you remove terms with negative indices?
- What is the difference between $3^3 3^4$ and $(3^3)^4$?
- Compare 3^4 and $81^{\frac{1}{4}}$. What is the relationship between these two powers? How could you use it to solve the following equations:
  - $x^3 = 27$
  - $x^4 = 16$
  - $x^{\frac{1}{3}} = 2$
  - $x^{\frac{1}{2}} = 4$

Reflection

- We have seen a relationship between
  - raising to the power of n, and
  - raising to the power of $\frac{1}{n}$
- We used these powers to find the number x such that x to some power was the number of interest.
  - For example, given $x^3 = 27$, we found $x = 3$ by
- We will now look at logarithms. A logarithm $\log_a(b)$ asks what is the power n such that $a^n = b$
  - For example, given base 2 what power gives us 8, that is , $\log_2 8 = 3$, as $2^3 = 8$.

Logarithms

First we will revise some properties of logs.

$\log_a(m) + \log_a(n) = \log_a(mn)$	eg $\log_3(10) + \log_3(4) = \log_3(40)$
$\log_a(m) - \log_a(n) = \log_a(\frac{m}{n})$	eg $\log_3(10) - \log_3(2) = \log_3(5)$
$c\log_a(b) = \log_a(b^c)$	eg $3\log_2(4) = \log_2(b^{4^3})$
$\log_a(1) = 0$	eg $\log_5(1) = 0$
$\log_a(a) = 1$	eg $\log_{17} 17 = 1$
$\log_a(b) = \frac{\log_c(b)}{\log_c(a)}$	eg $\log_4(32) = \frac{\log_2(32)}{\log_2(4)}$

Logarithms

Before we continue, your band should try to simplify the following expressions:

1. $\log_3(12) - \log_3(4)$
2. $3\log_5 2 - \log_5(16) + 2\log_5(10)$
3. $\log_4(16) + \log_4(1)$
4. $\log_2(3x) + 2\log_2(x) - \log_2(x)$

Logarithms

- You may wonder why we are in Ancient Mesopotamia, when John Napier 'discovered' the logarithm in the early 1600s.
- However, the concept of logarithm is said to have originated in Old Babylonian tablets.
- Tablets contained successive powers of integers.
- Some of these records include a question 'To what power must a certain number be raised in order to yield a given number?'
- This is equivalent to $\log_a(b)$, that is what power n gives us $a^n = b$?

Refernce: R.C. Pierce Jr. A Brief History of Logarithms. *The Two-Year College Mathematics Journal* , Vol. 8, pp. 22-26, 1977.
`http://www.jstor.com/stable/3026878`

Archimedes

Archimedes (ca 287-212 BCE)

- Geometric Sequence[1]: $a^0, a^1, a^2, \ldots$
- Eg $1, 2, 4, 8, 16, \ldots$
- Arithmetic Sequence[2]: $0, 1, 2, 3, 4, \ldots$
- Relationship between multiplying two numbers in the geometric sequence eg $a^2 \times a^3$ and adding in the arithmetic sequence.

If you had a table of powers of 2, which would be the easier method to find $2^{12} \times 2^{15}$

1. Look up 2^{12} and 2^{15} and multiply the answers (4096×32768), or
2. Add 12 and 15 and look up 2^{27}.

Remember, they would not have had a calculator.

[1] successive terms have the same ratio

[2] successive terms have the same difference

Slide Rules

The Rise and Fall of the Slide Rule: 350 Years of Mathematical Calculators urlhttps://library.stanford.edu/spc/exhibitspublications/past-exhibits/rise-and-fall-slide-rule-350-years-mathematical-calculators

- Small Calculators became available in the 1970's or later
- Graphing Calculator (1985)
- Slide rule - a mechanical device to perform calculations
- For Multiplication - the ruler was marked in logarithmic scale
- Centre would 'slide' to multiply two numbers
  - For example, 3×2
  - Slide centre to position marked '3' (distance is log(3)) on the lower section
  - Finding position 2 in (distance log(2)) in centre section
  - This adding of 3 and 2 corresponds to the distance $\log(2 \times 3) = \log(6)$ and so the answer is 6.

Babylonian Multiplication

Your band is curious to find how the Babylonians performed multiplication when the numbers were not powers or had different bases.

- Two tablets found at Sekerah (2000 BCE) give the squares of numbers up to 59 and cubes up to 32.
- The Babylonians used the formula $xy = ((x+y)^2 - (x-y)^2)/4$ and these type of tables for multiplication.
- Instead of long division they multiplied by the reciprocal of 4. They had tables of reciprocals too.

