Discrete Mathematics Homework 3

Abraham Murciano

December 11, 2019

1. (a) In order for a relation to be an equivalence relation, it must be reflexive, symmetric, and transitive.

   i. In order to prove that R is reflexive, the following must hold true.

   $$\forall(a,b) \in A, (a,b)R(a,b) \quad \Leftrightarrow \quad \forall(a,b) \in A, a \times b = b \times a$$

   And since multiplication is commutative, we know that $a \times b = b \times a$, therefore $(a,b)R(a,b)$ and R is reflexive.

   ii. Now we must show that R is symmetric. To show symmetry, we must prove that

   $$(a,b)R(c,d) \Rightarrow (c,d)R(a,b)$$

   We can show this as follows:

   $$(a,b)R(c,d) \quad \Rightarrow \quad ad = bc \quad \Rightarrow \quad cb = da \quad \Rightarrow \quad (c,d)R(a,b)$$

   Thus we see that R is in fact symmetric.

   iii. Finally, we must show that R is transitive. To prove that R is transitive, it must be shown that if $(a,b)R(c,d)$ and $(c,d)R(e,f)$, then $(a,b)R(e,f)$.

   $$(a,b)R(c,d) \quad \Rightarrow \quad a \times d = b \times c \tag{1}$$
   $$(c,d)R(e,f) \quad \Rightarrow \quad c \times f = d \times e \tag{2}$$

   As shown above, if $(a,b)R(c,d)$ and $(c,d)R(e,f)$ are true, then so are $ad = bc$ and $cf = de$. Therefore we can multiply each side of (1) by a side of (2), to obtain

   $$a \times d \times c \times f = b \times c \times d \times e$$

   If we assume that $c \times d \neq 0$, then we can divide each side by $c \times d$, and we get

   $$a \times f = b \times e \quad \Rightarrow \quad (a,b)R(e,f)$$

   However, if $c \times d = 0$, then it must be that $c = 0$ since we know that $c \in \mathbb{Z}$ and $d \in \mathbb{N}$, so d cannot be 0. Thus it follows that $a = 0$ since if $(a,b)R(0,d)$, then $a \times d = 0 \times b = 0$, and we know that $d \neq 0$. Similarly we know that $e = 0$, since we are also given $(c,d)R(e,f)$. Now that we know that if $c \times d = 0$, then $a = e = 0$, we can also come to the same conclusion as above:

   $$a \times f = b \times e \quad \Rightarrow \quad (a,b)R(e,f)$$

   (b) The equivalence class of the element $(-1,2)$ is:

   $$\{(a,b) \in A : 2a = -b\}$$

   In order to find the 'simplest' representative of any equivalence class in this relation, I would find the element in the equivalence class with the smallest second number. The simplest representative of this equivalent class would be $(-1,2)$.

   (c) The equivalence class of the element $(0,5)$ is:

   $$\{(a,b) \in A : 5a = 0\} \quad = \quad \{(a,b) \in A : a = 0\}$$

   The simplest representative of this equivalent class would be $(0,1)$.

(d) The quotient set A/R can be defined using a set we are familiar with, which is the rational numbers $\mathbb{Q}$. Specifically, each rational number in the set $\mathbb{Q}$ can represent an equivalence class of A, such that if $\frac{a}{b} \in \mathbb{Q}$ and $b > 0$, then $\frac{a}{b}$ represents the equivalence class of the ordered pair (a, b). More formally:

$$A/R = \{\{(a,b) \in A : \frac{a}{b} = x\} : x \in \mathbb{Q}\}$$

4. Let $A = \mathbb{N} \times \mathbb{N}$. We define the relation R on A as follows:

$$R = \{((m,n),(p,q)) : m + q = n + p\}$$

(a) To show that the relation R is an equivalence relation we must show that it is reflexive, symmetric, and transitive.

i. To show that it is reflexive, we must show that $(a,b)R(a,b)$.

$$(a,b)R(a,b) \quad \Leftrightarrow \quad a + b = b + a$$

We know that $a + b = b + a$ is true due to the commutative property of addition, therefore $(a,b)R(a,b)$ and R is reflexive.

ii. In order to show that R is symmetric, we must show that if $(a,b)R(c,d)$ then $(c,d)R(a,b)$.

