The Long and the Short on Counting Sequences

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1. INTRODUCTION. Consider the sequence of positive integers $S_0 = 2, 1, 1, 4$. S_0 consists of two 1's, one 2, and one 4, so let us define S_1 to be this description: $S_1 = 2, 1, 1, 2, 1, 4$. Repeating this process, S_1 consists of three 1's, two 2's and one 4, so set $S_2 = 3, 1, 2, 2, 1, 4$. Continuing in this way for several more steps produces

$$S_3 = 2,1,2,2,1,3,1,4$$
$$S_4 = 3,1,3,2,1,3,1,4$$
$$S_5 = 3,1,1,2,3,3,1,4$$
$$S_6 = 3,1,1,2,3,3,1,4.$$

In general, given any finite sequence of positive numbers S_0, this process of constructing S_{i+1} to be the sequence that counts how many times each number in S_i appears in S_i creates a *counting sequence* $\{S_i\}_{i \geq 0}$. As the reader certainly noticed, in our counting sequence we have $S_5 = S_6 = S_7 = \cdots$. In fact, in any counting sequence, because S_{i+1} is uniquely determined by S_i, if there exist numbers p and i such that $S_i = S_{i+p}$, then $S_{i'} = S_{i'+p}$ for all $i' \geq i$. We then say that $\{S_i\}_{i \geq 0}$ is *ultimately periodic*. The rather surprising main result of [**1**] is

Theorem 1. *For any finite sequence of positive integers S_0, the associated counting sequence $\{S_i\}_{i \geq 0}$ is ultimately periodic. In other words, given S_0 there are integers p_0 and p so that $S_{i+p} = S_i$ for all $i \geq p_0$.*

The smallest p_0 and smallest p satisfying Theorem 1 are called the *pre-period* and the *period* of the counting sequence $\{S_i\}$. Then a *periodic counting sequence* of period p, or simply a *p-cycle*, is a counting sequence of pre-period 0 and period p. For example, the counting sequence corresponding to $S_0 = 2, 1, 1, 4$ has pre-period 5 and period 1, that is, it "ends" in a 1-cycle. Similarly, the counting sequence corresponding to $S_0 = 5, 6$ ends in a two-cycle, and the counting sequence corresponding to $S_0 = 6, 7$ ends in a three-cycle.

Several different types of counting sequences have been studied in recent years (see [**1**], [**5**], [**6**], [**7**], [**8**], and M4779 in [**9**]). In this paper we consider these counting sequences, bring out their connections, and explore the periodic behavior of each. To expand on this, the questions we answer are:

1) What are the possible periods p? For each p, how many p-cycles are there? In Section 3 we find all possible periods and classify all cycles. Partial answers to these questions are given in [**6**].
2) A puzzle of Raphael Robinson [**3**, pp. 389–390] asks the reader to place numbers in the blanks so that the following is true: "In this sentence, the number of occurrences of 0 is __, of 1 is __, of 2 is __, of 3 is __, of 4 is __, of 5 is __, of 6 is __, of 7 is __, of 8 is __, and of 9 is __." To find such a

sentence we must find a one-cycle that contains all of the numbers in base 10, as opposed to the infinite base consisting of all the natural numbers implicitly used in the preceding paragraphs. More generally, one can build counting sequences in base k for any $k \geq 2$. Are such counting sequences also eventually periodic? In Section 4 we show that they are, and determine exactly how many different cycles there are in each base. This expands upon the results of **[6]**.

3) What happens when S_0 is replaced by an infinite sequence? It is very easy to give infinite sequences S_0 such that $\{S_i\}_{i \geq 0}$ is not well-defined. In Section 5 we show how to construct examples of infinite sequences S_0 so that $\{S_i\}_{i \geq 0}$ is well-defined and is ultimately periodic. We also give two different methods for constructing infinite sequences S_0 so that $\{S_i\}_{i \geq 0}$ is well-defined and but not ultimately periodic.
4) The second term, fourth term, sixth term, etc., in each sequence S_i of a counting sequence do little more than serve as place holders. Assuming there is a way to tell which integer each number is describing, what happens if we form counting sequences without these place holders? One can then ask questions similar to those in 1) for these sequences. These questions have, for the most part, been answered in **[5]**, **[7]**, and **[8]**. We see in Section 6 that the answers also follow as very simple corollaries of our work in Sections 2 and 3. Robinson's question can also be asked in this context; see **[10]**.

In each of the various methods we use to construct counting sequences, the successor sequence lists the number of appearances of a particular digit throughout the entire previous sequence. It is also possible to construct counting sequences in which the successor lists the number of consecutive appearances of a digit: if $C_0 = 2, 1, 1, 4$, then $C_1 = 1, 2, 2, 1, 1, 4$ and $C_2 = 1, 1, 2, 2, 2, 1, 1, 4$. See **[2]** for Conway's analysis of such counting sequences.

2. BASIC PROPERTIES OF COUNTING SEQUENCES.

We begin by giving several important properties of the sequences making up a counting sequence, and then give a simple proof of Theorem 1. Fix a finite sequence of positive integers S_0 and let $\{S_i\}_{i \geq 0}$ be the corresponding counting sequence. For $i \geq 1$ we write S_i as

$$S_i = m_{i,1}, f_{i,1}, m_{i,2}, f_{i,2}, \cdots, m_{i,n_i}, f_{i,n_i}.$$

We assume the $f_{i,j}$'s are in increasing order and leave out commas to unclutter the notation when there is no risk of confusion. The positive integer $m_{i,j}$ is called a *multiplier* of S_i and indicates that the integer $f_{i,j}$, called a *factor* of S_i, appears exactly $m_{i,j}$ times in S_{i-1}. Let $|S_i| = 2n_i$ be the total number of terms in S_i. The following observations about the S_i's are used often, and frequently without mention. Similar facts are proved in **[1]** and **[6]**.

