Supplementary Document

for

The k-RF Measures for Labeled Trees

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Contents

1 Proof of Propositions 1–3 in Section 4

Proposition 1. *Let T and S be two 1-labeled trees over $L(T)$ and $L(S)$, respectively.*

(a) *Let $|L(S) \cap L(T)| \leqslant 2$, $k \geqslant 1$, and $|E(T)| \geqslant 2$. Then,*

$$d_{k\text{-RF}}(S,T) = |E(S)| + |E(T)|.$$

(b) *Assume that $L(S) \neq L(T)$. Then, for $k < \min\{diam(T), diam(S)\}$, we have $k+1 \leqslant d_{k\text{-RF}}(S,T) \leqslant |E(S)| + |E(T)|$.*
In addition, if $k \geqslant \min\{diam(T), diam(S)\}$ and $|L(S)| = |L(T)|$, we have $d_{k\text{-RF}}(S,T) = |E(S)| + |E(T)|$.

(c) *Renaming each node with its label, we have $d_{0\text{-RF}}(S,T) = |E(S) \triangle E(T)|$.*

(d) *If $k \geqslant \max\{\mathrm{diam}(T), \mathrm{diam}(S)\} - 1$, then $d_{k\text{-RF}}(S,T) = d_{RF}(S,T)$.*

Proof. Note that if $k \geqslant 1$ and $|E(T)| \geqslant 2$, then each $P_T(e,k)$ involves at least three labels. Thus, it is obvious that if $|L(S) \cap L(T)| \leqslant 2$, then for every $e \in E(T), \acute{e} \in E(S)$, we have $P_T(e,k) \neq P_S(\acute{e},k)$. Thus, we have $\mathcal{P}_k(S) \cap \mathcal{P}_k(T) = \emptyset$, which implies that

$$d_{k\text{-RF}}(S,T) = |(\mathcal{P}_k(S) \cup \mathcal{P}_k(T)) \setminus (\mathcal{P}_k(S) \cap \mathcal{P}_k(T))| = |E(S)| + |E(T)|.$$

Thus, **(a)** is true.

For part **(b)**, if $k < \min\{\mathrm{diam}(T), \mathrm{diam}(S)\}$, without loss of generality, we can assume that $e = (u,v) \in E(T)$ such that $\ell(v) \not\subseteq L(S)$. Obviously, $P_T(e,k)$ does not belong to $\mathcal{P}_k(S)$. Thus, $d_{k\text{-RF}}(S,T) \geqslant 1$. Now, if $\mathrm{diam}(T) > k \geqslant 1$, we have either $deg(u) \geqslant 2$ or $deg(v) \geqslant 2$. Thus, we can find another edge, namely $e_1 = \{u_1, u\}$ such that $P_T(e_1,k)$ does not belong to $P_k(S)$. If $k = 1$, we have $k+1 \leqslant d_{k\text{-RF}}(S,T)$. Otherwise, since $\mathrm{diam}(T) > k \geqslant 2$, we have either $deg(v) \geqslant 2$ or $deg(u_1) \geqslant 2$ or $deg(u) \geqslant 3$ and degree of the third node adjacent to u is at least 2. Hence, we can find another edge, namely e_2 such that $P_T(e_2,k)$ does not belong to $\mathcal{P}_k(S)$. Continuing this process, we can find $e_1, ..., e_k$ such that $P_T(e_1,k), ..., P_T(e_k,k)$ do not belong to $\mathcal{P}_k(S)$. Thus, we have $k+1 \leqslant d_{k\text{-RF}}(S,T)$. Finally, since $d_{k\text{-RF}}(S,T) \leqslant |E(S)| + |E(T)|$, the first statement holds. For the second statement, note that if $|L(S)| = |L(T)|$ and $k \geqslant \min\{\mathrm{diam}(T), \mathrm{diam}(S)\}$, for the tree with the minimum diameter, all edge-induced pairs contain a label which is in neither of the edge-induced pairs of the other tree. Hence, the statement follows.

