Discrete Math HW5

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January 6, 2020

1. (a)

$$\{(1,a),(2,b)\}$$
$$\{(1,b),(2,a)\}$$
$$\{(1,a),(2,c)\}$$
$$\{(1,c),(2,a)\}$$
$$\{(1,b),(2,c)\}$$
$$\{(1,c),(2,b)\}$$

(b)

$$\{(1,a),(2,a)\}$$
$$\{(1,b),(2,b)\}$$
$$\{(1,c),(2,c)\}$$

2. (a) Given that $f(n) = 2g(n) - 1$ and f is one to one, we must show that g is also one to one.

Proof by contradiction:

Assume g is not one to one.

$$\exists a, b \in \mathbb{N}, g(a) = g(b) \land a \neq b$$
$$\Rightarrow \exists a, b \in \mathbb{N}, 2g(a) - 1 = 2g(b) - 1 \land a \neq b$$
$$\Rightarrow \exists a, b \in \mathbb{N}, f(a) = f(b) \land a \neq b$$
$$\Rightarrow f \text{ is not one to one.}$$

This is a contradiction, so g must be one to one.

(b) If f were onto, then

$$\forall b \in \mathbb{N}, \exists a \in \mathbb{N}, f(a) = b$$

But we know that $f(n) = 2g(n) - 1$. Since $g(n) \in \mathbb{N}$, therefore $2g(n)$ must always be even; so $2g(n) - 1$ must be odd. Therefore for any even number e,

$$\nexists a \in \mathbb{N}, f(a) = e$$

So f cannot be onto.

(c)

$$g \circ f = \begin{cases} \frac{f(n)}{2} & (f(n) \text{ is even}) \\ \frac{f(n)+1}{2} & (f(n) \text{ is odd}) \end{cases}$$
$$= \frac{2g(n) - 1 + 1}{2} \quad \because \text{ f(n) is always odd.}$$
$$= g(n)$$

4. (a) If f and g are one to one, then

$$f(a) = f(b) \Rightarrow a = b$$
$$g(a) = g(b) \Rightarrow a = b$$

We must show that $g \circ f$ is one to one. Then we must prove

$$(g \circ f)(a) = (g \circ f)(b) \Rightarrow a = b$$

Assuming

$$(g \circ f)(a) = (g \circ f)(b),$$
$$\Rightarrow g(f(a)) = g(f(b))$$
$$\Rightarrow f(a) = f(b)$$
$$\Rightarrow a = b$$
$$QED$$

(b) If $g \circ f$ is one to one, then f is one to one.

Proof by contradiction:

Assume f is not one to one. Therefore

$$\exists a, b \in A, a \neq b \wedge f(a) = f(b)$$
$$\Rightarrow g(f(a)) = g(f(b)) \wedge a \neq b$$
$$\Rightarrow g \circ f \text{ is not one to one.}$$

This is a contradiction, so f must be one to one.

5. (b) If $g \circ f$ is onto then g must be onto.

Proof by contradiction:

Assume g is not onto

$$\Rightarrow \exists c \in C, \forall b \in B, g(b) \neq c$$
$$\Rightarrow \exists c \in C, \forall a \in A, g(f(b)) \neq c$$
$$\Rightarrow g \circ f \text{ is not onto.}$$

This is a contradiction, so g must be onto.

(c) Even if $g \circ f$ is onto, it is not necessarily true that f is onto. For example, let

$$A = \{1, 2\}, \quad B = \{a, b, c\}, \quad C = \{\alpha, \beta\}$$
$$f : A \to B = \{(1, a), (2, b)\}, \quad g : B \to C = \{(a, \alpha), (b, \beta), (c, \alpha)\}$$

Here, f is not onto since $\nexists (x, c) \in f$, but $g \circ f$ is onto because every element in C is mapped from A by $g \circ f$.

