Discrete Mathematics HW4

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1. (b)

$$R \circ (S \cap T) = \{(x,y) \in A \times C : \exists b \in B : xRb \wedge bSy \wedge bTy\}$$
$$R \circ S = \{(x,y) \in A \times C : \exists b \in B : xRb \wedge bSy\}$$
$$R \circ T = \{(x,y) \in A \times C : \exists b \in B : xRb \wedge bTy\}$$
$$R \circ S \cap R \circ T = \{(x,y) \in A \times C : \exists b \in B : xRb \wedge bSy\} \cap \{(x,y) \in A \times C : \exists b \in B : xRb \wedge bTy\}$$

If $(x,y) \in R \circ (S \cap T)$, then $(x,y) \in R \circ S$, because if $\exists b \in B$ such that $xRb \wedge bSy \wedge bTy$, then b must also satisfy $xRb \wedge bSy$.

If $(x,y) \in R \circ (S \cap T)$, then $(x,y) \in R \circ T$, because if $\exists b \in B$ such that $xRb \wedge bSy \wedge bTy$, then b must also satisfy $xRb \wedge bTy$.

If $(x,y) \in R \circ S$ and $(x,y) \in R \circ T$, then $(x,y) \in R \circ S \cap R \circ T$

We have shown that if $(x,y) \in R \circ (S \cap T)$, then $(x,y) \in R \circ S \cap R \circ T$. Therefore, $R \circ (S \cap T) \subseteq R \circ S \cap R \circ T$.

2. (c) $R^2 \circ R^{-1} \neq R$. I will prove this by a counterexample. Let $A = \{1,2\}$, and $R = \{(1,2)\}$. $R^2 = R \circ R = \phi$. Therefore $R^2 \circ R^{-1} = \{(x,y) \in A \times A : \exists a \in A : (x,a) \in R^2 \wedge (a,y) \in R^{-1}\}$. However, in this case the condition $(x,a) \in R^2$ will never hold because $R^2 = \phi$, so $R^2 \circ R^{-1} = \phi \neq R$.

3. (b)

$$T \circ R = \{(x,y) \in C \times B : \exists a \in A : (x,a) \in T \wedge (a,y) \in R\}$$
$$T \circ S = \{(x,y) \in C \times B : \exists a \in A : (x,a) \in T \wedge (a,y) \in S\}$$

However, we know that if $(a,y) \in R$, then $(a,y) \in S$ because we are given that $R \subseteq S$. Therefore if $(x,y) \in T \circ R$ then $(x,y) \in T \circ S$, so we can conclude that $T \circ R \subseteq T \circ S$.

If we are given only that $T \circ R \subseteq T \circ S$, we cannot infer that $R \subseteq S$. To show this by a counterexample, let:

$$T = \{(1,2)\}, \quad R = \{(2,3),(4,5)\} \quad \text{and} \quad S = \{(2,3)\}.$$

In this case, $T \circ R = \{(1,3)\} \subseteq T \circ S = \{(1,3)\}$, but $R \not\subseteq S$ because $(4,5) \in R$ but $(4,5) \notin S$.

4. (a) Given that $R \subseteq S$, then every element $(x,y) \in R$ is also in S. R^2 is obtained by taking every two ordered pairs $a = (x_1,y_1), b = (x_2,y_2) \in R$ such that the second entry of a (y_1) is equal to the first entry of b (x_2). The ordered pair obtained would then be $(x_1,y_2) \in R^2$. But since $R \subseteq S$, therefore $a,b \in S$, so every element $(x_1,y_2) \in R^2$ would also be in S^2.

5. (b)

$$I_B = \{(b,b) : b \in B\}$$

For any pair $(a,b) \in R \subseteq A \times B$, it must be that $b \in B$. Which means that for all pairs $(a,b) \in R$, there is a corresponding pair $(b,b) \in I_B$. To form $R \circ I_B$, we take the first entry from each $(a,b) \in R$, which is simply a, together with the second entry of its corresponding $(b,b) \in I_B$, which is b, and we form a pair, $(a,b) \in R \circ I_B$.

Since this process maps each pair $(a,b) \in R$ to an identical pair $(a,b) \in R \circ I_B$, it follows that R and $R \circ I_B$ are both identical.