For part **(c)**, note that we may represent each node of a 1-labeled tree with its unique label. As a result, $P_T(e,0) = e$ and $P_S(\bar{e},0) = e$ for $e \in E(T)$

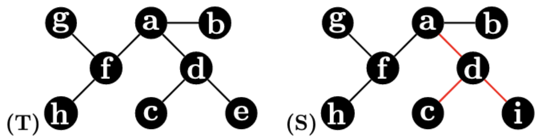

Figure 1: Two 1-labeled trees with the same size. The pairs induced by red edges in S are the only pairs different from all edge-induced pairs in T. Thus, the 1-RF score of S and T is 6.

and $\bar{e} \in E(S)$. Thus, **(c)** follows. Finally, Part **(d)** follows from the definition of the k-RF measure. □

Proposition 2. *Let $k \geqslant 0$ be an integer. The k-RF dissimilarity measure is a pseudometric on the space of all 1-labeled trees. In other words, it satisfies the non-negativity, symmetry and triangle inequality conditions.*

Proof. Let T_1, T_2, and T_3 be three 1-labeled trees. Then, we show that the following inequalities hold.

$$\begin{aligned}
&d_{k\text{-RF}}(T_1, T_2) \leqslant d_{k\text{-RF}}(T_1, T_3) + d_{k\text{-RF}}(T_3, T_2),\\
&d_{k\text{-RF}}(T_1, T_2) \geqslant 0,\\
&d_{k\text{-RF}}(T_1, T_2) = d_{k\text{-RF}}(T_2, T_1).
\end{aligned}$$

Let $\mathcal{P}_k(T_1)$, $\mathcal{P}_k(T_2)$ and $\mathcal{P}_k(T_3)$ be the three sets corresponding to T_1, T_2, T_3, respectively. The first inequality follows from the following relation:

$$\mathcal{P}_k(T_1) \triangle \mathcal{P}_k(T_2) \subseteq (\mathcal{P}_k(T_1) \triangle \mathcal{P}_k(T_3)) \cup (\mathcal{P}_k(T_3) \triangle \mathcal{P}_k(T_2)).$$

Also, the second and third inequalities follow from the definition of the k-RF measure. □

Remark 1. *Propositions 1 and 2, by the same proof, also hold for 1-labeled rooted trees.*

Lemma 1. *Let $k \geqslant 0$ and let T be a 1-labeled rooted tree with n nodes. For all w and all $i \leqslant k$, $D_i(w) = \{w\} \cup \{x \in D_T(w) | d(w, x) \leqslant i\}$ and $L(D_i(w))$ can be computed in $O(kn)$ set operations, where $D_T(w)$ consists of all the descendants of w.*

Proof. There are n labels in T. By ordering the n labels, we represent each label subset as a n-bit string, in which the i-th bit is 1 if and only if the i-th label is in the subset.

We prove the statement by induction. In the case $k = 0$, $D_0(w) = \{w\}$ and $L(D_0(w)) = \{\ell(w)\}$. Clearly, all the $D_0(w)$ $(w \in V(T))$ can be computed in $O(n)$ set operations.

Assume that all the $D_{k-1}(w)$ $(w \in V(T))$ can be computed in at most $2kn$ set operations. Assume w has d_w children $u_1, u_2, \ldots, u_{d(w)}$. Then,

$$D_k(w) = \{w\} \cup \left(\cup_{i=1}^{d_w} D_{k-1}(u_i)\right)$$

and so

$$L(D_k(w)) = \{\ell(w)\} \cup \left(\cup_{i=1}^{d_w} L(D_{k-1}(u_i))\right).$$

This implies that all $L(D_k(w))$ can be computed from all $\ell(w)$ and $L(D_{k-1}(w))$ $(w \in V(T))$ using $\sum_{v \in V(T)}(1 + d_w) = 2n - 1$ set operations. In total, we can compute all $L(D_k(w))$ in at most $2n - 1 + 2kn = 2(k+1)n$ set operations. Note that this proof implies a dynamic programming algorithm for computing the label subsets in $2(k+1)n = O(kn)$ set operations. $\square$

