7

Integration

Introduction

This topic covers the fundamental relationship between the concepts of differentiation and integration. This fundamental relationship permits the evaluation of both indefinite and definite integrals, applications of which abound in engineering, technology and the sciences. Indefinite integrals are used to solve problems concerning the motion of an object in a straight line, and definite integrals are used to find areas of regions in the plane. After studying this topic you should be able to:

- understand the relationship between integration and differentiation;
- integrate polynomial, trigonometric, and exponential functions;
- evaluate definite integrals of polynomial, trigonometric, and exponential functions;
- find areas contained between curves;
- understand the distinction between definite integrals and areas.

7.1 Anti-derivatives

1. If the derivative of a function is known, the original function can be found (almost) by **anti-differentiating (integrating)**. The anti-derivative **always** contains an arbitrary constant C, which cannot be determined unless extra information is given.
2. When applied to sketches of graphs, anti-differentiation is the process of finding the equation to a curve from the slope of that curve. In the simple case of a linear function, if the slope m is known, e.g. $m = 3$, then the equation of the straight line is $y = 3x + C$ .The same arbitrary constant C appears in the anti-derivative of any function.
   In general, given the slope of a curve, the equation of that curve is known, except for its y-intercept. If, however, a particular point on the curve is specified, the equation is completely determined.
3. The anti-derivative is more usually called the **indefinite integral**, where the word 'indefinite' indicates the presence of the constant C in the answer.
4. The notation $\int f(x)dx$ must be strictly observed. In particular, the dx cannot be omitted.
5. After the answer to an indefinite integral has been found, it can always be checked by differentiation. For instance, the answer to $\int 3x^2dx$ is $x^3 + C$ because $\frac{d}{dx}(x^3 + C) = 3x^2$.

Examples

1. Write the following statements about derivatives as statements about indefinite integrals

   (i) $\frac{d}{dx}(x^5) = 5x^4$ (ii) $\frac{d}{dx}(e^{2x}) = 2e^{2x}$

(iii) $\frac{d}{dx}(\sin 2x) = 2\cos 2x$ (iv) $\frac{d}{dx}(\cos 3x) = -3\sin 3x$.

1. (i) $\int 5x^4 dx = x^5 + C$ (ii) $\int 2e^{2x} dx = e^{2x} + C$

(iii) $\int 2\cos 2x dx = \sin 2x + C$ (iii) $\int -3\sin 3x dx = \cos 3x + C$.

Problems

1. Write the following statements about derivatives as statements about indefinite integrals:

(i) $\frac{d}{dx}(x^7) = 7x^6$ (ii) $\frac{d}{dx}(e^{5x}) = 5e^{5x}$

(iii) $\frac{d}{dx}(\sin 6x) = 6\cos 6x$ (iv) $\frac{d}{dx}(\cos 2x) = -2\sin 2x$.

Answers

1. (i) $\int 7x^6 dx = x^7 + C$ (ii) $\int 5e^{5x} dx = e^{5x} + C$

(iii) $\int 6\cos 6x dx = \sin 6x + C$ (iii) $\int -2\sin 2x dx = \cos 2x + C$.

7.2 Rules for integration

1. Since differentiation and integration are inverse operations, every rule for derivatives can be converted into a corresponding rule for integrals. In particular, since,

   if $y = \frac{x^{n+1}}{n+1}$, then $\frac{dy}{dx} = (n+1) \times \frac{x^n}{n+1} = x^n$, it follows that:

   $$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad \text{(provided } n + 1 \neq 0\text{, i.e. } n \neq -1\text{).}$$

2. Note that, in the answer above, the power of x is the same as the denominator, and is found by adding 1 to the original power of x. In terms of the power of x, **differentiation** takes the power **down**, whereas **integration** takes the power **up**.

3. The results in the table of integrals (below) can all be proved by converting a rule for differentiation into a corresponding rule for integration. For instance, the result:

   if $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$ leads to $\int \frac{1}{x} dx = \ln|x| + C$.

   Note that the **modulus (absolute value)**, $|x|$ ensures that the logarithm is defined for both positive and negative values of x, as, by definition $|x|$

is the distance of x from the origin, and is never negative. As examples, $|-3| = 3$; $|-8| = 8$; and $|6| = 6$.

4. More general results for integrals of trigonometric, exponential and logarithmic functions can be derived, as follows.

   Since, if $y = \frac{-1}{k}\cos kx$, then $\frac{dy}{dx} = \frac{-1}{k} \times (-k\sin kx) = \sin kx$

   $$\int \sin kx dx = \frac{-1}{k}\cos kx + C.$$

   Similarly, if $y = \frac{1}{k}\sin kx$, then $\frac{dy}{dx} = \frac{1}{k} \times (k\cos kx) = \cos kx$

   $$\therefore \int \cos kx dx = \frac{1}{k}\sin kx + C.$$

   Also, if $y = \frac{1}{k}e^{kx}$, then $\frac{dy}{dx} = \frac{1}{k} \times (ke^{kx}) = e^{kx}$

   $$\therefore \int e^{kx} dx = \frac{1}{k}e^{kx} + C.$$

5. In summary, the 5 major results for the indefinite integrals of the commonly used functions are shown in the table below (k is a constant).

$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C \; (n \neq -1)$
$\sin kx$	$\frac{-1}{k}\cos kx + C$
$\cos kx$	$\frac{1}{k}\sin kx + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$
$\frac{1}{x}$	$\ln\|x\| + C$

Examples

1. Find:

   (i) $I = \int x^6 dx$ (ii) $I = \int x^{-6} dx$

   (iii) $I = \int \sin 6x dx$ (iv) $I = \int \cos 6x dx$

(v) $I = \int e^{6x} dx$ (vi) $I = \int e^{-6x} dx$.

2. Find:

(i) $I = \int \left(2x + \frac{5}{x^2} - 7\right) dx$ (ii) $I = \int \left(4\sqrt{x} - \frac{3}{\sqrt{x}} + 2\right) dx$

(iii) $I = \int (2x^{-1/3} + 5x^{2/3}) dx$ (iv) $I = \int 3x(x-8) dx$

(v) $I = \int (x+3)^2 dx$ (vi) $I = \int \left(\frac{4}{x} - 5\right) dx$.

3. Find:

(i) $I = \int (3e^{-9x} + 8e^{4x}) dx$ (ii) $I = \int 5\cos\left(\frac{x}{2}\right) dx$

(iii) $I = \int (5\sin 2x + 3\cos 8x) dx$ (iv) $I = \int \frac{2}{e^{3x}} dx$.