(d)

$$I_A = \{(a,a) : a \in A\}$$

For all pairs $(a, a) \in I_A$, that same pair is the only one that has a first entry a. Therefore when ${I_A}^2$ is calculated, for each pair $(a, a) \in I_A$, it can only be matched with itself, to form exactly (a, a). Meaning that each element $(a, a) \in I_A$ directly corresponds to an identical element in ${I_A}^2$. Therefore $I_A = {I_A}^2$.

For the remainder of the questions, we are asked to use the claims from the previous questions. Below is a list of all of those claims.

$$R \circ (S \cup T) = R \circ S \cup R \circ T \tag{1}$$
$$R \circ (S \cap T) = R \circ S \cap R \circ T \tag{2}$$
$$(S \cup T)^{-1} = S^{-1} \cup T^{-1} \tag{3}$$
$$(S \cap T)^{-1} = S^{-1} \cap T^{-1} \tag{4}$$
$$(S \circ T)^{-1} = S^{-1} \circ T^{-1} \tag{5}$$
$$(R \circ S) \circ T = R \circ (S \circ T) \tag{6}$$
$$R^a \circ R^b = R^{a+b} \tag{7}$$
$$(R^a)^b = R^{ab} \tag{8}$$
$$(R^{-1})^{-1} = R \tag{9}$$
$$(R^{-1})^2 = (R^2)^{-1} \tag{10}$$
$$R_1 \subseteq R_2 \wedge S_1 \subseteq S_2 \Rightarrow R_1 \circ S_1 \subseteq R_2 \circ S_2 \tag{11}$$
$$R \subseteq S \Rightarrow T \circ R \subseteq T \circ S \tag{12}$$
$$R \subseteq S \Rightarrow R^2 \subseteq S^2 \tag{13}$$
$$R \subseteq S \Rightarrow R^{-1} \subseteq S^{-1} \tag{14}$$
$$I_A \circ R = R \tag{15}$$
$$R \circ I_B = R \tag{16}$$
$$I^{-1} = I \tag{17}$$
$$I^2 = I \tag{18}$$

6. (b) Let $A = \{1, 2\}, R = \{(1, 1)\}, R^2 = \{(1, 1)\}$. R is not reflexive, since $I_A = \{(1, 1), (2, 2)\} \nsubseteq R$, but $R \subseteq R^2$, since $R = R^2 = \{(1, 1)\}$.

7. (a) If $I_A \subseteq R \circ R^{-1}$, then $R \circ R^{-1}$ is reflexive. In order for $R \circ R^{-1}$ to be reflexive, R must satisfy both $Dom(R) = A$ and $Im(R) = B$.

8. (a) $R \cap R^{-1}$ is symmetric, only if $(R \cap R^{-1})^{-1} = R \cap R^{-1}$.

$$\begin{aligned} (R \cap R^{-1})^{-1} &= R^{-1} \cap (R^{-1})^{-1} && \text{claim (4)} \\ &= R^{-1} \cap R && \text{claim (9)} \end{aligned}$$

10. (a) If R is transitive, then $R^2 \subseteq R$. And by claim (14), that means that $(R^2)^{-1} \subseteq R^{-1}$. By claim (10), that implies that $(R^{-1})^2 \subseteq R^{-1}$, which means that R^{-1} is transitive.

12. (b) Given that R is an order relation on A, it must be reflexive, antisymmetric, and transitive.

    i. Since R is reflexive, therefore $I_A \subseteq R$. Therefore by claim (14), ${I_A}^{-1} \subseteq R^{-1}$. However by claim (17), ${I_A}^{-1} = I_A \subseteq R^{-1}$. Therefore R^{-1} is reflexive.

    ii. Since R is antisymmetric, therefore $R^{-1} \cap R \subseteq I_A$. But R^{-1} is only antisymmetric if $(R^{-1})^{-1} \cap R^{-1} \subseteq I_A$. However, we know from claim (9) that $(R^{-1})^{-1} \cap R^{-1} = R \cap R^{-1} \subseteq I_A$. Therefore R^{-1} is also antisymmetric.

    iii. Since R is transitive, therefore R^{-1} is also transitive, as we have proven in question 10 (a).