Lemma 2. *Let $k \geqslant 0$ and T be a 1-labeled rooted tree with n nodes. Using $L(D_i(w))$ $(w \in V(T), 0 \leqslant i \leqslant k)$, we can compute $L(B_k(w))$ for all w in $O(kn)$ set operations, where*

$$B_k(w) = \{x \in V(T) \mid \exists y \in A_T(w) \cup \{w\} : d(y, w) + d(y, x) \leq k\}.$$

Proof. Let T be a 1-labeled tree and r be its root. For any node $w \in V(T)$, we assume the unique path from r to w be:

$$w_0 = r, w_1, \ldots, w_t = w.$$

Then, we have that

$$B_k(w_t) = \cup_{i=0}^{\min(k,t)} D_{k-i}(w_{t-i}).$$

Given all $L(D_i(w))$, $L(B_k(w))$ is a union of at most k known label subsets and so can be computed in at most k set operations for each node w. In total, we can compute all $L(B_k(w))$ in $O(kn)$ set operations. $\square$

Proposition 3. *Let S and T be two 1-labeled trees each with n nodes and $k \geqslant 0$. Then, $d_{k\text{-RF}}(S, T)$ can be computed in $O(kn)$ time.*

Proof. We first consider the rooted tree case. Let T and S be two 1-labeled rooted trees with n nodes. Without loss of generality, we may assume that S and T are labeled with the same set L, $|L| = n$. By Lemma 1 and Lemma 2, We can compute all $P_X(e, k)$ $(e \in E(X))$ in $O(kn)$ set operations for $X = S, T$. Since each edge-induced is an ordered pair of label subsets and we represent each label subset using a n-bit string, we consider $P_X(e,k)$ as a $2n$-bit string. In this way, we sort all the edge-induced partitions for each tree in $O(n)$ time by radix sort (that is, indexing) and then compute the symmetric difference of the two set of edge-induced partitions in $O(n)$. This concludes the proof.

In the unrooted case, we first root the trees at a leaf. In this way, we can compute all the edge-induced partitions in the derived rooted trees in $O(kn)$ time. Since the edge-induced partitions are unordered pairs of label subsets in the original trees, we rearrange the two label subsets of each obtained partition in such a way that the smallest label in the first subset is smaller than every label in the second one. After the rearrangement, we can use radix-sort the partitions and compute the k-RF score in linear time. □

Remark 2. *Let $k \geqslant 0$ be an integer. If T is a 1-labeled rooted tree over $L(T)$, then we have $P_T(e,k) = P_T(\acute{e},k)$ if and only if $e = \acute{e}$. Moreover, each $P_T(e,k)$ consists of two disjoint sets.*

Theorem 1. *The 0-RF dissimilarity measure is a metric on the space of all 1-labeled rooted trees. In other words, besides satisfying non-negativity, symmetry and triangle inequality conditions, it is zero for two trees if and only if the trees are isomorphic.*

Proof. Let S and T be two 1-labeled rooted trees. By the proof of Proposition 2, it is enough to show that if $d_{0\text{-RF}}(S,T) = 0$ (equivalently, $\mathcal{O}P_0(T) = \mathcal{O}P_0(S)$), then $S \cong T$. Clearly, the equality $\mathcal{O}P_0(T) = \mathcal{O}P_0(S)$ implies that for each $e = (u,v) \in E(T)$, there is one and only one edge $\bar{e} = (\bar{u},\bar{v}) \in E(S)$ such that $\ell(u) = \ell(\bar{u})$ and $\ell(v) = \ell(\bar{v})$ (note that $P_T(e,0) = (\ell(v),\ell(u))$ and $P_S(\bar{e},0) = (\ell(\bar{v}),\ell(\bar{u}))$). Now, we define $f : V(T) \to V(S)$, which maps v and u to $\bar{v}$ and $\bar{u}$, respectively. Obviously, f defines an isomorphism between S and T. □

