Computer Arithmetic

What is the difference between `signed int` and `unsigned int`?

Pat Hanrahan

cs107e

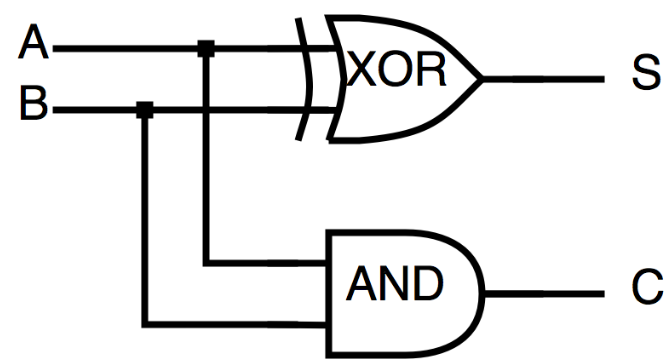
Addition

Adding 2 1-bit numbers

a	b	sum
0	0	00
0	1	01
1	0	01
1	1	10

Adding 2 1-bit numbers (Half Adder)

a	b	sum
0	0	00
0	1	01
1	0	01
1	1	10



bit 0 of sum: S = a^b
bit 1 of sum: C = a&b

Have reduced addition to logical operations!

Adding 2 8-bit numbers

```
             Carry
  00000111 A
 +00001011 B
 ---------
             Sum
```

Adding 2 8-bit numbers

```
        1  Carry
  00000111 A
 +00001011 B
 ---------
         0 Sum
```

Adding 2 8-bit numbers

```
      11  Carry
  00000111 A
 +00001011 B
 ---------
        10 Sum
```

Adding 2 8-bit numbers

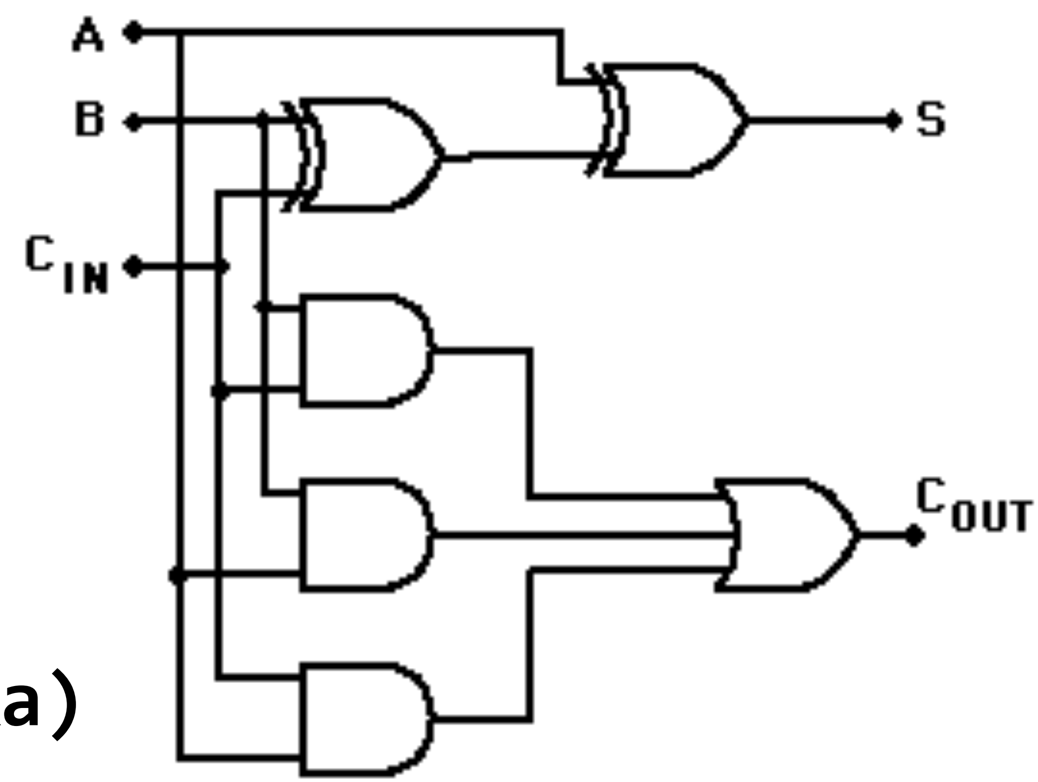
```
 00001111  Carry
  00000111 A
 +00001011 B
 ---------
  00010010 Sum
```

Adding 3 1-bit numbers

a	b	c	=	c	s
0	0	0		0	0
0	1	0		0	1
1	0	0		0	1
1	1	0		1	0
0	0	1		0	1
0	1	1		1	0
1	0	1		1	0
1	1	1		1	1

Adding 3 1-bit numbers (Full Adder)

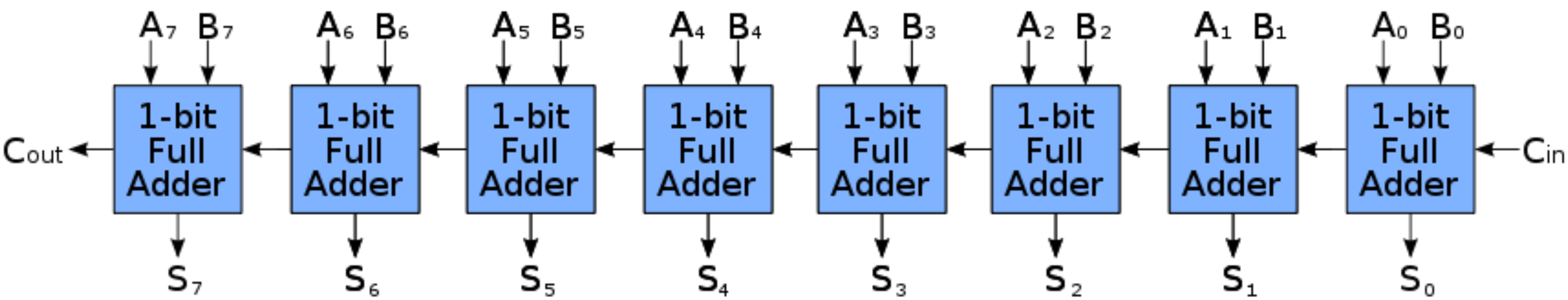
a	b	ci	=	co	s
0	0	0		0	0
0	1	0		0	1
1	0	0		0	1
1	1	0		1	0
0	0	1		0	1
0	1	1		1	0
1	0	1		1	0
1	1	1		1	1



s = a^b^ci

co = (a&b)|(b&c)|(c&a)

8-bit Ripple Adder



Note `Cin` **and** `Cout`

```
// Multiple precision addition
// http://godbolt.org/g/HMYrme

uint64_t add64(uint64_t a, uint64_t b)
{
  return a + b;
}

add64:
  adds r0, r0, r2
  adc  r1, r1, r3
  bx   lr
```