1. (i) $I = \int x^6 dx = \frac{x^7}{7} + C \qquad (n = 6).$

(ii) $I = \int x^{-6} dx = \frac{x^{-5}}{-5} + C = \frac{-x^{-5}}{5} + C = \frac{-1}{5x^5} + C \qquad (n = -6).$

(iii) $I = \int \sin 6x dx = \frac{-1}{6}\cos 6x + C \qquad (k = 6).$

(iv) $I = \int \cos 6x dx = \frac{1}{6}\sin 6x + C \qquad (k = 6).$

(v) $I = \int e^{6x} dx = \frac{1}{6}e^{6x} + C \qquad (k = 6).$

(vi) $I = \int e^{-6x} dx = \frac{1}{-6}e^{-6x} + C = \frac{-1}{6}e^{-6x} + C \qquad (k = -6)$

2. (i) $I = \int \left(2x + \frac{5}{x^2} - 7\right) dx = \int (2x^1 + 5x^{-2} - 7x^0) dx$

$(n = 2, -1, 1$ respectively$)$

$$= \frac{2x^2}{2} + \frac{5x^{-1}}{-1} - \frac{7x^1}{1} + C \qquad (n = 2, -1, 1 \text{ respectively})$$

$\therefore I = x^2 - 5x^{-1} - 7x + C$.

(ii) $I = \int \left(4\sqrt{x} - \frac{3}{\sqrt{x}} + 2\right) dx = \int (4x^{1/2} - 3x^{-1/2} + 2x^0) dx$

$(n = \frac{1}{2}, \frac{-1}{2}, 0$ respectively$)$

$$= \frac{4x^{3/2}}{3/2} - \frac{3x^{1/2}}{1/2} + \frac{2x^1}{1} + C = 4 \times \frac{2}{3}x^{3/2} - 3 \times 2x^{1/2} + 2x + C$$

$$\therefore I = \frac{8x^{3/2}}{3} - 6x^{1/2} + 2x + C.$$

(iii) $I = \int (2x^{-1/3} + 5x^{2/3})dx \qquad (n = \frac{-1}{3}, \frac{2}{3}$ respectively)

$$= \frac{2x^{2/3}}{2/3} + \frac{5x^{5/3}}{5/3} + C = 2 \times \frac{3}{2}x^{2/3} + 5 \times \frac{3}{5}x^{5/3} + C$$

$$\therefore I = 3x^{2/3} + 3x^{5/3} + C.$$

(iv) $I = \int 3x(x-8)dx = \int (3x^2 - 24x^1)dx \quad (n = 2, 1$ respectively)

$$= \frac{3x^3}{3} - \frac{24x^2}{2} + C$$

$$\therefore I = x^3 - 12x^2 + C$$

(v) $I = \int (x+3)^2 dx = \int (x^2 + 6x^1 + 9x^0)dx$

($n = 2, 1, 0$ respectively)

$$= \frac{x^3}{3} + \frac{6x^2}{2} + \frac{9x^1}{1} + C$$

$$\therefore I = \frac{x^3}{3} + 3x^2 + 9x + C$$

(vi) $I = \int \left(\frac{4}{x} - 5\right)dx = \int (4x^{-1} - 5x^0)dx \ (n = -1, 0$ respectively)

$$= 4\ln|x| - \frac{5x^1}{1} + C$$

$$\therefore I = 4\ln|x| - 5x + C.$$

Note: Since the constant k can always be written as $k = kx^0$,

$$\int k dx = \int kx^0 dx = \frac{kx^1}{1} + C = kx + C$$

3. (i) $I = \int (3e^{-9x} + 8e^{4x})dx \qquad (k = -9, 4$ respectively)

$$= \frac{3e^{-9x}}{-9} + \frac{8e^{4x}}{4} + C$$

$$\therefore I = \frac{-e^{-9x}}{3} + 2e^{4x} + C$$

(ii) $I = \int 5\cos\left(\frac{x}{2}\right)dx \qquad (k = \frac{1}{2})$

$$= \frac{5}{1/2}\sin\left(\frac{x}{2}\right) + C = 5 \times 2\sin\left(\frac{x}{2}\right) + C$$

$$\therefore I = 10\sin\left(\frac{x}{2}\right) + C$$

(iii) $I = \int (5\sin 2x + 3\cos 8x)dx$ $\qquad$ ($k = 2, 8$ respectively)

$$\therefore I = \frac{-5}{2}\cos 2x + \frac{3}{8}\sin 8x + C$$

(iv) $I = \int \frac{2}{e^{3x}}\,dx = \int 2e^{-3x}dx$ $\qquad$ ($k = -3$)

$$= \frac{2e^{-3x}}{-3} + C$$

$$\therefore I = \frac{-2e^{-3x}}{3} + C.$$

Problems

1. Find

(i) $I = \int x^4 dx$ $\qquad$ (ii) $I = \int x^{-4} dx$

(iii) $I = \int \sin 4x\, dx$ $\qquad$ (iv) $I = \int \cos 4x\, dx$

(v) $I = \int e^{4x} dx$ $\qquad$ (vi) $I = \int e^{-4x} dx$.

2. Find

(i) $I = \int \left(4x + \frac{6}{x^3} - 1\right) dx$ $\qquad$ (ii) $I = \int \left(6\sqrt{x} - \frac{1}{\sqrt{x}} + 3\right) dx$

(iii) $I = \int (3x^{-1/4} + 5x^{1/4})dx$ $\qquad$ (iv) $I = \int 2x(3x-2)dx$

(v) $I = \int (x-2)^2 dx$ $\qquad$ (vi) $I = \int \left(\frac{7}{x} - 2\right) dx$.

3. Find

(i) $I = \int (5e^{-5x} + 8e^{2x})dx$ $\qquad$ (ii) $I = \int 2\cos\left(\frac{x}{3}\right) dx$

(iii) $I = \int (7\sin 3x + 3\cos 5x)dx$ $\qquad$ (iv) $I = \int \frac{3}{e^{2x}}\, dx$.

Answers

1. (i) $I = \int x^4 dx = \dfrac{x^5}{5} + C$

(ii) $I = \int x^{-4} dx = \dfrac{x^{-3}}{-3} + C = \dfrac{-x^{-3}}{3} + C = \dfrac{-1}{3x^3} + C$

(iii) $I = \int \sin 4x dx = \dfrac{-1}{4}\cos 4x + C$

(iv) $I = \int \cos 4x dx = \dfrac{1}{4}\sin 4x + C$

(v) $I = \int e^{4x} dx = \dfrac{1}{4}e^{4x} + C$

(vi) $I = \int e^{-4x} dx = \dfrac{1}{-4}e^{-4x} + C = \dfrac{-1}{4}e^{-4x} + C$.

