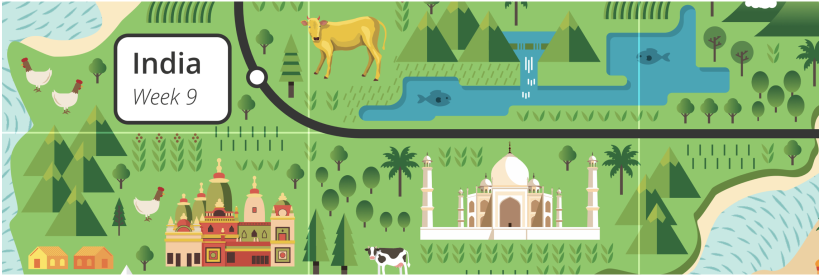
SIT190 - WEEK 9



- Antidifferentiation or Integration
  - Power rule
  - Rules for logarithmic and exponential functions
  - Rules for trigonometric functions
- Finding the constant
- Indefinite integral
- Kinematics

Your band has safely arrived in Kerala, India and must begin the search for the missing notebook. A scribe at the school introduces you to some of their achievements.

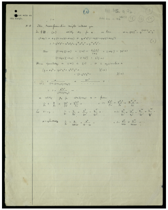


Image: The First Page of the So-called Lost Notebook, Add.ms.a.94, page 1 by Trinity College, Cambridge

Kerala school of astronomy and mathematics

Madhava, a mathematician and astronomer, was considered a founder of the school. Some of his contributions include:

- Infinite series
  - $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$
  - $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
  - $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Discover

Your band applies your differentiation skills to demonstrate that you too can prove the cosine result.

> Beginning with $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$ you take the derivative of both sides.

(Remember the $\frac{x^n}{n!} = \frac{1}{n!}x^n$ so you can apply the power rule to the terms on the right hand side.)

Circumference of a circle

Madhava is credited with the following method for finding the circumference c of a circle with diameter d:

$$c = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \dots$$

Now, the circumference of a circle of radius r is $2\pi r = \pi d$, so

$$\pi d = \frac{4d}{1} - \frac{4d}{3} + \frac{4d}{5} - \frac{4d}{7} + \dots \Rightarrow \pi = \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots$$

and so

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

‘*Madhavan invented calculus . . . the key conceptual step in this was the recognition that local approximation by linear functions (tangents) in other words differentiation, and their subsequent summing up, integration, . . . , the earliest version of what came to be known . . . as the fundamental theorem of calculus.* ’

P.P. Divakaran. The First Textbook of Calculus: ‘Yuktibhasa’. Journal of Indian Philosophy. 35: p. 417–443, 2007.

Undoing the Derivative

- In previous workshops, we have *differentiated* a function to find the *derivative*.
- We *anti-differentiate* or *integrate* the derivative to find the original function.
- Power rule: $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$.
  1. Multiply by the power n
  2. Reduce the power to $n - 1$
- Inverse of the power rule: $\int x^n dx = \frac{x^{n+1}}{n+1} + C$, $n \neq -1$.
  For example: $\int x^3 dx = \frac{x^4}{4} + C$
  1. Increase the power to $n + 1$
  2. Divide by the new power, that is, $n + 1$

Can you explain why this rule does not work for the case where $n = -1$?

Undoing the Derivative

- The inverse power rule can be used to integrate a term with a constant factor k.
  For example, $\int 3x^2 dx = \frac{3x^3}{3} + C = x^3 + C$.
- A function that consists of the sum of terms, can be integrated term by term.
  For example,
  $\int (x^2 + x + 1) dx = \int x^2 dx + \int x dx + \int 1 dx = \frac{x^3}{3} + \frac{x^2}{2} + x + C$.
- An example, with constant factors and multiple terms:
  $\int (3x^2 + 4x - 1) dx = \int 3x^2 dx + \int 4x dx - \int 1 dx$
  $= x^3 + 2x^2 - x + C$.

Undoing the Derivative

Identify which of the following functions have derivative $f'(x) = 3x^2 - 4x + 6$.

1. $f(x) = x^3 - 4x^2 + 6x - 10$
2. $f(x) = x^3 - 2x^2 + 6x + 10$
3. $f(x) = x^3 - 2x^2 - 6x - 10$
4. $f(x) = x^3 - 2x^2 + 6x - 2$
5. $f(x) = x^3 - 2x^2 + 6x - 24$
6. $f(x) = x^3 - 2x^2 + 6x - e^2$

Reflection

1. Can you explain why there is a constant C in the integral $\int(3x^2 - 4x + 6)dx$?
2. Why do you think we write
   $\int(6x^2 - 3x + 2)dx = \int 6x^2dx - \int 3xdx + \int 2dx$
   $= 2x^3 - \frac{3}{2}x^2 + 2x + C$
   and not
   $\int(6x^2 - 3x + 2)dx = \int 6x^2dx - \int 3xdx + \int 2dx =$
   $2x^3 + C_1 - \frac{3}{2}x^2 + C_2 + 2x + C_3$?

Integration - identifying the constant

Anti-differentiation recovers a family of functions (one for every value of C).

> Consider the integral $\int(2x+2)dx = x^2 + 2x + C$.
> Sketch the graph when $C = -3$, $C = 0$, and $C = 1$.
> What effect does C have on the graph in terms of shape and position?

- To find the value of C, we need additional information.

Finding C

Solve $y = \int(3x^2 + 14x + 2)dx$. Your band randomly identifies a point (x, y) by randomly selecting two cards.

For example, if you select first the 2 of hearts and then the 3 of diamonds, then your point is $(2, 3)$.

Use this information to determine the constant C.