Theorem 2. *Let $k \geqslant 1$ be an integer. The k-RF dissimilarity measure is a metric on the space of all 1-labeled rooted trees. In other words, besides satisfying non-negativity, symmetry and triangle inequality conditions, it is zero for two trees if and only if the trees are isomorphic.*

Proof. Let S and T be two 1-labeled rooted trees. By the proof of Proposition 2, it is enough to show that if $d_{k\text{-RF}}(S,T) = 0$ (equivalently, $\mathcal{O}P_k(T) = \mathcal{O}P_k(S)$), then $S \cong T$. We have $V(T) = \cup^0_{l=depth(T)-1} V_l(T) \cup \mathit{Leaf}(T)$ and $V(S) = \cup^0_{l=depth(T)-1} V_l(S) \cup \mathit{Leaf}(S)$, where

$$V_l(T) = \{v \in V(T) \mid v \text{ is in level } l\}.$$

($V_l(S)$ is defined similarly.) Now, we define $f : V(T) \to V(S)$ as follows.

Clearly, $v \in \mathit{Leaf}(T)$ (respectively, $w \in \mathit{Leaf}(S)$) if and only if $P_T(e,k) = (\ell(v), A)$ (respectively, $P_S(\bar{e},k) = (\ell(w), B)$), where e (resp. $\bar{e}$) is the edge incident to v (respectively, w), $A \subseteq L(T)$ (resp. $B \subseteq L(S)$), and $\ell(v)$ (respectively, $\ell(w)$) is a set with only one element. Now, we define $f(v) = u$ where $u \in \mathit{Leaf}(S)$ and $\ell(v) = \ell(u)$. Thus, f is a one-to-one correspondence between $\mathit{Leaf}(T)$ and $\mathit{Leaf}(S)$, where corresponding nodes have the same labels.

Suppose we have defined

$$f : \cup_{l=depth(T)-1}^{n} V_l(T) \cup Leaf(T) \to \cup_{l=depth(S)-1}^{n} V_l(S) \cup Leaf(S)$$

such that for each $n \leqslant l \leqslant \mathit{depth}(T)-1$, $f|_{V_l(T)} : V_l(T) \to V_l(S)$ and $f|_{\mathit{Leaf}(T)} : \mathit{Leaf}(T) \to Leaf(S)$ is a one-to-one correspondence and corresponding nodes have the same labels. In addition, suppose that $(v_1, v_2) \in E(T)$ if and only if $(f(v_1), f(v_2)) \in E(S)$ for $v_1, v_2 \in \cup_{l=\mathit{depth}(T)-1}^{n} V_l(T) \cup \mathit{Leaf}(T)$. Now, let $v \in V_{n-1}(T) \setminus \mathit{Leaf}(T)$, and consider the edge connecting v to its parent. The pair induced by this edge is in the form $(\ell(v) \cup (L(D_k(v)) \cap L(V_n(T))) \cup \cdots \cup (L(D_k(v)) \cap L(\mathit{Leaf}(T))), A)$, where $A \subseteq L(T)$. By our assumption, there is $(u_1, u) \in E(S)$ inducing the same pair. On the other side, by properties of f, there are $B_l \subseteq V_l(S), B \subseteq \mathit{Leaf}(S)$ with $L(B_l) = L(D_k(v)) \cap L(V_l(T))$ and $L(B) = L(D_k(v)) \cap L(\mathit{Leaf}(T))$ such that the first term in the pair is in the form $\ell(u) \cup L(B_n) \cup \cdots \cup L(B_{\mathit{depth}(T)-1}) \cup L(B)$. Thus, $\ell(u) = \ell(v)$, and by definition of pairs assigned to a rooted tree, each node in B_n must be adjacent to u and $u \in V_{n-1}(S)$. Now, we define $f(v) = u$. Clearly, $f : \cup_{l=\mathit{depth}(T)-1}^{n-1} V_l(T) \cup \mathit{Leaf}(T) \to \cup_{l=\mathit{depth}(S)-1}^{n-1} V_l(S) \cup \mathit{Leaf}(S)$ such that $f|_{V_{n-1}(T)} : V_{n-1}(T) \to V_{n-1}(S)$ is a one-to-one correspondence and corresponding nodes have the same labels. Moreover, we have $(v_1, v_2) \in E(T)$ if and only if $(f(v_1), f(v_2)) \in E(S)$ for $v_1, v_2 \in \cup_{l=\mathit{depth}(T)-1}^{n-1} V_l(T) \cup \mathit{Leaf}(T)$.

