Discrete Math HW6

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2. In this question, x will be an element of one of the sets, $f(x)$ an element of the other, and f a function to map from one set to the other.

   (b)
   $$f(x) = -7x + 12$$

   (d)
   $$f(x) = \begin{cases} 2\left(1 + \frac{1}{2^{n+1}}\right) & \text{if } x = 1 + \frac{1}{2^n} \\ 2x & \text{otherwise} \end{cases}$$

3. (b) Let A amd B be two disjoint sets which have the same cardinality as $\mathbb{R}$, specifically $\aleph_1$.

   Since every interval also has the cardinality $\aleph_1$, there is a bijective function $f : A \to (-\infty, 0)$ and a bijective function $g : B \to [0, \infty)$.

   Therefore we can define, using these functions, a function $h : A \cup B \to \mathbb{R}$ such that:
   $$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

   Therefore $|A \cup B| = |\mathbb{R}| = \aleph_0$.

4.
$$\begin{aligned} |A - B| &= |B - A| \\ |A - (B \cap A)| &= |B - (A \cap B)| && \because A - B = A - (B \cap A) \\ |A| &= |B| && \because B \cap A = A \cap B \end{aligned}$$

5. Given that $|A| = |B|$ and $|C| = |D|$:

   (b) It is **not** necessarily true that
   $$|A \cup C| = |B \cup D|.$$

   To show this via a counterexample, Let
   $$A = B = C = \{a\} \quad \text{and} \quad D = \{b\}$$

   The cardinality of all these four sets is 1, so this example satisfies the premise; but
   $$A \cup C = \{a\} \quad \text{and} \quad B \cup D = \{a, b\}$$
   $$\therefore |A \cup C| = 1 \quad \text{and} \quad |B \cup D| = 2$$

   so the example doesn't satisfy the conclusion. Therefore the conclusion is false.

   (c) It is **not** necessarily true that
   $$|A - C| = |B - D|.$$

To show this via a counterexample, Let

$$A = B = C = \{a\} \quad \text{and} \quad D = \{b\}$$

The cardinality of all these four sets is 1, so this example satisfies the premise; but

$$A - C = \phi \quad \text{and} \quad B - D = \{a\}$$
$$\therefore |A - C| = 0 \quad \text{and} \quad |B - D| = 1$$

so the example doesn't satisfy the conclusion. Therefore the conclusion is false.

(e)

$$|K \times T| = |K| \times |T| = |T| \times |K| = |T \times K|$$

Alternatively, this can be shown by describing a bijective function $f : |K \times T| \rightarrow |T \times K|$ such that

$$f = \{(t, k) \in T \times K : (k, t) \in K \times T\}$$

7. (b) In order to show that $|(0, 1)| = |(1, \infty)|$ we must find a bijective function $f : (0, 1) \rightarrow (1, \infty)$.

$$f(x) = \tan\left(\pi x + \frac{\pi}{2}\right)$$

This is a bijective function. To prove that it is bijective, we must show that it is both injective and surjective. In order for f to be injective, the following must be true.

$$f(x) = f(y) \Rightarrow x = y$$

$$\begin{aligned}
f(x) &= f(y) \\
\tan\left(\pi x + \frac{\pi}{2}\right) &= \tan\left(\pi y + \frac{\pi}{2}\right) \\
\pi x + \frac{\pi}{2} &= \pi y + \frac{\pi}{2} + k\pi \qquad k \in \mathbb{Z} \\
\pi x &= \pi y + k\pi \\
x &= y + k
\end{aligned}$$

But since x and y are both between 0 and 1, it must be that $k = 0$. Therefore, $x = y$ and f is injective. In order for f to be surjective, the following must be true.

$$\forall y \in (1, \infty), \exists x \in (0, 1), f(x) = y$$

$$\begin{array}{ccccccc}
1 & < & y & = & \tan\left(\pi x + \frac{\pi}{2}\right) & & \\
\frac{\pi}{4} & < & \tan^{-1}(y) & = & \pi x + \frac{\pi}{2} - \pi & < & \frac{\pi}{2} \\
\frac{\pi}{4} & < & \tan^{-1}(y) & = & \pi\left(x - \frac{1}{2}\right) & < & \frac{\pi}{2} \\
\frac{1}{4} & < & \frac{\tan^{-1}(y)}{\pi} & = & x - \frac{1}{2} & < & \frac{1}{2} \\
\frac{3}{4} & < & x & = & \frac{\tan^{-1}(y)}{\pi} + \frac{1}{2} & < & 1
\end{array}$$

Hence for any $y \in (1, \infty)$ choose an x according to the last equation, and that will satisfy $f(x) = y$. Furthermore, since we have shown that $\frac{3}{4} < x < 1$, therefore $x \in (0, 1)$. Thus we can conclude that

$$|(0, 1)| = |(1, \infty)|$$

8. (b) To show that $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$, we must find a bijective function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$.

