Data Structures II

Theoretical Homework 4

Question 1

A. $T(n) = 2T(n/2) + 1$

B. $a = 2,\ b = 2,\ f(n) = 1,\ log_b a = 1,\ n^{log_b a} = n$

$f(n) = O(n^{1-\varepsilon}) \Rightarrow T(n) = \Theta(n)$

Question 2

A. $T(n) = T(n-1) + n$

B. Claim: $T(n) = O(n^2)$

$\therefore\ T(n) \le cn^2$

Proof by induction:

$\forall\ c \ge 1,\ T(1) = 1 \le c * 1^2$

Assume $T(n-1) \le c(n-1)^2$

$$\begin{aligned} T(n) &= T(n-1) + n \\ &\le c(n-1)^2 + n \\ &= c(n^2 - 2n + 1) + n \\ &= cn^2 - 2cn + c + n \\ &\le cn^2 \text{ for any c such that } 2cn + c + n \ge 0 \end{aligned}$$

$2cn + c \ge n$

$c(2n+1) \ge n$

$c \ge n/(2n+1) \approx 1/2$

Question 3

A. $T(n) = 2T(n/2) + n^3$

$a = 2,\ b = 2,\ f(n) = n^3,\ log_b a = 1,\ n^{log_b a} = n$

$f(n) = \Omega(n^{1+\varepsilon}) \Rightarrow T(n) = \Theta(n^3)$

B. $T(n) = T(n/3) + 2log(n)$

$a = 1,\ b = 3,\ f(n) = 2log(n),\ log_b a = 0,\ n^{log_b a} = 1$

$f(n) = \Omega(n^{0+\varepsilon}) \Rightarrow T(n) = \Theta(2log(n)) = \Theta(log(n))$

C. $T(n) = T(n/8) + nlog(n)$

$a = 1,\ b = 3,\ f(n) = nlog(n),\ log_b a = 0,\ n^{log_b a} = 1$

$f(n) = \Omega(n^{0+\varepsilon}) \Rightarrow T(n) = \Theta(nlog(n))$

D. $T(n) = 2T(n/3) + n^{3/2}$

$a = 2,\ b = 3,\ f(n) = n^{3/2},\ log_b a = 0.631,\ n^{log_b a} = n^{0.631}$

$f(n) = O(n^{0.631-\varepsilon}) \Rightarrow T(n) = \Theta(n^{0.631})$

E. $T(n) = 3T(n/2) + n^2 log(n)$

$a = 3,\ b = 2,\ f(n) = n^2 log(n),\ log_b a = 1.585,\ n^{log_b a} = n^{1.585}$

$f(n) = \Omega(n^{1.585+\varepsilon}) \Rightarrow T(n) = \Theta(n^2 log(n))$

F. $T(n) = 2T(n/4) + \sqrt{n}$

$a = 2,\ b = 4,\ f(n) = \sqrt{n},\ log_b a = 1/2,\ n^{log_b a} = \sqrt{n}$

$f(n) = \Theta(\sqrt{n}) \Rightarrow T(n) = \Theta(\sqrt{n} log(n))$

G. $T(n) = T(\sqrt{n}) + 2log(n)$

Claim: $T(n) = O(log(n))$

$\therefore\ T(n) \leq c * log(n)$

Proof by induction:

$\forall\ c,\ T(1) \leq c * log(1)$

Assume $T(\sqrt{n}) \leq c * log(\sqrt{n})$

$$
\begin{aligned}
T(n) &= T(\sqrt{n}) + 2log(n) \\
&\leq c * log(\sqrt{n}) + 2log(n) \\
&= c/2 * log(n) + 2log(n) \\
&= (c/2 + 2) * log(n)
\end{aligned}
$$

Choose $c = 4$.

H. $T(n) = 4T(n/4) + n/lg(n)$

$a = 4,\ b = 4,\ f(n) = n/lg(n),\ log_b a = 1,\ n^{log_b a} = n$

$f(n) = O(n^{1-\varepsilon}) \Rightarrow T(n) = \Theta(n)$