On the Properties of A Self Counting Sequence

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Abstract

We present a self referential rule for sequence generation, parametrized by a postive integer k, originally proposed in a Reddit post. We prove that for each k the sequence is eventually periodic, and the period must be greater than or equal to $k+1$. We further conjecture that a repeating cycle of length $k+1$ exists for all k-sequences, and include constructions for several infinite families of integers.

1 Self Counting Sequence

Throughout the paper, let $k \in \mathbb{Z}^+$ be the paramatrizing integer chosen to generate the sequence. As originally posed, the sequence can be generated by the following rules:

1. $\mathcal{S}_k(0) = 1$,
2. $\mathcal{S}_k(n) := \sum_{i=n-k}^{n-1} [\mathcal{S}_k(i) = \mathcal{S}_k(n-1)]$ where we use the Iverson bracket notation, i.e. $[P]$ returns 1 if P is true and 0 if P is false.

To describe the sequence in words, we begin with a 1, and each subsequent term is the number of occurences of the last term in the last k terms. In (1) we show the first 30 terms of the sequence for $k = 5$.

$$\mathcal{S}_5 : 1, 1, 2, 1, 3, 1, 3, 2, 1, 2, 2, 3, 1, 2, 3, 2, 2, 3, 2, 3, 2, 3, 3, 3, 4, 1, 1, 2, 1, 3, \ldots \tag{1}$$

For the first k terms of the sequence, we only count the occurence of a term from the beginning of the sequence. Equivalently, we may pad the beginning of the sequence with a set of $k-1$ zeros. Further, each term in the sequence is treated as a single digit, i.e. a term of 17 does not count towards the counting of 1s or 7s.

Throughout this work, unless otherwise noted, we have excluded the trivial possibility that

$$\mathcal{S}_k = ..., \overbrace{k, k, k, ..., k, k}^{\text{k occurences}}, ... \tag{2}$$

It can be easily shown that the trivial sequence of repeating k only occurs if the sequence began with repeating terms of k, i.e. it cannot arise from the sequence beginning with 1.

Definition 1. A sequence $\mathcal{S}_k$ is ultimately periodic if there exist $t_0, T \in \mathcal{Z}^+$ such that $\mathcal{S}_k(t_0 + n) = \mathcal{S}_k(t_0 + T + n)\ \forall\ n \geq 0$. We call T the period of the cycle.

Note that for a periodic sequence there are infinitely many valid periods T (all integer multiples), we will always refer to the *least* value as the period.

Theorem 2. *For any k, the self counting sequence is ultimately periodic with period $T \leq k^k$.*

Proof. The n-th term of the sequence is a function of the k previous terms, each of which are drawn from the set $\{1, 2, 3..., k\}$. Therefore, there are at most k^k unique subsequences, and there exists a t_0 such that $S_k(t_0 + n) = S_k(k^k + n)\ \forall n \in \{1, 2, ..., k\}$. Therefore, given the rules to generate the sequence only depend on the previous k terms, we have $S_k(t_0 + n) = S_k(k^k + n)\ \forall n \geq 0$ with period of $T \leq k^k$. □

Clearly k^k is an extremely conservative bound, for example the $\mathcal{S}_5$ sequence (1) has period $T_5 = 25 << 5^5$. Our main results are concerned with the eventually periodic nature of these sequences, and are presented in Theorem **todo** and Conjecture **todo**. First we provide a few lemmas and a theorem bounding the value of the sequence terms.

Lemma 3. *A term $a \in \mathcal{S}_k$ cannot occur more than $a + 1$ times consecutively within $\mathcal{S}_k$.*

Proof. For any element $a \in \mathcal{S}_k$, assume there exists an index $t > 0$ such that $\mathcal{S}_k(t + i) = a$ for $0 \leq i \leq a + 1$, i.e. a has occured $a + 1$ times consecutively. Clearly, as $a \in \mathcal{S}_k$, we have $a \leq k$, and therefore all $a + 1$ terms are counted for the subsequent term, which must then be $a + 1$. □

Lemma 4. *For a given $\mathcal{S}_k$, no term $a \geq \lfloor \frac{k}{2} \rfloor + 1$ occurs twice consecutively.*

