Motor Modeling Notes

Ben Katz

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In the process of writing a motor simulator, I found it frustratingly difficult to find easily-understandable references which made assumptions similar to those I care about, and easily allowed for things like funny flux-linkages, non-sinusoidaly varying inductance, and computing everything from the stator reference frame, rather than the D/Q frame. These notes are a neater version of the chicken-scratch I generated while working through the relevant equations on my own. Maybe someone will read this and find it useful. This work was done in the process of building a motor-simulation tool, the source of which is available at `https://github.com/bgkatz/motor-modeling`

Full warning, these notes assume some basic motor and E&M familiarity. I'm not starting with a diagram of a motor and explaining from scratch how it works. This is mostly for myself, really, so I can look back and see how I approached this problem. And writing things down is a good way to clarify them to myself.

Also, if you spot any egregious errors please let me know.

1 Modeling Goals

Here are the goals of this motor modeling exercise:

- Model everything at the phases, rather than in D/Q coordinates
- Make no assumptions about phase inductances (i.e. they don't have to be constant or vary sinusoidally, phases can saturate)
- Make no assumptions about the shape of the rotor-flux linkage (i.e. it doesn't need to be sinusoidal, it can very with current, etc.)
- Account for mutual inductance between phases

2 Voltage Equations

2.1 A Single Phase

The first step is to look at the voltage equation for a single phase of the motor. This phase has a resistance R, a current through it i, and a flux linkage λ. Voltage across that phase is:

$$v = R \cdot i + \frac{d\lambda}{dt} \tag{1}$$

The first trick is decomposing the $\frac{d\lambda}{dt}$ term. First of all, what is λ, equal to, let alone its time derivative? Depending on what effects you choose to model, the λ term can become *very* large. In this case, I'm choosing to consider:

- The flux linked by the the rotor permanent magnets, λ_r
- The flux linked by the phase itself, which is equal to the self-inductance of the phase times the current through the phase: $L_a \cdot i_a$, where L_a is the self-inductance of phase a.
- The flux linked to the phase by all the other phases. For every other phase, the flux linked to the phase of interest is their mutual inductance times the current through the other phase: For phase a, this is the sum of $L_{an} \cdot i_n$ for every other phase n.

To put everything into context, here's flux linked by phase a in a 3-phase motor, which has phases b and c as well.

$$\lambda_a = \lambda_{ra} + L_a \cdot i_a + L_{ab} \cdot i_b + L_{ac} \cdot i_c \tag{2}$$

In general, none of these values are constants! λ_r, L_a, L_{ab}, and L_{ac} are typically functions of the rotor position and current (θ, and i_a, i_b, and i_c), λ_r and a function of rotor position and phase current. For special motors under certain operating conditions, some terms might be pretty much constant, causing their time derivatives to be zero and drop out. For example, in a surface permanent magnet motor operating at low-ish currents, (i.e. a typical "brushless" motor), the inductances are basically constant, so their derivatives are equal to zero (if you ignore current-varying inductance, i.e. saturation). And for a reluctance motor, the PM flux is zero, so the derivative of λ_r is zero.

Since none of the values in (2) are constants, taking the time derivative of that expression makes it even longer:

$$\frac{d\lambda_a}{dt} = \frac{d\lambda_r}{dt} + L_a\frac{di_a}{dt} + i_a\frac{dL_a}{dt} + L_{ab}\frac{di_b}{dt} + i_b\frac{dL_{ab}}{dt} + L_{ac}\frac{di_c}{dt} + i_c\frac{dL_{ac}}{dt} \tag{3}$$

Then the expanded voltage equation for phase a is:

$$v_a = L_a\frac{di_a}{dt} + L_{ab}\frac{di_b}{dt} + L_{ac}\frac{di_c}{dt} + i_a\frac{dL_a}{dt} + i_b\frac{dL_{ab}}{dt} + i_c\frac{dL_{ac}}{dt} + \frac{d\lambda_{ra}}{dt} + R_a i_a \tag{4}$$