Proposition 2. *Fix S_0 and let $\{S_i\}_{i \geq 0}$ be the corresponding counting sequence. Let $i \geq 1$.*

1) *For each factor $f_{i,j}$ of S_i there are $m_{i,j} - 1$ or $m_{i,j}$ multipliers of S_{i-1} with the value $f_{i,j}$, depending on whether or not the value $f_{i,j}$ appears as a factor in S_{i-1}.*
2) *We have $|S_{i-1}| = \sum_{j=1}^{n_i} m_{i,j}$ and $|S_{i-1}| \leq |S_i|$, because every factor of S_{i-1} is also a factor of S_i.*
3) *If $\{S_i\}_{p_0 \leq i \leq p}$ constitutes a p-cycle, then $|S_i| = |S_{i+1}|$ for all i and each S_i in the cycle has exactly the same factors. Further, $|S_{i-1}| = \sum_{j=1}^{n_i}(m_{i,j} - 1)f_{i,j}$.*

To show how these facts will be used we provide the following proof of Theorem 1.

Proof of Theorem 1: Fix S_0, and let $\max(S_i)$ be the value of the largest term in S_i. Clearly either $\max(S_i) = \max(S_2)$ for all $i \geq 2$, or there is some i such that $\max(S_{i+1}) > \max(S_i)$. First assume the former. For $i \geq 2$, the number of sequences S_i with $\max(S_i) \leq n$ for any particular n is at most $(n+1)^n$, and so is finite. Since S_{i+1} is completely determined by S_i, we then see that the counting sequence $\{S_i\}_{i \geq 0}$ must eventually repeat, and so enters a cycle.

So now suppose $\max(S_{i+1}) > \max(S_i)$ for some $i \geq 2$, and choose n so that $n+1$ is a term in S_{i+1} and is larger than every term in S_i. Since $n+1$ can appear in S_{i+1} only as a multiplier, S_i has at least $n+1$ equal terms. But clearly $|S_i| \leq 2n$, and since $i \geq 2$, the factors in S_i are distinct. It must therefore be the case that all of the multipliers of S_i are equal, $|S_i| = 2n$, and each of the integers from 1 to n appears as a multiplier in S_i. Write $S_i = m, 1, m, 2, \ldots, m, n$ for some $m \geq 1$. Then $mn = \Sigma_{j=1}^{n} m = |S_{i-1}| \leq |S_i| = 2n$ shows $m \leq 2$.

If $m = 2$ then

$$2n \geq |S_{i-1}| \geq \sum_{j=1}^{n} (m-1) f_j = \sum_{j=1}^{n} f_j \geq \sum_{j=1}^{n} j$$

shows that $n \leq 3$, and that S_i must be $2, 1$ or $2, 1, 2, 2$ or $2, 1, 2, 2, 2, 3$. A counting sequence containing any of these is easily shown to converge to $2, 1, 3, 2, 2, 3, 1, 4$, a one-cycle. A similar argument shows that if $m = 1$ and $i \geq 2$, then $S_{i-1} = 1, 2$ or $1, 2, 3, 4$ or $1, 2, 3, 4, 5, 6$, all of which lead to periodic counting sequences. ■

3. CYCLES AND THEIR TRUNCATIONS. Theorem 1 ensures that no matter what finite sequence S_0 of positive integers we begin with, the counting sequence associated to S_0 is ultimately periodic, that is, ends in a cycle of some period p. We now determine the possible periods, and for each p classify the p-cycles. As the word "classify" hints, there are actually infinitely many different cycles, and the sequences in these cycles may be arbitrarily long. Fortunately there are only three possible periods, and each cycle has a companion cycle made up of very short sequences. We will use these truncated sequences to make our classification.

Fix a p-cycle, and for ease, rename the sequences in it $S_1, S_2, \ldots, S_p$. We first show that 1 occurs as a term in each S_i, unless the cycle is the one-cycle $S_1 = 2, 2$. This implies that the multiplier of the factor 1 will play an important role in our classification.

Lemma 3. *Either* 1 *occurs at least twice in each* S_i, *or* $p = 1$ *and* $S_1 = 2, 2$.

Proof: First suppose no S_i has 1 as a factor, so all of the multipliers in each S_i have values larger than 1. Let $|S_i| = 2n$. Since the sum of the n multipliers of S_i equals $|S_{i-1}| = |S_i| = 2n$, all of the multipliers of S_i must equal 2. This is true for all i. But then all of the S_i's have exactly the same multipliers, all of value 2, and exactly the same factors, so this cycle is the one-cycle $S_1 = 2, 2$.

Next, when one S_i has 1 as a factor, then each S_i does. If S_{i+1} contains exactly one 1, for some i, then none of S_i's multipliers equal 1. Again, $\Sigma_{j=1}^{n} m_{i,j} = |S_{i-1}| = |S_i| = 2n$ then implies that all of the multipliers of S_i have the value 2. However, as in the proof of Theorem 1, a counting sequence containing such an element converges to the one-cycle $2, 1, 3, 2, 2, 3, 1, 4$, which is not equal to S_{i+1}, contradicting our assumption. Thus each S_i contains at least two 1's, as desired. ■

Next consider the factors whose multipliers are equal to 1. If the factor f of S_i has multiplier 1, then f appears in S_{i-1} only as a factor, so it plays a relatively unimportant role in the creation of S_i. This leads us to consider the *truncation* S_i' of S_i formed by deleting all the multiplier-factor pairs of S_i whose multipliers are 1. For example, if $S_1 = 6,1,2,2,1,3,1,4,1,5,2,6,1,7$, then $S_1' = 6,1,2,2,2,6$. We will see that there are rather few sequences that arise as the truncation of a sequence in a cycle, and so will be able to use truncation to classify the cycles.