Continuing the process, we infer that $f : \cup_{\mathit{depth}(T)-1}^{0} V_l(T) \cup \mathit{Leaf}(T) \to \cup_{l=\mathit{depth}(S)-1}^{0} V_l(S) \cup \mathit{Leaf}(S)$ is an isomorphism which implies that $S \cong T$. (Note that $\mathit{root}(T) \in V_0(T)$ is mapped to $\mathit{root}(S) \in V_0(S)$ by f.) $\square$

Corollary 1. *Let $k \geqslant 0$ be an integer. The k-RF dissimilarity measure is a metric on the space of all 1-labeled trees.*

Proof. If $k = 0$, the statement follows from the same proof as for Proposition 1. Now, let S and T be two 1-labeled trees and $k \geqslant 1$. By Proposition 2, it is enough to show that if $d_{k\text{-RF}}(S,T) = 0$ (equivalently, $\mathcal{P}_k(T) = \mathcal{P}_k(S)$), then $S \cong T$. Using the same strategy stated in the proof of Proposition 2, we can infer that there is a one-to-one correspondence between $\mathit{Leaf}(T)$ and $\mathit{Leaf}(S)$, where corresponding nodes have the same labels. Let $u \in \mathit{Leaf}(T)$

and $v \in Leaf(S)$ be such that $\ell(u) = \ell(v)$. Then, we root T and S at u and v respectively. We call the induced rooted trees $\acute{T}$ and $\acute{S}$.

Now, we show that $\mathcal{O}P_k(\acute{T}) = \mathcal{O}P_k(\acute{S})$. Let $e_u \in E(T)$ and $e_v \in E(S)$ are the edges leaving u and v, respectively. Since $\ell(u) = \ell(v)$ and $\mathcal{P}_k(T) = \mathcal{P}_k(S)$, we have $P_{\acute{T}}(e_u, k) = P_{\acute{S}}(e_v, k)$. Now, suppose that the end points of e_u and e_v are u_1 and v_1, respectively. Let e_{u_1} leave u_1. Then, we can find one and only one edge, namely e_{v_1} which leaves v_1 such that $P_{\acute{T}}(e_{u_1}, k) = P_{\acute{S}}(e_{v_1}, k)$ and $l(u_1) = l(v_1)$. Continuing this process, we have $\mathcal{O}P_k(\acute{T}) = \mathcal{O}P_k(\acute{S})$. Finally, the statement follows from Proposition 2. $\square$

2 Supplementary figures for the frequency distribution of pairwise k-RF scores

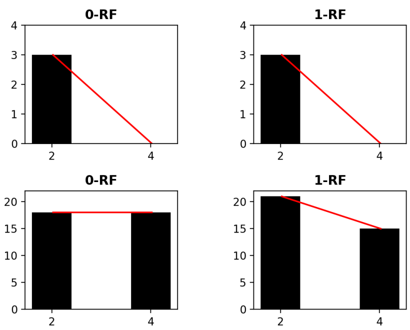

Figure 2: The frequency distribution of k-RF scores for the 1-labeled 3-node unrooted (top row) and rooted (bottom row) trees.