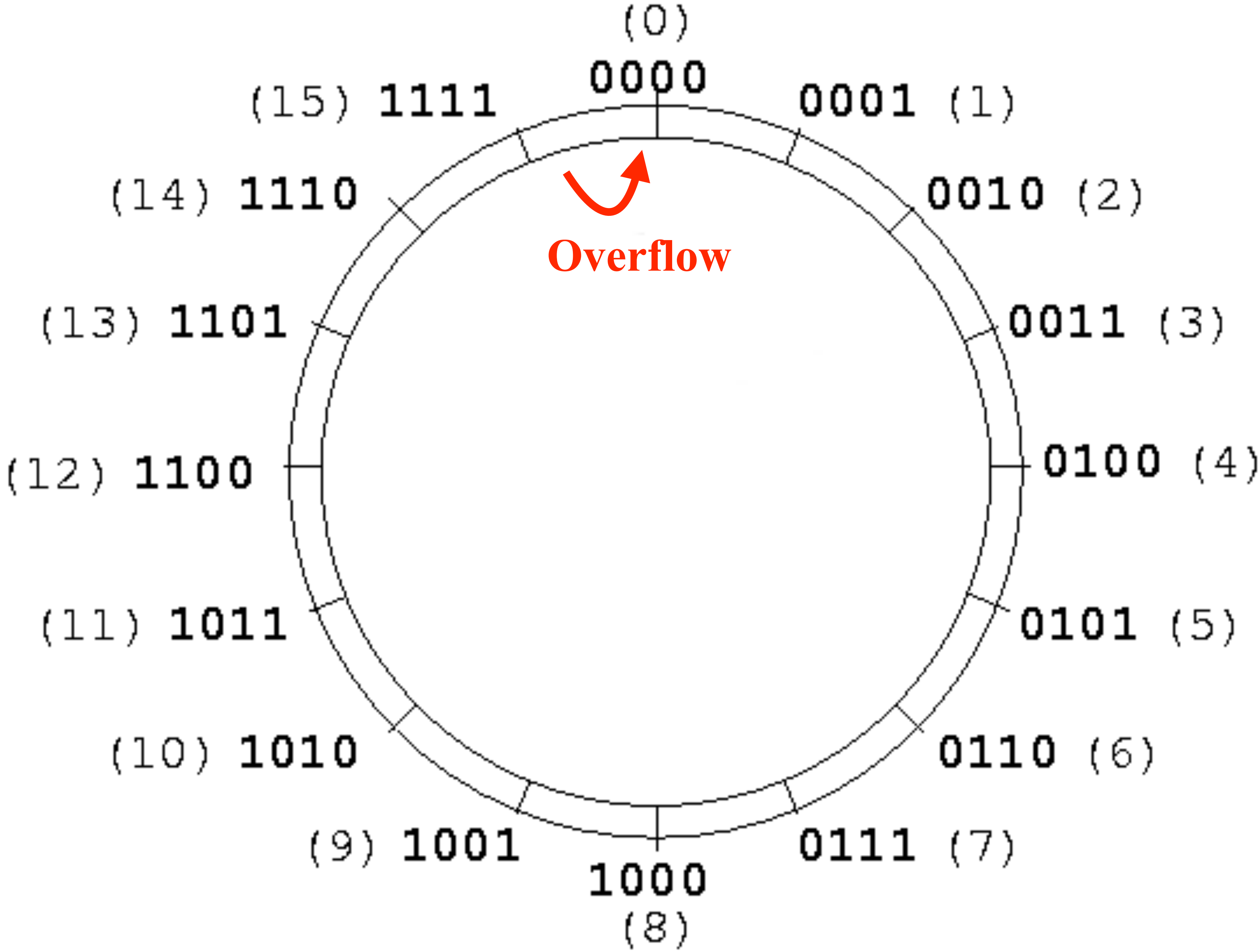
Binary Addition

```
 11111111  Carry
  11111111 A
 +00000001 B
 ---------
 100000000 Sum
```

To represent the result of adding two n-bit numbers with full precision requires n+1 bits

But we only have 8-bits!

```
  sum = 0b00000000 = (A+B)%256
```


(0)
0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000
(8)
(9) 1001
(10) 1010
(11) 1011
(12) 1100
(13) 1101
(14) 1110
(15) 1111
Overflow


999999
99999
000000

Gangnam Style overflows INT_MAX, forces YouTube to go 64-bit

Psy's hit song has been watched an awful lot of times.

PETER BRIGHT - 12/3/2014, 2:32 PM

- In two's complement, when you exceed the maximum value of int (2,147,483,647), you "wrap around" to negative numbers:

- Here is the link after Google upgraded to 64-bit integers:

Subtraction

BIG IDEA: Define subtraction using addition

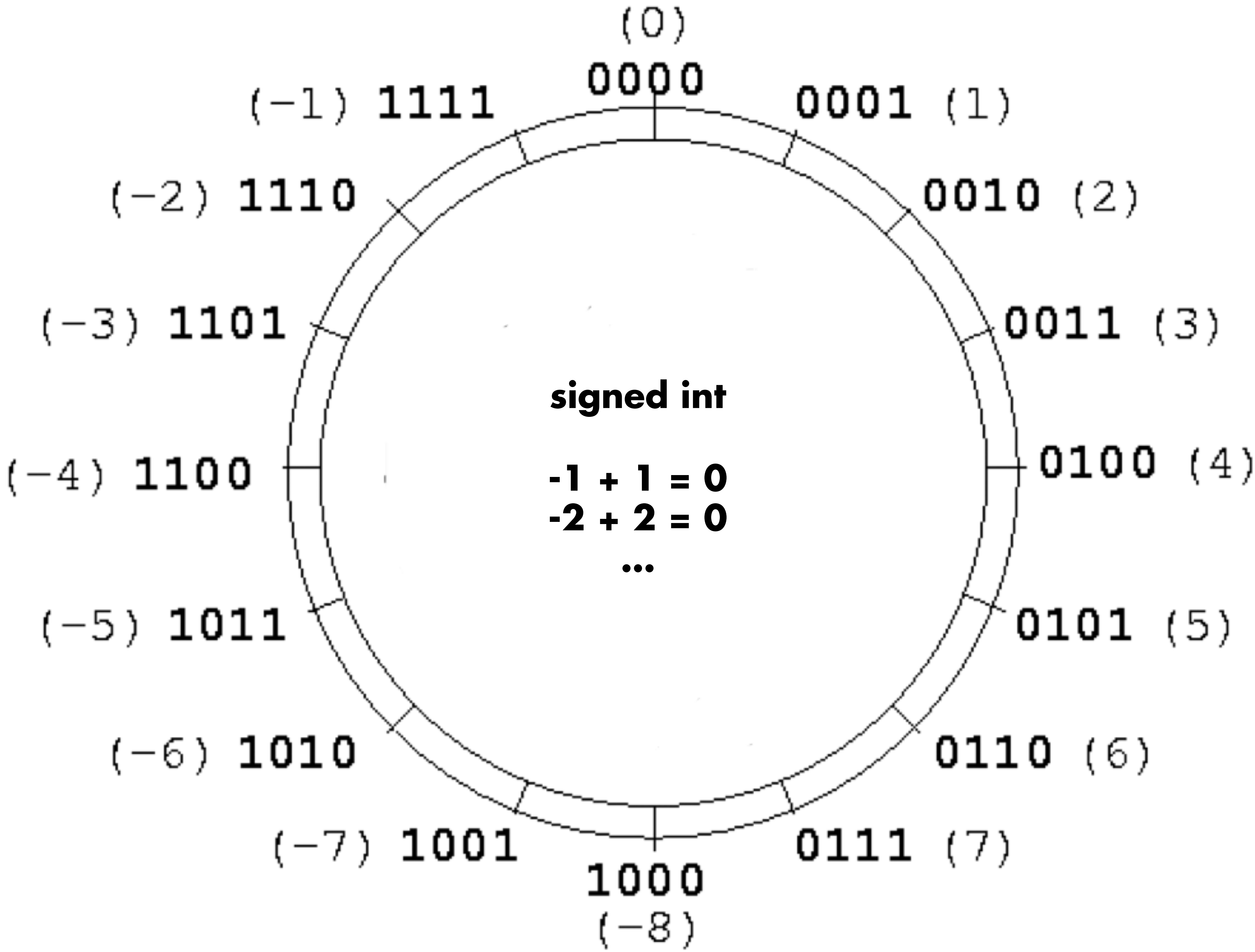
A clever way of defining subtraction by 1 is to find a number to add that yields the same result as the subtract by 1.