2. (i) $I = \int\left(4x + \dfrac{6}{x^3} - 1\right)dx = 2x^2 - \dfrac{3}{x^2} - x + C$

(ii) $I = \int\left(6\sqrt{x} - \dfrac{1}{\sqrt{x}} + 3\right)dx = 4x^{3/2} - 2x^{1/2} + 3x + C$

(iii) $I = \int(3x^{-1/4} + 5x^{1/4})dx = 4x^{3/4} + 4x^{5/4} + C$

(iv) $I = \int 2x(3x-2)dx = 2x^3 - 2x^2 + C$

(v) $I = \int(x-2)^2 dx = \dfrac{x^3}{3} - 2x^2 + 4x + C$

(vi) $I = \int\left(\dfrac{7}{x} - 2\right)dx = 7\ln|x| - 2x + C$.

3. (i) $I = \int(5e^{-5x} + 8e^{2x})dx = -e^{-5x} + 4e^{2x} + C$

(ii) $I = \int 2\cos\left(\dfrac{x}{3}\right)dx = 6\sin\left(\dfrac{x}{3}\right) + C$

(iii) $I = \int(7\sin 3x + 3\cos 5x)dx = \dfrac{-7}{3}\cos 3x + \dfrac{3}{5}\sin 5x + C$

(iv) $I = \int \dfrac{3}{e^{2x}} dx = \dfrac{-3e^{-2x}}{2} + C$.

7.3 Finding the constant *C*

1. If $\dfrac{dy}{dx} = f(x)$, then $y = \int f(x)dx$.

   The solution for y contains an unknown arbitrary constant C, which can only be determined if extra information is given. Such extra information is usually given in the form of one point on the solution curve, e.g. when $x = 0$, $y = 2$. The given point is sometimes written in function notation, e.g. $y(0) = 2$, where the x-value appears in the brackets.

2. For many applications of integration, the variable t (time) is used, rather than x. In such cases, a given point on the solution curve is called an **initial condition**.

3. Equations of the form $\frac{dy}{dx} = f(x)$, or $\frac{dy}{dt} = f(t)$ are called differential equations.

4. For motion in a straight line, with the usual notation, i.e.

   ***s* is the displacement of the object;**

   ***t* is the time;**

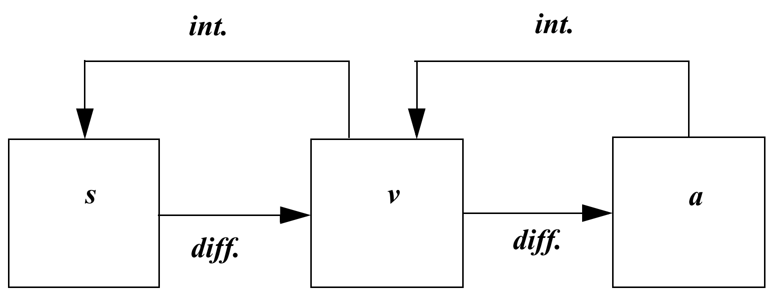
   ***v* is the velocity;**

   ***a* is the acceleration.**

   the relations $v = \frac{ds}{dt}$, and $a = \frac{dv}{dt}$ can be re-written as:

   $s = \int v dt$ and $v = \int a dt$.

   If any one of the displacement, velocity or acceleration is known, the other two can be found using differentiation and integration, as shown in the diagram below.



Examples

1. Find y given that:

   (i) $\frac{dy}{dx} = 4x - 5$ and $y(2) = 3$

   (ii) $\frac{dy}{dx} = 6x^2 + 8x - 5$ and $y(0) = -1$

   (iii) $\frac{dy}{dx} = 2e^{-6x}$ and $y(0) = 0$.

2. The acceleration of an object is given by:
   $a = 12t + 2$ (m/sec/sec.), for $0 \le t \le 8$.
   If the object is initially at rest, find the following:
   (i) the velocity after 5 seconds
   (ii) the distance travelled in the first 2 seconds
   (iii) the acceleration when the velocity is 8 (m/sec).

1. (i) $\dfrac{dy}{dx} = 4x - 5$ and $y(2) = 3$

$$\therefore y = \int (4x-5)dx = \frac{4x^2}{2} - 5x + C$$

$$\therefore y = 2x^2 - 5x + C$$

Since $y = 3$ when $x = 2$,

$$3 = 2 \times 2^2 - 5 \times 2 + C = 8 - 10 + C = -2 + C$$

$$\therefore C = 3 + 2 = 5$$

$$\therefore y = 2x^2 - 5x + 5$$

(ii) $\dfrac{dy}{dx} = 6x^2 + 8x - 5$ and $y(0) = -1$

$$\therefore y = \int (6x^2 + 8x - 5)dx = \frac{6x^3}{3} + \frac{8x^2}{2} - 5x + C$$

$$\therefore y = 2x^3 + 4x^2 - 5x + C$$

Since $y = -1$ when $x = 0$,

$$-1 = 0 + 0 - 0 + C$$

$$\therefore C = -1$$

$$\therefore y = 2x^3 + 4x^2 - 5x - 1$$

(iii) $\dfrac{dy}{dx} = 2e^{-6x}$ and $y(0) = 0$

$$\therefore y = \int 2e^{-6x}dx = \frac{2e^{-6x}}{-6} + C$$

$$\therefore y = \frac{-e^{-6x}}{3} + C$$

Since $y = 0$ when $x = 0$,

$$0 = \frac{-e^0}{3} + C = \frac{-1}{3} + C$$

$$\therefore C = \frac{1}{3}$$

$$\therefore y = \frac{-e^{-6x}}{3} + \frac{1}{3} = \frac{1 - e^{-6x}}{3}.$$

2. $a = 12t + 2$, for $0 \le t \le 8$.

As the object is initially at rest, $v(0) = 0$

(i) Since $v = \int a\,dt = \int (12t+2)dt = \dfrac{12t^2}{2} + 2t + C$,

$$v = 6t^2 + 2t + C.$$

Since $v = 0$ when $t = 0$,

$$0 = 0 + 0 + C \qquad \therefore C = 0.$$

$$\therefore v = 6t^2 + 2t.$$

When $t = 5$, $v = 6 \times 5^2 + 2 \times 5 = 150 + 10 = 160$.