Babylonian Multiplication

$$xy = \frac{(x+y)^2 - (x-y)^2}{4}$$

1. Look up the square of $x+y$ and the square of $x-y$ in the table.
2. Subtract the latter from the former.
3. Multiply by $\frac{1}{4}$ (that is the reciprocal of four)

For example, to calculate 34×26:

1. $(34+26)^2 = 3600$ and $(34-26)^2 = 64$
2. $3600 - 64 = 3536$
3. $3536 \times \frac{1}{4} = 884$

The Babylonians tables enabled them to solve mathematical problems even though they had not calculator.

Inside Larsa

Your band enters Larsa and you follow the tablet's directions to the ruins of an old ziggurat. Thankfully, you did not use all your supplies on the way, because you must now barter some with the doorkeeper to be allowed to enter into the building. There are three gateways, and if you choose the wrong one, you will have to barter more supplies to try another gate. In order to find the correct gate, simplify the following mathematical expression:

$$3\log_3(4) - \log_3(256) + \log_3(12)$$

.

10 Points for the band who finds the correct gate first.

Which floor?

Your band correctly identifies the gate and enters the building. The tablet instructs you to go to the following floor:

$$\log_9(243) + \log_9(3)$$

10 Points for the band who identifies the correct floor first.

Locating the tablet

Your band clambers up the stairs to the correct floor. To your horror, Mush-hushshu Dragon appears as you round the corner of the stairs.

Figure: Image: (604?562 BCE). Ishtar Gate, wall frieze detail, Mush-hushshu Dragon, Symbol of the God Marduk [Das Ischtar-Tor unter Nebukadnezar II. errichtet; Detail des Wandfrieses - Dr Schreitender

Fighting the Dragon

The Mush-hushshu Dragon attacks your band by bombarding them with some mathematical problems. Sneakily, he uses the natural logarithm and the number e hoping to undermine your newfound skills in this area. However, your band is confident that they can apply their skills and refuses to be daunted by a mere change of $\log_e(x)$ to $\ln(x)$.

Figure: Image: (604?562 BCE). Ishtar Gate, wall frieze detail, Mush-hushshu Dragon, Symbol of the God Marduk [Das Ischtar-Tor - unter Nebukadnezar II. errichtet; Detail des Wandfrieses - Dr Schreitender

Fighting the Dragon

The challenge is to solve these problems before the dragon overcomes your band.

1. Simplify $\frac{e^{2x}-4}{e^x+2}$
2. Expand $(e^{3x}-1)^2$
3. Simplify $\ln(3) - \ln(2) + 2\ln(2)$
4. Solve for x, $\ln(3x-4) = 5$
5. Solve for x, $\sqrt{e^x - 1} = 4$

10 points to the band who solves these first and conquers the dragon.

Do you aspire to be a Magi? Can you solve $e^{2x^2-3x} = 5$ for x?

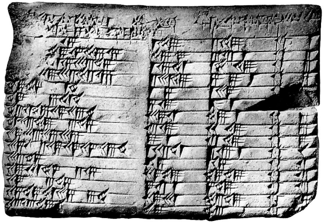
Reflection

You overcome the dragon and continue down the hallway.
But first we reflect on the way you attacked these problems.

- Using expansion and factorisation techniques from previous weeks:
  - $(a \pm b)^2 = a^2 \pm 2ab + b^2$
  - $(a - b)(a + b) = a^2 - b^2$
- Applying rules for powers and logs
- Recognising $\ln(x)$ is the same as $\log_e(x)$ and that we can use the usual log rules.
- Recognising e is just a number (albeit a special one)
- Recognising 'pairs' of operations that are 'inverses' of each other
  - Addition and subtraction eg $3 + 7 - 7 = 3$
  - Multiplication and division eg $3 \times 7 \div 7 = 3$ (or, multiplication and multiplication by the reciprocal eg $3 \times 7 \times \frac{1}{7}$)
  - Integer powers and n-th roots eg $(3^4)^{\frac{1}{4}} = \sqrt[4]{3^4} = 3$

Capturing the artefact

Your band moves down the hallway and locate a cupboard full of fragments of clay tablets. Nearby is an elderly scribe. She says that she has been confined to the ziggurat until she can solve a problem about interest. But she has been unable to find a tablet with tables of figures to help her calculation. She has however, found the missing fragment of the Plimpton Tablet and is happy to give it to you providing you solve her problem.



Bablyonian Interest Calculations

- Compound interest or *sibat sibtim* (or "interest on interest" in Akkadian) known in Old Babylonian Period 2000-1600 BCE.
- Possibly earlier in pre-Sargonic period (2600-2350 BCE) where there are references to *mas ur-ra* or interest bearing loans.
- Lagash ruler of Lagash leant barley at interest to the city of Umma. ($\approx$ 2400BCE)
- Umma did not repay the loan
- Rate of interest on barley in Mesopotamia was $33\frac{1}{3}\%$
- How long would it take for the amount owed to increase to 7.5 times the principal?