Assume S_i is a sequence belonging to a cycle, and by Lemma 3 that S_i' has the form

$$S_i' = m_{i,1}\, 1\, m_{i,2}\, f_{i,2} \cdots m_{i,k_i}\, f_{i,k_i}$$

with $m_{i,j} \geq 2$ for all j. In studying S_i', the first step is to establish a property similar to part 2 of Proposition 2.

Lemma 4. *In a cycle we have* $k_i - 1 = |S_i'|/2 - 1 \leq \Sigma_{j=2}^{k_i}(m_{i,j} - 1) = |S_{i-1}'|/2$ *for all* i. *In particular,* $|S_i'| \leq |S_{i-1}'| + 2$ *for all* i.

Proof: The first equality and the following inequality are trivial, since $m_{i,j} \geq 2$. For the last equality, since $m_{i,j} - 1$ is the number of multipliers in S_{i-1} with value $f_{i,j}$, the number of multipliers in S_{i-1} that are not equal to 1 is $\Sigma_{j \geq 2}(m_{i,j} - 1)$. But the multipliers in S_{i-1}' are exactly the multipliers in S_{i-1} that do not equal 1. Thus the sum equals the number of multipliers in S_{i-1}', or $|S_{i-1}'|/2$. ■

Therefore, in a cycle either $|S_i'| = |S_{i-1}'|$ for all i, or $|S_i'| = |S_{i-1}'| + 2$ for some i. Since each multiplier in S_i' is larger than 1, if $|S_i'| = |S_{i-1}'|$ then Lemma 4 shows that $\{m_{i,j} : j \geq 2\}$ consists of all 2's except for possibly one 3, while if $|S_i'| = |S_{i-1}'| + 2$ then $m_{i,j} = 2$ for all $j \geq 2$. We next show that the first case corresponds to the one-cycles, and so the second case corresponds to the longer cycles.

Proposition 5. *Suppose* $\{S_i\}_{1 \leq i \leq p}$ *is a cycle such that* $|S_i'| = |S_{i+1}'|$ *for all* i. *Then* $p = 1$, *so* S_1 *is actually a one-cycle.*

Proof: Since all the S_i's have exactly the same factors, it suffices to show that the multiplier of any particular factor is the same in all of the S_i's. Because the set of multipliers of S_{i-1} is exactly the set of factors of S_i', and we might as well assume S_i is not $2,2$, the only factors of S_i whose multipliers are not 1 are $1,2,3$, and m, where m is the multiplier of 1 in S_{i-1}. We concentrate on these factors. First, $1 + (|S_i| - |S_i'|)/2$ is independent of i and gives the number of 1's in S_i. Thus $1 + (|S_i| - |S_i'|)/2 = m$, and each of the S_i's contains m 1's. Next, we have seen that each $\{m_{i,j} : j \geq 2\}$ consists of all 2's except for possibly one 3. Since the value of the sum $\Sigma_{j \geq 2}(m_{i,j} - 1) = |S_{i-1}'|/2$ is independent of i, as m is, we see that each S_i must contain the same number of 2's and the same number of 3's. Finally, for $m \geq 4$, m occurs in each S_i exactly twice, as the multiplier of 1 and as a factor. ■

This proposition allows us to find the truncations of all the one-cycles. Except for $S = 2,2$, the set of multipliers of a one-cycle consists of 2's, possibly one 3, and the multiplier m of 1. We point out the various cases and let the reader check the details. If $m = 2$ then $S' = 2,1,3,2,2,3$, while if $m = 3$ then $S' = 3,1,2,2,3,3$ or $S' = 3,1,3,3$ depending on whether 2 is a multiplier of S or not. Since m is at least 2, the only other possibility is $m \geq 4$, and then $S' = m, 1, 3, 2, 2, 3, 2, m$. We have proved

Theorem 6. *If S is a one-cycle, then S' is $2, 2$ or $3, 1, 3, 3$ or $2, 1, 3, 2, 2, 3$ or $3, 1, 2, 2, 3, 3$ or $m, 1, 3, 2, 2, 3, 2, m$ for some $4 \leq m$.*

We next turn our attention to the cycles whose periods are longer than 1. From Lemma 4 and Proposition 5 we know that in such a cycle $|S'_i| = |S'_{i-1}| + 2 \geq 4$ for some i, and that the multipliers in S'_i all equal 2, except for possibly the multiplier of 1. In fact, since $2n = |S_i| = |S_{i-1}| = \Sigma_{j=1}^{n} m_{i,j}$ is a sum of 1's, at least one 2, and the multiplier of 1, we see the multiplier of 1 in S_i must be at least 3. If we write

$$S'_i = m_{i,1}, 1, 2, f_{i,2}, \cdots, 2, f_{i,k_i},$$

for $k_i \geq 2$, then

$$S'_{i+1} = m_{i+1,1}, 1, k_i, 2, 2, m_{i,1}$$

for some m_{i+1}. Thus if $|S'_i| = |S'_{i-1}| + 2$ for some i, then $|S'_{i+1}| = 6$. Because $|S'_i| \leq |S'_{i-1}| + 2$ for all i, it must therefore be the case that $|S'_{i-1}| = 6$, $|S'_i| = 8$, and $|S'_{i+1}| = 6$.

Now write $S'_i = m, 1, 2, a, 2, b, 2, c$ for $m \geq 3$ and some a, b, and c. Clearly S_{i-1} must have $m - 1$ multipliers equal to 1. Because $|S'_{i-1}| = 6$, we have $|S_i| = |S_{i-1}| = 2(m + 2)$, so the number of 1's in S_i is $1 + (|S_i| - |S'_i|)/2 = (m - 1)$. Thus the multiplier of 1 in S_{i+1} is $m - 1$. Similarly, the multiplier of 1 in S_{i-1} is m or $m - 1$, depending on whether $|S_{i-2}|$ is 6 or 8, so S_i contains either two m's or two $(m - 1)$'s.