So, the velocity after 5 seconds is 160 (m/sec).

(ii) Since $s = \int v\,dt = \int (6t^2 + 2t)dt = \dfrac{6t^3}{3} + \dfrac{2t^2}{2} + C$,

$$s = 2t^3 + t^2 + C$$

Since $s = 0$ when $t = 0$,

$$0 = 0 + 0 + C \qquad \therefore C = 0.$$

$$s = 2t^3 + t^2.$$

When $t = 2$, $s = 2 \times 2^3 + 2^2 = 16 + 4 = 20$

So, the distance travelled in the first 2 seconds is 20 (m).

(iii) When $v = 8$, $6t^2 + 2t = 8$. Dividing by 2 gives

$3t^2 + t = 4$, i.e. $3t^2 + t - 4 = 0$

Using the Quadratic Formula (with $a = 3$, $b = 1$, $c = -4$),

$$t = \frac{-1 \pm \sqrt{1^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

$$\therefore t = \frac{-1 \pm \sqrt{1 + 48}}{6} = \frac{-1 \pm \sqrt{49}}{6}$$

$$\therefore t = \frac{-1 \pm 7}{6}. \qquad \text{So, } t = \frac{-8}{6} = \frac{-4}{3}, \text{ and } t = \frac{6}{6} = 1.$$

As $0 \le t \le 8$, the only permissible value of t is $t = 1$.

Since $a = 12t + 2$, when $t = 1$, $a = 12 + 2 = 14$

So, the acceleration when the velocity is 8 (m/sec), i.e. after 1 sec, is 14 (m/sec/sec).

Problems

1. Find y given that

   (i) $\frac{dy}{dx} = 8x - 1$ and $y(2) = 5$

   (ii) $\frac{dy}{dx} = 9x^2 + 12x - 4$ and $y(1) = 8$

   (iii) $\frac{dy}{dx} = 2e^{8x}$ and $y(0) = 1$.

2. The acceleration of an object is given by:
   $a = 72t - 12t^2 + 1$ (m/sec/sec.), for $0 \le t \le 6$.
   If the object is initially at rest, find the following

   (i) the velocity after 5 seconds

   (ii) the distance travelled in the first 2 seconds

   (iii) the velocity when the acceleration is 109 (m/sec/sec).

Answers

1. (i) $y = 4x^2 - x - 9$ (ii) $y = 3x^3 + 6x^2 - 4x + 3$

   (iii) $y = \frac{3 + e^{8x}}{4}$.

2. Velocity $v = 36t^2 - 4t^3 + t$; displacement $s = 12t^3 - t^4 + \frac{t^2}{2}$.

   (i) 405 (m/sec) (ii) 82 (m)

   (iii) When $a = 109$, $72t - 12t^2 + 1 = 109$

   $\therefore 12t^2 - 72t + 108 = 0$ Dividing by 12 gives

   $t^2 - 6t + 9 = 0$, i.e. $(t - 3)^2 = 0$ $\therefore t = 3$.

   When $t = 3$,

   $v = 36 \times 3^2 - 4 \times 3^3 + 3 = 324 - 108 + 3 = 219$.

   $\therefore v = 219$ (m/sec).

7.4 The definite integral

1. $\int_a^b f(x)dx$ is called the **definite integral** of f for $a \le x \le b$. The constant ***a* and *b* are called the terminals of integration**. By definition,

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a),$$

where $F(x)$ is any anti-derivative of $f(x)$.

Note that **the definite integral is a number** (not a function), whose value is obtained by evaluating $F(x)$ at the top terminal b, and the bottom terminal a, before subtracting in the correct order, $F(b) - F(a)$.

2. For definite integrals, the constant C is not needed, as it cancels after subtraction, i.e.

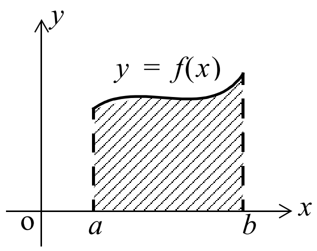
$$\int_a^b f(x)dx = [F(x) + C]_a^b = \{F(b) + C\} - \{F(a) + C\}$$

$$= F(b) + C - F(a) - C = F(b) - F(a).$$

3. It is important to note that the definite integral $\int_a^b f(x)dx$ represents an **area only if the curve** $y = f(x)$ **is non-negative**, i.e.

$$f(x) \geq 0 \quad \text{for } a \leq x \leq b.$$

The relevant area is shaded in the diagram below.



4. In general, **the definite integral may be positive, negative or zero**, and evaluating a definite integral should not be confused with finding an area.

5. Since the definite integral is a number, the variable of integration is a 'dummy' variable. So,

$\int_a^b f(x)dx$, $\int_a^b f(t)dt$, and $\int_a^b f(z)dz$ all represent the same number.

Examples

1. Evaluate the following definite integrals

(i) $I = \int_0^6 x^2 dx$ (ii) $I = \int_1^9 \frac{1}{\sqrt{x}} dx$

(iii) $I = \int_0^3 (x^2 - 4)dx$ (iv) $I = \int_1^3 (2 - x)dx$

(v) $I = \int_0^1 (2 - 3x^2)dx$ (vi) $I = \int_0^6 3\,dx$

(vii) $I = \int_1^9 \left(\frac{3}{\sqrt{x}} - 2\right)dx$ (viii) $I = \int_1^3 \frac{1}{x^4}dx$.

2. Evaluate the following definite integrals:

(i) $I = \int_0^{\pi/2} \sin x\,dx$ (ii) $I = \int_0^{\ln 2} 8e^{4x}dx$

(iii) $I = \int_0^{\pi} (2\cos x - \sin 2x)dx$ (iv) $I = \int_2^8 \frac{1}{4x}dx$.

1. (i) $I = \int_0^6 x^2 dx \quad = \left[\frac{x^3}{3}\right]_0^6 \quad = \frac{1}{3}[x^3]_0^6$

$\therefore I = \frac{1}{3}[6^3 - 0^3] \quad = \frac{1}{3}(216 - 0) \quad = 72$

$\therefore I = 72$

(ii) $I = \int_1^9 \frac{1}{\sqrt{x}}dx \quad = \int_1^9 x^{-1/2}dx$

$\therefore I = \left[\frac{x^{1/2}}{1/2}\right]_1^9 \quad = 2[x^{1/2}]_1^9$

$\therefore I = 2[9^{1/2} - 1^{1/2}] \quad = 2(3 - 1) \quad = 2 \times 2 = 4$

$\therefore I = 4$

(iii) $I = \int_0^3 (x^2 - 4)dx \quad = \left[\frac{x^3}{3} - 4x\right]_0^3$

$\therefore I = \left(\frac{3^3}{3} - 4 \times 3\right) - (0 - 0) \quad = \frac{27}{3} - 12 - 0$

$\therefore I = 9 - 12$

$\therefore I = -3$.

