Discrete Math HW1

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1. (b) True
   (c) True
   (e) False
   (f) False
   (i) True
   (k) False

   (g) $x \notin y, x \subseteq y$
   (h) $x \in y, x \subseteq y$

3. (a)
$$P(A) = \{\phi, \{0\}, \{\phi\}, \{\{\phi\}\}, \{0, \phi\}, \{0, \{\phi\}\}, \{\phi, \{\phi\}\}, \{0, \phi, \{\phi\}\}\}$$
   (b)
$$2^{2^8} = 2^{256}$$
   (c) Statement: If $A \subseteq B$ then $P(A) \subseteq P(B)$.
   Proof: If $A \subseteq B$, then all elements of A are also elements of B. $P(A)$ is the set containing all sets made from only elements in A, but since all those elements are also elements of B, therefore it follows that every set in $P(A)$ is also an element of $P(B)$, so $P(A) \subseteq P(B)$.

4. (b)
$$(A - B) - C = A - (B \cup C)$$
$$(A \cap \overline{B}) - C = (A \cap \overline{B}) \cap \overline{C} = A \cap (\overline{B} \cap \overline{C}) = A \cap \overline{(B \cup C)} = A - (B \cup C)$$
   (c)
$$(A \cup B) - C = (A - C) \cup (B - C)$$
$$(A - C) \cup (B - C) = (A \cap \overline{C}) \cup (B \cap \overline{C}) = (A \cup B) \cap \overline{C} = (A \cup B) - C$$

5. (a)
$$(A - B) \cup (A - C) = A - (B \cap C)$$
$$(A \cap \overline{B}) \cup (A \cap \overline{C}) = A \cap (\overline{B} \cup \overline{C}) = A \cap \overline{(B \cap C)} = A - (B \cap C)$$
   (c)
$$(A - B) - C = (A - C) - (B - C)$$
$$(A \cap \overline{C}) - (B \cap \overline{C}) = (A \cap \overline{C}) \cap \overline{(B \cap \overline{C})} = (A \cap \overline{C}) \cap (\overline{B} \cup C)$$
$$= A \cap (\overline{C} \cap (\overline{B} \cup C)) = A \cap ((\overline{C} \cap \overline{B}) \cup (\overline{C} \cap C)) = A \cap \overline{C} \cap \overline{B} = (A - B) - C$$

7. (a) True

8. To prove that ① and ② are equivalent, we must show that ① ⇒② and that ② ⇒ ①.

To show that ① $\Rightarrow$ ②,

$$A \subseteq B \subseteq C$$

$$A \subseteq B \Rightarrow A \cup B = B$$

$$B \subseteq C \Rightarrow B \cap C = B$$

$$\Rightarrow A \cup B = B \cap C$$

To show that ② $\Rightarrow$ ①,

$$A \cup B = B \cap C \Rightarrow$$

$$B \cap C \subseteq B \Rightarrow A \cup B \subseteq B \Rightarrow A \cup B = B \Rightarrow A \subseteq B$$

$$B = B \cap C \Rightarrow B \subseteq C \Rightarrow A \subseteq B \subseteq C$$

11. $P(X) = \{\phi\} \Rightarrow X = \phi$.

If $S - T = \phi$ then there are no elements in S which are not in T. Therefore every element in S is also in T. So $S \subseteq T$.

