CS 270 - Lab 11

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1 Introduction

You may work in teams of **one** or **two** students. Submit one copy for the entire group.

Write your answers on this lab sheet. Only what is written on this lab sheet will be graded.

This lab is due at the end of the class period. You may not continue to work on it once class has ended.

This lab contains 4 questions.

Grading

- 25 points - Putting everyones names on this page
- 20 points - Earned for each correct question (Answer is fully correct)
- 5 points - Earned for **partial credit** any question

No additional point amounts can be earned. You cannot earn 7 points on a question for example.

The maximum score for a lab is 100. If you get everything correct, that adds up to 105 points but will be reduced to 100.

A question will be marked correct as long as it covers all requirements of the question. It does not need to be perfect, but must be fully correct. A single typo or very minor issue where the intention is clear and all requirements are met would still earn full points.

We want you to complete questions fully, not try to earn partial credit on multiple questions. You may ask your Professor/Course assistant questions during lab.

Enter the name of the student in the group

Member 1: Elan Rubin

Member 2: Josh Koo

Score (out of 100): ______ **Graded By**: ______

Question 1 :

Before starting a Proof by Mathematical Induction, it is important to identify the goals of the proof. This question asks you to identify the parts of a proof based on the statement.

You will use the following function in this question.

```
;input-contract: integer x is greater than or equal to 1
;output-contract: an integer answer to 2x^2-3x
(define (h x)
  (if (= x 1) -1 (+ (- (* 4 x) 5) (h (- x 1)))))
```

We want to prove that $\forall x \in \mathbb{N}\left((x \geq 1) \implies (\text{h x}) = 2x^2 - 3x\right)$

Note: You will not prove this statement, only answer questions about how the proof would work. An entire proof is not required for this question.

(a) All Induction Proofs need one or more Anchors. We need to identify the Anchor. What value of x is the anchor in this question?

For this question, the anchor would be 1, because the input contract says the input number has to be greater than or equal to 1

(b) To prove the Base Case, we need to identify the **premise** of the Left-Hand Side (LHS) and Right-Hand Side (RHS).

What is the premise of the Base Case LHS? (Do not prove it, just give the premise)

(h x)

(c) What is the premise of the Base Case RHS? (Do not prove it, just give the premise)

2x^2 -3x

(d) For the Leap Case, we need to identify the correct **Inductive Hypothesis**. What is the **Inductive Hypothesis**?

the inductive hypothesis (IH) is

(h (+ k 1)) = 2(k+1)^2 -3(k+1)

(e) To prove the Leap Case, we need to identify the **premise** of the Left-Hand Side (LHS) and Right-Hand Side (RHS).

What is the premise of the Leap Case LHS? (Do not prove it, just give the premise.)

(h (+ k 1))

(f) What is the premise of the Leap Case RHS? (Do not prove it, just give the premise.)

2(k+1)^2 -3(k + 1)

(g) When we complete a proof, we normally write a sentence at the end as a conclusion. Fill in the blank to finish this conclusion.

Both the anchor case and leap case have been demonstrated, thus by induction, we have $\forall x \in \mathbb{N}\left((x \geq 1) \implies (\text{h x}) = 2x^2 - 3x\right)$

Question 2 :

Determining the correct Inductive Hypothesis is key to a Proof by Induction. Writing an incorrect hypothesis can cause an entire proof to fail.

In this question, you will be given three formulas. For each part, give the Inductive Hypothesis. Note that you are not being asked to prove or justify anything, you merely need to state the IH.

(a) Give the Inductive Hypothesis to prove $\forall x \in \mathbb{N}\left((x \geq 1) \implies (\mathrm{M}\ \mathrm{x}) = \frac{1}{2}x(x+1)\right)$ The function M is defined below.

```
;input-contract: integer x is greater than or equal to 1
;output-contract: an integer answer to 1/2 x(x+1)
(define (M x) (if (= x 1) 1 (+ x (M (- x 1)))))
```

the inductive hypothesis (IH) is

(M k) = 1/2k(k + 1)

(b) Give the Inductive Hypothesis to prove $\forall x \in \mathbb{N}\left((x \geq 7) \implies (\mathrm{W}\ \mathrm{x}) < 2\right)$ The function W is defined below.

```
;input-contract: integer x is greater than or equal to 7
;output-contract: an fraction that is always less than 2
(define (W x) (if (= x 7) (/ 127 64) (+ (/ 1 (expt 2 x)) (W (- x 1)))))
```

(W k) < 2

(c) When doing an inductive proof, we sometime justify a step using the inductive hypothesis. What is the justification we write when making use of the inductive hypothesis?

invoke the IH

(d) When is it permitted to do algebraic reasoning in a proof with Racket Equational Reasoning?