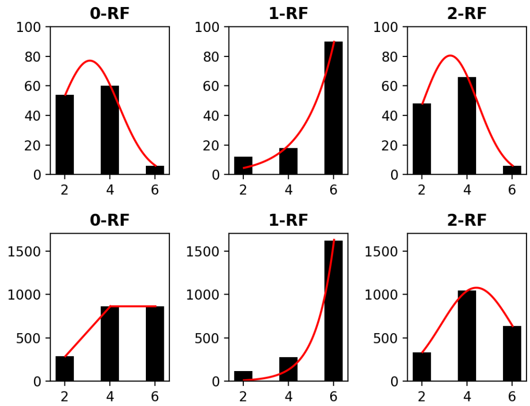

Figure 3: The frequency distribution of k-RF scores for the 1-labeled 4-node unrooted (top row) and rooted (bottom row) trees.

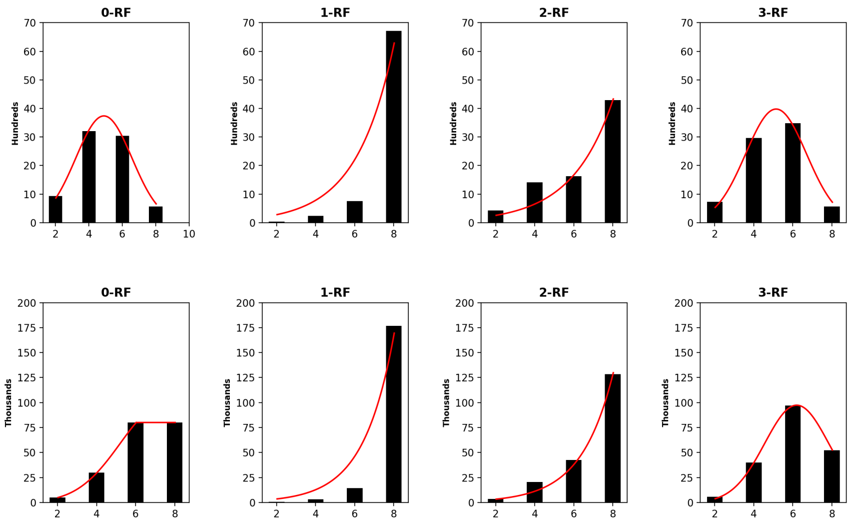

Figure 4: The frequency distribution of k-RF scores for the 1-labeled 5-node unrooted (top row) and rooted (bottom row) trees.