Before considering these two cases, we show how to construct S'_{i+1} directly from S'_i. Any integer $f \neq 1$ appears as a factor in S'_{i+1} if and only if it appears as a multiplier in S'_i, and then its multiplier in S'_{i+1} is one more than the number of times it appears as a multiplier in S'_i. Lemma 3 shows that $f = 1$ also appear as a factor in S'_{i+1}. The next lemma shows how to compute its multiplier.

Lemma 7. *The number of 1's in S_i is $1 + \Sigma\big((m_{i,j} - 1)(f_{i,j} - 1) - 1\big)$, where the sum is over the multipliers of S'_i.*

Proof: Let m be the number of 1's appearing as multipliers in S_i. Then

$$\begin{aligned} 1 + &\sum_{m_{i,j} \in S'_i} \big((m_{i,j} - 1)(f_{i,j} - 1) - 1\big) \\ &= 1 + m + \sum_{m_{i,j} \in S_i} \big((m_{i,j} - 1)(f_{i,j} - 1) - 1\big) \\ &= 1 + m + \sum_{m_{i,j} \in S_i} (m_{i,j} - 1) f_{i,j} - \sum_{m_{i,j} \in S_i} m_{i,j}. \end{aligned}$$

The two last sums both equal $|S_{i-1}|$, and $1 + m$ is the number of 1's in S_i. ∎

Consider again $S'_i = m, 1, 2, a, 2, b, 2, c$. If S_i contains two $(m - 1)$'s, with $m \geq 3$, then $S'_i = m, 1, 2, a, 2, b, 2, m - 1$, for some a and b, and so $S'_{i+1} = m - 1, 1, A, 2, 2, m$, for some A. It is then clear that m cannot be 3. By Lemma 7, $a + b = 6$, so $a = 2$ and $b = 4$. Thus $S'_i = m, 1, 2, 2, 2, 4, 2, m - 1$ and $m \neq 5$. Using Lemma 7 again we see

$$\begin{aligned} &S'_{i+1} = m - 1, 1, 4, 2, 2, m \\ \text{and} \quad &S'_{i+2} = m, 1, 2, 2, 2, 4, 2, m - 1 = S'_i, \end{aligned}$$

for $m = 4$ or $m \geq 6$, and so this is a two-cycle. Notice that when $m \geq 6$ the multiplier-factor pair $1, m$ must appear in S_i and the pair $1, m-1$ in S_{i+1}.

If S_i contains two m's, then $S'_i = m, 1, 2, a, 2, b, 2, m$ for some a and b, and so $S'_{i+1} = m-1, 1, A, 2, 2, m$ for some A. By Lemma 7, $a + b = 5$, so $a = 2$, $b = 3$ and $S'_i = m, 1, 2, 2, 2, 3, 2, m$. Thus $S'_{i+1} = m-1, 1, 4, 2, 2, m$. If m is not equal to 5, we are led to one of the cycles above. If m does equal 5, then using Lemma 7 we have

$$S'_i = 5, 1, 2, 2, 2, 3, 2, 5$$

$$S'_{i+1} = 4, 1, 4, 2, 2, 5$$

$$S'_{i+2} = 5, 1, 2, 2, 3, 4$$

and $S'_{i+3} = S'_i$, so this is a three-cycle. Notice that the pair $1, 3$ must appear in S_{i+1} and $1, 5$ must appear in S_{i+2}. We have proved

Theorem 8. *Suppose $\{S_i\}_{1 \leq i \leq p}$ is a periodic counting sequence with period $p > 1$. Then either $p = 2$ or $p = 3$. In fact,*

i. *If $p = 2$ then the truncated form of $\{S_i\}$ is* $\begin{cases} S'_1 = m, 1, 2, 2, 2, 4, 2, m-1 \\ S'_2 = m-1, 1, 4, 2, 2, m \end{cases}$ *with $m = 4$ or $m \geq 6$*

ii. *If $p = 3$ then the truncated form of $\{S_i\}$ is* $\begin{cases} S'_1 = 5, 1, 2, 2, 2, 3, 2, 5 \\ S'_2 = 4, 1, 4, 2, 2, 5 \\ S'_3 = 5, 1, 2, 2, 3, 4. \end{cases}$

It is a simple matter to rebuild a cycle from its truncation just by picking reasonable factors. For instance, if $S' = 3, 1, 2, 2, 3, 3$ then $S = 3, 1, 2, 2, 3, 3, 1, 4, 1, 5$ gives a one-cycle. Of course, so does $S = 3, 1, 2, 2, 3, 3, 1, 5, 1, 20$ and, if we expand our possible choice of factors, so does $S = 1, -4, 1, 0, 3, 1, 2, 2, 3, 3$. In the sequel it will be useful to allow 0 as a factor.

Notice that no three-cycle can contain more than seven factors, or more than two factors larger than 5. Thus, if S_0 contains eight or more distinct numbers, or two or more distinct numbers larger than 5, then the cycle to which its counting sequence converges cannot have period 3. Nor can it converge to any one-cycle, except for one whose truncation has the form $m, 1, 3, 2, 2, 3, 2, m$. Since most finite sequences S_0 contain three different numbers larger than 5, we see most counting sequences converge to a one-cycle of the form $m, 1, 3, 2, 2, 3, 2, m$, or to a two-cycle. In fact, the multiplier of 2 can be used to distinguish between these last two cases, but unfortunately we do not have methods to predict the multipliers of 2. It would be quite interesting to have a more precise answer.

Similarly, we would like to have a method to determine the pre-period of a given S_0, that is, to be able to measure how far S_i is from entering a cycle.