2.2 Back-EMF

You'd expect there to be some term in the phase voltage equation that looks like your classic brushed-motor "Back-EMF", i.e. a voltage proportional to the speed and torque constant of the motor. Since all the derivatives are with respect to time, these terms are a little less obvious. Ignoring magnetic saturation (inductance depending on current), generally, λ_r and L are functions of rotor position, θ. So saying $L = L(\theta)$ and $\lambda_r = \lambda_r(\theta)$ and taking the time-derivatives gives you:

$$\frac{d}{dt}\lambda_r(\theta) = \frac{d\theta}{dt}\frac{d\lambda_r}{d\theta} = \omega\frac{d\lambda_r}{d\theta} \tag{5}$$

$$\frac{d}{dt}L(\theta) = \frac{d\theta}{dt}\frac{dL}{d\theta} = \omega\frac{dL}{d\theta} \tag{6}$$

where ω is angular velocity.

Plugging those into (5) gives you two voltages proportional to angular velocity! The λ term is your "normal" back-emf from the permanent magnets, and the L term is why pure reluctance motors (with no magnets) still have back-emf.

2.3 Model <u>All</u> the Phases!

By looking at (5), you can just write down the voltage equations for the remaining phases by symmetry. These equations lend themselves to being written in matrix form. In it's entirety, the system of equations is as follows. I think it's important to fully write it out once:

$$\begin{bmatrix} v_a \\ v_b \\ v_c \end{bmatrix} = \begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ab} & L_b & L_{bc} \\ L_{ac} & L_{bc} & L_c \end{bmatrix} \begin{bmatrix} \frac{di_a}{dt} \\ \frac{di_b}{dt} + \\ \frac{di_c}{dt} \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} L_a & L_{ab} & L_{ac} \\ L_{ab} & L_b & L_{bc} \\ L_{ac} & L_{bc} & L_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} + \frac{d}{dt}\begin{bmatrix} \lambda_{ra} \\ \lambda_{rb} \\ \lambda_{rc} \end{bmatrix} + \begin{bmatrix} R_a & 0 & 0 \\ 0 & R_b & 0 \\ 0 & 0 & R_c \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_c \end{bmatrix} \tag{7}$$

That's extremely long, so from now on, I'll use the following abbreviations.

- $\left[L\right]$ is the matrix of inductances.
- $\left[\frac{dL}{dt}\right]$ is the matrix of inductance derivatives.

- $\left[R\right]$ is the resistance matrix.
- $\left[v\right]$ is the column vector of phase voltages.
- $\left[i\right]$ is the column vector of currents.
- $\left[\frac{di}{dt}\right]$ is the column vector of current derivatives.
- $\left[\frac{d\lambda_r}{dt}\right]$ is the column vector of PM flux linkage derivatives.

Given those abbreviations, the voltage equation is:

$$\left[v\right] = \left[L\right]\left[\frac{di}{dt}\right] + \left[\frac{dL}{dt}\right]\left[i\right] + \left[\frac{d\lambda_r}{dt}\right] + \left[R\right]\left[i\right] \tag{8}$$

Since I'm trying to model motors on a computer, the order of things needs some re-scrambling. In my case, $\left[v\right]$ is a known (set by the inverter), and I'm trying to solve for the currents and current derivatives. The way about this is to assume that over a short time period, currents are constant, and solve for their derivatives, $\left[\frac{di}{dt}\right]$. Numerically integrating these derivatives gives currents.

So the actual equation getting solved by the computer is:

$$\left[\frac{di}{dt}\right] = \left[L\right]^{-1}\left(\left[v\right] - \left[\frac{dL}{dt}\right]\left[i\right] - \left[\frac{d\lambda_r}{dt}\right] - \left[R\right]\left[i\right]\right) \tag{9}$$

3 Winding Termination

The previously described voltage equations solve for the voltages across each phase. This is not the same thing as the terminal voltages of the motor. Since an inverter sets the terminal voltages, not the phase voltages, to actually simulate a motor we need to get the equations in terms of the terminal voltages (u, v, w) rather than the phase voltages (v_a, v_b, v_c).