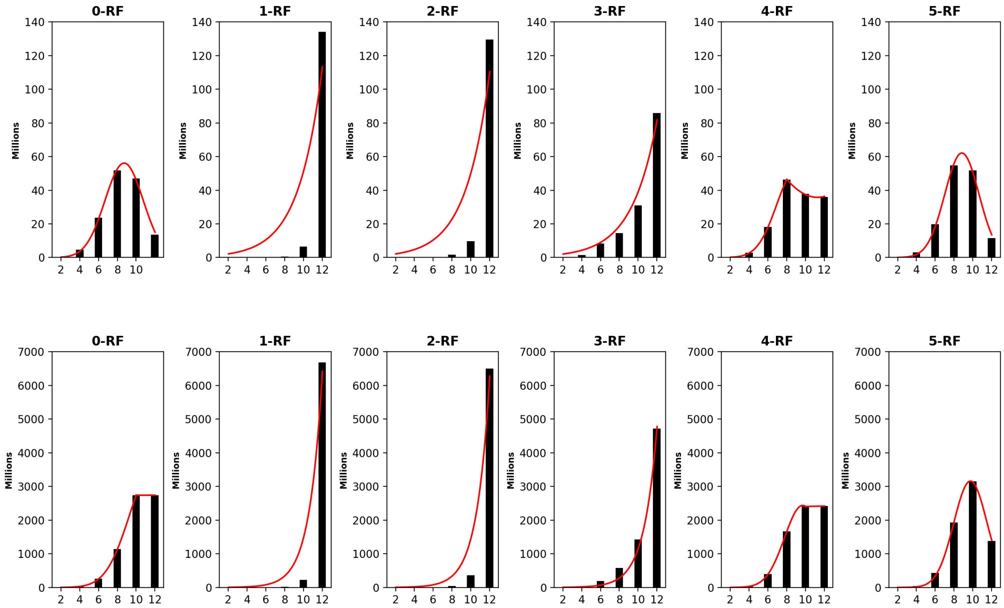

Figure 5: The frequency distribution of k-RF scores for the 1-labeled 7-node unrooted (top row) and rooted (bottom row) trees.

3 Proof of Propositions 4 and 5

Proposition 4. *Let $k \geqslant 1$ be an integer. The k-RF dissimilarity measure is a pseudometric on the space of all trees whose nodes are labeled by multi-subsets of their corresponding label sets. In other words, it satisfies the non-negativity, symmetry and triangle inequality conditions.*

Proof. Let T_1, T_2, and T_3 be three labeled trees. Then, we have

$$d_{k\text{-RF}}(T_1, T_2) \leqslant d_{k\text{-RF}}(T_1, T_3) + d_{k\text{-RF}}(T_3, T_2),$$
$$d_{k\text{-RF}}(T_1, T_2) \geqslant 0,$$
$$d_{k\text{-RF}}(T_1, T_2) = d_{k\text{-RF}}(T_2, T_1).$$

Let $\mathcal{P}_k(T_1)$, $\mathcal{P}_k(T_2)$ and $\mathcal{P}_k(T_3)$ be the three multisets corresponding to T_1, T_2, T_3, respectively. We show that the first inequality holds.

If $x^{m(x)} \in \mathcal{P}_k(T_1) \triangle_m \mathcal{P}_k(T_2)$, we have either $x^{m(x)} \in \mathcal{P}_k(T_1) \setminus_m \mathcal{P}_k(T_2)$ or $x^{m(x)} \in \mathcal{P}_k(T_2) \setminus_m \mathcal{P}_k(T_1)$. Assume $x^{m(x)} \in \mathcal{P}_k(T_1) \setminus_m \mathcal{P}_k(T_2)$. Then, we have $m_{\mathcal{P}_k(T_1)}(x) > m_{\mathcal{P}_k(T_2)}(x)$. Now, if $x \notin \text{Supp}(\mathcal{P}_k(T_3) \setminus_m \mathcal{P}_k(T_2))$, we have $m_{\mathcal{P}_k(T_1)}(x) > m_{\mathcal{P}_k(T_2)}(x) \geqslant m_{\mathcal{P}_k(T_3)}(x)$, which implies that $x \in$

$\mathrm{Supp}(\mathcal{P}_k(T_1) \setminus_m \mathcal{P}_k(T_3))$ and

$$m_{\mathcal{P}_k(T_1)\setminus_m \mathcal{P}_k(T_3)}(x) = m_{\mathcal{P}_k(T_1)}(x) - m_{\mathcal{P}_k(T_3)}(x) \geqslant m_{\mathcal{P}_k(T_1)}(x) - m_{\mathcal{P}_k(T_2)}(x) = m(x).$$

Thus, we have

$$m(x) \leqslant m_{\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_3)}(x) + m_{\mathcal{P}_k(T_3)\triangle_m \mathcal{P}_k(T_2)}(x).$$