Image: `https://arxiv.org/pdf/1510.00330.pdf`

Babylonian Interest Calculations

This problem is equivalent to asking you to solve the following problem for x:

$$(1 + \frac{1}{3})^x = 7.5$$

Can your band solve this problem for the scribe and so gain the artefact?

10 points for the first band to solve this problem.

Reflection

Your band has solved the problem and gained the artefact. You have also seen an example where logarithms can be used to solve a modern day problem.

Compound interest:

- Invest principal P at annual rate r compounded m times a year.
- How much will you have after t years?
- Amount $A = P(1 + \frac{r}{m})^{mt}$

If you invest $P = \$10,000$ at rate $r = \frac{1}{5}$ compounding $m = 4$ times a year, how long will it take you to double your money?

$$2P = P(1 + \frac{1}{20})^{4t} \Rightarrow 2 = \left(\frac{21}{20}\right)^{4t}$$

Reflection

How long?

$$\begin{aligned}
2P &= P\left(1+\frac{1}{20}\right)^{4t} \\
\Rightarrow 2 &= \left(\frac{21}{20}\right)^{4t} = \left(\left(\frac{21}{20}\right)^{4}\right)^{t} \\
\Rightarrow \ln(2) &= \ln\left(\left(\frac{21}{20}\right)^{4}\right)^{t} = t\ln\left(\left(\frac{21}{20}\right)^{4}\right) \\
\Rightarrow t &= \frac{\ln(2)}{\ln\left(\left(\frac{21}{20}\right)^{4}\right)} \\
\Rightarrow t &= \frac{\ln(2)}{4\ln\left(\frac{21}{20}\right)} \approx 3.552 \text{ years}
\end{aligned}$$

Reflection

An interesting connection with the number e.
If you invested $P = 1$ with an annual rate $r = 1$ and compounded it n times annually, then

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = e.$$

So, our general formula: Amount $A = P(1+\frac{r}{m})^{mt}$ becomes

$$A = Pe^{rt}$$

when the interest is compounded **continuously.**

Other Applications of Exponentials

The function $y = Ae^{kx}$ arises in many areas where we have exponential growth ($k > 0$) or exponential decay ($k < 0$).

Domain: $x \in R$

Range: $y > 0$

Intercepts: $(0, A)$ is y-intercept. There are no x intercepts.

Some applications:

- Population growth
- Growth of bacteria
- Modelling epidemics
- Tracking growth of savings or debt

Logarithms

The function $y = \log_a(x)$:

Domain: $x > 0$

Range: $y \in R$

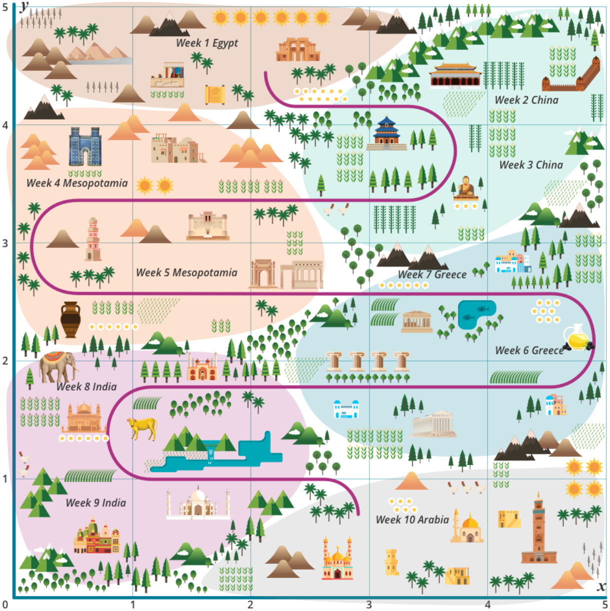
Intercepts: $(1, 0)$ is x-intercept. There are no y intercepts.

Graph: Reflection of $y = a^x$ with respect to line $y = x$

Next Week

You have completed this week's quest and found a missing fragment of the Plimpton tablet. More importantly, we have applied index and log laws to solve and simplify mathematical expressions.
In order to move to the next location of the quest, your band needs to find the location where you will be transported to Greece.
The point is (x, x) on the map where $x = \log_9(243)$.
Your band should solve this by using a change of basis.

Map



y
x
0
1
2
3
4
5
1
2
3
4
5
Week 1 Egypt
Week 2 China
Week 3 China
Week 4 Mesopotamia
Week 5 Mesopotamia
Week 6 Greece
Week 7 Greece
Week 8 India
Week 9 India
Week 10 Arabia