Discrete Mathematics

Homework 8

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1. (b) We must recursively define a formula f_n as the number of distinct strings composed of n digits, where each digit is either '0', '1', or '2', such that no string has two consecutive '2's. We will refer to such a string as an n-digit valid string.

   We will first define the initial condition as

   $$f_0 = 1,$$

   meaning there is only one unique zero-digit valid string; i.e. the empty string ''.

   To define f_n recursively, we must find a way of counting all the valid strings of n digits, given that f_{n-1} is the number of $(n-1)$-digit valid strings. To calculate this number, we can multiply f_{n-1} by 3, to count the number of strings with n digits, because we can append either a '0', '1', or '2' to any of the strings with $n-1$ digits. However this will result is some extra invalid strings, namely those which ended with a '2' and we appended a '2' onto.

   In order to continue, we must have a way of knowing how many of these invalid strings we will have after we multiply f_{n-1} by 3, so that we may subtract them. The number of invalid strings is equal to the number of valid strings with $n-1$ digits which end with a '2'. We will now declare the formula g_n as the number of valid strings of n digits which end with a '2'.

   We can now complete our definition for f_n.

   $$f_0 = 1$$
   $$f_n = 3f_{n-1} - g_{n-1}$$

   All that remains now is to define g_n, or the number of valid n-digit strings ending with a '2'. Since to get n-digit strings, we append a digit to $(n-1)$-digit strings, and we can only append a '2' to those strings which do not end with '2', therefore g_n is equal to the number of valid $(n-1)$-digit strings, f_{n-1}, minus the number of valid $(n-1)$-digit strings ending with '2', g_{n-1}. Or more formally:

   $$g_0 = 0$$
   $$g_n = f_{n-1} - g_{n-1}$$

2. The number of sequences of 1's and 2's whose sum is n can be obtained by taking any such sequence whose sum is $n-1$ and appending a 1, or by taking a sequence whose sum is $n-2$ and appending a 2. Furthermore, there is one such sequences whose sum is 0 — namely the sequence with no terms — and one sequence whose sum is 1 — namely the sequence with just a single 1.

Thus we can define this as a recursive formula

$$f_0 = 1$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2}$$

This formula is identical to the Fibonacci sequence.

4. The number of n-letter words with the alphabet {0, 1, 2} and no repeating 1's or 2's can be found by taking all $n-1$ letter words, and appending any of our three symbols in our alphabet. We can append a 0 to all of the $n-1$ letter words; that is already f_{n-1} words. We can append a 1 or a 2 to anything of length $n-1$ ending in 0; this part gives us an additional $2 \times f_{n-2}$, because the number of $n-1$ letter words ending in 0 must be equal to the total number of $n-2$ letter words (since we can append a 0 to any word to make a legal word on the next length).

   All that remains is counting the number of words that we can make by appending a 1 to a shorter word ending with a 2, as well as the number of words we make by appending a 2 to the words ending in 1. The sum of both of these is equal to the number of $n-1$ letter words which end in either 1 or 2. This, in turn, is equal to the total number of $n-1$ letter words, minus the number of $n-1$ letter words ending in 0.

   Therefore we have

$$f_n = f_{n-1} + 2f_{n-2} + f_{n-1} - f_{n-2} = 2f_{n-1} + f_{n-2}$$

   Now we must specify the initial conditions f_1 and f_2. The number of one-letter words with no repeating 1's and 2's, f_1, is three. (The words are {0, 1, 2}.) The number of two-letter words, f_2 is seven. (These are {00, 01, 02, 10, 12, 20, 21}.) Finally we have

$$f_n = 2f_{n-1} + f_{n-2}$$
$$f_1 = 3$$
$$f_2 = 7$$

6. We must find a recursive formula and initial conditions to describe the number of n-letter words over {1, 2, 3, 4, 5} which contain no consecutive identical even digits.

   First of all, we can append a 1, 3, or 5 to any word of length $n-1$. That gives us $3f_{n-1}$ words so far.

   Secondly, we can append a 2 to any word ending in 1, 3, 4, or 5. Similarly we can append a 4 to any word not ending in 4. The number of words formed from these two groups is equal to f_{n-1} minus the number of $n-1$ letter words ending in 2, plus f_{n-1} minus the number of words ending in 4. In other words, this is equal to $2f_{n-1}$ minus the number of words ending in either 2 or 4.

   However, the number of words ending in 2 or 4 is equal to the total number of words of length $n-1$ minus the number of words ending in 1, 3, or 5. As we explained at earlier, the number of n-letter words ending in 1, 3, or 5 is equal to $3f_{n-1}$. So the number of $n-1$ letter words ending in 1, 3, or 5 would be $3f_{n-2}$. Therefore the number of $n-1$ letter words ending in 2 or 4 is

$$f_{n-1} - 3f_{n-2}$$

   Once we have this, we can calculate the number of words we can form by appending a 2 or 4. As we stated above, but this time substituting in the calculation we have arrived at, it is equal to

$$2f_{n-1} - (f_{n-1} - 3f_{n-2}) = f_{n-1} + 3f_{n-2}$$

Now we can add this to the number of words formed by appending a 1, 3, or 5, and we get the final formula

$$f_n = 3f_{n-1} + f_{n-1} + 3f_{n-2} = 4f_{n-1} + 3f_{n-2}$$

And once we work out the initial conditions we end up with

$$f_n = 4f_{n-1} + 3f_{n-2}$$
$$f_0 = 1$$
$$f_1 = 5$$

since the number of zero-letter words we can make is one (‘’), and the number of one-letter words is 5 (‘1’, ‘2’, ‘3’, ‘4’, and ‘5’).