$$(a,b)R(c,d) \quad \Rightarrow \quad a + d = b + c \quad \Rightarrow \quad c + b = d + a \quad \Rightarrow \quad (c,d)R(a,b)$$

iii. Finally, to show that R is transitive, we must show that if $(a,b)R(c,d)$ and $(c,d)R(e,f)$, then $(a,b)R(e,f)$. Given the following two relationships,

$$(a,b)R(c,d) \quad \Rightarrow \quad a + d = b + c \tag{3}$$
$$(c,d)R(e,f) \quad \Rightarrow \quad c + f = d + e \tag{4}$$

we can add the equations implied by (3) and (4) together, giving us

$$a + d + c + f = b + c + d + e \quad \Rightarrow \quad a + f = b + e \quad \Rightarrow \quad (a,b)R(e,f)$$

(b) The equivalence class of $(1,3)$ is

$$\{(a,b) \in A : (a,b)R(1,3)\} = \{(a,b) \in A : a + 3 = b + 1\} = \{(a,b) \in A : b - a = 2\}$$

(c) The quotient set A/R can be identified with the set of all integers $\mathbb{Z}$. Each integer x identifies exactly one equivalence class such that every element (a,b) in said equivalence class satisfies the equation $a - b = x$. Or more formally:

$$A/R = \{\{(a,b) \in A : a - b = x\} : x \in \mathbb{Z}\}$$

5. Let $A = \{1,2,3,4,5\}$. Let $S = \{(1,3),(2,3),(4,5)\} \cup I_A$.

(a) To check if S is an order relation, we must check three properties. Reflexivity, antisymmetry, and transitivity.

i. It is clear that S is reflexive, since I_A is reflexive, and every element in I_A is also in S.

ii. S is also antisymmetric, since there are no elements in $a, b \in A$ such that aSb and bSa and $a \neq b$. This is clearly visible from the definition of S, since every element $(a,b) \in I_A$ does not satisfy the condition $a \neq b$, and none of the other three ordered pairs $(a,b) \in S$ satisfy aSb and bSa.

iii. S is transitive as well, since every ordered pair in S satisfies the condition 'if aSb and bSc, then aSc'.

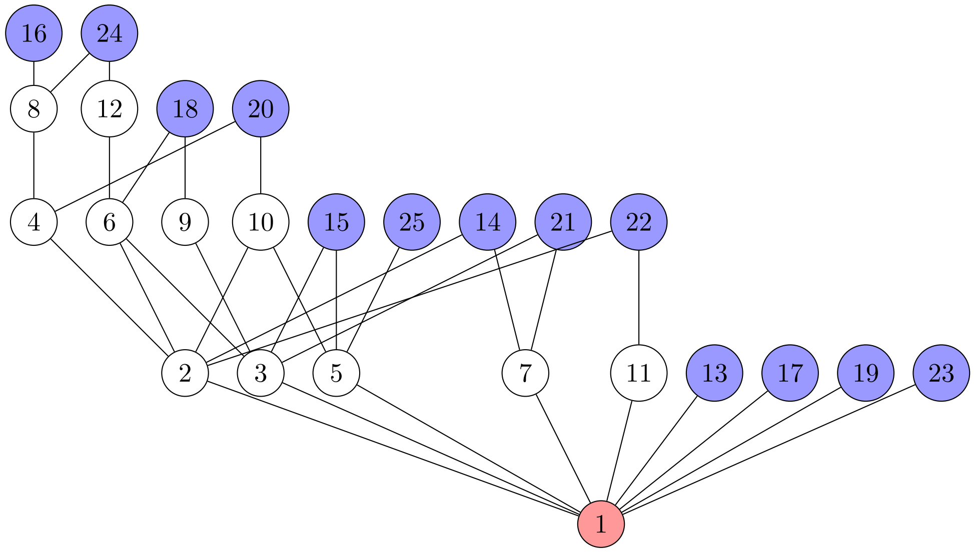
(b) In order for an order relation to be a total order relation, in addition to the reflexive, symmetric, and transitive properties, it must also have the connexity property, meaning that for all elements $a, b \in A$, either aSb or bSa.

In order to make S a total order relation different from $\leq$, we must add the following pairs.