(iv) $I = \int_1^3 (2 - x)dx \quad = \left[2x - \frac{x^2}{2}\right]_1^3$

$$\therefore I = \left(2 \times 3 - \frac{3^2}{2}\right) - \left(2 \times 1 - \frac{1^2}{2}\right) \qquad = 6 - \frac{9}{2} - \left(2 - \frac{1}{2}\right)$$

$$\therefore I = 6 - \frac{9}{2} - 2 + \frac{1}{2} \qquad = 4 + \frac{1}{2} - \frac{9}{2}$$

$$\therefore I = \frac{9}{2} - \frac{9}{2} = 0$$

$$\therefore I = 0.$$

(v) $$I = \int_0^1 (2 - 3x^2)dx \qquad = \left[2x - \frac{3x^3}{3}\right]_0^1$$

$$\therefore I = [2x - x^3]_0^1 \qquad = (2 \times 1 - 1^3) - (0 - 0)$$

$$\therefore I = 2 - 1 - 0 = 1$$

$$\therefore I = 1.$$

(vi) $$I = \int_0^6 3dx \qquad = [3x]_0^6 \qquad = (3 \times 6) - (3 \times 0)$$

$$\therefore I = 18 - 0 = 18$$

$$\therefore I = 18.$$

(vii) $$I = \int_1^9 \left(\frac{3}{\sqrt{x}} - 2\right)dx \qquad = \int_1^9 (3x^{-1/2} - 2)dx$$

$$\therefore I = \left[\frac{3x^{1/2}}{1/2} - 2x\right]_1^9$$

$$\therefore I = [6x^{1/2} - 2x]_1^9 = (6 \times 9^{1/2} - 2 \times 9) - (6 \times 1^{1/2} - 2 \times 1)$$

$$\therefore I = 6 \times 3 - 2 \times 9 - 6 \times 1 + 2 \times 1$$

$$\therefore I = 18 - 18 - 6 + 2$$

$$\therefore I = -4.$$

(viii) $$I = \int_1^3 \frac{1}{x^4}dx \qquad = \int_1^3 x^{-4}dx$$

$$\therefore I = \left[\frac{x^{-3}}{-3}\right]_1^3 = -\frac{1}{3}\left[\frac{1}{x^3}\right]_1^3$$

$$\therefore I = -\frac{1}{3}\left(\frac{1}{3^3} - \frac{1}{1^3}\right) = -\frac{1}{3}\left(\frac{1}{27} - \frac{1}{1}\right) = -\frac{1}{3} \times -\frac{26}{27}$$

$$\therefore I = \frac{26}{81}.$$

2. (i) $$I = \int_0^{\pi/2} \sin x\,dx \qquad = [-\cos x]_0^{\pi/2}$$

$$\therefore I = -[\cos x]_0^{\pi/2} \qquad = -\left(\cos\frac{\pi}{2} - \cos 0\right)$$

$$\therefore I = -(0-1) = 0+1 = 1$$

$$\therefore I = 1.$$

(ii) $$I = \int_0^{\ln 2} 8e^{4x}dx \qquad = \left[\frac{8e^{4x}}{4}\right]_0^{\ln 2}$$

$$\therefore I = [2e^{4x}]_0^{\ln 2} \qquad = 2[e^{4x}]_0^{\ln 2}$$

$$\therefore I = 2(e^{4\ln 2} - e^0)$$

Using log. rules, $4\ln 2 = \ln(2^4) = \ln 16$

$$\therefore e^{4\ln 2} = e^{\ln 16} = 16.$$

$$\therefore I = 2(16-1) = 2\times 15 = 30$$

$$\therefore I = 30.$$

(iii) $$I = \int_0^{\pi} (2\cos x - \sin 2x)dx$$

$$\therefore I = \left[2\sin x - \left(\frac{-1}{2}\cos 2x\right)\right]_0^{\pi} \qquad = \left[2\sin x + \frac{1}{2}\cos 2x\right]_0^{\pi}$$

$$\therefore I = \left(2\sin\pi + \frac{1}{2}\cos 2\pi\right) - \left(2\sin 0 + \frac{1}{2}\cos 0\right)$$

$$\therefore I = \left(2\times 0 + \frac{1}{2}\times 1\right) - \left(2\times 0 + \frac{1}{2}\times 1\right)$$

$$\therefore I = 0 + \frac{1}{2} - 0 - \frac{1}{2} = 0$$

$$\therefore I = 0.$$

(iv) $$I = \int_2^8 \frac{1}{4x}dx \qquad = \frac{1}{4}\int_2^8 \frac{1}{x}dx$$

$$\therefore I = \frac{1}{4}[\ln|x|]_2^8 \qquad = \frac{1}{4}(\ln|8| - \ln|2|)$$

$$\therefore I = \frac{1}{4}(\ln 8 - \ln 2) \qquad = \frac{1}{4}\ln\left(\frac{8}{2}\right) = \frac{1}{4}\ln 4$$

$$\therefore I = \frac{1}{4}\ln 4 .$$

Problems

1. Evaluate the following definite integrals

(i) $I = \int_0^4 x^3 dx$ (ii) $I = \int_4^{16} \frac{2}{\sqrt{x}} dx$

(iii) $I = \int_3^6 (2x^2 - 9)dx$ (iv) $I = \int_1^2 (3 - 2x)dx$

(v) $I = \int_0^2 (1 - 6x^2)dx$ (vi) $I = \int_1^2 \frac{1}{x^3} dx.$

2. Evaluate the following definite integrals:

(i) $I = \int_0^{\pi/2} \cos x dx$ (ii) $I = \int_0^{\ln 2} 4e^{2x} dx$

(iii) $I = \int_0^{\pi} (3\cos 3x - \sin x)dx$ (iv) $I = \int_3^9 \frac{1}{5x} dx .$