On the other hand, if $x \in \mathrm{Supp}(\mathcal{P}_k(T_3)\setminus_m \mathcal{P}_k(T_2))$ and $m_{\mathcal{P}_k(T_3)}(x) \geqslant m_{\mathcal{P}_k(T_1)}(x)$, then

$$m_{\mathcal{P}_k(T_3)\setminus_m \mathcal{P}_k(T_2)}(x) = m_{\mathcal{P}_k(T_3)}(x) - m_{\mathcal{P}_k(T_2)}(x) \geqslant m_{\mathcal{P}_k(T_1)}(x) - m_{\mathcal{P}_k(T_2)}(x) = m(x).$$

If $x \in \mathrm{Supp}(\mathcal{P}_k(T_3)\setminus_m \mathcal{P}_k(T_2))$ and $m_{\mathcal{P}_k(T_3)}(x) < m_{\mathcal{P}_k(T_1)}(x)$, we have $m_{\mathcal{P}_k(T_1)}(x) > m_{\mathcal{P}_k(T_3)}(x) > m_{\mathcal{P}_k(T_2)}(x)$, which implies that $x \in \mathrm{Supp}(\mathcal{P}_k(T_1) \setminus_m \mathcal{P}_k(T_3))$. Thus, we have

$$m(x) = m_{\mathcal{P}_k(T_1)\setminus_m \mathcal{P}_k(T_3)}(x) + m_{\mathcal{P}_k(T_3)\setminus_m \mathcal{P}_k(T_2)}(x) \leqslant m_{\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_3)}(x) + m_{\mathcal{P}_k(T_3)\triangle_m \mathcal{P}_k(T_2)}(x).$$

Similarly, if $x^{m(x)} \in \mathcal{P}_k(T_2) \setminus_m \mathcal{P}_k(T_1)$, then we obtain the same result.

To summarize, we have

$$\mathrm{Supp}(\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_2)) \subseteq \mathrm{Supp}(\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_3)) \cup \mathrm{Supp}(\mathcal{P}_k(T_3)\triangle_m \mathcal{P}_k(T_2)).$$

In addition, for each $x \in \mathrm{Supp}(\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_2))$, we have

$$m_{\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_2)}(x) \leqslant m_{\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_3)}(x) + m_{\mathcal{P}_k(T_3)\triangle_m \mathcal{P}_k(T_2)}(x).$$

Therefore, we have

$$|\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_2)| \leqslant |\mathcal{P}_k(T_1)\triangle_m \mathcal{P}_k(T_3)| + |\mathcal{P}_k(T_3)\triangle_m \mathcal{P}_k(T_2)|.$$

Thus, the first inequality holds. Also, the second inequality and the third equality follow from the definition of the k-RF measures. $\square$

Proposition 5. *Let $k \geqslant 0$ and S, T be two (rooted) trees whose nodes are labeled by $L(S)$ and $L(T)$, respectively. Then, $d_{k\text{-RF}}(S,T)$ can be computed in $O(n(k + |L(S)| + |L(T)|))$ if the total multiplicity of each label is upper bounded by a constant.*

Proof. An algorithm in the 1-labeled case can be modified as follows for computing k-RF multiset-labeled unrooted and rooted trees:

- Represent each label multiset as a $(|L(S)| + |L(T)|)$-digit number, in which the digit at position j is the multiplicity of the j-th label. Compute all edge-induced partitions in $O(kn)$ set operations.

- Radix-sort all the edge-induced partitions in each tree in $O(n(|L(S)| + |L(T)|))$ time, where we use the assumption that the total multiplicity of each label is bounded by a constant B.
- Compute the symmetric difference of the set of the edge-induced partitions in the two input trees in $O(n)$ time.

□

Theorem 3. *Let $k \geqslant 1$ be an integer. The k-RF dissimilarity measure is a pseudometric on the space of all multiset-labeled rooted trees . In other words, it satisfies the non-negativity, symmetry and triangle inequality conditions.*

Proof. It is exactly the same as proof of Proposition 4. □