This number is the *negative* of the number.

More precisely, this number is the number that when added to 1, results in 0 (mod 16)

0x1 - 0x1 = 0x1 + 0xf = 0x10 % 16 = 0x0

0xf can be *interpreted* as -1


(0)
0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000
(−8)
(−7) 1001
(−6) 1010
(−5) 1011
(−4) 1100
(−3) 1101
(−2) 1110
(−1) 1111
signed int
-1 + 1 = 0
-2 + 2 = 0
...

```
Signed 4-bit numbers,

0x0 =  0
0xf = -1
0xe = -2
…
0x8 = -8 (could be interpreted as 8)
0x7 =  7
…
0x1 =  1
0x0 =  0
```

if we choose to *interpret* 0x8 as -8, then the most-significant bit of the number indicates that it is negative (n)

`signed int` **vs as** `unsigned int`

Are just different interpretations of the bits comprising the number

`0xff` **vs** `-1`

Negation

How do we negate an 8-bit number?

Find a number -x, s.t. (x + (-x)) % 256 = 0

Subtract it from 256 = 2^8 = 100000000

-x = 100000000 - x

Since then (x + (-x)) % 256 = 0

```
 11111111  Borrow     100000000 Carry
 100000000             00000001
 -00000001            +11111111
 ---------            ---------
  11111111             00000000
```

This is called two's complement

```
Another way to negate

Rewrite 100000000 = (11111111 + 1)

-x = (11111111+1)-x
   = (11111111-x)+1 (one's complement)
   = ~x+1 // ~ is invert

E.g. -1 = 11111111

~00000001 = 11111111
           -00000001
           ---------
            11111110

 11111110 + 00000001 = 11111111
```

Subtraction is converted negation + addition

-B is implemented using ~B+1

A - B = A + ~B + 1

```
01 - 00 = 01 + ff + 01 = 01 + c
01 - 01 = 01 + fe + 01 = 00 + c
01 - 02 = 01 + fd + 01 = ff
```

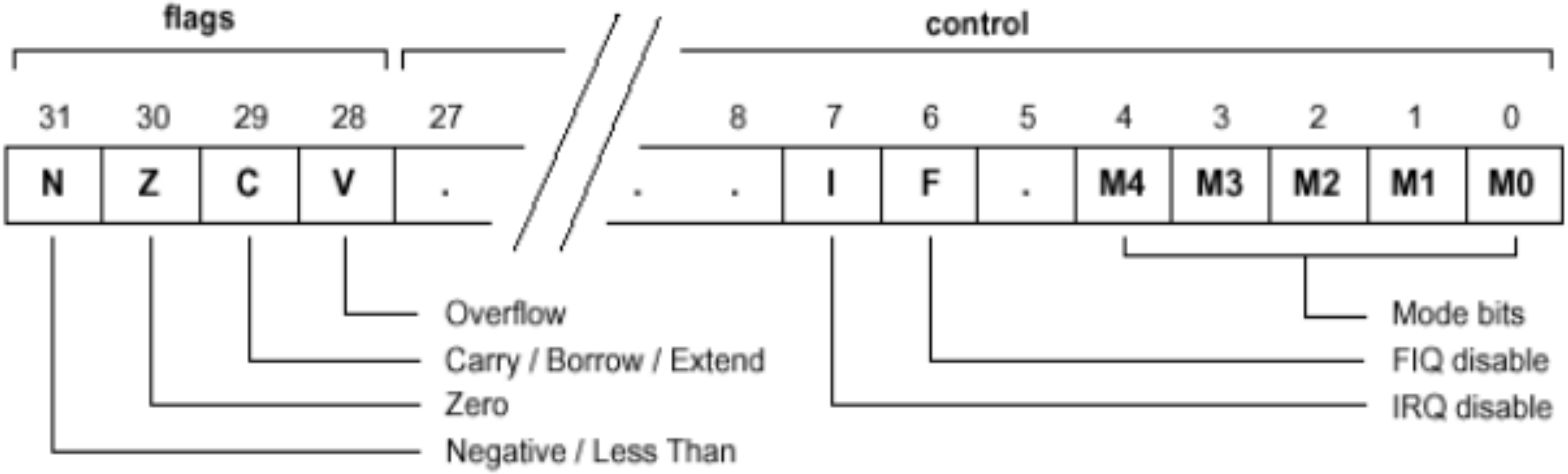
Note the carry out bit c

The +1 can be done by setting Cin to 1!

```
unsigned int timer_get_ticks(void)
{
  return *SYSTIMERCLO;
}

void timer_delay_us(unsigned int usecs)
{
  unsigned int start=timer_get_ticks();
  while (timer_get_ticks()-start) < usecs);
}

// The timer continuously ticks.

// Does this code work if the timer overflows?
```