Answers

1. (i) $I = \int_0^4 x^3 dx = \; = \left[\frac{x^4}{4}\right]_0^4 = 64$

(ii) $I = \int_4^{16} \frac{2}{\sqrt{x}} dx = 4[x^{1/2}]_1^{16} = 12$

(iii) $I = \int_3^6 (2x^2 - 9)dx = \left[\frac{2x^3}{3} - 9x\right]_3^6 = 99$

(iv) $I = \int_1^2 (3 - 2x)dx = [3x - x^2]_1^2 = 0$

(v) $I = \int_0^2 (1 - 6x^2)dx = [x - 2x^3]_0^2 = -14$

(vi) $I = \int_1^2 \frac{1}{x^3} dx = \left[\frac{x^{-2}}{-2}\right]_1^2 = \frac{3}{8}.$

2. (i) $I = \int_0^{\pi/2} \cos x dx = \quad = [\sin x]_0^{\pi/2} = 1$

(ii) $I = \int_0^{\ln 2} 4e^{2x} dx = \left[\frac{4e^{2x}}{2}\right]_0^{\ln 2} = 6$

(iii) $I = \int_0^{\pi} (3\cos 3x - \sin x) dx = [\sin 3x + \cos x]_0^{\pi} = -2$

(iv) $I = \int_3^9 \frac{1}{5x} dx = \frac{1}{5}[\ln|x|]_3^9 = \frac{1}{5}\ln 3.$

7.5 Area between curves

1. Given the curve $y = f(x)$, the definite integral $\int_a^b f(x) dx$ represents an area only if $f(x) \geq 0$ for $a \leq x \leq b$.

   If, however, $f(x) \leq 0$ for $a \leq x \leq b$, the definite integral $\int_a^b f(x) dx$ is negative, and does not represent an area.

   In general, evaluating $\int_a^b f(x) dx$ is not the same as finding the area contained between the curve $y = f(x)$ and the x-axis, for $a \leq x \leq b$.

   To find a required area, the curve (or curves) must be sketched first, before forming the definite integral corresponding to the area.

2. The area contained between two curves can be found using the following rule.

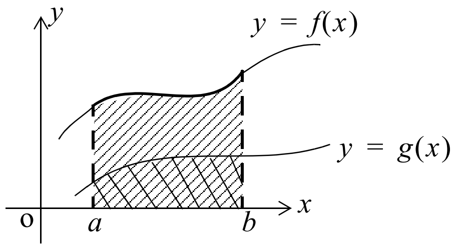
AREAS

The area enclosed between $y = f(x)$ and $y = g(x)$

(where $f(x) \geq g(x)$ for $a \leq x \leq b$) is given by:

$$A = \int_a^b [f(x) - g(x)] dx.$$

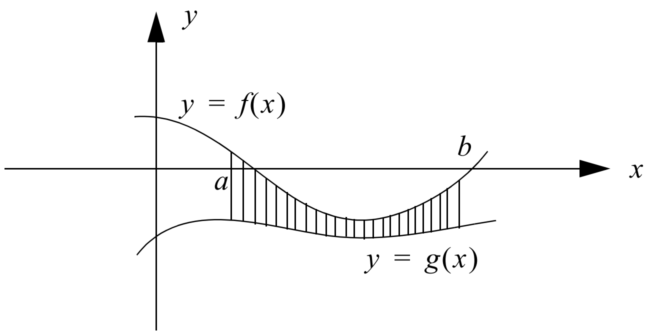
The formula is derived by subtracting the area underneath the curve $y = g(x)$ from the area underneath the curve $y = f(x)$, as shown in the following diagram.



i.e. $A = \int_a^b f(x)dx - \int_a^b g(x)dx$

$\therefore A = \int_a^b [f(x) - g(x)]dx.$

3. The above integral determines the required area in all cases where $f(x) \geq g(x)$ for $a \leq x \leq b$, irrespective of whether $f(x)$ and $g(x)$ themselves take negative values. Provided $f(x) \geq g(x)$, the difference $f(x)–g(x)$ is positive, and the area is given by $A = \int_a^b [f(x) - g(x)]dx$, as shown below.



4. Once the curves $y = f(x)$ and $y = g(x)$ have been sketched, **the function to be integrated is simply the difference between the y value on the top curve and the y value on the bottom curve.**

5. When sketching the curves $y = f(x)$ and $y = g(x)$, **all intersections should be found**. Often, the terminals of integration are determined by the intersections of the curves.

Examples

1. Find the area bounded by the curve $y = x^2 - 4$ and the x-axis for $0 \leq x \leq 2$.

2. (i) Evaluate $I = \int_0^2 (1 - x^2)dx$

(ii) Find the area bounded by the curve $y = 1 - x^2$ and the x-axis for $0 \le x \le 2$.

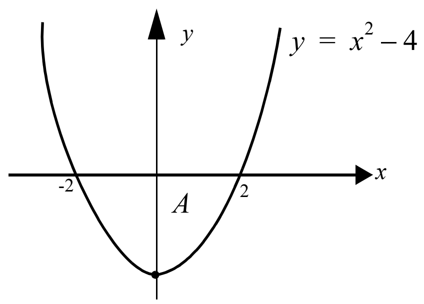
3. Find the area bounded by the curve $y = 6x^2 + 7$ and the x-axis for $1 \le x \le 3$

4. Find the area bounded by the curve $y = 3x^2$ and the straight line $y = 6x$

5. Find the area bounded by the curve $y = x^2 + 2x + 1$ and the straight line $y = 5x + 1$.

1. The curve $y = x^2 - 4$ is a cup shaped parabola with a y-intercept of -4.

   For the x-intercepts, $x^2 - 4 = 0$ $\quad \therefore x^2 = 4$ $\quad \therefore x = \pm 2$.



The top 'curve' is $y = 0$, and the bottom curve is $y = x^2 - 4$.