12. (b)

$$A \oplus B = (A - B) \cup (B - A) = (A \cap \overline{B}) \cup (B \cap \overline{A}) = (\overline{B} \cap A) \cup (\overline{A} \cap B)$$

$$= (\overline{B} - \overline{A}) \cup (\overline{A} - \overline{B}) = (\overline{A} - \overline{B}) \cup (\overline{B} - \overline{A}) = \overline{A} \oplus \overline{B}$$

(c)

$$A \oplus B = (A - B) \cup (B - A) = \{x : (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin A)\}$$

$$\begin{gathered} A \oplus (B \oplus C) = \{x : (x \in A \wedge x \notin (B \oplus C)) \vee (x \in (B \oplus C) \wedge x \notin A)\} \\ = \{x : ((x \in A \wedge x \in B \wedge x \in C) \vee (x \in A \wedge x \notin B \wedge x \notin C)) \\ \vee((x \notin A \wedge x \in B \wedge x \notin C) \vee (x \notin A \wedge x \notin B \wedge x \in C))\} \\ = \{x : (x \in A \wedge x \in B \wedge x \in C) \\ \vee(x \in A \wedge x \notin B \wedge x \notin C) \\ \vee(x \notin A \wedge x \in B \wedge x \notin C) \\ \vee(x \notin A \wedge x \notin B \wedge x \in C)\} \end{gathered}$$

$$\begin{gathered} (A \oplus B) \oplus C = \{x : (x \in (A \oplus B) \wedge x \notin C) \vee (x \in C \wedge x \notin (A \oplus B))\} \\ = \{x : ((x \in A \wedge x \notin B \wedge x \notin C) \vee (x \notin A \wedge x \in B \wedge x \notin C)) \\ \vee((x \in A \wedge x \in B \wedge x \in C) \vee (x \notin A \wedge x \notin B \wedge x \in C))\} \\ = \{x : (x \in A \wedge x \notin B \wedge x \notin C) \\ \vee(x \notin A \wedge x \in B \wedge x \notin C) \\ \vee(x \in A \wedge x \in B \wedge x \in C) \\ \vee(x \notin A \wedge x \notin B \wedge x \in C)\} \end{gathered}$$

Since the conditions in the set builder notation for $A \oplus (B \oplus C)$ are identical to those in the set builder notation for $(A \oplus B) \oplus C$ but in different order, and since the logical and operator $\wedge$ is commutative, therefore this proves that $A \oplus (B \oplus C) = (A \oplus B) \oplus C$, and therefore symmetric difference is associative.

(e)

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

Proof:

$$\begin{aligned}
A \cap (B \oplus C) &= A \cap ((B - C) \cup (C - B)) \\
&= (A \cap (B - C)) \cup (A \cap (C - B)) \\
&= (A \cap (B \cap \overline{C})) \cup (A \cap (C \cap \overline{B})) \\
&= (A \cap B \cap \overline{C}) \cup (A \cap C \cap \overline{B}) \\
&= (B \cap (A \cap \overline{C})) \cup (C \cap (A \cap \overline{B})) \\
&= (B \cap (\phi \cup (A \cap \overline{C}))) \cup (C \cap (\phi \cup (A \cap \overline{B}))) \\
&= (B \cap ((A \cap \overline{A}) \cup (A \cap \overline{C}))) \cup (C \cap ((A \cap \overline{A}) \cup (A \cap \overline{B}))) \\
&= (B \cap A \cap (\overline{A} \cup \overline{C})) \cup (C \cap A \cap (\overline{A} \cup \overline{B})) \\
&= ((A \cap B) \cap (\overline{A} \cup \overline{C})) \cup ((A \cap C) \cap (\overline{A} \cap \overline{B})) \\
&= ((A \cap B) \cap \overline{(A \cap C)}) \cup ((A \cap C) \cap \overline{(A \cap B)}) \\
&= ((A \cap B) - (A \cap C)) \cup ((A \cap C) - (A \cap B)) \\
&= (A \cap B) \oplus (A \cap C)
\end{aligned}$$

14. (b)

$$(A \times A) \cup (B \times C) \neq (A \cup B) \times (A \cup C)$$

Proof by counterexample:
Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$.

$$(A \times A) \cup (B \times C)$$

$$\begin{aligned}
&= (\{1\} \times \{1\}) \cup (\{2\} \times \{3\}) \\
&= \{(1,1)\} \cup \{(2,3)\} \\
&= \{(1,1),(2,3)\}
\end{aligned}$$

$$(A \cup B) \times (A \cup C)$$

$$\begin{aligned}
&= (\{1\} \cup \{2\}) \times (\{1\} \cup \{3\}) \\
&= \{1,2\} \times \{1,3\} \\
&= \{(1,1),(1,3),(2,1),(2,3)\}
\end{aligned}$$

15. (a) i.

$$\bigcup_{i=1}^{100} A_i = A_1 \cup A_2 \cup ... \cup A_{100}$$

.