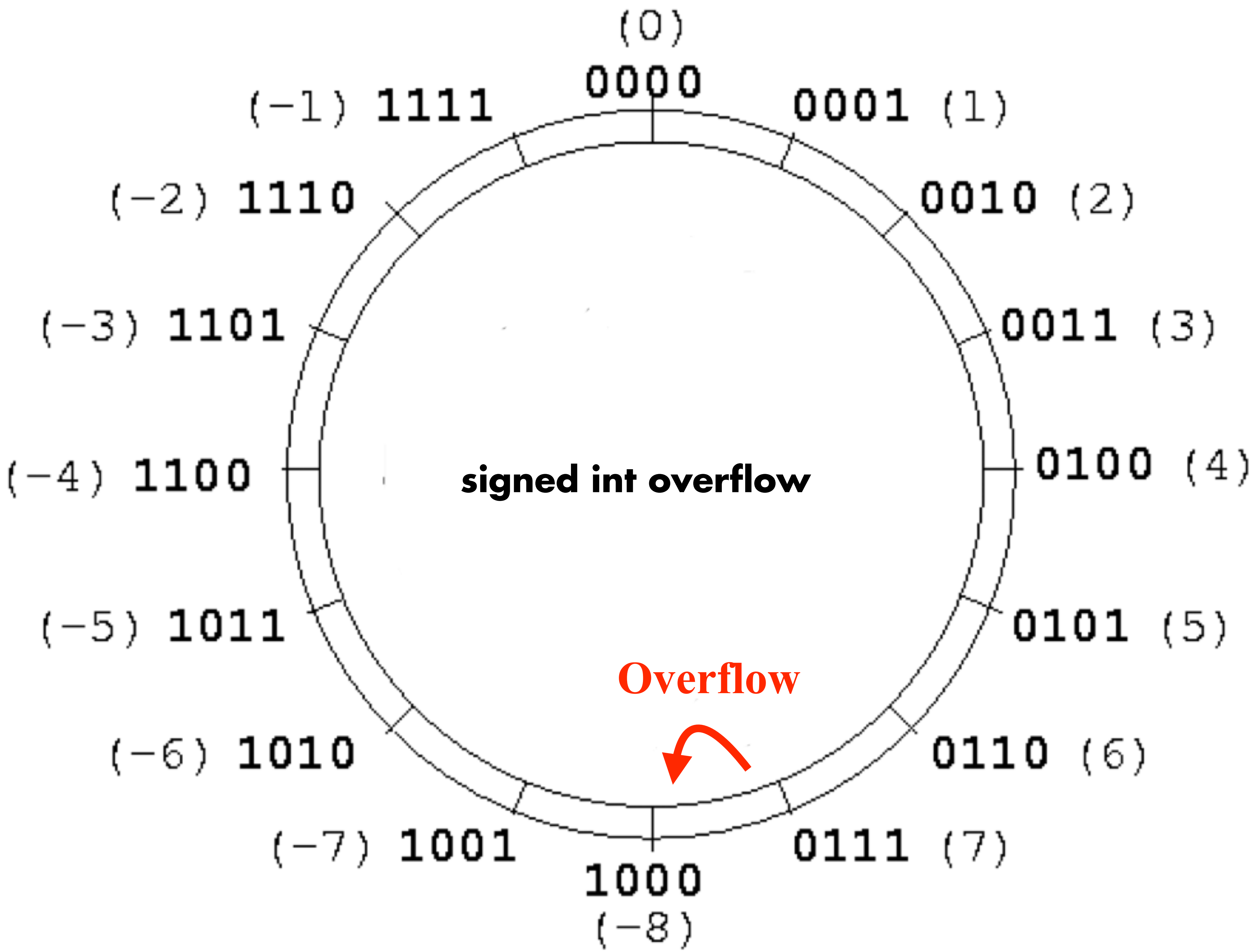
Addition and Subtraction of `signed` and `unsigned` numbers are the same!

