Mathematical Logic
Homework 9

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1. Define the following three propositions.

$$\varphi_1 = \forall x \exists y (R(x,y) \vee R(y,x))$$
$$\varphi_2 = \forall x (\exists y R(x,y) \vee \exists y R(y,x))$$
$$\varphi_3 = (\forall x \exists y R(x,y)) \vee (\forall x \exists y R(y,x))$$

(a) i. First we will show that $\varphi_1 \to \varphi_2$.

1.	$\forall x \exists y (R(x,y) \vee R(y,x))$	premise
2.	$\exists y (R(a,y) \vee R(y,a))$	1, universal elimination
3.	$R(a,b) \vee R(b,a)$	assumption
4.	$R(a,b)$	assumption
5.	$\exists y R(a,y)$	4, existential introduction
6.	$\exists y R(a,y) \vee \exists y R(y,a)$	5, disjunction introduction
7.	$R(b,a)$	assumption
8.	$\exists y R(y,a)$	7, existential introduction
9.	$\exists y R(a,y) \vee \exists y R(y,a)$	8, disjunction introduction
10.	$\exists y R(a,y) \vee \exists y R(y,a)$	3, 4–6, 7–9 disjunction elimination
11.	$\exists y R(a,y) \vee \exists y R(y,a)$	2, 3–10, existential elimination
12.	$\forall x (\exists y R(x,y) \vee \exists y R(y,x))$	11, universal introduction

Therefore by the deduction theorem we have

$$\forall x \exists y (R(x,y) \vee R(y,x)) \to \forall x (\exists y R(x,y) \vee \exists y R(y,x))$$

ii. Now we must show that $\varphi_2 \to \varphi_1$.

1.	$\forall x (\exists y R(x,y) \vee \exists y R(y,x))$	premise
2.	$\exists y R(a,y) \vee \exists y R(y,a)$	1, universal elimination
3.	$\exists y R(a,y)$	assumption
4.	$R(a,b)$	assumption
5.	$R(a,b) \vee R(b,a)$	4, disjunction introduction
6.	$\exists y (R(a,y) \vee R(y,a))$	5, existential introduction
7.	$\exists y (R(a,y) \vee R(y,a))$	3, 4–6, existential elimination
8.	$\exists y R(y,a)$	assumption
9.	$R(b,a)$	assumption
10.	$R(a,b) \vee R(b,a)$	9, disjunction introduction
11.	$\exists y (R(a,y) \vee R(y,a))$	10, existential introduction
12.	$\exists y (R(a,y) \vee R(y,a))$	8, 9–11, existential elimination
13.	$\exists y (R(a,y) \vee R(y,a))$	2, 3–7, 8–12 disjunction elimination
14.	$\forall x \exists y (R(x,y) \vee R(y,x))$	13, universal introduction

Therefore by the deduction theorem we have

$$\forall x(\exists y R(x,y) \vee \exists y R(y,x)) \rightarrow \forall x \exists y (R(x,y) \vee R(y,x))$$

Since $\varphi_1 \rightarrow \varphi_2$ and $\varphi_2 \rightarrow \varphi_1$, therefore φ_1 and φ_2 are equivalent.

(b) i. Now we will demonstrate that $\varphi_2 \not\rightarrow \varphi_3$ by stating a counterexample model. Let the universe for x and y be the real numbers in the interval $[0,1]$, and let the predicate $R(x,y)$ be $x < y$.

In this example, φ_2 means that "for all numbers in $[0,1]$, there is a number in $[0,1]$ which is smaller than it, or there is a number in $[0,1]$ which is greater than it." In this case, φ_2 is true, because 0 is smaller than every number in the interval except for 0 itself, and every number in the interval is greater than 0, except for 0 itself.

However, φ_3 means that "either every number in $[0,1]$ has a number greater than it, or every number in $[0,1]$ has a number smaller than it." However this is not true, because there is no number in $[0,1]$ which is greater than 1, so the first part of the proposition is false. The second part is also false, because there is no number in $[0,1]$ which is less than 0. Therefore $\varphi_2 \not\rightarrow \varphi_3$.

ii. Now we will show that $\varphi_3 \rightarrow \varphi_2$.