However, $A_{i-1} \subseteq A_i$. This can be proven by induction since $A_1 = \{-1, 0, 1\}$ and $A_2 = \{-2, -1, 0, 1, 2\}$, showing that $A_1 \subseteq A_2$.

Then in general, $A_{i-1} = \{-(i-1), -(i-1)+1, -(i-1)+2, ..., i-1\}$ which is equal to $\{-i+1, -i+2, -i+3, ..., i-1\}$. This is a set containing all the elements in A_i except for $-i$ and i. Therefore, any set $A_{i-1} = A_i \cup \{-i, i\}$. This implies $A_{i-1} \subseteq A_i$.

And by the transitive property of subsets, $A_{i-1} \subseteq A_i \Rightarrow A_j \subseteq A_i$ for all $j < i$.

Therefore $A_1 \cup A_2 \cup ... \cup A_{100} = A_{100}$, because $A_1, A_2, A_3, ..., A_{99}$ are all subsets of A_{100}, and if $A \subseteq B \Rightarrow A \cup B = B$. Therefore:

$$\bigcup_{i=1}^{100} A_i = A_{100}$$

.

ii. Assuming $0 \notin \mathbb{N}$, then

$$\bigcap_{i \in \mathbb{N}} A_i = A_1 \cap A_2 \cap ...$$

It was shown in 15. (a) i. that for all i and j, such that $j < i$, $A_j \subseteq A_i$. Since $A \subseteq B \Rightarrow A \cap B = A$, it follows that $A_1 \cap A_2 \cap ... = A_1 = \{-1, 0, 1\}$

iii.

$$\bigcap_{i \in \mathbb{N}} \overline{A_i} = \overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap ...$$

$$A_{i-1} \subseteq A_i \Rightarrow \overline{A_{i-1}} \supseteq \overline{A_i} \Rightarrow \overline{A_{i-1}} \cap \overline{A_i} = \overline{A_i}$$

Therefore, since the union takes $\overline{A_i}$ for all i to ∞, it follows that

$$\bigcap_{i \in \mathbb{N}} \overline{A_i} = \overline{A_\infty} = \phi$$

where A_∞ denotes $\lim_{n \to \infty} A_n$.

(b) i.

$$A_i = \{x \in \mathbb{N} : x \leq i\}$$

$$\bigcap_{i \in \mathbb{N}} A_i = \{1\}$$

ii.

$$B_i = \{2x : x \in \mathbb{N} \land x \leq i\}$$

$$\bigcup_{j \in \mathbb{N}} \overline{B_j} = \mathbb{N} - \{2\}$$

18.

$$\bigcup_{k=1}^{\infty} \left(-\frac{1}{k}, \frac{1}{k}\right) = (-1, 1)$$

$$\bigcup_{k=1}^{\infty} \left(-\frac{1}{k}, \frac{1}{k}\right) = (-1, 1) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) \cup ...$$

$$\forall n > 1, \frac{1}{n} < 1 \Rightarrow \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq (-1, 1) \Rightarrow (-1, 1) \cup \left(-\frac{1}{n}, \frac{1}{n}\right) = (-1, 1)$$

$$\Rightarrow (-1, 1) \cup \left(-\frac{1}{2}, \frac{1}{2}\right) \cup \left(-\frac{1}{3}, \frac{1}{3}\right) \cup ... = (-1, 1)$$