CPSR

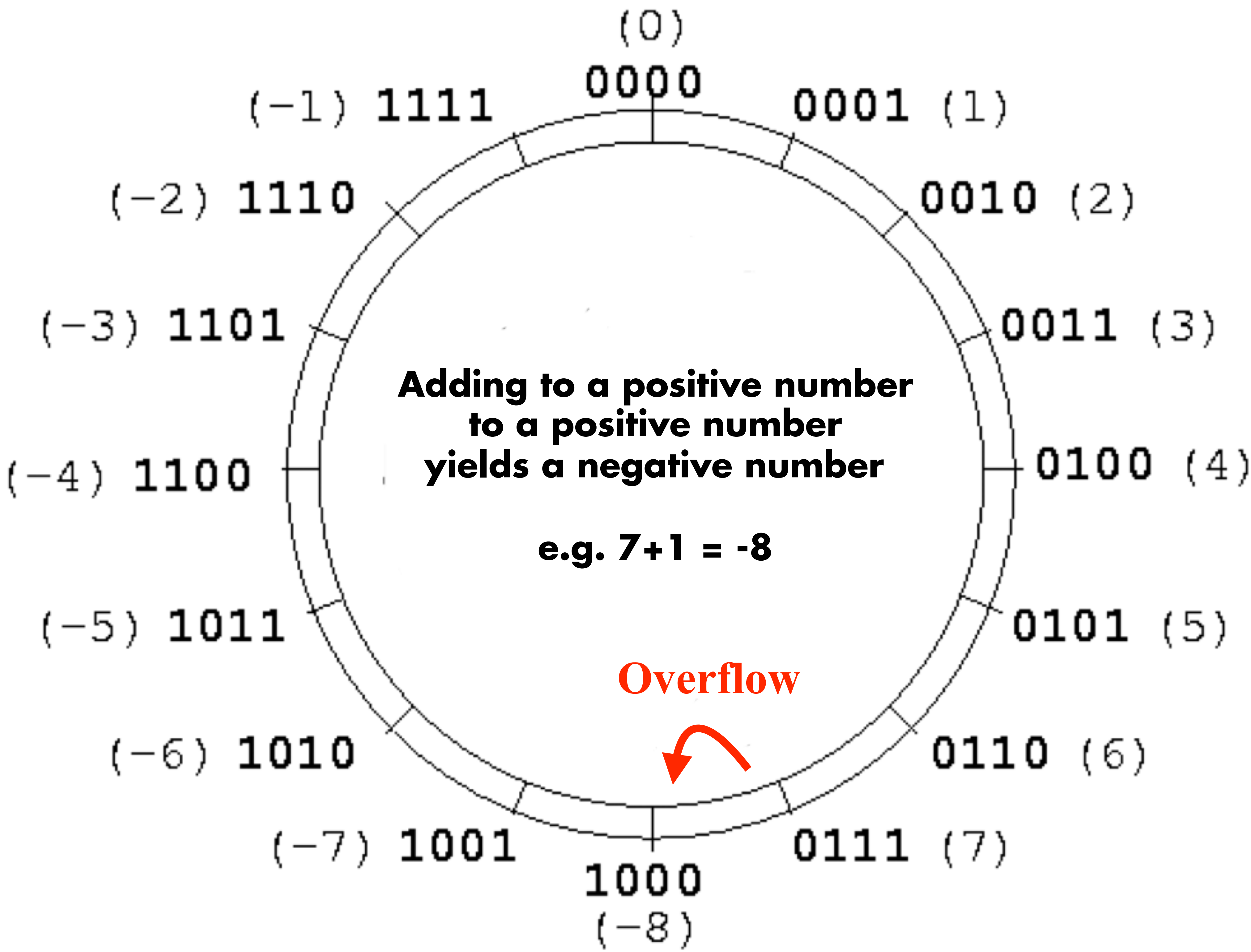


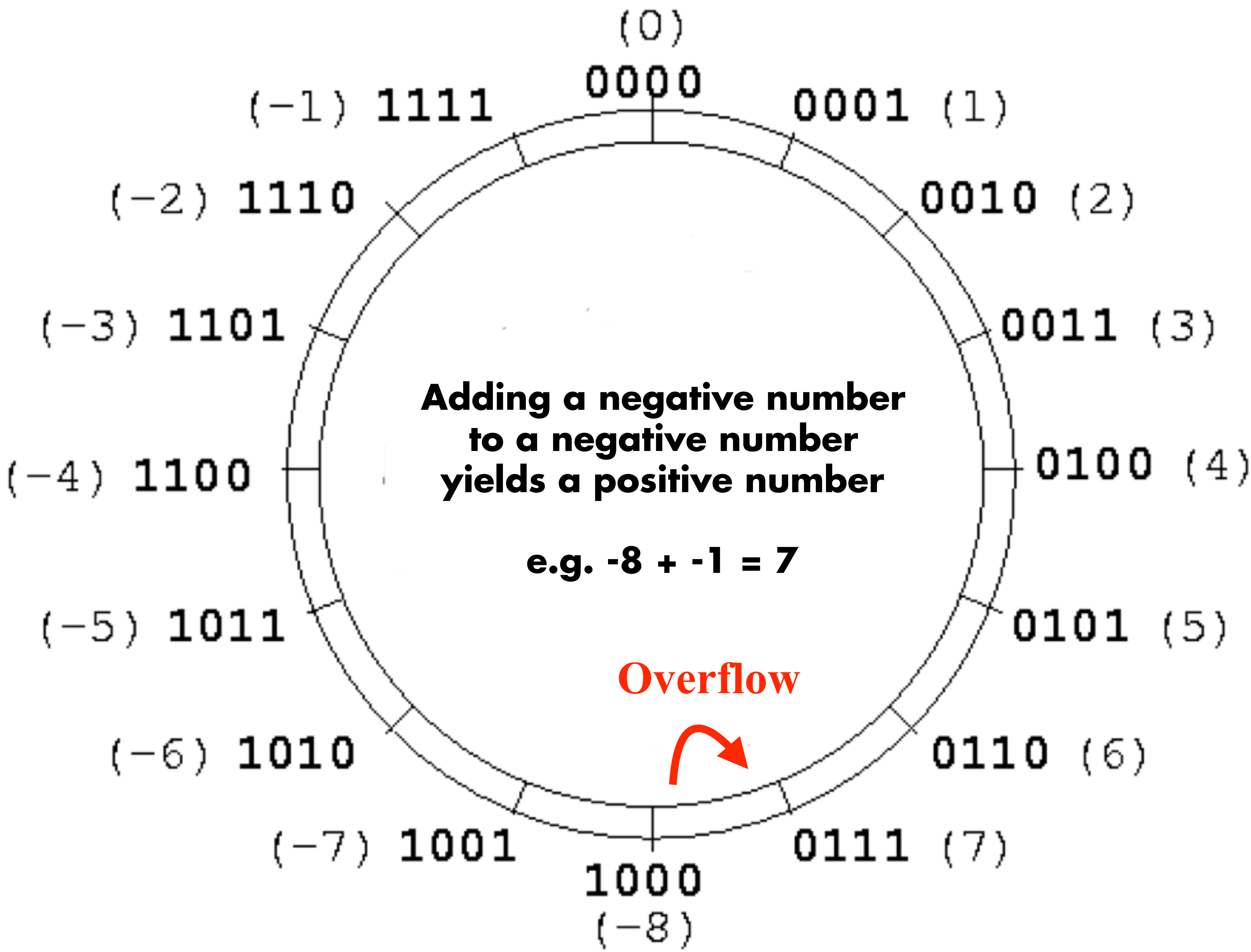
Arithmetic instructions set N, Z, C, V
Logic instructions just set N, Z

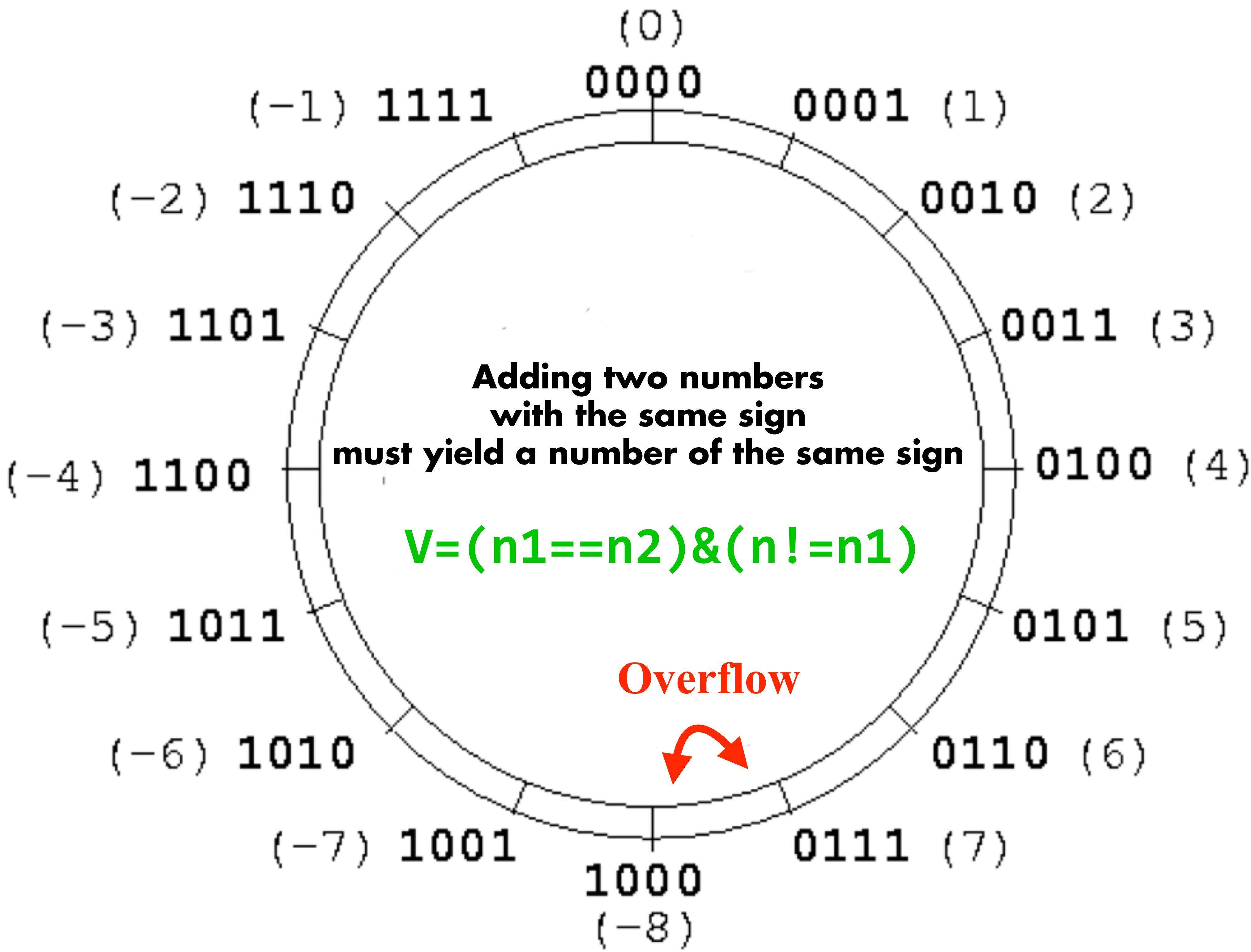
What is V?