So, $f(x) - g(x) = 0 - (x^2 - 4) = -x^2 + 4$.

$$\therefore A = \int_a^b [f(x) - g(x)]dx = \int_0^2 (-x^2 + 4)dx$$

$$\therefore A = \left[\frac{-x^3}{3} + 4x\right]_0^2 \qquad = \left(\frac{-2^3}{3} + 4 \times 2\right) - (0 + 0)$$

$$\therefore A = \frac{-8}{3} + 8 = \frac{-8 + 24}{3} = \frac{16}{3}$$

$$\therefore A = \frac{16}{3}$$

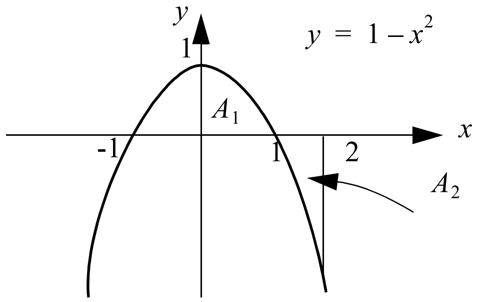
2. (i) $$I = \int_0^2 (1 - x^2)dx \qquad = \left[x - \frac{x^3}{3}\right]_0^2$$

$$\therefore I = \left(2 - \frac{2^3}{3}\right) - (0 - 0) \qquad = 2 - \frac{8}{3} = \frac{6 - 8}{3}$$

$$\therefore I = \frac{-2}{3}$$

(ii) The curve $y = 1 - x^2$ is a frown shaped parabola with a y-intercept of 1 .

For the x-intercepts, $1 - x^2 = 0 \qquad \therefore x^2 = 1 \qquad \therefore x = \pm 1$.



The area consists of 2 distinct areas, A_1 (for $0 \le x \le 1$), and A_2 (for $1 \le x \le 2$).

For $0 \le x \le 1$, the top curve is $y = 1 - x^2$, and the bottom 'curve' is $y = 0$.

So, $f(x) - g(x) = 1 - x^2 - 0 = 1 - x^2$.

$$\therefore A_1 = \int_0^1 (1 - x^2)dx \qquad = \left[x - \frac{x^3}{3}\right]_0^1$$

$$\therefore A_1 = \left(1 - \frac{1^3}{3}\right) - (0 - 0) \qquad = 1 - \frac{1}{3} = \frac{2}{3}$$

For $1 \le x \le 2$, the top 'curve' is $y = 0$, and the bottom curve is $y = 1 - x^2$.

So, $f(x) - g(x) = 0 - (1 - x^2) = -1 + x^2$.

$$\therefore A_2 = \int_1^2 (-1 + x^2)dx \qquad = \left[-x + \frac{x^3}{3}\right]_1^2$$

$$\therefore A_2 = \left(-2 + \frac{2^3}{3}\right) - \left(-1 + \frac{1^3}{3}\right) \qquad = -2 + \frac{8}{3} + 1 - \frac{1}{3}$$

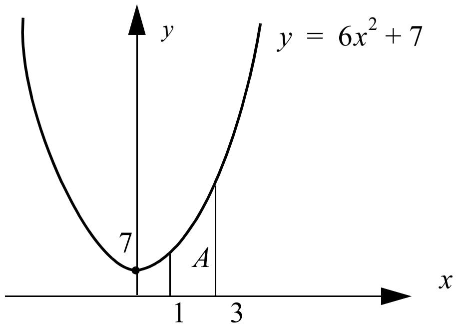
$$\therefore A_2 = -1 + \frac{7}{3} = \frac{-3 + 7}{3} = \frac{4}{3} .$$

$$\therefore A = A_1 + A_2 = \frac{2}{3} + \frac{4}{3} = \frac{6}{3} = 2$$

$$\therefore A = 2 .$$

3. The curve $y = 6x^2 + 7$ is a cup shaped parabola with a y-intercept of 7 .

As $y = 6x^2 + 7 \geq 0$ for all values of x, there are no x-intercepts.



The top curve is $y = 6x^2 + 7$, and the bottom 'curve' is $y = 0$.

So, $f(x)–g(x) = 6x^2 + 7 - 0 = 6x^2 + 7$.

$$\therefore A = \int_1^3 (6x^2 + 7)dx \qquad = \left[\frac{6x^3}{3} + 7x\right]_1^3$$

$$\therefore A = [2x^3 + 7x]_1^3 \qquad = (2 \times 3^3 + 7 \times 3) - (2 \times 1^3 + 7 \times 1)$$

$$\therefore A = (54 + 21) - (2 + 7) \qquad = 75 - 9 = 66$$

$$\therefore A = 66 .$$

4. The curve $y = 3x^2$ is a cup shaped parabola with a y-intercept of 0 .

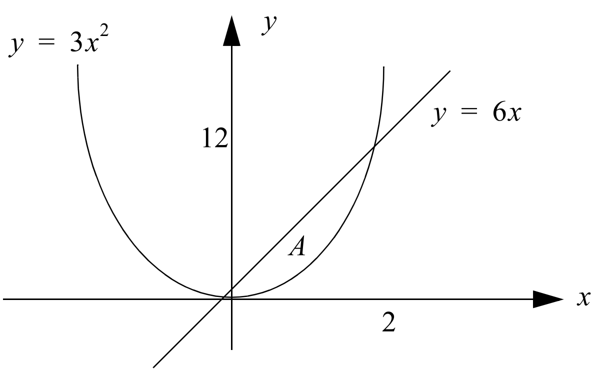
   For the x-intercepts, $3x^2 = 0 \qquad \therefore x^2 = 0 \qquad \therefore x = 0$.

   The straight line $y = 6x$ has slope 6, and passes through the origin.

   The curve and the straight line intersect when $3x^2 = 6x$, i.e. when

   $3x^2 - 6x = 0 \qquad \therefore 3x(x - 2) = 0 \qquad \therefore x = 0$ and $x = 2$.

   The graphs can be sketched without finding the corresponding y-values at the points of intersection, but can be found easily using either equation. When $x = 0$, $y = 6 \times 0 = 0$, and, when $x = 2$, $y = 6 \times 2 = 12$.



From the diagram, the terminals of integration are $x = 0$ and $x = 2$. The top 'curve' is $y = 6x$, and the bottom curve is $y = 3x^2$.

So, $f(x) - g(x) = 6x - 3x^2$.

$$\therefore A = \int_0^2 (6x - 3x^2)\,dx \qquad = \left[\frac{6x^2}{2} - \frac{3x^3}{3}\right]_0^2$$

$$\therefore A = [3x^2 - x^3]_0^2 \qquad = (3 \times 2^2 - 2^3) - (0 - 0)$$

$$\therefore A = 12 - 8 = 4$$

$$\therefore A = 4.$$

5. The curve $y = x^2 + 2x + 1$ is a cup shaped parabola with a y-intercept of 1.

For the x-intercepts,

$$x^2 + 2x + 1 = 0 \qquad \therefore (x+1)^2 = 0 \qquad \therefore x = -1.$$

The straight line $y = 5x + 1$ has slope 5, and a y-intercept of 1.

When $y = 0$, $5x + 1 = 0$ $\qquad \therefore 5x = -1$ $\qquad \therefore x = \frac{-1}{5}$.

