Mathematical Logic HW2

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Throught this document I will use T to represent a tautology, and F to represent a contradiction.

1. (b) $x \to x$ is a tautology, since $\to$ is only false when the left operand is true but the right operand is false. Since in $x \to x$ both operands are the same, it cannot occur that the left operand be true and the right one be false. Therefore $(x \to x) \to x = T \to x$. However, $T \to x$ is only true when x is true, so $(x \to x) \to x = x$. Therefore:

$$\begin{aligned}&(((x \to x) \to x) \to x) \to x \\ &= (x \to x) \to x \\ &= x\end{aligned}$$

which is neither a tautology or a contradiction.

(d) As the truth table below shows, when A is false and B and C are true, the proposition is false, otherwise it is true. Therefore this proposition is neither a tautology nor a contradiction.

A	B	C	$((A \vee \neg B) \to (A \vee (B \wedge C))) \to (C \to A)$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	T

2. (a) "Yesterday, the sun rose in the west" is simply an atomic proposition. It is not a contradiction, even though we know that factually it is incorrect.

(b) "It's rainy only if it's cloudy." is neither a tautology nor a contradiction, since although we know that in the world we live in we need clouds for there to be rain, regarding mathematical logic, we can in theory assign a truth value to "It's rainy" and a false value to "it's cloudy", thus giving our proposition a value of falsehood in that case.

(c) "If my name is Ron, then my name is not Ron." is neither a tautology nor a contradiction. Let R be the atomic proposition "my name is Ron". The former proposition is equivalent to $R \to \neg R$. If R takes the value 'True', then $\neg R$ takes the value 'False', in which case $R \to \neg R = T \to F = F$, proving that the proposition in question is not a tautology. However, if R takes the value 'False', then $R \to \neg R = F \to T$, which evaluates to 'True', so the proposition is also not a contradiction.

3. (b)

$$(A \to B) \wedge (C \to B) \Leftrightarrow (A \vee C) \to B$$

Proof:

$$\begin{aligned}&(A \to B) \wedge (C \to B) \\ =&(\neg(A \wedge \neg B)) \wedge (\neg(C \wedge \neg B)) \\ =&(\neg A \vee B) \wedge (\neg C \vee B) \\ =&(\neg A \wedge \neg C) \vee B \\ =&\neg(A \vee C) \vee B \\ =&\neg((A \vee C) \wedge \neg B) \\ =&(A \vee C) \to B\end{aligned}$$

4. (b) Let $\alpha =$ "a is larger than b", $\beta =$ "a is larger than 0", $\gamma =$ "b is larger than 0", $\delta =$ "b is equal to 0". We want to determine if

$$(\alpha \wedge (\gamma \vee \delta)) \rightarrow \beta \quad \Leftrightarrow \quad (\alpha \wedge \neg\beta) \rightarrow (\neg\gamma \wedge \neg\delta)$$

$$\begin{aligned}
&(\alpha \wedge (\gamma \vee \delta)) \rightarrow \beta \\
=&\neg(\alpha \wedge (\gamma \vee \delta) \wedge \neg\beta) \\
=&\neg\alpha \vee \neg(\gamma \vee \delta) \vee \beta \\
=&\neg\alpha \vee \beta \vee (\neg\gamma \wedge \neg\delta) \\
=&\neg(\alpha \wedge \neg\beta) \vee (\neg\gamma \wedge \neg\delta) \\
=&\neg((\alpha \wedge \neg\beta) \wedge \neg(\neg\gamma \wedge \neg\delta)) \\
=&(\alpha \wedge \neg\beta) \rightarrow (\neg\gamma \wedge \neg\delta)
\end{aligned}$$

5. (b)

$$\begin{aligned}
&(X \vee Y) \wedge (X \vee \neg Y) \\
=&X \vee (Y \wedge \neg Y) \\
=&X \vee F \\
=&X
\end{aligned}$$

(d)

$$\begin{aligned}
&(X \wedge Y \wedge Z) \vee (X \wedge Y \wedge \neg Z) \vee (X \wedge \neg Y) \\
=&X \wedge ((Y \wedge Z) \vee (Y \wedge \neg Z) \vee \neg Y) \\
=&X \wedge ((Y \wedge (Z \vee \neg Z)) \vee \neg Y) \\
=&X \wedge ((Y \wedge T) \vee \neg Y) \\
=&X \wedge (Y \vee \neg Y) \\
=&X \wedge T \\
=&X
\end{aligned}$$

6. (b) Let P be false, and R and S be true. $P \vee R$ is true because R is true. $P \rightarrow S$ is true because P is false. $R \rightarrow S$ is true because both R and S are true. This is an example where all three given propositions hold, yet P is false. Therefore the given implications do not hold.