4. CYCLES IN A FINITE BASE. From Theorem 6 the numerical portion of an answer Raphael Robinson's puzzle is

$$1, 0, 1, 7, 1, 3, 2, 2, 3, 1, 4, 1, 5, 1, 6, 2, 7, 1, 8, 1, 9.$$

Is this answer unique? No, there is another:

$$1, 0, 11, 1, 2, 2, 1, 3, 1, 4, 1, 5, 1, 6, 1, 7, 1, 8, 1, 9,$$

if we read 11 as two 1's. This makes sense only if we represent the value *eleven* as $1 \cdot 10^1 + 1 \cdot 10^0$, i.e., if we write our numbers in base 10. This example reveals the basic and interesting difference between the counting sequences over finite bases and those over the infinite base: when we use a finite base the multipliers can consist of multiple digits, which, by definition, is impossible over the infinite base.

What cycles are possible if we must choose the factors from the digits 0 through $k-1$ and consider the multipliers in base k? If the factors are kept smaller than k, then Theorems 6 and 8 provide examples of cycles in base k. In base 5, for instance,

$$1,0,3,1,1,2,3,3 \quad \text{and} \quad 2,1,3,2,2,3,1,4$$

are one-cycles. However $(11)_5, 1,1,2,1,3,1,4$ is also a one-cycle in base 5, where $(11)_5$ is the representation of the number *six* in base 5, so these theorems do not list all of the cycles. It thus remains for us to find the cycles that contain at least one multiplier with multiple digits in base k.

We first show that given any sequence S_0 the counting sequence $\{S_i\}_{i\geq 0}$ formed in base k is eventually periodic. As in Section 2, it suffices to show that S_i, $i \geq 1$, can take on only finitely many forms. We now write $|S_i|_k$ for the total number of digits appearing in S_i when it is written in base k. For example, $|(11)_3, 1, 1, 2|_3 = 5$.

Lemma 9. *In base $k \geq 4$, we have $|S_i|_k \leq 2k+1$ for all sufficiently large i.*

Proof: We simply show that if $|S_{i-1}|_k \leq |S_i|_k$ for some i, then $|S_i|_k \leq 2k+1$. As in Proposition 2 we have $|S_{i-1}|_k = \sum_{j\geq 1} m_j$, where the m_j's are the multipliers of S_i. Letting $\#m_j$ be the number of digits of m_j in base k, we then have

$$|S_i|_k = \text{the number of factors in } S_i + \sum_{m_j \in S_i} \#m_j \leq k + \sum_{m_j \in S_i} \#m_j. \tag{4.1}$$

Using $\sum m_j = |S_{i-1}|_k \leq |S_i|_k$, we see that

$$\sum_{m_j \in S_i} (m_j - \#m_j) \leq k. \tag{4.2}$$

But $m_j - \#m_j$ is at least $k-2$ if $m_j \geq k \geq 4$. Thus, for $k \geq 5$ there can be at most one multiplier of S_i consisting of multiple digits in base k, and its value can be at most $k+2$. When $k=4$, one shows easily that a sequence in a counting sequence with two multipliers larger than 3 must have multipliers $4 = (10)_4$, $4 = (10)_4$, 1, and 1, and its counting sequence converges to $1, 0, (11)_4, 1, 1, 2, 2, 1, 3$. Therefore $|S_i|_k \leq 2k+1$, as desired. ■

Thus, when k is at least 4 there are only finitely many sequences that may appear in any given counting sequence. Inequality (4.2), which holds in any base, shows that 5 is the largest possible value of a multiplier in a counting sequence in base 2 or 3, and so also over these two bases a sequence in any given counting sequence may take on only finitely many forms. Therefore, all counting sequences in base k are eventually periodic for all k.

Checking the possibilities, which we leave to the reader, in base 2 the only cycles are $(11)_2, 1$ from Theorem 8, and $(11)_2, 0, (100)_2, 1$. In base 3, Theorem 6 gives only 2, 2, while Theorem 8 gives three one-cycles. The only other one-cycles in base 3 are

$$(10)_3, 0, (10)_3, 1, 2, 2 \quad \text{and} \quad 2, 0, 2, 1, (10)_3, 2 \quad \text{and} \quad (10)_3, 0, (10)_3, 1,$$

and the only longer cycle is $\begin{cases} S_1 = 1, 0, (10)_3, 1, (10)_3, 2 \\ S_2 = (10)_3, 0, (11)_3, 1, 1, 2 \\ S_3 = 2, 0, (12)_3, 1, 1, 2. \end{cases}$

Now suppose k is at least 4, and, by Lemma 9, that S_i is a sequence with one multiplier M such that $k \leq M \leq k + 2$. If S_i has f factors, then S_{i-1} has at most f factors, and since they each have no more than one multiplier with two digits, we see that $|S_{i-1}|_k \leq |S_i|_k$. Thus inequality (4.2) may be more accurately stated as

$$k - 2 \leq \sum_{m_j \in S_i} (m_j - \#m_j) \leq (|S_i|_k - 1)/2 \leq k. \tag{4.3}$$

If $M = k + 2$, then all of the other multipliers in S_i must equal 1, and $|S_i|_k = 2k + 1$. One then sees that S_{i+1} has the form

$$S_{i+1} = 1, 0, (11)_k, 1, 2, 2, 1, 3, \ldots, 1, k\text{-}1, \tag{4.4}$$

which constitutes a one-cycle. If $M = k + 1$, then the other multipliers in S_i equal 1, except for possibly one 2. When 2 is a multiplier of S_i, we have $|S_i|_k = 2k + 1$ and then S_{i+1} is as given in (4.4). When 2 is not a multiplier in S_i, then $|S_i|_k = 2k - 1$ and

$$S_{i+1} = 1, 0, (11)_k, 1, 1, 2, \ldots, \widehat{1, l}, \ldots, 1, k - 1$$

for some $0 \leq l \leq k - 1$, $l \neq 1$, where $\widehat{1, l}$ means that this pair does not appear in S_{i+1}. Clearly S_{i+1} forms a one-cycle. Finally, if $M = k = (10)_k$, then it is not difficult to use part 2) of Proposition 2 to show that $\{S_i\}_{i \geq 0}$ converges to a one-cycle consisting of terms having one base k digit each. We have proved

Proposition 10. *The only cycles in base $k \geq 4$ that have multipliers with two or more digits are one-cycles. Further, if S' is the truncated version of one of these sequences, then S' is either $(11)_k, 1$ or $(11)_k, 1, 2, 2$.* ■

We have now discovered all possible cycles in base k. Since k is finite, the number of cycles is finite and can be counted.