Proof. Assume there exists an index t_0 and elements $a, b \in \mathcal{S}_k$ such that

$$\mathcal{S}_k = \overbrace{...}^{t_0}, b, a, a, ... \tag{3}$$

with $a \geq \lfloor \frac{k}{2} \rfloor + 1$ and $b \neq a$. From indices $t_0 - k + 1$ to $t_0 + 1$, the element b must occur a times, and occur at least $a - 1$ times between indices $t_0 - k + 2$ to $t_0 + 1$. Similarly, the element a must occur a times between indices $t_0 - k + 2$ to $t_0 + 2$. We thus have $a - 1 + a$ occurences of elements a and b between indices $t_0 - k + 2$ to $t_0 + 2$

From the pigeonhole principle, if we have a sub-sequence $X^{\neq s}, s, s$ where $s > \lceil \frac{k}{2} \rceil$, then the previous k terms (before $X^{\neq s}$) must have contained s occurences of $X^{\neq s}$. Therefore, including the first occurence of s, we have $s - 1$ occurances of $X^{\neq s}$ and s occurences of s, which gives $s - 1 + s \geq \lceil \frac{k}{2} \rceil + \lceil \frac{k}{2} \rceil + 1 > k$, which is a contradiction. □

Note that Lemma 4 includes runs of longer than two, a fortiriori.

Theorem 5. *All elements of a k-foggy sequence have an upper bound of $\lceil \frac{k}{2} \rceil + 1$.*

Proof. Let $S_k = a_k(t-k+1:t)$ be the subsequence of length k preceeding $a_k(t)$. We know that S_k must contain $a_k(t) > \lceil \frac{k}{2} \rceil + 1$ copies of $a_k(t-1)$. Let $a^* = a_k(t-1)$ be the repeated element. From Lemma 4, we know that $a^* \leq \lceil \frac{k}{2} \rceil$, as otherwise it could appear only at every other position in S_k, which only $\lceil frack2 \rceil$ occurences. From Lemma 3, we know that there cannot be more than a^*+1 occurences of a^* in a row within S_k. After the initial set of a^*+1 occurences in S_k, all occurences of a^* must be singletons, as the subsequent element is the count of occurences of a^* which will be greater than a^*+1. Therefore, if $a_k(t) > \lceil \frac{k}{2} \rceil + 2$, then there must be an a^* in S_k followed by $\lceil \frac{k}{2} \rceil + 2$. It suffices then to prove that it is impossible to construct a sequence yeilding $a_k(t) = \lceil \frac{k}{2} \rceil + 2$.

We begin by assuming the maximal value $a^* = \frac{k}{2}$. We first partition the subseqeunce S_k as shown in (4),

$$S_k = [\ \overbrace{s_1, s_2, ..., s_{l_1}}^{\mathcal{C}_{a^*}(S_k(1:l_1))=a^*+1}\ , X^{\neq a^*}, a^*]. \tag{4}$$

In (4), we have partitioned the first a^*+1 occurences of s into the first l_1 elements, followed by an element not equal to a^*, followed by an a^*. In order for the $X^{\neq a^*}$ to be followed by an a^*, there must be a^* occurences in the previous k elements, along with the a^*+1 occurences of a^* itself in the first l_1 elements. Therefore, we have that

$$a^* + (a^*+1) \leq k \tag{5}$$
$$a^* \leq \frac{k}{2} - \frac{1}{2} \tag{6}$$

which contradicts the assumption $a^* = \frac{k}{2}$.

We generalize now by assuming s has value $a^* = \frac{k}{2} - i$. We then partition the subseqeunce S_k as shown in (7),

$$S_k = [\ \overbrace{s_1, s_2, ..., s_{l_1}}^{\mathcal{C}_{a^*}(S_k(1:l_1))=a^*+1}\ , (X^{\neq a^*}, a^*)^i, X^{\neq a^*}, a^*]. \tag{7}$$

In order for the final $X^{\neq a^*}$ to be followed by an a^*, there must be a^* occurences of it in the previous k elements. Note that after every occurence of a^*, after the initial l_1 partition, the $X^{\neq a^*}$ is the running total of occurences of a^* in S_k. Therefore, the terms $X^{\neq a^*}$ are unique, and summing the number of elements required to generate the final occurence of a^* gives

$$a^* + 1 + 2i + a^* \leq k \tag{8}$$
$$a^* \leq \frac{k}{2} - i - \frac{1}{2} \tag{9}$$

which contradicts the assumption that $a^* = \frac{k}{2} - i$.

This proof holds if the terms $X^{\neq a^*}$ are substituted for partitions of length l_i, containing any number of elements not equal to a^*. The scenario in (7) is the lower bound of the sizes of the intervening $X^{\neq a^*}$ partitions, and thus holds, as there still must be at least one unique element following every occurence of an a^* (after the initial partition). $\square$

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References

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