The Taxicab Numbers

$$y = \int (3x^2 + 14x + 2)dx = x^3 + 7x^2 + 2x + C$$

If the point had been $(10, 1729)$ then you would have found

$$1729 = 1000 + 700 + 20 + C \Rightarrow C = 9.$$

Mathematician G.H. Hardy related the story of telling Ramanujan that he had travelled in Taxi Number 1729 and remarking that the number was a dull one. Ramanujan replied that it was a very interesting number as it was the smallest number that could be expressed as the sum of two cubes in two different ways:

$$1729 = 10^3 + 9^3 = 1^3 + 12^3.$$

Locating the room containing the lost notebook

Your band is advised that finding the constants in the following integrals will identify the floor and room number where the missing notebook is hidden.

- **Floor:** Find $y = \int (x^{\frac{-1}{2}} - 3x^{\frac{1}{2}} - 2x^{\frac{-3}{2}})dx$ where $(1, 6)$ is a point on the curve.
- **Room Number:** Find $y = \int (\frac{9}{x^4} - \frac{8}{x^3})dx$ where $(2, \frac{21}{8})$ is a point on the curve.

More Integration Rules

In previous weeks, we found the derivative of exponentials, logarithms and trigonometric functions.

Complete the rules for the following integrals:

1. $\int e^{kx} dx =$
2. $\int \frac{1}{x} dx =$
3. $\int \sin(kx) dx =$
4. $\int \cos(kx) dx =$

Entering the room

Your band enters the room and begins to search for the pages of the missing notebook. The constants in the following integrals will identify the bookshelf and shelf number where the missing notebook is hidden.

- **Bookshelf:** Find $y = \int (12\cos(3x) - 4\sin(2x))$ where $(\frac{\pi}{6}, 7)$ is a point on the curve.
- **Shelf Number:** Find $y = \int (\frac{4}{e^{2x}} + 4e^{2x})dx$ where the y intercept is $(0, 4)$

The integral of $\frac{1}{x}$

The integral

$$\int \frac{1}{x} dx = \ln(|x|) + C$$

uses the absolute value of x denoted $|x|$, where $|x| = x$, if $x \geq 0$, and $|x| = -x$ if $x < 0$.

For example, $|-3| = 3$ and $|4| = 4$.

Find $y = \int 3x^{-1} dx$ where $(-e, 6)$ is a point on the curve.

An Aside

Why is the function $f(x) = |x|$ not differentiable at the point $x = 0$?
Informally, because the graph has a sharp point when $x = 0$ and so we cannot find a nice tangent line. The gradient of the function is -1 on one side of $x = 0$ and 1 on the other side.

An Aside

More formally, the limit (if it exists) at this point is

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} (\frac{|x|}{x}).$$

But the left limit

$$\lim_{x \to 0^-} \left(\frac{x}{-x} \right) = -1$$

does not equal the right limit

$$\lim_{x \to 0^+} \left(\frac{x}{x} \right) = 1.$$

Found

Your band discovers some pages that appear to be from the missing notebook under some old manuscripts on the shelf. You must now leave the building with the pages before you are discovered.

Rules for Integration

Function $f(x)$	Integral $\int f(x)dx$	Function $f(x)$	Integral $\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} + C,\ n \neq -1$	x^{-1}	$\ln\lvert x\rvert + C$
$\sin(kx)$	$\frac{-1}{k}\cos(kx) + C$	$\cos(kx)$	$\frac{1}{k}\sin(kx) + C$
e^{kx}	$\frac{1}{k}e^{kx} + C$		

Kinematics

Before your band can work out their travel to the final destination, they review their knowledge about the relationship between displacement, velocity and acceleration.

- Given a function that gives the displacement s of an object at time t.
- The velocity $v = \frac{ds}{dt}$.
- The acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

The functions v and a can be obtained by *differentation*. But now you have a new tool, *integration*.

Select the correct operation (differentiation or integration) in each of the following statements:

1. Velocity can be obtained by [differentiating/integrating] the displacement function.
2. Velocity can be obtained by [differentiating/integrating] the acceleration function.
3. Acceleration can be obtained by [differentiating/integrating] the velocity function.
4. Acceleration can be obtained by [differentiating/integrating] the displacement function twice.
5. Displacement can be obtained by [differentiating/integrating] the velocity function.
6. Displacement can be obtained by [differentiating/integrating] the acceleration function twice.

Some Examples

The velocity of an object at time t seconds is given by:
$v = 6t^2 - 24t,\ t \geq 0.$

1. Find the displacement s given the displacement was 64 metres at time $t = 0$.
2. Find the acceleration a.
3. What is the displacement when $a = 30$?

Some Examples

The velocity of particle at time t seconds is given by $10t - 6t^2 - 5$, $t \geq 0$.

1. Find the displacement s at time t given the displacement was 3 at time $t = 0$ seconds.
2. Find the displacement when the acceleration is $-2\ m/s^2$
3. Find the displacement when the velocity is $\frac{-3}{2}\ m/s$.

Some Examples

Suppose there are $V(t)$ litres of water in a container at time t minutes.

If $V'(t) = -50t$ and the container is empty when $t = 4$, find the original amount of water in the container.

Leaving India

Your band must now select the best mode of transport. You wish to select the mode that takes you to $s = 110$km in the least time.

- Transport Mode 1:
  - Has acceleration $a = 22 - 6t$, $t \geq 0$.
  - Has velocity $v = 9$ km/hr when $t = 1$.
  - Has displacement $s = 10$ when $t = 0$.
  - Reaches $s = 110$km when t is an integer and $3 < t < 9$.
- Transport Mode 2:
  - Has velocity $v = -2t + 4$, $t \geq 0$.
  - Has displacement $s = 53$ when $t = 13$.

> Which mode of transport does your team take? At what time will they arrive at the point with $s = 110$km?