CS 107e
Floating Point Numbers

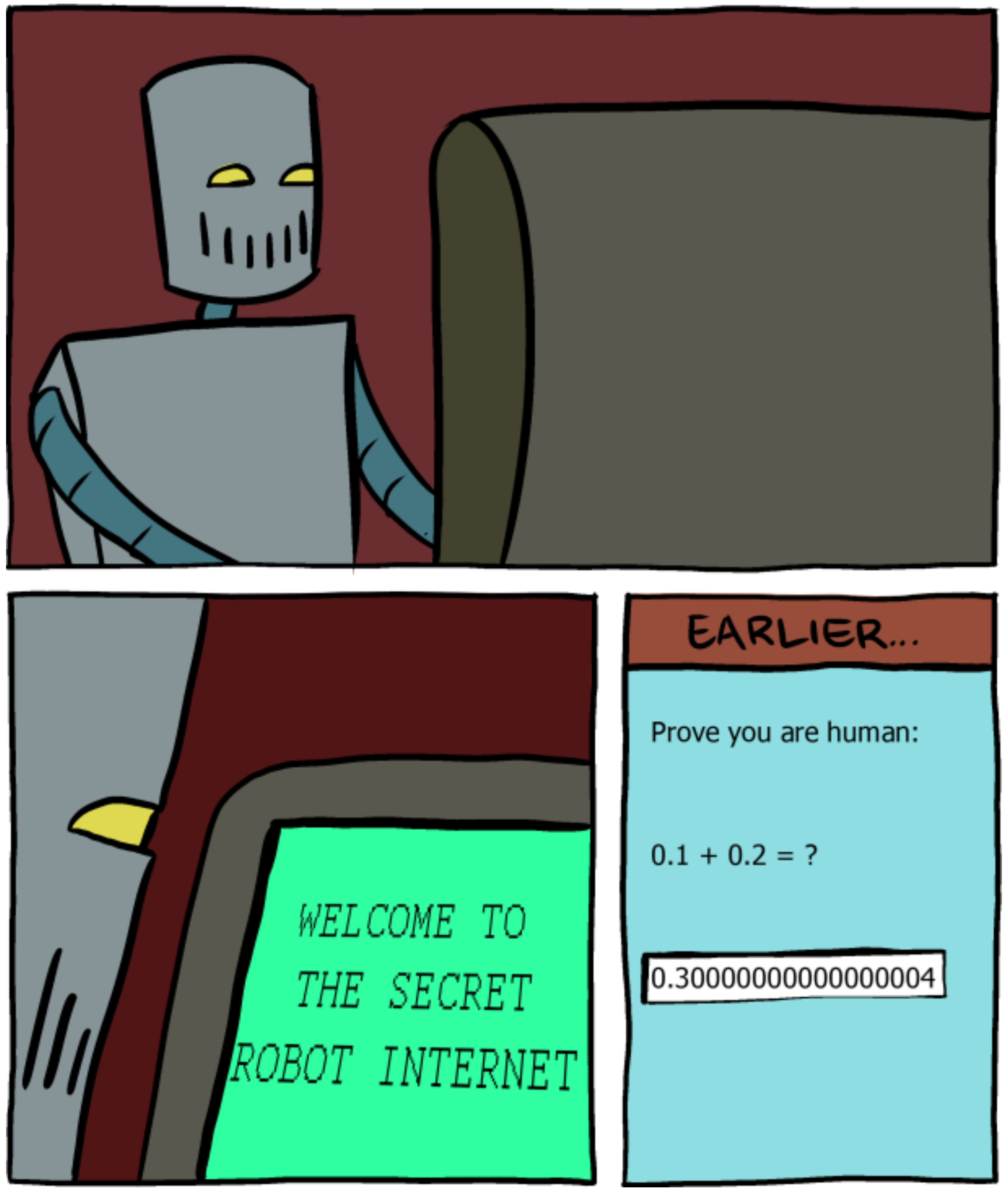
Friday, November 30, 2018

Computer Systems from the Ground Up
Fall 2018
Stanford University
Computer Science Department

$$V = (-1)^s \times M \times 2^E$$

Single precision (`float`)