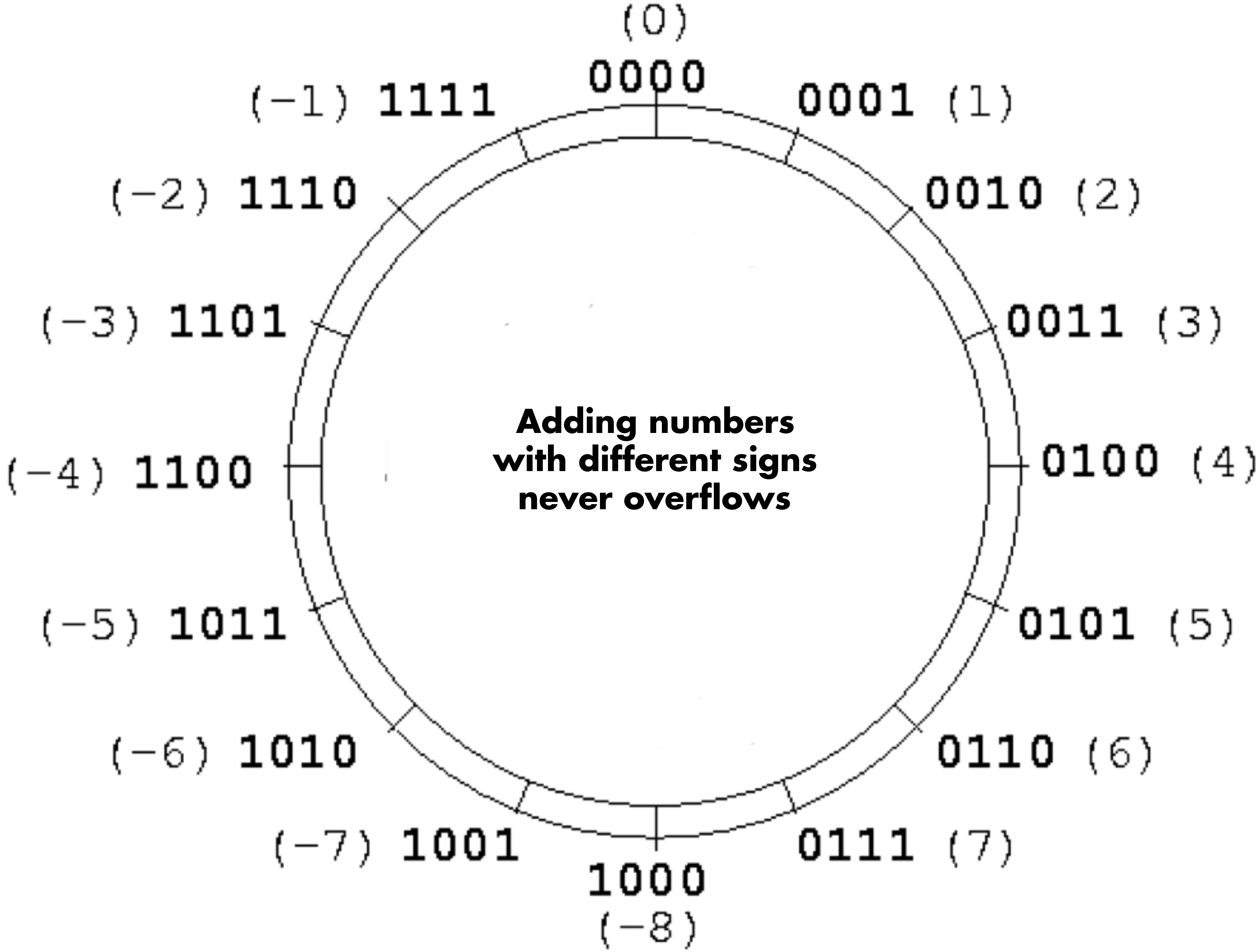


(0) 0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000 (-8)
(-7) 1001
(-6) 1010
(-5) 1011
(-4) 1100
(-3) 1101
(-2) 1110
(-1) 1111
signed int overflow
Overflow


(0) 0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000 (-8)
(-7) 1001
(-6) 1010
(-5) 1011
(-4) 1100
(-3) 1101
(-2) 1110
(-1) 1111
Adding to a positive number
to a positive number
yields a negative number
e.g. 7+1 = -8
Overflow


(0) 0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000 (-8)
(-7) 1001
(-6) 1010
(-5) 1011
(-4) 1100
(-3) 1101
(-2) 1110
(-1) 1111
Adding a negative number to a negative number yields a positive number
e.g. -8 + -1 = 7
Overflow


(0)
0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000
(-8)
(-7) 1001
(-6) 1010
(-5) 1011
(-4) 1100
(-3) 1101
(-2) 1110
(-1) 1111
Adding two numbers
with the same sign
must yield a number of the same sign
V=(n1==n2)&(n!=n1)
Overflow


(0)
0000
0001 (1)
0010 (2)
0011 (3)
0100 (4)
0101 (5)
0110 (6)
0111 (7)
1000
(-8)
(-7) 1001
(-6) 1010
(-5) 1011
(-4) 1100
(-3) 1101
(-2) 1110
(-1) 1111
Adding numbers
with different signs
never overflows

Comparison

=

Subtract and Look at Flags

Code	Suffix	Flags	Meaning
0000	EQ	Z set	equal
0001	NE	Z clear	not equal
0010	CS	C set	unsigned higher or same
0011	CC	C clear	unsigned lower
0100	MI	N set	negative
0101	PL	N clear	positive or zero
0110	VS	V set	overflow
0111	VC	V clear	no overflow
1000	HI	C set and Z clear	unsigned higher
1001	LS	C clear or Z set	unsigned lower or same
1010	GE	N equals V	greater or equal
1011	LT	N not equal to V	less than
1100	GT	Z clear AND (N equals V)	greater than
1101	LE	Z set OR (N not equal to V)	less than or equal
1110	AL	(ignored)	always

Methods used to compare `signed` and `unsigned` numbers are NOT the same!

