Discrete Math HW2

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1.

$$\begin{aligned}
&A = \{1, 2, 3, 4\}\\
&R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}\\
&S = \{(4, 1), (4, 2), (4, 3), (4, 4), (3, 2), (3, 3), (3, 4), (2, 3), (2, 4), (1, 4)\}
\end{aligned}$$

(b) i.

$$\begin{aligned}
R^{-1} &= \{(1, 1), (2, 1), (3, 1), (4, 1), (2, 2), (3, 2), (4, 2), (3, 3), (4, 3), (4, 4)\}\\
&= \{(1, 1), (2, 1), (2, 2), (3, 1), (3, 2), (3, 3), (4, 1), (4, 2), (4, 3), (4, 4)\}
\end{aligned}$$

ii.

$$\begin{aligned}
R \circ S = \{&(1, 4), (1, 3), (1, 2), (1, 1),\\
&(2, 3), (2, 4), (2, 2), (2, 1),\\
&(3, 2), (3, 3), (3, 4), (3, 1),\\
&(4, 1), (4, 2), (4, 3), (4, 4)\}
\end{aligned}$$

$$R \circ S = A \times A$$

iii.

$$\begin{aligned}
&S \circ R = \{(4, 1), (4, 2), (4, 3), (4, 4), (3, 2), (3, 3), (3, 4), (2, 3), (2, 4), (1, 4)\}\\
&S \circ R = S
\end{aligned}$$

(c) i. $R \cap S = \{(1, 4), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\}$

ii. $R \cap S$ is not reflexive. 1 and 2 are elements of A, but there is no element $(1, 1)$ or $(2, 2)$ in $R \cap S$.

iii. $R \cap S$ is not symmetric. For example, $(1, 4) \in R \cap S$, but $(4, 1) \notin R \cap S$.

iv. $R \cap S$ is antisymmetric since $\forall (a, b) \in R \cap S, \exists (b, a) \notin R \cap S$.

v. $R \cap S$ in transitive since $\forall (a, b), (b, c) \in R \cap S, \exists (a, c) \in R \cap S$.

2. (b) Assuming that we have a set S with n elements, the number of relations that can be defined on S is equal to the number of subsets that can be made from the set $S \times S$ (this is equal to $|P(S \times S)|$). This is because every subset of $S \times S$ is a relation on S.

$$|P(S \times S)| = 2^{n^2}$$

Given that $|A| = 2$, that implies that $|A \times A| = 4$. So $|P(A \times A)| = 2^4 = 16$. Therefore the number of relations that can be defined on $P(A \times A) = 2^{16^2} = 2^{256}$.

4. (a) False. If $R \subseteq A \times A$, then every element in R is an ordered pair of elements in A. So R^2 is a set of ordered pairs of elements in R, meaning R^2 is a set of ordered pairs of ordered pairs of elements in A. Therefore there is no element in R which is also in R^2.

(c) False. For example, $(3, 1) \in R$ and $(1, 2) \in R$, but $(3, 2) \notin R$ and $(3, 2) \notin R^2$, so $(3, 2) \notin R \cup R^2$. Therefore, $R \cup R^2$ is not transitive.

6. (c) Given that S and R are reflexive on A, for every element $x \in A$, both sets S and R contain the ordered pair (x, x). Or more formally,

$$\forall\, x \in A, \quad \exists\,(x, x) \in S \wedge \exists\,(x, x) \in R$$

By the definition of relational compositions, an ordered pair (a, b) is in the relation $S \circ R$ on A, if and only if there exists an element $z \in A$ such that $(a, z) \in S$ and $(z, b) \in R$.
Therefore, since for every element $x \in A$, both sets S and R contain the ordered pair (x, x), then $S \circ R$ must also contain (x, x), proving that $S \circ R$ is in fact reflexive.

7. (c) i. For any relation R on A, R^{-1} is defined as

$$R^{-1} = \{(y, x) : (x, y) \in R\}$$

Therefore, for any ordered pair $(x, y) \in R$, there exists $(y, x) \in R^{-1}$, and vice versa. Thus:

$$\forall\,(x, y) \in R \cup R^{-1}, \quad \exists\,(y, x) \in R \cup R^{-1}$$

So $R \cup R^{-1}$ is in fact symmetric.

ii. Using the above definition of R^{-1}, $R \cap R^{-1}$ can be expressed as follows.

$$R \cap R^{-1} = \{(x, y) : (x, y) \in R \wedge (x, y) \in R^{-1}\}$$

This is equivalent to saying

$$R \cap R^{-1} = \{(x, y) : (x, y) \in R \wedge (y, x) \in R\}$$

And if any (x, y) satisfies the condition, so does (y, x), therefore:

$$\forall\,(x, y) \in R \cap R^{-1}, \quad \exists\,(y, x) \in R \cap R^{-1}$$

So $R \cap R^{-1}$ is in fact symmetric.

8. Let S and T be relations on $\mathbb{Z}$ defined as follows:

$$S = \{(k, m) \in \mathbb{Z}^2 : \frac{k - m}{5} \in \mathbb{Z}\}$$
$$T = \{(k, m) \in \mathbb{Z}^2 : \frac{k + m}{5} \in \mathbb{Z}\}$$

i. In order to show that S is an equivalence relation, we must show that it is reflexive, symmetric and transitive.

- Since the following statement is true, all $x \in \mathbb{Z}$ is related to itself, so S is reflexive.

$$\forall\, x \in \mathbb{Z}, \quad \frac{x - x}{5} = 0 \in \mathbb{Z}$$

- If x is related to y, then $\frac{x-y}{5} \in \mathbb{Z}$. Let $x - y = a$; if $\frac{a}{5} \in \mathbb{Z}$, then $-\frac{a}{5} = \frac{y-x}{5} \in \mathbb{Z}$, so y is related to x, and thus S is symmetric.
- Let $x \sim y$, and $y \sim z$; We have $\frac{x-y}{5} \in \mathbb{Z}$ and $\frac{y-z}{5} \in \mathbb{Z}$. Therefore it is transitive, as shown below:

$$\frac{x - y}{5} + \frac{y - z}{5} \in \mathbb{Z} \quad \Rightarrow \quad \frac{(x - y) + (y - z)}{5} \in \mathbb{Z} \quad \Rightarrow \quad \frac{x - z}{5} \in \mathbb{Z} \quad \Rightarrow \quad x \sim z$$

9.

$$R = \{(A, B) \in P(\mathbb{Z})^2 : |A \cap B| = 1\}$$

1) $|A \cap A| = 1$ only if $|A| = 1$. For example, let $A = \{1, 2\}$. $|A \cap A| = 2 \neq 1$. Therefore R is not reflexive.
2) Given that $A \sim B$, we know that $|A \cap B| = 1$. Since set intersection is commutative, $A \cap B = B \cap A$, so $|B \cap A| = 1$. Therefore $B \sim A$ and R is symmetric.
3) R is not transitive. For example, let $A = \{1, 2\}, B = \{2, 3\}, C = \{3, 4\}$. We have $A \sim B$ and $B \sim C$. However, $A \cap C = \phi$ and $|\phi| = 0$, so $A \nsim C$.

12. Let R be a relation on a set A. Prove or disprove: If R isn't reflexive then either R isn't symmetric or R isn't transitive.

False. For example, let $A = \mathbb{N}, R = \phi$. R is not reflexive, since $\{(1, 1), (2, 2), \dots\} \nsubseteq R$. However, R is symmetric since $\nexists\,(a, b) \in R : (b, a) \notin R$; and R is transitive since $\nexists\,(a, b), (b, c) \in R : (a, c) \notin R$.