3.1 Independent Phases

Here's the simplest case. Each phase is independently controlled. This looks like a wye-terminated motor with the center point grounded. The phase voltages are equal to the terminal voltages.

$$\begin{aligned} v_a &= u \\ v_b &= v \\ v_c &= w \end{aligned} \tag{10}$$

3.2 Delta

The delta case is also easy. The phase voltages are just the differences of the terminal voltages:

$$\begin{aligned} v_a &= u - v \\ v_b &= v - w \\ v_c &= w - u \end{aligned} \tag{11}$$

3.3 Wye

Things get a bit more unpleasant when dealing with a wye-terminated motor. To solve wye motors, I introduced another unknown variable: N, the neutral voltage (at the center point). The phase voltages are then equal to:

$$\begin{aligned} v_a &= u - N \\ v_b &= v - N \\ v_c &= w - N \end{aligned} \tag{12}$$

The extra unknown comes with another constraint, $i_a + i_b + i_c = 0$, i.e. KCL at the center node.

To solve, I substituted the above into the voltage equations (8), so there were terminal voltages on the left, and neutral voltages on the right. Then I re-scrambled the matrices to be 4x4 to add the current constraint. Intuitively, I'm fairly sure there's a more elegant way to enforce the currents-sum-to-zero constraint without explicitly solving for N, (like a reaction force in a mechanical dynamics problem, usually you can solve the dynamics without solving for the reaction force) but this was the method that I went with. Also, I actually enforce the constraint on the current derivatives (which must also sum to zero).

Here's the equation I end up solving for a wye motor:

$$\begin{bmatrix} u \\ v \\ w \\ 0 \end{bmatrix} = \begin{bmatrix} & & & 1 \\ & [L] & & 1 \\ & & & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} [\frac{di}{dt}] \\ N \end{bmatrix} + \begin{bmatrix} & & & 0 \\ & [\frac{dL}{dt}] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [i] \\ 0 \end{bmatrix} + \begin{bmatrix} [\frac{d\lambda_r}{dt}] \\ 0 \end{bmatrix} + \begin{bmatrix} & & & 0 \\ & [R] & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} [i] \\ 0 \end{bmatrix} \tag{13}$$

To solve, like (10), move things to the left side of the equation and multiply by the inverse of the new, modified inductance matrix. The resulting vector is the current derivatives and the neutral voltage.

4 Power and Torque

To find power from the voltage equations, both sides get dotted with the current vector, since power is current times voltage. The left side of the equation is electrical power going into the terminals of the motor (phase voltages times phase currents). The right side of the equation takes a little interpretation. First, the easiest term is the one with resistances. Doing the dot product, the term is

$$R_a i_a^2 + R_b i_b^2 + R_c i_c^2 \tag{14}$$

which is the power dissipated in the stator resistance.

Next up is the inductance term, which comes out to

$$\left[L\right] \left[\frac{di}{dt}\right] \cdot \left[i\right] \tag{15}$$

which is the power going into the inductors (i.e. rate of energy getting stored in the magnetic fields)
Finally, there are the remaining Back-EMF terms:

$$\left[\frac{dL}{dt}\right] \left[i\right] \cdot \left[i\right] + \left[\frac{d\lambda_r}{dt}\right] \cdot \left[i\right] \tag{16}$$

which correspond to the mechanical power of the motor. The two summed terms in (16) come from the two different torque sources - permanent-magnet torque, and reluctance torque.

To get torque, just divide this term by the angular velocity of the motor, since power is also equal to torque times angular velocity. Keep in mind that these equations have all been *per pole-pair*, so to get actual speed and torque, divide and mulitply by number of pole-pairs respectively.

5 Useful References

Kirtley, J.L. *6.061 Class Notes, Ch 12 Permanent Magnet "Brushless" DC Motors*
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