Theorem 11. *For $k \geq 4$, the number of one-cycles in base k is $2^{k-4} + k(k - 1)/2$. In bases 4 and 5 there are no longer cycles, while in base $k \geq 6$ there are $2^{k-5} - 1$ two-cycles, $\binom{k-5}{2}$ three-cycles, and no longer cycles.*

The proof of Theorem 11 is just a matter of undoing the truncation process, and then using the binomial theorem. For example, for each $m \geq 4$ and $k \geq 6$ there are $\binom{k-4}{m-1}$ one-cycles S with $S' = m, 1, 3, 2, 2, 3, 2, m$, so there are

$$\sum_{m=4}^{k-1} \binom{k-4}{m-1} = 2^{k-4} - 1 - (k - 4) - \binom{k-4}{2}$$

one-cycles S with S' having the form $m, 1, 3, 2, 2, 3, 2, m$. We leave the rest of the proof to the reader.

5. INFINITE SEQUENCES AND INFINITE CYCLES. From Theorem 1 we know that every counting sequence beginning with a finite sequence S_0 is ultimately periodic. Is this true when S_0 is an infinite sequence? In this section we show that it is not and provide two methods for constructing counter-examples.

If one chooses an infinite sequence S_0 at random, its associated counting sequence may fail to exist. For example, if we choose $S_0 = 1, 2, 3, 4, 5, 6, \ldots,$ then $S_1 = 1, 1, 1, 2, 1, 3, 1, 4, 1, 5, 1, 6, \ldots,$ but S_2 is not well-defined and so $\{S_i\}_{i \geq 0}$ does not exist. It would be interesting to have necessary or sufficient conditions on S_0 so that its counting sequence exists. We will not concern ourselves here with general existence and convergence questions but instead concentrate on supplying a variety of examples.

We begin by constructing infinite sequences whose associated counting sequences are actually one-cycles. First, let $S_0^0 = 4, 4$, and define $S_0^1 = 4, 4, 4, 5, 4, 6$. Notice that there are four 4's in S_0^1, which fits the description given in S_0^0. Next, create S_0^2 to fit the description given in S_0^1 and to have consecutive factors, and then similarly create S_0^3 by the description implicit in S_0^2:

$$S_0^2 = 4,4,4,5,4,6,5,7,5,8,5,9,6,10,6,11,6,12$$

$$\begin{aligned} S_0^3 = {} & 4,4,4,5,4,6,5,7,5,8,5,9,6,10,6,11,6,12,7,13,7,14,7,15,7,16,8,17,8,\\ & 18,8,19,8,20,9,21,9,22,9,23,9,24,10,25,10,26,10,27,10,28,10,29,11,\\ & 30,11,31,11,32,11,33,11,34,12,35,12,36,12,37,12,38,12,39. \end{aligned}$$

Finally, define S_0 to be the limit of the finite sequences $\{S_0^k\}$. It is then clear that S_0 forms a one-cycle, and so each element of the counting sequence $\{S_i\}$ exists. We adopt the terminology of [3] to call the process that takes the finite sequence S_0^0 and produces the infinite sequence S_0 the *self-generating process*.

Proposition 12. *Let $S = m_1, f_1, m_2, f_2, \dots, m_n, f_n$ be a sequence of positive integers such that the f_i are strictly increasing, f_i appears no more than m_i times in S, and each m_i also appears as an f_j. Then, setting $S_0^0 = S$, the sequences S_0^k can be constructed using the self-generating process, $S_0 = \lim_k S_0^k$ exists, and S_0 forms a one-cycle.*

To give the relative sizes of the factors and multipliers of our particular example $S_0 = 4,4,4,5,4,6,\dots$ we introduce an integer sequence constructed and studied first by Golomb [3]. This sequence

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, \cdots$$

consists of the values of the function $G(n)$ defined on the natural numbers by

(i) $G(1) = 1$
(ii) $G(n) = \#\{\text{integers } m : G(m) = n\}$
(iii) $G(n)$ is non-decreasing.

Golomb proved the asymptotic formula $G(n) \sim \phi(n/\phi)^{\phi - 1}$, where $\phi = (\sqrt{5} + 1)/2$ is the golden ratio. If we replace the first three terms of Golomb's sequence by a 3, and then add 1 to each term, the resulting sequence consists of the multipliers of S_0. Thus, the multiplier of f in S_0 is approximately $G(f)$. Inverting the asymptotic formula for $G(f)$ then gives

Proposition 13. *Let m_f be the multiplier of f in $S_0 = 4, 4, 4, 5, \dots$. Then*

$$f \sim \phi\left(\frac{m_f}{\phi}\right)^{\phi}.$$

It is a simple matter to modify S_0 to create counting sequences consisting of infinite sequences that converge to longer cycles. For instance, define $S_0(24)$ to be the sequence that is identical to S_0 except that the multiplier-factor pair $9, 24$, is replaced by $10, 24$, i.e.,

$$\begin{aligned} S_0(24) = {} & 4,4,4,5,4,6,5,7,5,8,5,9,6,10,6,11,6,12,7,13,7,14,7,15,7,16,8,17,\\ & 8,18,8,19,8,20,9,21,9,22,9,23,\underline{10,24},10,25,10,26,10,27,10,28,\dots. \end{aligned}$$

