Statistics

Homework 3 – Estimation

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1. (a) Given a sample $x_1, \ldots, x_n$, we are to prove that $\forall k \in \{1, \ldots, n\}$,

$$\hat{\mu}_k = \frac{\sum_{i=1}^{k} x_i}{k}$$

is an unbiased estimator of the average μ. To demonstrate this, we must show that $\mathrm{E}[\hat{\mu}_k] = \mu$.

$$\mathrm{E}[\hat{\mu}_k] = \frac{\sum_{i=1}^{k} \mathrm{E}[x_i]}{k} = \frac{\sum_{i=1}^{k} \mu}{k} = \frac{k\mu}{k} = \mu$$

(b) The larger the value of k, the better the estimator $\hat{\mu}_k$ is. This is because when an estimator's variance from its parameter is small, it is a better estimator. The variance of $\hat{\mu}_k$ is

$$\mathrm{Var}(\hat{\mu}_k) = \mathrm{Var}\left(\frac{\sum_{i=1}^{k} x_i}{k}\right) = \frac{1}{k^2}\mathrm{Var}\left(\sum_{i=1}^{k} x_i\right)$$

It is clear from this that the larger k is, the smaller $\frac{1}{k^2}$ is, so the smaller $\mathrm{Var}(\hat{\mu}_k)$ is.

2. With respect to a random variable X having mean μ and variance σ^2, we have 2 independent samples of sizes n_1 and n_2 respectively, whose means are $\bar{x}_1$ and $\bar{x}_2$. We are given the following estimator.

$$\hat{\mu} = a\bar{x}_1 + (1-a)\bar{x}_2$$

(a) We are to show that $\hat{\mu}$ is an unbiased estimator for μ for all values of a.

To achieve this we must show that $\mathrm{E}[\hat{\mu}] = \mu$.

$$\begin{aligned}
\mathrm{E}[\hat{\mu}] &= \mathrm{E}[a\bar{x}_1 + (1-a)\bar{x}_2] \\
&= \mathrm{E}[a\bar{x}_1] + \mathrm{E}[(1-a)\bar{x}_2] \\
&= a\,\mathrm{E}[\bar{x}_1] + (1-a)\,\mathrm{E}[\bar{x}_2] \\
&= a\mu + (1-a)\mu \\
&= a\mu + \mu - a\mu \\
&= \mu
\end{aligned}$$

(b) The value of a for which we have the best possible estimator (within this class of estimators) occurs when the variance of the estimator is lowest.

4. We are given a random variable X with a mean μ and a variance σ^2, as well as the following two estimators for μ.

$$\begin{aligned}
\hat{\mu}_1 &= \frac{\sum_{i=1}^{7} x_i}{7} \\
\hat{\mu}_2 &= \frac{2x_1 - x_6 + x_4}{2}
\end{aligned}$$

(a) $\hat{\mu}_1$ is unbiased since $\mathrm{E}[\hat{\mu}_1] = \mu$.

$$\mathrm{E}[\hat{\mu}_1] = \mathrm{E}\left[\frac{\sum_{i=1}^{7} x_i}{7}\right] = \frac{\sum_{i=1}^{7} \mathrm{E}[x_i]}{7} = \frac{7\,\mathrm{E}[X]}{7} = \mathrm{E}[X] = \mu$$

$\hat{\mu}_2$ is also unbiased since $\mathrm{E}[\hat{\mu}_2] = \mu$.

$$\mathrm{E}[\hat{\mu}_2] = \mathrm{E}\left[\frac{2x_1 - x_6 + x_4}{2}\right] = \frac{2\,\mathrm{E}[x_1] - \mathrm{E}[x_6] + \mathrm{E}[x_4]}{2} = \frac{2\,\mathrm{E}[X]}{2} = \mathrm{E}[X] = \mu$$

(b) Of these two estimators, the preferred one would be the one with the smaller variance.

$$\mathrm{Var}(\hat{\mu}_1) = \mathrm{Var}\left(\frac{\sum_{i=1}^{7} x_i}{7}\right) = \frac{1}{49}\sum_{i=1}^{7} \mathrm{Var}(x_i) = \frac{1}{49}7\sigma^2 = \frac{\sigma^2}{7}$$

$$\mathrm{Var}(\hat{\mu}_2) = \mathrm{Var}\left(\frac{2x_1 - x_6 + x_4}{2}\right) = \frac{\mathrm{Var}(2x_1) - \mathrm{Var}(x_6) + \mathrm{Var}(x_4)}{4} = \frac{4\sigma^2 - \sigma^2 + \sigma^2}{4} = \sigma^2$$

Since $\mathrm{Var}(\hat{\mu}_1) \leq \mathrm{Var}(\hat{\mu}_2)$, $\hat{\mu}_1$ is a better estimator for μ.