$$f(x,y) = \begin{cases} 0 & \text{if} \quad x = 0 \quad \wedge \quad y = 0 \\ -f(1-y,0) & \text{if} \quad x = 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(1-y,0) > 0 \\ 1 - f(1-y,0) & \text{if} \quad x = 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(1-y,0) \leq 0 \\ -f(-y,0) & \text{if} \quad x = 0 \quad \wedge \quad y < 0 \quad \wedge \quad f(-y,0) > 0 \\ 1 - f(-y,0) & \text{if} \quad x = 0 \quad \wedge \quad y < 0 \quad \wedge \quad f(-y,0) < 0 \\ -f(1-x,-b) & \text{if} \quad x > 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(1-x,-b) > 0 \\ 1 - f(1-x,-b) & \text{if} \quad x > 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(1-x,-b) < 0 \\ -f(1-x,1-b) & \text{if} \quad x > 0 \quad \wedge \quad y \leq 0 \quad \wedge \quad f(1-x,1-b) > 0 \\ 1 - f(1-x,1-b) & \text{if} \quad x > 0 \quad \wedge \quad y \leq 0 \quad \wedge \quad f(1-x,1-b) < 0 \\ -f(-x,-b) & \text{if} \quad x < 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(-x,-b) > 0 \\ 1 - f(-x,-b) & \text{if} \quad x < 0 \quad \wedge \quad y > 0 \quad \wedge \quad f(-x,-b) < 0 \\ -f(-x,1-b) & \text{if} \quad x < 0 \quad \wedge \quad y \leq 0 \quad \wedge \quad f(-x,1-b) > 0 \\ 1 - f(-x,1-b) & \text{if} \quad x < 0 \quad \wedge \quad y \leq 0 \quad \wedge \quad f(-x,1-b) < 0 \end{cases}$$

This rather complex recursive function is a bijection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Essentially, it maps each tuple of integers in the order obtained from laying $\mathbb{Z} \times \mathbb{Z}$ out in a grid alternating between positive and negative rows and columns, then traversing them diagonally in a downwards left direction, to a corresponding integer in the order $0, 1, -1, 2, -2, 3, -3 \ldots$

Since there is such a bijective function, therefore $|\mathbb{Z} \times \mathbb{Z}| = |\mathbb{Z}|$.

(d) We must show that $|\mathbb{C}| = |\mathbb{R}| = \aleph_1$. Each complex number z, however, can be represented in the form $z = x+yi$, where $x, y \in \mathbb{R}$. Using this representation, we can easily find a bijective function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ such that

$$f(x,y) = x + yi.$$

Thus it is clear that $|\mathbb{C}| = |\mathbb{R} \times \mathbb{R}|$.

Now we must show that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}| = \aleph_1$. We know that $|P(\mathbb{N})| = 2^{\aleph_0} = \aleph_1$. We also know that $|A \times B| = |A| \times |B|$, which tells us that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|^2$. Therefore we have as follows:

$$|\mathbb{C}| = |\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|^2 = {\aleph_1}^2 = \left(2^{\aleph_0}\right)^2 = 2^{2\aleph_0} = 2^{\aleph_0} = \aleph_1$$

10. (b) Each increasing arithmetic sequence of rational numbers can be uniquely identified with two numbers, $a, d \in \mathbb{Q}$, where a is the first term of the sequence, and $d \geq 0$ is the common difference between each term. Therefore the cardinality of the set of all such sequences is equal to

$$|\mathbb{Q} \times \mathbb{Q}| = |\mathbb{N} \times \mathbb{N}| = |\mathbb{N}| = \aleph_0.$$

11. Define a relation S on $\mathbb{R}$ as follows: xSy whenever $x - y \in \mathbb{Z}$.

(a) To prove that S is an equivalence relation, we must show that it is reflexive, symmetric, and transitive.

i. S is reflexive, because $\forall a \in \mathbb{R}, aSa$, since $a - a = 0 \in \mathbb{Z}$.

ii. S is symmetric, because if aSb, then $a - b = n \in \mathbb{Z}$, therefore $b - a = -n \in \mathbb{Z}$, so bSa.

iii. S is transitive, because if aSb and bSc, then $a - b = n \in \mathbb{Z}$ and $b - c = m \in \mathbb{Z}$, therefore $a - c = (a - b) + (b - c) = n + m \in \mathbb{Z}$.

(b) The cardinality of each equivalence class of S, $[x]$, is $\aleph_0$. This is because $[x]$ has exactly one element corresponding to each integer. $[x]$ can be described as follows.

$$[x] = \{x + n : n \in \mathbb{Q}\}$$

(c) The cardinality of the quotient set $\mathbb{R}/S$ is $\aleph_1$. This is because each real number $x \in [0, 1)$ can represent a *different* equivalence class $[x]$. And we know that $|[0, 1)| = \aleph_1$, so $|\mathbb{R}/S| = \aleph_1$

14. (b) There are countably infinite finite subsets of $\mathbb{N}$. A procedure for counting them could go as follows. Define

$$N_i = \{n \in \mathbb{N} : n \leq i\}$$

Start by counting all the subsets of N_0. Then continue by counting all of the subsets of N_1, ignoring any sets that have already been counted. Subsequently, count all the subsets of N_2, N_3, etc.

Now that we know the cardinality of the set of finite subsets of $\mathbb{N}$ is $\aleph_0$, and we know that each finite subset of $\mathbb{N}$ corresponds to a unique co-finite subset of $\mathbb{N}$, therefore we can conclude that the cardinality of the set of co-finite subsets of $\mathbb{N}$ is equal to that of the finite subsets of $\mathbb{N}$, namely $\aleph_0$.