Type Conversion

Jedi Job Interview Questions

```
#include <stdint.h>

uint16_t x = 0xffff;
uint32_t y = x;

// x = 0xffff
// y = ?
```

```c
#include <stdint.h>

uint16_t x = 0xffff;
uint32_t y = x;

// x = 0xffff
// y = 0x0000ffff
```

Type Hierarchy

uint32 0 … 4294967295 (0xffffffff)

↑

uint16 0 … 65535 (0xffff)

↑

uint8 0 … 255 (xff)

Types are *sets* of allowed values

Arrow indicate *subsets*: uint16 ⊂ uint32

uint32

uint16

uint8

Type Conversion is Safe
(values preserved)

```
int16_t x = -1;
int32_t y =  x;

// x = -1
// y = ?
```

```
int16_t x = -1;
int32_t y =  x;

// x = -1
// y = -1
```

int32 -2,147,483,648 … 2,147,483,647

int16 -32768 … 32767

int8 -128 … 127

Type Conversion is Safe
(values preserved)

```
int16_t x = -1;
int32_t y =  x;

// x = -1 = 0xffff
// y = -1 = 0xffffffff

int16_t x =  1;
int32_t y =  x;

// x = 1 = 0x0001
// y = 1 = 0x00000001
```

```
// Sign extension

int8_t 0xfe -> int32_t 0xfffffffe
int8_t 0x7e -> int32_t 0x0000007e


// Sign extend instructions:
//
//  sxtb - sign extend byte to word
//  sxth - sign extend half word to word
//
```

```
int32_t x = 0x80000;
int16_t y = x;

// x = 0x80000
// y = ?
```

```
int32_t x = 0x80000;
int16_t y = x;

// x = 0x80000
// y =  0x0000
```

value has changed

int32

↓

int16

↓

int8

Defined (remove most significant bits)

Dangerous (doesn't preserve all values)

```
int32_t  x = -1;
uint32_t y =  x;

// x = -1
// y =  ?
```

```
int32_t  x = -1;
uint32_t y =  x;

// x = -1
// y = 0xffffffff = 4294967295
```

value has changed

(x is negative, y is positive!)

uint32 ← int32

uint16 ← int16

uint8 ← int8

Defined (copies bits)

uint32 ← int32

uint16 ← int16

uint8 ← int8

Dangerous! (neg maps to pos)

uint32 → int32

uint16 → int16

uint8 → int8

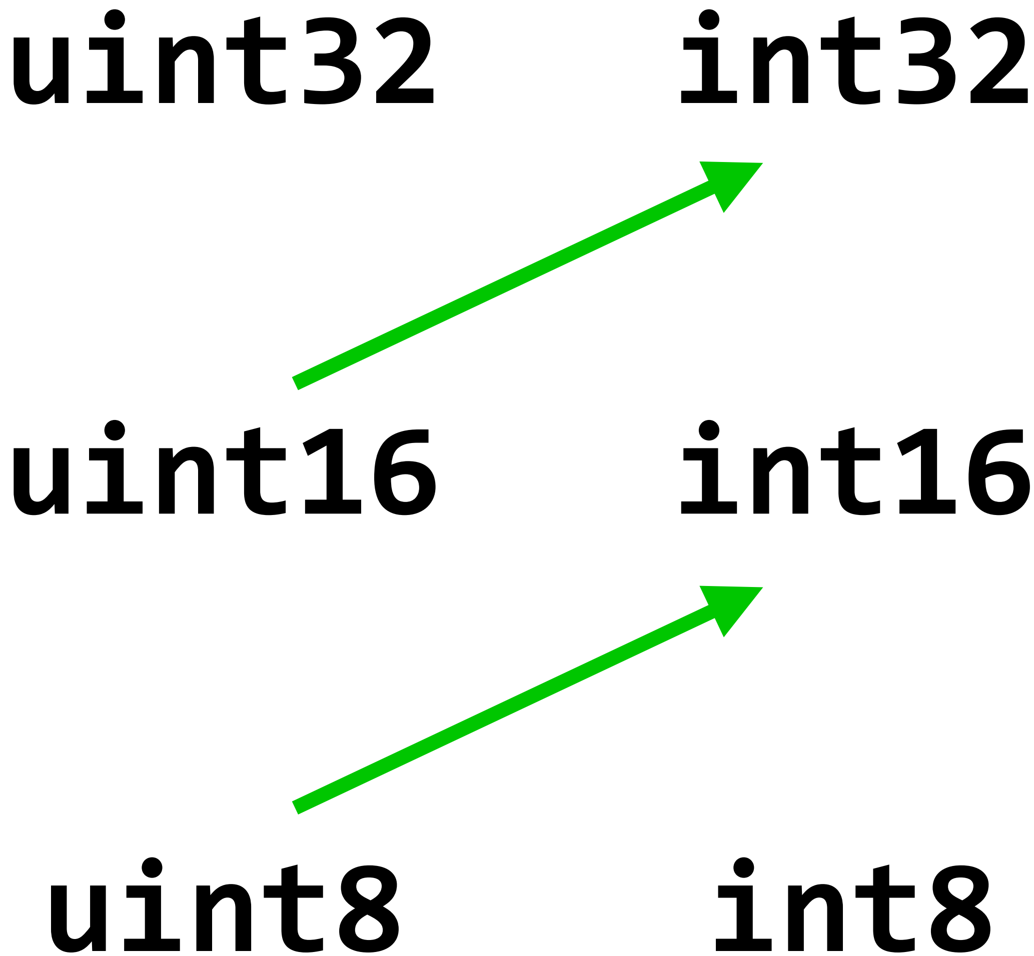
Technically Not Defined
(arm: copies bits)

uint32 → int32

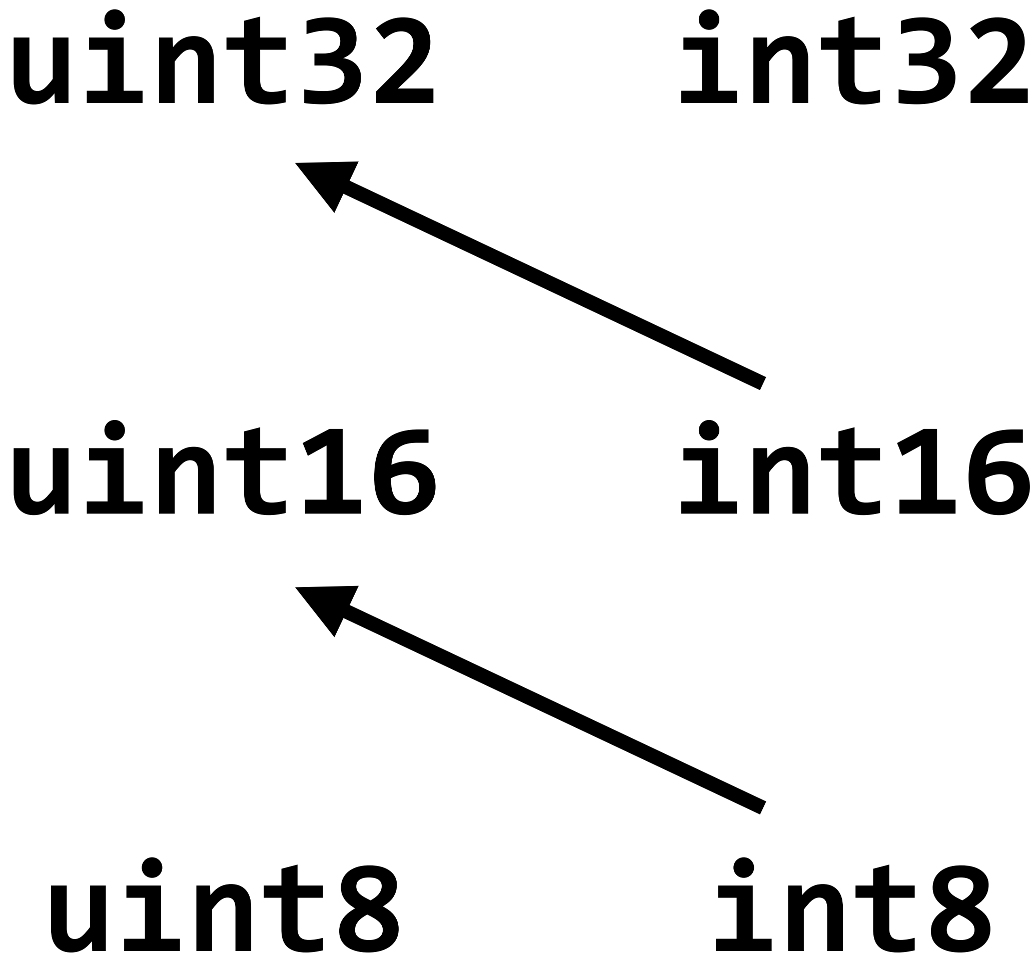
uint16 → int16

uint8 → int8

Dangerous!
(large positive numbers change)



Safe?