We have underlined the multiplier-factor pairs of $S_0(24)$ that do not agree exactly with S_0, i.e., the positions of $S_0(24)$ that are in "error" when compared to S_0. If $S_1(24)$ is the usual description of the sequence $S_0(24)$, then $S_1(24)$ contains one more 10 but one fewer 9 than S_0 contains, so

$$S_1(24) = 4,4,4,5,4,6,5,7,5,8,\underline{4,9,7,10},6,11,6,12,7,13,7,14,7,15,7,16,8,17,$$
$$8,18,8,19,8,20,9,21,9,22,9,23,9,24,10,25,10,26,10,27,10,28,\cdots,$$

and

$$S_2(24) = \underline{5,4,3,5,3,6,6,7},5,8,5,9,6,10,6,11,6,12,7,13,7,14,7,15,7,16,8,17,$$
$$8,18,8,19,8,20,9,21,9,22,9,23,9,24,10,25,10,26,10,27,10,28,\cdots.$$

Notice that the multiplier-factor pair in error in $S_0(24)$ has been "repaired" in $S_1(24)$, and that the errors in $S_1(24)$ are repaired in $S_2(24)$. Also notice that the numerical values of the multipliers in error in S_0 and factors in error in S_1 are very close. The same is true for the multipliers in error in S_1 and the factors in error in S_2. If we continue, the counting sequence converges to

$$S_{10}(24) = \underline{1,1,4,2,2,3,2,4,5,5},4,6,5,7,5,8,5,9,6,10,\cdots$$
$$S_{11}(24) = \underline{2,1,3,2,1,3,3,4,5,5},4,6,5,7,5,8,5,9,6,10,\cdots$$
$$S_{12}(24) = \underline{2,1,2,2,3,3,2,4,5,5},4,6,5,7,5,8,5,9,6,10,\cdots$$
$$S_{13}(24) = S_{10}(24) = \underline{1,1,4,2,2,3,2,4,5,5},4,6,5,7,5,8,5,9,6,10,\cdots.$$

So we have constructed an example of an infinite three-cycle.

We can abstract two facts from this example. Suppose we create an infinite sequence $S_0(f)$ that is identical to an infinite one-cycle S_0 except that the multiplier of f in S_0 has been increased by one. Then, (1), at the beginning of the counting sequence $\{S_i(f)\}_{i \geq 0}$ the errors move quickly to the "left", and (2) once the errors have reached the beginning of the sequences, they (relatively) quickly settle into a cycle. It is not too difficult to convince oneself of these facts, because Proposition 13 tells us that a multiplier in S_0 is far smaller than its factor.

More generally, one may define $S_0(f_1, f_2, \ldots, f_m)$ to be identical to an infinite one cycle S_0 except that the multipliers of the f_j's in S_0 have been increased by one, and then consider the counting sequence $\{S_i(f_1, f_2, \ldots, f_m)\}_{i \geq 0}$. Each such counting sequence ends in a cycle. It would be interesting to classify the cycles that arise in this manner.

Using these ideas we can describe the construction of an infinite counting sequence that is *not* ultimately periodic. For a given integer f_j let n_j be the pre-period of $\{S_i(f_j)\}_{i \geq 0}$. That is, $S_i(f_j)$ is part of a cycle if $i \geq n_j$. Choose an infinite set of factors f_j, $j \geq 1$, growing fast enough in j so that for all $i \leq n_j$ and $f < f_{j-1}$, the multipliers of the factor f in S_0 and $S_i(f_j)$ are equal. In other words, choose f_j so that it takes more than n_j steps for the errors in $S_i(f_j)$ to move themselves to the point of the initial error in $S_0(f_{j-1})$. Once we fix such an infinite sequence, then $\{S_i(f_1, f_2, f_3, \cdots)\}_{i \geq 0}$ will be a non-periodic counting sequence. To actually construct such a family of f_j's one needs to use Proposition 13 to give a careful study of the rates at which the errors in $\{S_i(f_j)\}$ spread and move to the left.

As this study would occupy the better part of several pages, we instead end this section with a very simple method for constructing infinite counting sequences that are both well-defined and not ultimately periodic. Define $\{S_i\}_{i \geq 1}$ by the following rules:

(i) The multiplier of i in S_i has value at least i.
(ii) Every natural number occurs as a factor in each S_i.
(iii) The multipliers in each S_i form a non-decreasing sequence.
(iv) S_{i+1} is the description of S_i for $i \geq 1$.

Rule (i) insures that the terms below the main diagonal are not influenced by those above the main diagonal. For instance, taking the multiplier of i to be $i+1$ gives the following:

