Mathematical Logic HW8

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1. (b)

$$\begin{aligned}
&\neg(\exists x(P(x) \land Q(x)) \to \forall x(P(x) \to Q(x)))\\
&= \exists x(P(x) \land Q(x)) \land \neg\forall x(P(x) \to Q(x))\\
&= \exists x(P(x) \land Q(x)) \land \exists x\neg(P(x) \to Q(x))\\
&= \exists x(P(x) \land Q(x)) \land \exists x(P(x) \land \neg Q(x))
\end{aligned}$$

2. (b) We must prove $\exists x Q(x)$.

1.	$\forall x(P(x) \lor Q(x))$	premise
2.	$\forall x \neg P(x)$	premise
3.	$P(a) \lor Q(a)$	1, universal elimination
4.	$\neg P(a)$	2, universal elimination
5.	$Q(a)$	3, 4, disjunctive syllogism
6.	$\exists x Q(x)$	5, existential introduction QED

(d) We must prove $\neg\forall x Q(x)$.

1.	$\neg\forall x(P(x) \land Q(x))$	premise
2.	$\forall x P(x)$	premise
3.	$\exists x\neg(P(x) \land Q(x))$	1, change of quantifier
4.	$\neg(P(a) \land Q(a))$	3, assumption
5.	$\neg P(a) \lor \neg Q(a)$	4, negation of conjunction
6.	$P(a)$	2, universal elimination
7.	$\neg Q(a)$	5, 6, disjunctive syllogism
8.	$\exists x\neg Q(x)$	7, existential introduction
9.	$\neg\forall x Q(x)$	8, change of quantifier
10.	$\neg\forall x Q(x)$	3, 4-9, existential elimination QED

(f) We must prove $Q(y) \land \exists x(P(x) \land R(x))$.

1.	$\forall x(P(x) \to (Q(y) \land R(x)))$	premise
2.	$\exists x P(x)$	premise
3.	$P(a)$	2, assumption
4.	$P(a) \to (Q(y) \land R(a))$	1, universal elimination
5.	$Q(y) \land R(a)$	3, 4, modus ponens
6.	$Q(y)$	5, conjunction elimination
7.	$R(a)$	5, conjunction elimination
8.	$P(a) \land R(a)$	3, 7, conjunction introduction
9.	$\exists x(P(x) \land R(x))$	8, existential introduction
10.	$Q(y) \land \exists x(P(x) \land R(x))$	6, 9 conjunction introduction
11.	$Q(y) \land \exists x(P(x) \land R(x))$	2, 3-10, existential elimination QED

3. (b) $(\exists x P(x) \wedge \exists x Q(x)) \Rightarrow \exists x(P(X) \wedge Q(x))$ is not a valid implication.

A counterexample can be demonstrated as follows. Let the predicate $P(x)$ be "$x = 0$", the predicate $Q(x)$ be "$x = 1$", and the universe for x be $\mathbb{Z}$. There does exist an integer which is equal to zero, namely zero; so $\exists x P(x)$ is true in this interpretation. Similarly, there is an integer equal to one, so $\exists x Q(x)$ is true as well.

If the implication were valid, we would be able to say $\exists x(P(X) \wedge Q(x))$. Or in English, "there is an integer which is equal to both one and zero". However this consequent is false in our interpretation, since there is no such integer.

(d) $\forall x(P(x) \to Q(x)) \Rightarrow \forall x P(x) \to \forall x Q(x)$ is a valid implication. Below is the inference procedure.

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$P(a) \to Q(a)$	1, universal elimination
3.	$\forall x P(x)$	assumption
4.	$P(a)$	3, universal elimination
5.	$Q(a)$	2, 4, modus ponens
6.	$\forall x Q(x)$	5, universal introduction
7.	$\forall x P(x) \to \forall x Q(x)$	3-6, deduction theorem QED

4. We are to determine whether the following implication is valid or not.

$$\forall x \exists y P(x, y) \models \exists y \forall x P(x, y)$$

The 'proof' provided in the question is not sound. In line 4 of the 'proof', the universal quantifier is introduced, violating one of the restrictions of the rule of inference 'Universal Introduction'. The restriction in question states that from $\varphi(\beta/\alpha)$ we can infer $\forall x \varphi$, only if "β is not mentioned in any hypothesis or undischarged assumptions". In the question, z takes the position of β, but z *is* mentioned in an undischarged hypothesis on line 3. (Although not clear in the provided proof, line 3 should be an *assumption* for the start of a new sub-proof.)

The above implication is not sound. A counterexample can be shown as follows. Let the binary predicate $P(x, y)$ be defined as $y = x$, and the universe for x and y be $\mathbb{Z}$. In this interpretation, our premise $\forall x \exists y P(x, y)$ is true, because for every integer x, there is an integer y which is equal to it. However, the 'conclusion' $\exists y \forall x P(x, y)$ is false, since there is no such integer y such that all other integers are equal to it.