

On the Properties of A Self Counting Sequence

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Abstract

We present a self referential rule for sequence generation, parametrized by a positive integer k , originally proposed in a [Reddit post](#). We prove that for each k the sequence is eventually periodic, and the period must be greater than or equal to $k + 1$. We further conjecture that a repeating cycle of length $k + 1$ exists for all k -sequences, and include constructions for several infinite families of integers.

1 Self Counting Sequence

Throughout the paper, let $k \in \mathbb{Z}^+$ be the parametrizing integer chosen to generate the sequence. As [originally posed](#), the sequence can be generated by the following rules:

1. $\mathcal{S}_k(0) = 1$,
2. $\mathcal{S}_k(n) := \sum_{i=n-k}^{n-1} [\mathcal{S}_k(i) = \mathcal{S}_k(n-1)]$ where we use the Iverson bracket notation, i.e. $[P]$ returns 1 if P is true and 0 if P is false.

To describe the sequence in words, we begin with a 1, and each subsequent term is the number of occurrences of the last term in the last k terms. In [\(1\)](#) we show the first 30 terms of the sequence for $k = 5$.

$$\mathcal{S}_5 : 1, 1, 2, 1, 3, 1, 3, 2, 1, 2, 2, 3, 1, 2, 3, 2, 2, 3, 2, 3, 2, 3, 3, 3, 4, 1, 1, 2, 1, 3, \dots \quad (1)$$

For the first k terms of the sequence, we only count the occurrence of a term from the beginning of the sequence. Equivalently, we may pad the beginning of the sequence with a set of $k - 1$ zeros. Further, each term in the sequence is treated as a single digit, i.e. a term of 17 does not count towards the counting of 1s or 7s.

Throughout this work, unless otherwise noted, we have excluded the trivial possibility that

$$\mathcal{S}_k = \dots, \overbrace{k, k, k, \dots, k, k}^{k \text{ occurrences}}, \dots \quad (2)$$

It can be easily shown that the trivial sequence of repeating k only occurs if the sequence began with repeating terms of k , i.e. it cannot arise from the sequence beginning with 1.

Definition 1. A sequence \mathcal{S}_k is ultimately periodic if there exist $t_0, T \in \mathbb{Z}^+$ such that $\mathcal{S}_k(t_0 + n) = \mathcal{S}_k(t_0 + T + n) \forall n \geq 0$. We call T the period of the cycle.

Note that for a periodic sequence there are infinitely many valid periods T (all integer multiples), we will always refer to the *least* value as the period.

Theorem 2. For any k , the self counting sequence is ultimately periodic with period $T \leq k^k$.

Proof. The n -th term of the sequence is a function of the k previous terms, each of which are drawn from the set $\{1, 2, 3, \dots, k\}$. Therefore, there are at most k^k unique subsequences, and there exists a t_0 such that $\mathcal{S}_k(t_0 + n) = \mathcal{S}_k(k^k + n) \forall n \in \{1, 2, \dots, k\}$. Therefore, given the rules to generate the sequence only depend on the previous k terms, we have $\mathcal{S}_k(t_0 + n) = \mathcal{S}_k(k^k + n) \forall n \geq 0$ with period of $T \leq k^k$. \square

Clearly k^k is an extremely conservative bound, for example the \mathcal{S}_5 sequence (1) has period $T_5 = 25 \ll 5^5$. Our main results are concerned with the eventually periodic nature of these sequences, and are presented in Theorem **todo** and Conjecture **todo**. First we provide a few lemmas and a theorem bounding the value of the sequence terms.

Lemma 3. A term $a \in \mathcal{S}_k$ cannot occur more than $a + 1$ times consecutively within \mathcal{S}_k .

Proof. For any element $a \in \mathcal{S}_k$, assume there exists an index $t > 0$ such that $\mathcal{S}_k(t + i) = a$ for $0 \leq i \leq a + 1$, i.e. a has occurred $a + 1$ times consecutively. Clearly, as $a \in \mathcal{S}_k$, we have $a \leq k$, and therefore all $a + 1$ terms are counted for the subsequent term, which must then be $a + 1$. \square

Lemma 4. For a given \mathcal{S}_k , no term $a \geq \lfloor \frac{k}{2} \rfloor + 1$ occurs twice consecutively.