$$S_1 = \mathbf{2}, \mathbf{1}, 2, 2, 3, 3, 3, 4, 4, 5, 4, 6, 5, 7, 5, 8, 5, 9, 6, 10, 6, 11, 6, 12, 7, 13, 7, 14, 7, 15, \cdots$$
$$S_2 = 1, 2, \mathbf{3}, \mathbf{2}, 3, 3, 3, 4, 4, 5, 4, 6, 4, 7, 5, 8, 5, 9, 5, 10, 6, 11, 6, 12, 6, 13, 7, 14, 7, 15, \cdots$$
$$S_3 = 1, 1, 2, 2, \mathbf{4}, \mathbf{3}, 4, 4, 4, 5, 4, 6, 5, 7, 5, 8, 5, 9, 5, 10, 6, 11, 6, 12, 6, 13, 6, 14, 7, 15, \cdots$$
$$S_4 = 2, 1, 2, 2, 1, 3, \mathbf{5}, \mathbf{4}, 5, 5, 5, 6, 5, 7, 5, 8, 6, 9, 6, 10, 6, 11, 6, 12, 6, 13, 7, 14, 7, 15, \cdots$$
$$S_5 = 2, 1, 3, 2, 1, 3, 1, 4, \mathbf{6}, \mathbf{5}, 6, 6, 6, 7, 6, 8, 6, 9, 6, 10, 7, 11, 7, 12, 7, 13, 7, 14, 7, 15, \cdots$$
$$S_6 = 3, 1, 2, 2, 2, 3, 1, 4, 1, 5, \mathbf{7}, \mathbf{6}, 7, 7, 7, 8, 7, 9, 7, 10, 7, 11, 7, 12, 8, 13, 8, 14, 8, 15, \cdots$$
$$S_7 = 3, 1, 3, 2, 2, 3, 1, 4, 1, 5, 1, 6, \mathbf{8}, \mathbf{7}, 8, 8, 8, 9, 8, 10, 8, 11, 8, 12, 8, 13, 8, 14, 9, 15, \cdots$$
$$S_8 = 4, 1, 2, 2, 3, 3, 1, 4, 1, 5, 1, 6, 1, 7, \mathbf{9}, \mathbf{8}, 9, 9, 9, 10, 9, 11, 9, 12, 9, 13, 9, 14, 9, 15, \cdots.$$

In S_{i+1} the multiplier of i is either 2 or 1, depending on whether the multiplier of i in S_i is i or greater than i. Therefore $\{S_i\}_{i \geq 1}$ is well-defined but is not ultimately periodic.

6. FACTOR-FREE COUNTING SEQUENCES. We end this paper the way we began it: by using the sequence $2, 1, 1, 4$ to build a type of counting sequence. Because $2, 1, 1, 4$ consists of 2 ones, 1 two, 0 threes, and 1 four, let us define R_1 to be the numbers making up this description: $R_1 = 2, 1, 0, 1$. Repeating this process, R_1 consists of 1 zero, 3 ones, 1 two, 0 threes, and 0 fours, so set $R_2 = 1, 2, 1, 0, 0$. Continuing we have

$$R_3 = 2, 2, 1, 0, 0$$
$$R_4 = 2, 1, 2, 0, 0$$
$$R_5 = 2, 1, 2, 0, 0.$$

We call the sequence $\{R_i\}_{i \geq 1}$ a *factor-free* counting sequence. The cycles of factor-free sequences are called *self-descriptive* and *co-descriptive strings* in **[5]**, **[7]**, and **[8]**.

Since a factor-free counting sequence is built without the explicit benefit of place-keeping factors, we need a method for indicating which integer each term in each R_i describes. For $i \geq 1$ we assume that the j-th entry of R_i gives the number of times $j-1$ appears in R_{i-1}, and that this entry is 0 if $j-1$ does not appear in R_{i-1} but some integer at least as large as $j-1$ appears in some $R_{i'}$, $1 \leq i' < i$. Then, just as the first number in an element S_i of a counting sequence almost always describes the number of 1's in S_{i-1}, the first number in an element R_i of a factor-free counting sequence describes the number of 0's in R_{i-1}.

Of course, one may allow the first digit of a sequence to describe numbers other than 0. The only finite example of this is the one-cycle 1, which is the factor-free version of the one-cycle 2, 2. There are, however, many infinite examples. For instance, Golomb's sequence can be thought of as an infinite factor-free one-cycle that begins by describing the number of 1's it contains.

Using techniques similar to those in Section 2 it is easy to show that if R_0 is a finite sequence of non-negative integers, then the factor-free counting sequence $\{R_i\}_{i\geq 0}$ is ultimately periodic. To find all of the possible cycles we will relate the factor-free and "ordinary" counting sequences. Following **[1]**, say that an element S_i of a counting sequence is *complete* if its factors are consecutive and the smallest factor is 1. If $S = m_1, f_1, m_2, f_2, \ldots, m_k, f_k$ is a complete element of a counting sequence, then defining $R = n_0, n_1, \ldots, n_k$ by $n_j = m_{j+1} - 1$ gives a factor-free sequence. Similarly, given a sequence $R = n_0, n_1, \ldots, n_k$ of a factor-free counting sequence, defining S by $f_j = j + 1$ and $m_j = n_{j-1} + 1$ gives a factor-containing sequence. Notice that the sequence S corresponding to R is complete. While it is not true that this process allows one to convert between counting sequences $\{S_i\}_{i\geq 1}$ and factor-free counting sequences $\{R_i\}_{i\geq 1}$, it is very easy to show that there is a one-to-one correspondence between the cycles of factor-free counting sequences and the cycles of complete counting sequences. Since there is also a one-to-one correspondence between complete cycles and the truncations appearing in Section 3, Theorems 6 and 8 give our final result.

Corollary 18. *Other than* 1, *the cycles of factor-free counting sequences all contain zeros, and have length one, two, or three. The one-cycles are* $2, 0, 2, 0$ *and* $1, 2, 1, 0$ *and* $2, 1, 2, 0, 0$ *and* $m + 3, 2, 1, (m0\text{'s}), 1, 0, 0, 0$ *for* $m \geq 0$. *The two-cycles are*

$$\begin{cases} 3,1,1,1,0,0 \\ 2,3,0,1,0,0 \end{cases} \quad \textit{and} \quad \begin{cases} m+3,1,0,1,(m0\text{'s}),1,0 \\ m+2,3,0,0,(m0\text{'s}),0,1, \end{cases}$$

for $m \geq 2$. *Finally, the only other cycle of any length is the three-cycle*

$$\begin{cases} 4,1,1,0,1,0,0 \\ 3,3,0,0,1,0,0 \\ 4,1,0,2,0,0,0. \end{cases}$$

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