7. (b) We seek a recursive formula, t_n, to describe the number of distinct ways to tile a $2 \times n$ board using identical tiles of dimensions 1×2 and 2×2.

   To begin, we can juxtapose a vertical 2×1 tile with any of the t_{n-1} board configurations of size $2 \times (n-1)$, as shown in Figure 1. That gives us so far t_{n-1} ways.

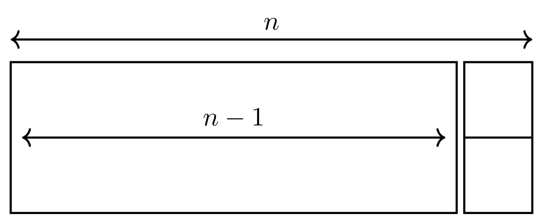

Figure 1: A board of size $2 \times n$ made by appending a vertical 2×1 tile to an arbitrary board of size $2 \times (n-1)$.

   Besides for these t_{n-1} ways, we also can exchange the last tile of any of the $2 \times (n-1)$ sized boards which ended with a vertical 2×1 tile with a 2×2 tile or with two horizontal 2×1 tiles. The number of such tiles which ended with a vertical 2×1 tile is precisely the same as the number of tiles of size $2 \times (n-2)$, since each of those was appended with a vertical tile to obtain a $2 \times (n-1)$ sized tile. This is equal to t_{n-2}. Since we can exchange any of these t_{n-2} tiles in two ways to form two distinct tilings, we must multiply t_{n-1} by two.

   Thus we have our recursive formula

$$t_n = t_{n-1} + 2t_{n-2}$$
$$t_0 = 1$$
$$t_1 = 1$$

12. (b) We seek a closed form formula for the recursive formula

$$a_n = 7a_{n-1} - 12a_{n-2}$$
$$a_0 = 1$$
$$a_1 = 1$$

We begin by assuming a solution of the form x^n, and substituting that into our recursive definition.

$$\begin{aligned}
a_n &= x^n \\
x^n &= 7x^{n-1} - 12x^{n-2} \\
x^{n-2} \times x^2 &= x^{n-2}(7x - 12) \\
x^2 &= 7x - 12 \\
0 &= x^2 - 7x + 12 \\
x &= \frac{7 \pm \sqrt{49 - 4 \times 12}}{2} \\
&= 3 \text{ and } 4
\end{aligned}$$

Now that we have solved the characteristic equation, we know that the closed form formula is of the form

$$a_n = A \cdot 3^n + B \cdot 4^n$$

Solving for A and B using the initial conditions we can obtain our closed form formula

$$\begin{aligned}
1 &= A \cdot 3^0 + B \cdot 4^0 \\
1 &= A \cdot 3^1 + B \cdot 4^1 \\
A + B &= 1 \\
3A + 4B &= 1 \\
3A + 3B &= 3 \\
B &= -2 \\
A - 2 &= 1 \\
A &= 3
\end{aligned}$$

Finally, our closed form formula is

$$\begin{aligned}
a_n &= 3 \cdot 3^n - 2 \cdot 4^n \\
&= 3^{n+1} - 2^{2n+1}
\end{aligned}$$

(d) Given the recursive formula below, a corresponding closed form formula must be found.

$$\begin{aligned}
&a_n + 3a_{n-1} = 0 \\
&a_0 = -2
\end{aligned}$$

Rearranging the formula and expanding each term yields the following.

$$
\begin{aligned}
a_n &= -3a_{n-1} \\
&= -3(-3a_{n-2}) \\
&= (-3)^2(a_{n-2}) \\
&= (-3)^2(-3a_{n-3}) \\
&= (-3)^3(a_{n-3}) \\
&= (-3)^3(-3a_{n-4}) \\
&= (-3)^4(a_{n-4}) \\
&\vdots \\
&= (-3)^k(a_{n-k}) \\
&\vdots \\
&= (-3)^n(a_{n-n}) \\
&= (-3)^n(a_0) \\
&= (-3)^n(-2)
\end{aligned}
$$

Thus the closed form formula is

$$a_n = -2(-3)^n$$

13. (b) The coefficient of the term

$$x_1 x_2 {x_3}^2 {x_4}^3$$

in the expansion of

$$(x_1 + x_2 + x_3 + x_4)^7$$

is equal to the multinomial coefficient

$$\binom{7}{1, 2, 2, 3} = \frac{7!}{1! \times 2! \times 2! \times 3!} = \frac{5040}{24} = 210$$