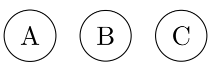
$$(2,1),(4,1),(5,1),(2,4),(2,5),(4,3),(5,3)$$

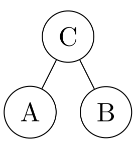
The order obtained after adding the new pairs is $2,4,5,1,3$.

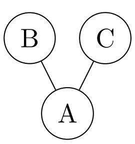
7. (c) Below is the Hasse diagram of the order relation described by the relation “is a divisor of”. Red nodes show minimal elements and blue nodes show maximal elements. 1 is the minimum element. There is no maximum element.

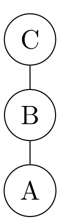


8. These are all the Hasse diagrams for an order relation on a set containing three elements, A, B, C.

(a) 

(b) 

(c) 

(d) 

9. T is defined as a relation on $\mathbb{R} \times \mathbb{R}$ as follows.

$$T = \{((a,b),(c,d)) : a^2 + b^2 \le c^2 + d^2\}$$

(a) T is reflexive, since $a^2 + b^2 \le a^2 + b^2$.

(b) T is antisymmetric since if $a^2 + b^2 < c^2 + d^2$, then $c^2 + d^2 \nless a^2 + b^2$.

(c) T is transitive since if $a^2 + b^2 < c^2 + d^2$, and $c^2 + d^2 < e^2 + f^2$, then $a^2 + b^2 < e^2 + f^2$.

Therefore T is an order relation.

10. S is defined as a relation on $\mathbb{N} \times \mathbb{N}$ as follows.

$$S = \{((x,y),(m,n)) : x|m \wedge y|n\}$$

(a) S is reflexive since every natural number divides itself.

(b) S is antisymmetric since if $a \ne b$ and $a|b$ then $b \nmid a$.

(c) S is transitive since if $a|b$ and $b|c$, then $a|c$. Another way to think about this is as follows. If $a|b$ then $b = a \times k_1$, where $k_1 \in \mathbb{N}$. If $b|c$, then $c = b \times k_2$, where $k_2 \in \mathbb{N}$. Therefore $c = a \times k_1 \times k_2$, so this all implies that a must divide c.

Therefore S is an order relation.

11. (a) Since R is an order relation, it must have the antisymmetric property, which states that $\forall x, y \in A$, if xRy and yRx, then it must be that $x = y$. Therefore, since all elements $(x, y) \in S$ satisfy both xRy and yRx, all element in S must be of the form (x, x).

    S is clearly reflexive, as we just explained. It is also symmetric, si S is clearly reflexive, as we just explained. It is also symmetric, si S is clearly reflexive, as we just explained. It is also symmetric, since elements can only be related to themselves. And for the same reason S must also be transitive.

    $$S = R \cap R^{-1}$$

    (b) S is not necessarily an equivalence relation, since we are not guaranteed that it is transitive. For example, if we have R such that aRc and bRc but $a \not R b$ and $b \not R a$, then S must be such that aSc, bSc, cSa, and cSb, but we will not have aSb or bSa.

    $$S = R \cup R^{-1}$$

    (c) S not an equivalence relation on A since it is not always transitive. Let a, b, c, d, e be elements of A. If we have R such that aRd and bRd, and we also have bRe and cRe, but $\nexists f \in A$ such that aRf and cRf, then we have a case where aSb and bSd but $a \not S c$.

    An example where this is the case is as follows. Let A and R be defined as in question 7(c).

    $$A = \{x \in \mathbb{N}_1 : x \leq 25\}$$
    $$R = \{(a, b) \in A \times A : a|b\}$$

    Here we have $4R24$ and $6R24$, so $4S6$. We also have $6R18$ and $9R18$, so $6R9$. But there is no element in A such that both 4 and 9 are related to it. Therefore $4 \not S 9$.