    Since we have shown that if R is an order relation, then R^{-1} must be reflexive, antisymmetric, and transitive, then if R is an order relation, then so is R^{-1}.

13. (a) If R and S are equivalence relations on A, then they must be reflexive, symmetric, and transitive.

    i. Since R and S are reflexive, therefore $I_A \subseteq R$ and $I_A \subseteq S$. Therefore $I_A \subseteq R \cap S$, so $R \cap S$ is reflexive.

    ii. Since R and S are symmetric, therefore $R^{-1} = R$ and $S^{-1} = S$. Therefore $R^{-1} \cap S^{-1} = R \cap S$. And by claim (4), $R^{-1} \cap S^{-1} = (R \cap S)^{-1} = R \cap S$, therefore $R \cap S$ is reflexive.

iii. Question 11 (a).

Since we have shown that if R is an order relation, then $R \cap S$ must be reflexive, antisymmetric, and transitive, then if R is an order relation, then so is $R \cap S$.

(d) If R, S and $R \cup S$ are equivalence relations on A, then they must be reflexive, symmetric, and transitive. Which means that we have the following premises:

$$I_A \subseteq R \tag{19}$$
$$I_A \subseteq S \tag{20}$$
$$I_A \subseteq R \cup S \tag{21}$$

$$R^{-1} = R \tag{22}$$
$$S^{-1} = S \tag{23}$$
$$(R \cup S)^{-1} = R \cup S \tag{24}$$

$$R^2 \subseteq R \tag{25}$$
$$S^2 \subseteq S \tag{26}$$
$$(R \cup S)^2 \subseteq R \cup S \tag{27}$$

i. To show that $R \circ S = R \cup S$, we will show that $R \circ S \subseteq R \cup S$ and $R \circ S \supseteq R \cup S$.

A. To show $R \circ S \subseteq R \cup S$:

$$\begin{aligned}
& (R \cup S)^2 \subseteq R \cup S && \text{from premise (27)} \\
\Rightarrow & (R \cup S) \circ (R \cup S) \subseteq R \cup S && \\
\Rightarrow & (R \cup S) \circ R \cup (R \cup S) \circ S \subseteq R \cup S && \text{from claim (1)} \\
\Rightarrow & (R \cup S) \circ S \subseteq R \cup S && \\
\Rightarrow & R \circ S \cup S \circ S \subseteq R \cup S && \text{from claim (1)} \\
\Rightarrow & R \circ S \subseteq R \cup S &&
\end{aligned}$$

B. To show $R \circ S \supseteq R \cup S$:

$$\begin{aligned}
& I_A \subseteq S && \text{from premise (20)} \\
\Rightarrow & R \circ I_A \subseteq R \circ S && \text{from claim (12)} \\
\Rightarrow & R \subseteq R \circ S && \text{from claim (16)}
\end{aligned}$$

$$\begin{aligned}
& I_A \subseteq R && \text{from premise (19)} \\
\Rightarrow & I_A \circ S \subseteq R \circ S && \text{from claim (12)} \\
\Rightarrow & S \subseteq R \circ S && \text{from claim (15)}
\end{aligned}$$

$$R \subseteq R \circ S \wedge S \subseteq R \circ S \quad \Rightarrow \quad R \cup S \subseteq R \circ S \quad \Rightarrow \quad R \circ S \supseteq R \cup S$$

Hence, $R \circ S \subseteq R \cup S$.

ii. To show that $R \circ S = S \circ R$, we can just show that $S \circ R = R \cup S$, since we already know that $R \circ S = R \cup S$. But in order to show that $S \circ R = R \cup S$, we can use the exact same procedure as in part i.

14. (b) If R, S are order relations on A, it is not necessarily true that $R \cup S$ be an order relation as well. To illustrate this, we will show a counterexample.

$$A = \{1, 2\}, \quad R = \{(1,1), (1,2), (2,2)\}, \quad S = \{(2,2), (2,1), (1,1)\}$$
$$R \cup S = \{(1,1), (1,2), (2,1), (2,2)\}$$

$R \cup S$ is not antisymmetric in this case, since it contains both (1, 2) and (2, 1), yet $1 \neq 2$. Therefore $R \cup S$ is not an order relation.