50.021 Artificial Intelligence
Homework 1

Due: every Monday, 4PM before class starts

[Q1.] Write down the distribution of $p(x, y)$ from in class coding exercise.

Solution:
Since the distribution depends on whether it comes from Gaussian 1 or 2, we can write,

$$p(y, x) = p(y, x, c(x) = 1) + p(y, x, c(x) = 2),$$

By definition of conditional probability, we can rewrite the above into,

$$p(y, x) = p(y|x, c(x) = 1)p(x, c(x) = 1) + p(y|x, c(x) = 2)p(x, c(x) = 2)$$

And we know that

$$\begin{aligned}
p(y = 0|x, c(x) = 1) = 0.3 \\
p(y = 1|x, c(x) = 2) = 0.7 \\
p(y = 0|x, c(x) = 1) = 0.6 \\
p(y = 1|x, c(x) = 2) = 0.4
\end{aligned}$$

We can further decompose the joint probability into the multiplication of the pdf (prob. density function) of the Gaussian and the probability of that Gaussian being selected,

$$p(x, c(x) = i) = f(x|c(x) = i) \cdot p(c(x) = i)$$

Therefore,

$$\begin{aligned}
p(y, x) &= p(y, x, c(x) = 1) + p(y, x, c(x) = 2), \\
&= p(y|x, c(x) = 1)p(x, c(x) = 1) + p(y|x, c(x) = 2)p(x, c(x) = 2), \\
&= p(y|x, c(x) = 1)f(x|c(x) = 1)p(c(x) = 1) + p(y|x, c(x) = 2)f(x|c(x) = 2) \cdot p(c(x) = 2), \\
&= (0.3 \cdot 1[y = 0] + 0.7 \cdot 1[y = 1])f(x|c(x) = 1)0.5 + (0.6 \cdot 1[y = 0] + 0.4 \cdot 1[y = 1])f(x|c(x) = 2)0.5
\end{aligned}$$

Note: $1[y = i]$ is an indicator function, where $[y = i]$ is 1 if y takes the value of i.