Mathematical Logic HW5

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December 21, 2019

1. (b) Prove $((P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot) \leftrightarrow ((P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C)$

1.	$(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot$	assumption
2.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \vee \bot$	1, material implication
3.	$\neg\bot$	axiom
4.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C)$	2, 3, disjunctive syllogism
5.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \vee C$	4, negation of conjunction
6.	$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C$	5, material implication
7.	$((P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot) \rightarrow ((P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C)$	1-6, deduction theorem
8.	$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C$	assumption
9.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \vee C$	8, material implication
10.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C)$	9, negation of conjunction
11.	$\neg(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \vee \bot$	10, disjunction introduction
12.	$(P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot$	11, material implication
13.	$((P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C) \rightarrow ((P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot)$	8-12, deduction theorem
14.	$((P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg C) \rightarrow \bot) \leftrightarrow ((P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow C)$	7, 13, biconditional introduction

2. (b) Define the following two premises.

$a =$ Bacon wrote plays which were attributed to Shakespeare
$b =$ Bacon was a great writer

We want to claim the following.

$$\begin{array}{l} a \rightarrow b \\ b \\ \hline \therefore a \end{array}$$

By the deduction theorem, however, the claim can be true if and only if $\vdash (a \rightarrow b) \rightarrow (b \rightarrow a)$. However, this is not a tautology, as shown in the truth table below.

a	b	$(a \rightarrow b) \rightarrow (b \rightarrow a)$
T	T	T
T	F	T
F	T	F
F	F	T

3. (a) We must prove that the following premises are inconsistent.

1.	$\neg(A \to B)$	premise
2.	$\neg(B \to C)$	premise
3.	$\neg(\neg A \vee B)$	1, material implication
4.	$\neg(\neg B \vee C)$	2, material implication
5.	$A \wedge \neg B$	3, negation of disjunction
6.	$B \wedge \neg C$	4, negation of disjunction
7.	$\neg B$	5, conjunction elimination
8.	B	6, conjunction elimination
9.	$\perp$	7, 8, contradiction introduction

(b) We must prove that the following premises are inconsistent.

1.	A	premise
2.	$B \to \neg(C \wedge A)$	premise
3.	$\neg(C \to \neg B)$	premise
4.	$\neg(\neg C \vee \neg B)$	3, material implication
5.	$C \wedge B$	4, negation of disjunction
6.	B	5, conjunction elimination
7.	$\neg(C \wedge A)$	2, 6, modus ponens
8.	$\neg C \vee \neg A$	7, negation of conjunction
9.	C	5, conjunction elimination
10.	$\neg A$	8, 9, disjunctive syllogism
11.	$\perp$	1, 10, contradiction introduction

(d) We must prove that the following premises are inconsistent.

1.	$A \leftrightarrow B$	premise
2.	$A \to C$	premise
3.	$\neg(B \to C)$	premise
4.	$B \to A$	1, biconditional elimination
5.	$B \to C$	2, 4, hypothetical syllogism
6.	$\perp$	3, 5, contradiction introduction

4. Define the following atomic propositions.

$a =$ Moses is a hero.
$b =$ Moses is happy with what he has.
$c =$ Moses shows weakness.
$d =$ Moses conquers his desires.

We must show that the following propositions are inconsistent.

1.	$a \to b$	premise
2.	$c \to \neg b$	premise
3.	$\neg a \to \neg d$	premise
4.	$d \wedge c$	premise
5.	c	4, conjunction elimination
6.	$\neg b$	2, 5, modus ponens
7.	$\neg a$	1, 6, modus tollens
8.	$\neg d$	3, 7, modus ponens
9.	d	4, conjunction elimination
10.	$\perp$	8, 9, contradiction elimination

5. (b) $\Gamma \nvdash C$ does not necessarily imply that $\Gamma \vdash \neg C$. We will prove this by a counterexample. Let $\Gamma = \phi$. We know that $\phi \nvdash C$, because C is a proposition, which could be true or false, since we have no premises. However, in this case we also know that $\phi \nvdash \neg C$, for the same reason.

6. (b) It is claimed that there exists a set Γ, such that for any proposition C, $\Gamma \vdash C$ and $\Gamma \vdash \neg C$. This claim is true, because for any set of premises which is inconsistent, any conclusion can be derived from that set. For example, the set of premises $\Gamma = \{A, \neg A\}$, $\Gamma \vdash C$, and $\Gamma \vdash \neg C$.