The curve and the straight line intersect when $x^2 + 2x + 1 = 5x + 1$,

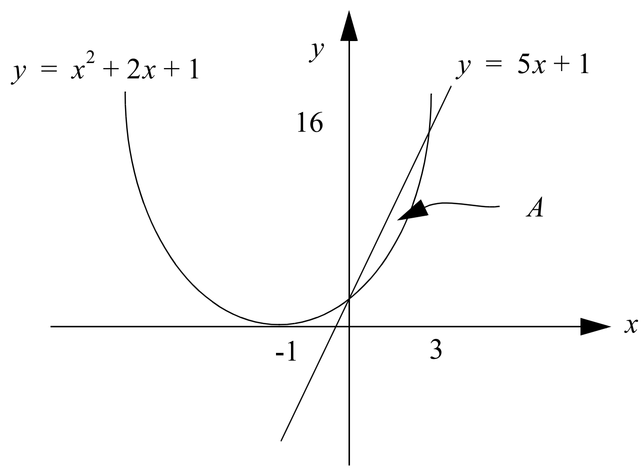
i.e. when $\quad x^2 + 2x = 5x \qquad \therefore x^2 - 3x = 0 \qquad \therefore x(x-3) = 0$,

i.e. $x = 0$ and $x = 3$.

When $x = 0$, $y = 0 + 1 = 1$.

When $x = 3$, $y = 5 \times 3 + 1 = 15 + 1 = 16$.

So, the intersections are $(0, 1)$, and $(3, 16)$.



From the diagram, the terminals of integration are $x = 0$ and $x = 3$.
The top 'curve' is $y = 5x + 1$, and the bottom curve is $y = x^2 + 2x + 1$.

So,

$$f(x) - g(x) = 5x + 1 - (x^2 + 2x + 1) = 5x + 1 - x^2 - 2x - 1.$$

$$f(x) - g(x) = 3x - x^2$$

$$\therefore A = \int_0^3 (3x - x^2)\,dx \qquad = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$\therefore A = \left(\frac{3 \times 3^2}{2} - \frac{3^3}{3}\right) - (0 - 0) \qquad = \frac{27}{2} - 9 = \frac{27 - 18}{2}$$

$$\therefore A = \frac{9}{2}.$$

Problems

1. Find the area bounded by the curve $y = 3x^2 + 1$ and the x-axis for $1 \le x \le 3$.

2. (i) Evaluate $I = \int_0^3 (4 - x^2)\,dx$

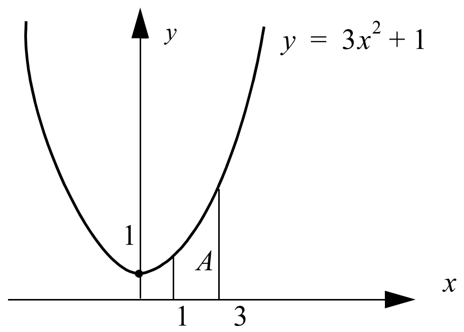
   (ii) Find the area bounded by the curve $y = 4 - x^2$ and the x-axis for $0 \le x \le 3$.

3. Find the area bounded by the curve $y = 12x^2 + 5$ and the x-axis for $1 \le x \le 2$

4. Find the area bounded by the curve $y = x^2$ and the straight line $y = 4x - 3$

Answers

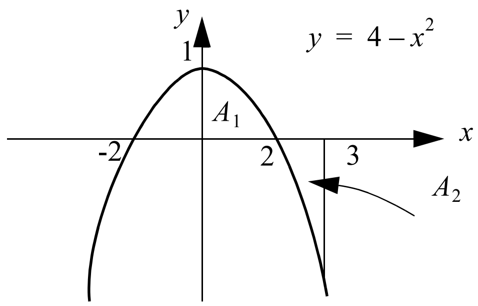
1.



$$A = \int_1^3 (3x^2 + 1)dx \qquad = [x^3 + x]_1^3 \qquad \therefore A = 28.$$

2. (i) $$I = \int_0^3 (4 - x^2)dx \qquad = \left[4x - \frac{x^3}{3}\right]_0^3 \qquad \therefore I = 3.$$

(ii)



$$A_1 = \int_0^2 (4 - x^2)dx \qquad = \left[4x - \frac{x^3}{3}\right]_0^2$$

$$\therefore A_1 = \frac{16}{3}.$$

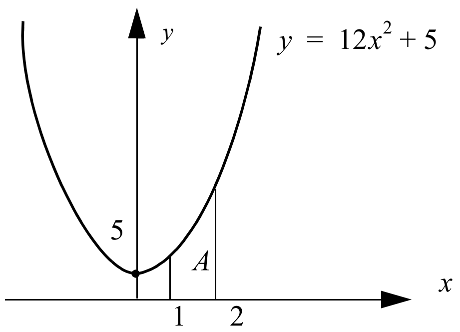
$$A_2 = \int_2^3 (-4 + x^2)dx \qquad = \left[-4x + \frac{x^3}{3}\right]_2^3$$

$$\therefore A_2 = \frac{7}{3}.$$

$$\therefore A = A_1 + A_2 = \frac{16}{3} + \frac{7}{3} = \frac{23}{3}$$

$$\therefore A = \frac{23}{3}.$$

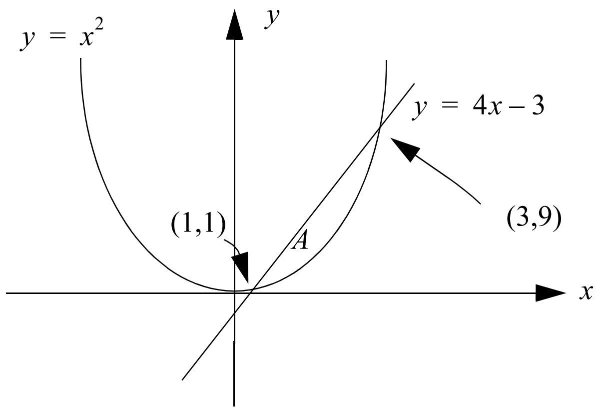
3.



$$A = \int_1^2 (12x^2 + 5)dx \qquad = [4x^3 + 5x]_1^2$$

$$\therefore A = 33\,.$$

4.



$$A = \int_1^3 (4x - 3 - x^2)dx \qquad = \left[2x^2 - 3x - \frac{x^3}{3}\right]_1^3$$

$$\therefore A = \frac{4}{3}\,.$$