# Mathematical Logic HW3

Abraham Murciano

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1. (b)

$$(A \leftrightarrow (\neg A \wedge B)) \rightarrow (B \wedge \neg B)$$

| $A$ | $B$ | $\neg A \wedge B$ | $A \leftrightarrow (\neg A \wedge B)$ | $(A \leftrightarrow (\neg A \wedge B)) \rightarrow (B \wedge \neg B)$ |
|---|---|---|---|---|
| $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ |

i. DNF:

$$(A \wedge B) \vee (A \wedge \neg B) \vee (\neg A \wedge B)$$

ii. CNF:

$$A \vee B$$

(d)

$$(x \rightarrow y) \rightarrow (z \wedge (y \downarrow x))$$

| $x$ | $y$ | $z$ | $x \rightarrow y$ | $y \downarrow x$ | $z \wedge (y \downarrow x)$ | $(x \rightarrow y) \rightarrow (z \wedge (y \downarrow x))$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $F$ | $F$ |
| $F$ | $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $F$ |

i. DNF:

$$(x \wedge y \wedge z) \vee (x \wedge \neg y \wedge z) \vee (x \wedge \neg y \wedge \neg z) \vee (\neg x \wedge y \wedge z)$$

ii. CNF:

$$(\neg x \vee \neg y \vee z) \wedge (x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z) \wedge (x \vee y \vee z)$$

2. (b)

$$((p \downarrow q) \rightarrow r) \uparrow p$$

| $p$ | $q$ | $r$ | $p \downarrow q$ | $(p \downarrow q) \rightarrow r$ | $((p \downarrow q) \rightarrow r) \uparrow p$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $F$ | $T$ |

As shown in the truth table, $((p \downarrow q) \rightarrow r) \uparrow p$ is neither a tautology nor a contradiction.

3. (b)

$$
\begin{aligned}
p \oplus q &= (p \wedge \neg q) \vee (\neg p \wedge q) \\
&= (\neg\neg p \wedge \neg q) \vee (\neg p \wedge \neg\neg q) \\
&= (\neg(p \downarrow p) \wedge \neg q) \vee (\neg p \wedge \neg(q \downarrow q)) \\
&= ((p \downarrow p) \downarrow q) \vee (p \downarrow (q \downarrow q)) \\
&= \neg(((p \downarrow p) \downarrow q) \downarrow (p \downarrow (q \downarrow q))) \\
&= (((p \downarrow p) \downarrow q) \downarrow (p \downarrow (q \downarrow q))) \downarrow (((p \downarrow p) \downarrow q) \downarrow (p \downarrow (q \downarrow q)))
\end{aligned}
$$

4. (b)

$$f(x, y, z) = \neg x \wedge (\neg y \to \neg z)$$

$$
\begin{aligned}
f(a, a, a) &= \neg a \wedge (\neg a \to \neg a) \\
&= \neg a \wedge T \\
&= \neg a
\end{aligned}
$$

$$
\begin{aligned}
f(a, a, f(a, a, a)) &= f(a, a, \neg a) \\
&= \neg a \wedge (\neg a \to a) \\
&= \neg a \wedge \neg(\neg a \wedge \neg a) \\
&= \neg a \wedge \neg\neg a \\
&= \neg a \wedge a \\
&= F
\end{aligned}
$$

$$
\begin{aligned}
f(f(a, a, f(a, a, a)), b, b) &= f(F, b, b) \\
&= T \wedge (\neg b \to \neg b) \\
&= T \wedge T \\
&= T
\end{aligned}
$$

5. (b) We must show that $\{\oplus, \wedge, \leftrightarrow\}$ are complete. One way to do that, is to show that with these operators we are able to make a $\downarrow$. We are able to make a $\downarrow$ as follows:

| $a$ | $b$ | $a \oplus b$ | $a \wedge b$ | $a \leftrightarrow b$ | $(a \oplus b) \leftrightarrow (a \wedge b)$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ |
| $F$ | $F$ | $F$ | $F$ | $T$ | $T$ |

As the table shows, $(a \oplus b) \leftrightarrow (a \wedge b) = a \downarrow b$, and we know that $\downarrow$ on its own is itself complete, thus the set of operators $\{\oplus, \wedge, \leftrightarrow\}$ is also complete.

(d) We must show that the functions $f$ and $g$ defined below are complete.

$$
\begin{aligned}
&f(x, y, z) = y \to ((x \wedge \neg z) \vee (z \wedge \neg x)) \\
&g(x, y) = \neg x \vee y
\end{aligned}
$$

| $x$ | $y$ | $z$ | $f(x, y, z)$ | $g(x, y)$ |
|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $T$ |
| $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $T$ | $F$ |
| $T$ | $F$ | $F$ | $T$ | $F$ |
| $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ |

$$
\begin{aligned}
g(a, a) &= \neg a \vee a \\
&= T
\end{aligned}
$$

$$
\begin{aligned}
f(g(a, a), a, b) &= f(T, a, b) \\
&= a \rightarrow ((T \wedge \neg b) \vee (b \wedge F)) \\
&= a \rightarrow (\neg b \vee F) \\
&= a \rightarrow \neg b \\
&= \neg(a \wedge \neg\neg b) \\
&= \neg(a \wedge b) \\
&= a \uparrow b
\end{aligned}
$$

We already know that $\uparrow$ is complete, and we can make a $\uparrow$ using the functions $f$ and $g$, so it follows that $f$ and $g$ are complete.