Proof. Assume there exists an index t_0 and elements $a, b \in \mathcal{S}_k$ such that

$$\mathcal{S}_k = \overbrace{\dots}^{t_0}, b, a, a, \dots \quad (3)$$

with $a \geq \lfloor \frac{k}{2} \rfloor + 1$ and $b \neq a$. From indices $t_0 - k + 1$ to $t_0 + 1$, the element b must occur a times, and occur at least $a - 1$ times between indices $t_0 - k + 2$ to $t_0 + 1$. Similarly, the element a must occur a times between indices $t_0 - k + 2$ to $t_0 + 2$. We thus have $a - 1 + a$ occurrences of elements a and b between indices $t_0 - k + 2$ to $t_0 + 2$

From the pigeonhole principle, if we have a sub-sequence $X^{\neq s}, s, s$ where $s > \lceil \frac{k}{2} \rceil$, then the previous k terms (before $X^{\neq s}$) must have contained s occurrences of $X^{\neq s}$. Therefore, including the first occurrence of s , we have $s - 1$ occurrences of $X^{\neq s}$ and s occurrences of s , which gives $s - 1 + s \geq \lceil \frac{k}{2} \rceil + \lceil \frac{k}{2} \rceil + 1 > k$, which is a contradiction. \square

Note that Lemma 4 includes runs of longer than two, a fortiori.

Theorem 5. *All elements of a k -foggy sequence have an upper bound of $\lceil \frac{k}{2} \rceil + 1$.*

Proof. Let $S_k = a_k(t - k + 1 : t)$ be the subsequence of length k preceeding $a_k(t)$. We know that S_k must contain $a_k(t) > \lceil \frac{k}{2} \rceil + 1$ copies of $a_k(t - 1)$. Let $a^* = a_k(t - 1)$ be the repeated element. From Lemma 4, we know that $a^* \leq \lceil \frac{k}{2} \rceil$, as otherwise it could appear only at every other position in S_k , which only $\lceil \frac{k}{2} \rceil$ occurrences. From Lemma 3, we know that there cannot be more than $a^* + 1$ occurrences of a^* in a row within S_k . After the initial set of $a^* + 1$ occurrences in S_k , all occurrences of a^* must be singletons, as the subsequent element is the count of occurrences of a^* which will be greater than $a^* + 1$. Therefore, if $a_k(t) > \lceil \frac{k}{2} \rceil + 2$, then there must be an a^* in S_k followed by $\lceil \frac{k}{2} \rceil + 2$. It suffices then to prove that it is impossible to construct a sequence yielding $a_k(t) = \lceil \frac{k}{2} \rceil + 2$.

We begin by assuming the maximal value $a^* = \frac{k}{2}$. We first partition the subsequence S_k as shown in (4),

$$S_k = [\overbrace{s_1, s_2, \dots, s_{l_1}}^{C_{a^*}(S_k(1:l_1))=a^*+1}, X^{\neq a^*}, a^*]. \quad (4)$$

In (4), we have partitioned the first $a^* + 1$ occurrences of s into the first l_1 elements, followed by an element not equal to a^* , followed by an a^* . In order for the $X^{\neq a^*}$ to be followed by an a^* , there must be a^* occurrences in the previous k elements, along with the $a^* + 1$ occurrences of a^* itself in the first l_1 elements. Therefore, we have that

$$a^* + (a^* + 1) \leq k \quad (5)$$

$$a^* \leq \frac{k}{2} - \frac{1}{2} \quad (6)$$

which contradicts the assumption $a^* = \frac{k}{2}$.

We generalize now by assuming s has value $a^* = \frac{k}{2} - i$. We then partition the subsequence S_k as shown in (7),

$$S_k = [\overbrace{s_1, s_2, \dots, s_{l_1}}^{C_{a^*}(S_k(1:l_1))=a^*+1}, (X^{\neq a^*}, a^*)^i, X^{\neq a^*}, a^*]. \quad (7)$$

In order for the final $X^{\neq a^*}$ to be followed by an a^* , there must be a^* occurrences of it in the previous k elements. Note that after every occurrence of a^* , after the initial l_1 partition, the $X^{\neq a^*}$ is the running total of occurrences of a^* in S_k . Therefore, the terms $X^{\neq a^*}$ are unique, and summing the number of elements required to generate the final occurrence of a^* gives

$$a^* + 1 + 2i + a^* \leq k \quad (8)$$

$$a^* \leq \frac{k}{2} - i - \frac{1}{2} \quad (9)$$

which contradicts the assumption that $a^* = \frac{k}{2} - i$.

This proof holds if the terms $X^{\neq a^*}$ are substituted for partitions of length l_i , containing any number of elements not equal to a^* . The scenario in (7) is the lower bound of the sizes of the intervening $X^{\neq a^*}$ partitions, and thus holds, as there still must be at least one unique element following every occurrence of an a^* (after the initial partition). \square

2 Acknowledgments

The authors would like to thank Reddit, for wasting their time and occasionally nerd-sniping them from the rafters. The authors also thank the anonymous referee for careful reading of the paper and suggestions which improved the paper.

References

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2010 *Mathematics Subject Classification*: Primary 11A41; Secondary 11N05.

Keywords:

(Concerned with sequences [A334539](#), and [A098168](#).)
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