Safe?

Implicit Type Promotion

in

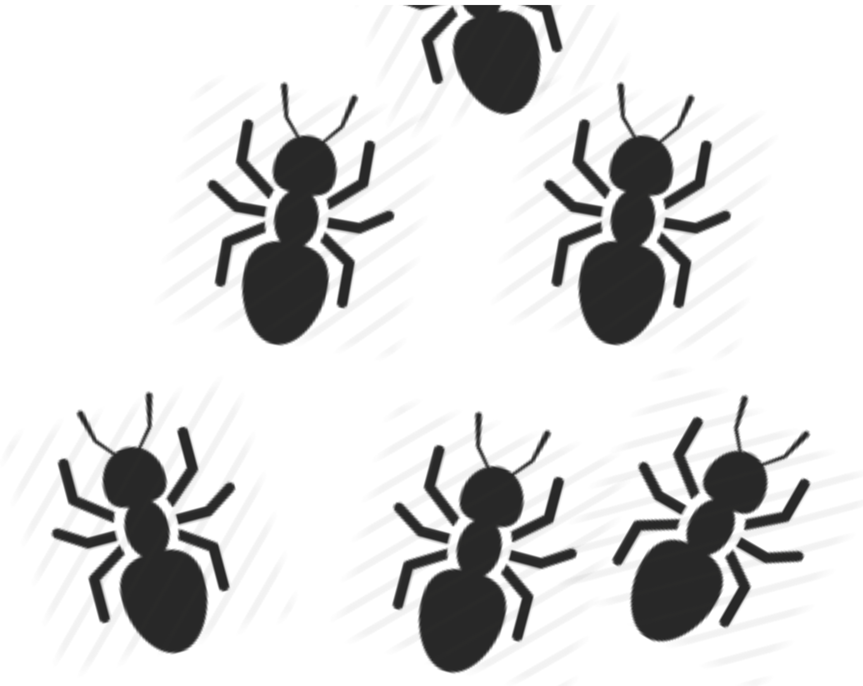
Binary Operators

Type promotions for binary operations

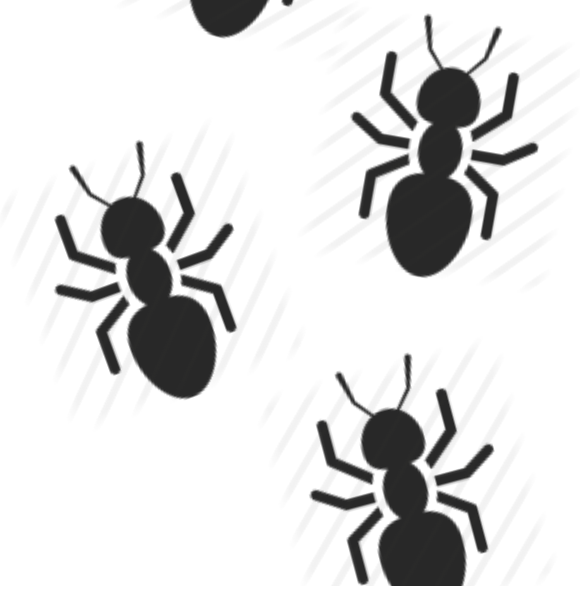
Note that the type of the result can be different than the type of the operands!

	u8	u16	u32	u64	i8	i16	i32	i64
u8	i32	i32	u32	u64	i32	i32	i32	i64
u16	i32	i32	u32	u64	i32	i32	i32	i64
u32	u32	u32	u32	u64	u32	u32	u32	i64
u64	u64	u64	u64	u64	u64	u64	u64	u64
i8	i32	i32	u32	u64	i32	i32	i32	i64
i16	i32	i32	u32	u64	i32	i32	i32	i64
i32	i32	i32	u32	u64	i32	i32	i32	i64
i64	i64	i64	i64	u64	i64	i64	i64	i64

arm-none-eabi-gcc type promotions



Bugs, Bugs, Bugs



```c
#include <stdio.h>

int main(void)
{
    int a = -20;
    unsigned int b = 6;

    if( a < b )
        printf("-20<6 - all is well\n");
    else
        printf("-20>=6 - omg \n");
}
```

Whenever you mix
signed and unsigned numbers
you get in trouble

Bjarne Stroustrup

Summary

Negation is performed by forming the two's complement

- Signed numbers are represented in two's complement ($-x = 2^n - x = \sim x + 1$)

In 2's complement,

- Arithmetic between signed and unsigned numbers is identical
- Comparison between signed and unsigned numbers is different

Know the rules for type conversion, watch out for implicit type conversions and promotions

C Type Conversion and Promotion Rules

The semantics of numeric casts are:

Casting from a larger integer to a smaller integer (e.g. u32 -> u8) will truncate

Casting from a smaller integer to a larger integer (e.g. u8 -> u32) will
zero-extend if the source is unsigned
sign-extend if the source is signed

Casting between two integers of the same size (e.g. i32 -> u32) is a no-op

6.3.1.3 Signed and unsigned integers conversions

1 When a value with integer type is converted to another integer type, if the value can be represented by the new type, it is unchanged.

2 Otherwise, if the new type is unsigned, the value is converted by repeatedly adding or subtracting one more than the maximum value that can be represented in the new type until the value is in the range of the new type.

3 Otherwise, if the new type is signed and the value cannot be represented in it; either the result is implementation-defined or an implementation-defined signal is raised.

6.3.1.8 Usual arithmetic conversions

1 If both operands have the same type, then no further conversion is needed.

2 Otherwise, if both operands have signed integer types or both have unsigned integer types, the operand with the type of lesser integer conversion rank is converted to the type of the operand with greater rank.

3 Otherwise, if the operand that has unsigned integer type has rank greater or equal to the rank of the type of the other operand, then the operand with signed integer type is converted to the type of the operand with unsigned integer type.

4 Otherwise, if the type of the operand with signed integer type can represent all of the values of the type of the operand with unsigned integer type, then the operand with unsigned integer type is converted to the type of the operand with signed integer type.

5 Otherwise, both operands are converted to the unsigned integer type corresponding to the type of the operand with signed integer type.