- only before invoking the Inductive Hypothesis
- (x) only after invoking the Inductive Hypothesis
- ◯ Always: algebraic rules are universally true in any math proof
- ◯ Never: algebraic rules cannot be used in a proof which involves Racket expressions

(e) Can we reference the input contract as part of a justification?

- (x) Yes
- ◯ No

Question 3 :

The following function will be used in this question.

```
;input-contract: integer x is greater than or equal to 3
;output-contract: an integer answer to  x^2 + x
(define (g x) (if (= x 3) 12 (+ (* 2 x) (g (- x 1)))))
```

Prove by Induction that $\forall x \in \mathbb{N}\big((x \geq 3) \implies (\text{g x}) = x^2 + x\big)$

Note that the unlike the previous questions, here you are being asked to *complete* the parts, which entails doing each step of the equational reasoning proof including stating the expression along with its justification.

Base Case: anchor is at $x = 3$

(a) Complete the Base Case LHS.

(g 3) ;base case LHS
(if (= 3 3) 12 (+ (* 2 3) (g (- 3 1))))) ;apply definition of g
(if #t 12 (+ (*2 3) (g (- 3 1))))) ;evaluation of =
12 ;evaluation of if

(b) Complete the Base Case RHS

3^2 + 3 ;base case RHS
9 + 3 ;evaluate exponent
12 ;evaluatie addition

Since LHS=RHS , the base case is established.

Leap Step IH: for a given $k \geq 3$, (g k) $= k^2 + k$

GOAL: (g (+ k 1)) $= (k + 1)^2 + (k + 1)$

(c) Complete the Leap Step LHS

(g (+ k 1)) ;premise of leap case LHS
(if (= (+ k 1) 3) 12 (+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))) ;apply definition of g
(if #f 12 (+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))) ;evaluate =
((+ (* 2 (+ k 1)) (g (- (+ k 1) 1))))) ;evaluate if
((+ (* 2 (+ k 1)) (g k))) ;evaluate subtraction
2(k+1)+k^2+k ;invoke IH
2x+2 + k^2+k ;apply distributive property
k^2 +3k +2 ;apply addition

RHS

```
(k+1)^2 + (k+1) ; premise of RHS at n=k+1
k^2+2k+1 + k + 1 ; algebra
k^2 + 3k + 2 ; combine like terms
```

Since the LHS = RHS (both k^2+3k+2), this establishes the Leap case. Consequently, since both the base case and the leap case have been demonstrated, by POMI, $\forall x \in \mathbb{N}(x \geq 3) \implies$ (g x) $= x^2 + x$.

Question 4 :

The following function will be used in this question.

```
;input-contract: integer x is greater than or equal to 1
;output-contract: an integer answer to 2x
(define (Q x) (if (= x 1) 2 (+ 2 (Q (- x 1)))))
```

Prove by Induction that $\forall x \in \mathbb{N}\,((x \geq 1) \implies (\text{Q x}) = 2x)$

(Q 1) ;premise of base case LHS
(If (= 1 1) 2 (+ 2 (Q (- 1 1))))) ;apply definition of Q
(if # 2 (+ 2 (Q (- 1 1)))); evaluate =
2; evaluate if

2(1); premise of RHS
2; apply algebra

(Q (+ k 1)) ;premise of LHS leap case
(if (= (+ k 1) 1) 2(+ 2 (Q (- (+ k 1) 1))) ;apply definition of Q
(if # 2 (+ 2 (Q (- (+ k 1) 1))) ;evaluate =
(+ 2 (Q(- (+ k 1) 1))); evaluate if
(+ 2 Q K) ;evaluation of + and -
2 + 2k ;invoke IH
2k + 2 ;evaluate algebra

2(k + 1) ;premise of RHS leap case
2k + 2 ; evaluate algebra