(d) Let R be false, and P and S be true. $P \vee R$ is true because P is true. $R \rightarrow S$ is true because R is false. $P \rightarrow S$ is true because both P and S are true. This is an example where all three given propositions hold, yet $R \vee \neg S$ is false. Therefore the given implications do not hold.

7.

$$\begin{aligned}
\alpha &= A \wedge B \wedge ((A \wedge B) \rightarrow C) \\
\alpha &= A \wedge B \wedge \neg((A \wedge B) \wedge \neg C) \\
\alpha &= A \wedge B \wedge (\neg(A \wedge B) \vee C) \\
\alpha &= A \wedge B \wedge (\neg A \vee \neg B \vee C) \\
\alpha &= (A \wedge B \wedge \neg A) \vee (A \wedge B \wedge \neg B) \vee (A \wedge B \wedge C) \\
\alpha &= F \vee F \vee (A \wedge B \wedge C) \\
\alpha &= A \wedge B \wedge C
\end{aligned}$$

$$\beta = A \wedge C \wedge (B \rightarrow (A \wedge C)) \wedge (C \rightarrow B)$$
$$\beta = A \wedge C \wedge \neg(B \wedge \neg(A \wedge C)) \wedge \neg(C \wedge \neg B)$$
$$\beta = A \wedge C \wedge (\neg B \vee (A \wedge C)) \wedge (\neg C \vee B)$$
$$\beta = A \wedge C \wedge (\neg B \vee A) \wedge (\neg B \vee C) \wedge (\neg C \vee B)$$
$$\beta = A \wedge (\neg B \vee A) \wedge (\neg B \vee C) \wedge ((C \wedge \neg C) \vee (C \wedge B))$$
$$\beta = A \wedge (\neg B \vee A) \wedge (\neg B \vee C) \wedge (F \vee (C \wedge B))$$
$$\beta = A \wedge B \wedge C \wedge (\neg B \vee A) \wedge (\neg B \vee C)$$
$$\beta = A \wedge C \wedge ((B \wedge \neg B) \vee (B \wedge A)) \wedge ((B \wedge \neg B) \vee (B \wedge C))$$
$$\beta = A \wedge C \wedge (F \vee (B \wedge A)) \wedge (F \vee (B \wedge C))$$
$$\beta = A \wedge C \wedge B \wedge A \wedge B \wedge C$$
$$\beta = A \wedge B \wedge C$$

$$\gamma = (C \wedge B) \wedge ((C \vee A) \wedge \neg(C \rightarrow \neg A))$$
$$\gamma = C \wedge B \wedge ((C \vee A) \wedge (C \wedge A))$$
$$\gamma = C \wedge B \wedge (C \vee A) \wedge C \wedge A$$
$$\gamma = A \wedge B \wedge C \wedge (C \vee A)$$
$$\gamma = A \wedge B \wedge C$$

$$\delta = C \wedge (B \vee (A \rightarrow C)) \wedge (C \rightarrow (A \vee B))$$
$$\delta = C \wedge (B \vee \neg(A \wedge \neg C)) \wedge \neg(C \wedge \neg(A \vee B))$$
$$\delta = C \wedge (B \vee \neg A \vee C) \wedge (\neg C \vee A \vee B)$$
$$\delta = C \wedge (\neg C \vee A \vee B)$$
$$\delta = (C \wedge \neg C) \vee (C \wedge \neg A) \vee (C \wedge \neg B)$$
$$\delta = (C \wedge \neg A) \vee (C \wedge \neg B)$$
$$\delta = C \wedge (\neg A \vee \neg B)$$
$$\delta = \neg(A \wedge B) \wedge C$$

$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge ((\neg B \wedge C) \rightarrow D) \wedge (A \rightarrow \neg(B \rightarrow \neg C)))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge \neg((\neg B \wedge C) \wedge \neg D) \wedge \neg(A \wedge \neg(B \wedge C)))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge (\neg(\neg B \wedge C) \vee D) \wedge (\neg A \vee (B \wedge C)))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge ((B \vee \neg C) \vee D) \wedge (\neg A \vee (B \wedge C)))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge (B \vee \neg C \vee D) \wedge (\neg A \vee (B \wedge C)))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee ((B \vee \neg C \vee D) \wedge ((A \wedge \neg A) \vee (A \wedge (B \wedge C))))$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee ((B \vee \neg C \vee D) \wedge A \wedge B \wedge C)$$
$$\epsilon = (A \wedge B \wedge C \wedge D) \vee (A \wedge B \wedge C)$$
$$\epsilon = A \wedge B \wedge C$$

α, β, γ and ϵ are all equivalent. δ is not equivalent to any of them.