1.	$(\forall x \exists y R(x,y)) \vee (\forall x \exists y R(y,x))$	premise
2.	$\forall x \exists y R(x,y)$	assumption
3.	$\exists y R(a,y)$	2, universal elimination
4.	$\exists y R(a,y) \vee \exists y R(y,a)$	3, disjunction introduction
5.	$\forall x \exists y R(y,x)$	assumption
6.	$\exists y R(y,x)$	5, universal elimination
7.	$\exists y R(a,y) \vee \exists y R(y,a)$	6, disjunction introduction
8.	$\exists y R(a,y) \vee \exists y R(y,a)$	1, 2–4, 5–8 disjunction elimination
9.	$\forall x(\exists y R(x,y) \vee \exists y R(y,x))$	8, universal introduction

Therefore by the deduction theorem we have

$$(\forall x \exists y R(x,y)) \vee (\forall x \exists y R(y,x)) \rightarrow \forall x(\exists y R(x,y) \vee \exists y R(y,x))$$

(c) i. Since $\varphi_1 \leftrightarrow \varphi_2$ and $\varphi_2 \not\rightarrow \varphi_3$, therefore $\varphi_1 \not\rightarrow \varphi_3$.

ii. Since $\varphi_3 \rightarrow \varphi_2$ and $\varphi_2 \rightarrow \varphi_1$, therefore $\varphi_3 \rightarrow \varphi_1$.

2. (b)

$$\begin{aligned}
&\exists x(P(x) \wedge Q(x)) \rightarrow \forall x(P(x) \rightarrow Q(x)) \\
&= \forall x((P(x) \wedge Q(x)) \rightarrow \forall y(P(y) \rightarrow Q(y))) \\
&= \forall x \forall y((P(x) \wedge Q(x)) \rightarrow (P(y) \rightarrow Q(y)))
\end{aligned}$$

3. Given the following propositions

$$\begin{aligned}
\alpha &= \exists x(P(x) \rightarrow Q(x)) \\
\beta &= \forall x P(x) \rightarrow \exists x Q(x)
\end{aligned}$$

(b) It is true that $\beta \rightarrow \alpha$. A proof goes as follows.

1.	$\forall x P(x) \to \exists x Q(x)$	premise
2.	$\neg\forall x P(x) \lor \exists x Q(x)$	1, material implication
3.	$\neg\forall x P(x)$	assumption
4.	$\exists x \neg P(x)$	3, universal negation
5.	$\neg P(a)$	assumption
6.	$\neg P(a) \lor Q(a)$	5, disjunction introduction
7.	$P(a) \to Q(a)$	6, material implication
8.	$\exists x(P(x) \to Q(x))$	7, existential introduction
9.	$\exists x(P(x) \to Q(x))$	4, 5–8 existential elimination
10.	$\exists x Q(x)$	assumption
11.	$Q(a)$	assumption
12.	$\neg P(a) \to Q(a)$	11, disjunction introduction
13.	$P(a) \to Q(a)$	12, material implication
14.	$\exists x(P(x) \to Q(x))$	13, existential introduction
15.	$\exists x(P(x) \to Q(x))$	10, existential elimination
16.	$\exists x(P(x) \to Q(x))$	2, 3–9, 10–16 existential elimination

4. (a) We must prove the following implication.

$$\exists x P(x) \to \forall x Q(x) \vdash \forall x(P(x) \to Q(x))$$

1.	$\exists x P(x) \to \forall x Q(x)$	premise
2.	$\neg\exists x P(x) \lor \forall x Q(x)$	1, material implication
3.	$\neg\exists x P(x)$	assumption
4.	$\forall x \neg P(x)$	3, existential negation
5.	$\neg P(a)$	4, universal elimination
6.	$\neg P(a) \lor Q(a)$	5, disjunction introduction
7.	$P(a) \to Q(a)$	6, material implication
8.	$\forall x(P(x) \to Q(x))$	7, universal introduction
9.	$\forall x Q(x)$	assumption
10.	$Q(a)$	9, universal elimination
11.	$\neg P(a) \lor Q(a)$	10, disjunction introduction
12.	$P(a) \to Q(a)$	11, material implication
13.	$\forall x(P(x) \to Q(x))$	12, universal introduction
14.	$\forall x(P(x) \to Q(x))$	2, 3–8, 9–13 disjunction elimination QED

(b) We must prove that the following implication does not hold. We will do so by demonstrating a counterexample model.

$$\forall x(P(x) \to Q(x)) \vdash \exists x P(x) \to \forall x Q(x)$$

Let the universe for x be $\mathbb{Z}$. Let the proposition $P(x)$ mean that "x is a multiple of 4", and the proposition $Q(x)$ mean that "x is even".

It is true that for every number, if said number is a multiple of four, then it must be even. However it is untrue that if there exists a multiple of four — which there does — then all integers must be even.

Therefore $\forall x(P(x) \to Q(x)) \nvdash \exists x P(x) \to \forall x Q(x)$.