8. (a) $\{-7, 2.5\}$

(c) $(-7, -2) \cup (0, 2.5) \cup [3, 6)$

9. (a) Proof by contradiction:

$$\text{Assume } C \nsubseteq f^{-1}(f(C))$$
$$\exists c \in C, c \notin f^{-1}(f(C))$$
$$\exists c \in C, f^{-1}(f(c)) \neq c$$

But this is a contradiction, since $f^{-1}(f(x)) = x$. Therefore $C \subseteq f^{-1}(f(C))$.

(b)

$$A = \{1, 2\}, \quad B = \{a\}, \quad C = \{1\}$$
$$f = \{(1, a), (2, a)\}$$
$$f^{-1}(f(C)) = \{1, 2\} \supset C$$

12. (a) Given that $f \circ g \circ f$ is bijective, that implies that f is bijective.

Proof by contradiction: Assume f is not bijective. Therefore f is neither one to one nor onto, because f maps from a set X to itself. Since f is not onto, then $f(g(f(x)))$ cannot map to every element in X so $f \circ g \circ f$ is not bijective. This is a contradiction to our premise, so our assumption must be false. Therefore f is bijective.

13. (a) True

(d) True

14. (a) Given $f : A \to A$ such that $f(f(x)) = x$, then f is onto.

Proof by contradiction:

Assume f is not onto

$$\begin{aligned}
&\Rightarrow \exists a \in A, \forall b \in A, f(b) \neq a \\
&\Rightarrow \exists a \in A, f(a) = c \wedge f(c) \neq a \\
&\Rightarrow \exists a \in A, f(f(a)) \neq a
\end{aligned}$$

This is a contradiction, so f must be onto.

(b) Given $f : A \to A$ such that $f(f(x)) = x$, then f is one to one.

Proof by contradiction:

$$\begin{aligned}
&(f \circ f)(x) = x \quad \Rightarrow \quad f \circ f = I_A \\
&\text{Assume } f \text{ is not one to one} \\
&\exists a, b \in A, f(a) = f(b) \wedge a \neq b \\
&\Rightarrow \exists a, b, \in A, f(f(a)) = f(f(b)) \wedge a \neq b
\end{aligned}$$

Therefore $f \circ f$ is not one to one, but we know that I_A is one to one, and $f \circ f = I_A$. This is a contradiction, so f must be one to one.

(c) The relation

$$R = \{(a, b) : a = b \vee f(a) = b\}$$

is an equivalence relation on A because it is reflexive, symmetric and transitive.

i. It is reflexive since

$$(a, a) \in R \ \because \ a = a$$

ii. It is symmetric because

$$\begin{aligned}
&(a, b) \in R \quad \Rightarrow \quad a = b \vee f(a) = b \\
&a = b \quad \Rightarrow \quad b = a \quad \Rightarrow \quad (b, a) \in R \\
&f(a) = b \quad \Rightarrow \quad f(b) = a \ \because \ f(f(a)) = a \quad \Rightarrow \quad (b, a) \in R
\end{aligned}$$

iii. It is transitive because if $(a, b) \in R$ and $(b, c) \in R$, then either $a = b$ or $f(a) = b$, and either $b = c$ or $f(b) = c$. Since $(a, b) \in R$ and $(b, c) \in R$, then if $a = b$ or if $b = c$, then $(a, c) \in R$.

Otherwise $f(a) = b$ and $f(b) = c$. So $f(f(a)) = c$. But we know that $f(f(x)) = x$, so $(a, c) \in R$.

(d) If $A = \{1, 2, 3, 4\}$ then a possible function definition for f can be

$$f = \{(1, 2), (2, 1), (3, 4), (4, 3)\}.$$

In this case, the quotient set of R would be

$$A/R = \{\{1, 2\}, \{3, 4\}\}.$$

18. (a) f is not one to one, because $f(0) = f(1) = 0$.

(b) If $f \circ g = I_{\mathbb{N}_0}$ then g must be f^{-1}. If $f \circ f^{-1} = I_{\mathbb{N}_0}$, then f would have to be invertible. Since f is not one to one, it cannot be invertible, so $f \circ g \neq I_{\mathbb{N}_0}$.