13. Let R and S be order relations on a set A.

    (a) $R \cap S$ is an order relation on A.

    i. In order for $R \cap S$ to be reflexive, it must contain every element $(a, a) \in A \times A$. But we know that both R and S are reflexive, so they both contain all of those elements, therefore $R \cap S$ also contains them, so $R \cap S$ is reflexive.

    ii. In order for $R \cap S$ to be antisymmetric, it must be true that if it contains an element (a, b) such that $a \neq b$, then it does not contain the element (b, a). We know, however, that if $R \cap S$ contains (a, b), then both R and S contain it as well. And since both R and S are antisymmetric, we also know that neither of them can contain (b, a). Therefore neither does $R \cap S$, so it is antisymmetric.

    iii. In order for $R \cap S$ to be transitive, it must be true that if it contains two elements (a, b) and (b, c), then it must contain the element (a, c). We also know that if $R \cap S$ contains both (a, b) and (b, c), then both R and S contain them as well. And since both R and S are transitive, we also know that both of them must contain (a, c). Therefore so does $R \cap S$, so it is transitive.

    (d) R^{-1} is an order relation on A.

    i. In order for R^{-1} to be reflexive, it must contain every element $(a, a) \in A \times A$. But we know that R is reflexive, so it contains all of those elements, therefore R^{-1} also contains them because the inverse of (a, a) is also (a, a), so R^{-1} is reflexive.

    ii. In order for R^{-1} to be antisymmetric, it must be true that if it contains an element (a, b) such that $a \neq b$, then it does not contain the element (b, a). We know, however, that if R^{-1} contains (a, b), then R contains (b, a). And since R is antisymmetric, we also know that it cannot contain (a, b). Therefore R^{-1} does not contain (b, a), so it is antisymmetric.

    iii. In order for R^{-1} to be transitive, it must be true that if it contains two elements (a, b) and (b, c), then it must contain the element (a, c). We also know that if R^{-1} contains both (a, b) and (b, c), then R must contain c, b and b, a. And since R is transitive, we also know that it must contain (c, a). Therefore R^{-1} contains (a, c), so it is transitive.

16 (b) R is reflexive and transitive. If R is *not* an order relation, it must be that R is not antisymmetric. Meaning, it must contain two elements of the form (a, b) and (b, a) such that $a \neq b$. Therefore S would also contain an element (a, b) such that $a \neq b$, so S would not be the identity relation on X.

However if R *is* an order relation, then it must be antisymmetric, so it does not contain any two elements a, b and b, a such that $a \neq b$. Therefore there is no element $(a, b) \in S$ such that $a \neq b$. Furthermore, since R is reflexive, it must contain the element (a, a) for all $a \in X$, which means that S would also contain those. Therefore S must be the identity relation on X.

17. (a) i. S is reflexive, since every set is a subset of itself, and since they contain the exact same elements they have the same Min.

    ii. S is antisymmetric, because if you have two sets A and B, the only way that they can both be subsets of each other is if they are the same set.

    iii. S is transitive, since 'is a subset of' and 'equals' are both transitive relations. So since every two sets related by S must satisfy those two relations, S must be transitive.

    (b)
```
   {1, 2, 3}
     /  \
{1, 2} {1, 3} {2, 3}
    \  /        |
    {1}        {2}    {3}
```

19. (a) There are six equivalence classes in $\approx$.

    (b) ${}^5\mathrm{C}_2 = {}^5\mathrm{C}_3 = 10$

24. (a) i. R is not an order relation, since it is not antisymmetric. If two X and Y are two distinct subsets of A such that they both have the same max, they will both be related to each other. For example, $\{1,3\}R\{2,3\}$ and $\{2,3\}R\{1,3\}$.

    ii.

$$
\begin{aligned}
&Max(\{2,5,7\}) = 7 \\
&Max(\{2,3,4,5,7\}) = 7 \\
&\Rightarrow \{2,3,4,5,7\}\ R\ \{2,5,7\} \\
&\Rightarrow \{2,5,7\}\ R^{-1}\ \{2,3,4,5,7\}
\end{aligned}
$$

    (b) i. R is not an order relation since it is not reflexive.

$$|X| = |X| \quad \Rightarrow \quad |X| \not< |X| \quad \Rightarrow \quad X \not R X$$

    ii.

$$
\begin{aligned}
&\{2,3,7\}\ R\ \{1,2,3,4\} \\
&\{1,2,3,4\}\ R\ \{1,3,4,5,8\} \\
&\Rightarrow \{2,3,7\}\ R \circ R\ \{1,3,4,5,8\}
\end{aligned}
$$