31	30 … 23	22 … 0
s	exp	frac



Today's Topics

- Logistics
  - Next week lab: discuss projects!
  - Project presentations: Friday, December 14th, 9am-11:30am
- Today: Floating Point Numbers
  - Real Numbers
  - Fixed Point
  - IEEE 754 Floating Point
    - Normalized values
    - Denormalized values
    - Exceptional values
    - Arithmetic

Real Numbers

$$\frac{1}{5}$$

Real Numbers

$$\frac{1}{5} = 0.2$$

Real Numbers

$$\frac{1}{3}$$

Real Numbers

$$\frac{1}{3} = 0.33333\ldots$$

Real Numbers

$$\frac{1}{3} = 0.33333\ldots$$

When I was in 6th grade, this was a mind blowing concept.

Fractions

$$1/3 = 0.33333\ldots$$

Especially this

Real Numbers

$$\pi = 3.14159\ldots$$

And don't get me started on this

Real Numbers

Once we leave the realm of integers, real numbers become … tricky.

- Some rational numbers, e.g., ⅕, can be represented exactly, by a fixed number of decimal digits (0.2 in this case)
- Some rational numbers, e.g., ⅓, can not be represented exactly, and we have this idea of "repeating indefinitely," which we represent with "…" (0.33333…)
- Irrational numbers can never be represented by a fixed number of decimal digits, and the meaning of "…" means "indefinitely" but loses the "repeating" part. Irrational numbers can never be represented exactly using a digit notation.

The big question: how do we represent real numbers in a computer?

Real Numbers in a Computer

The big question: how do we represent real numbers in a computer?

As always, we have choices. Here are some constraints:

1. We want to represent real numbers in a fixed number of bits. This means that we aren't going to be able to represent all real numbers exactly, nor even all rational numbers exactly. Furthermore, we can't even represent all rational numbers in a *range* exactly (there are infinitely many rational numbers in any fixed range).
2. We want to represent a large range of numbers.
3. We want to be able to perform calculations on the numbers.

Real Numbers in a Computer

One idea: Fixed Point

When we represented integers, we implicitly placed the decimal point (or binary point, in base 2) after the least significant digit, and we limited ourselves to positive powers of our base. E.g., 1234 is really 1234.0000…

We could just move the decimal place, and use that as our system for representing real numbers:

In decimal: $d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$

e.g., 123.45

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

What range of numbers can we represent now?

Fixed Point

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e.g., 123.45

What range of numbers can we represent now? 0 to 999.99

Fixed Point

$$d_2d_1d_0.d_{-1}d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

What range of numbers can we represent now? 0 to 999.99

What is the "precision" we can represent (i.e., how precise?)

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

What range of numbers can we represent now? 0 to 999.99

What is the "precision" we can represent (i.e., how precise?)

we can represent five decimal digits of precision, to the 100th place

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

We can't represent some rational numbers exactly:

123.456
123.333…
1000 (overflow? Also, 999.991, or 999.9901, or 999.99001, or …)
0.001 (underflow?)
We would have to round or truncate, or over/under-flow.

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

Fixed-point arithmetic is pretty easy:

```
 123.45
+678.90
 802.35
```

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

e.g., 123.45

Fixed-point arithmetic is pretty easy:

```
 123.45
+678.90
-------
 802.35
```

```
  100.22
*   1.08
--------
   80176
  000000
 1002200
--------
 1082376 = 108.2376
         = 108.24
         (rounded)
```

Fixed Point

$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

Fixed point has its uses, but it is somewhat limiting. We can do regular arithmetic, and we know how many decimal places of precision we get.

However, the range is set by where we fix the decimal place, and we had hoped for a large range. If we set the decimal place to be to the left of the most significant digit for a five-digit number, our range would only be 0 to 0.99999.

Floating Point

A different idea is to represent numbers in the form $V = x \times 2^y$

In this form, we will break our number into two parts (actually, three, including a sign bit), with an exponent (y) and a fractional value (x).

Before we get into the details, let's investigate what fractional values in binary look like. Recall, in decimal:

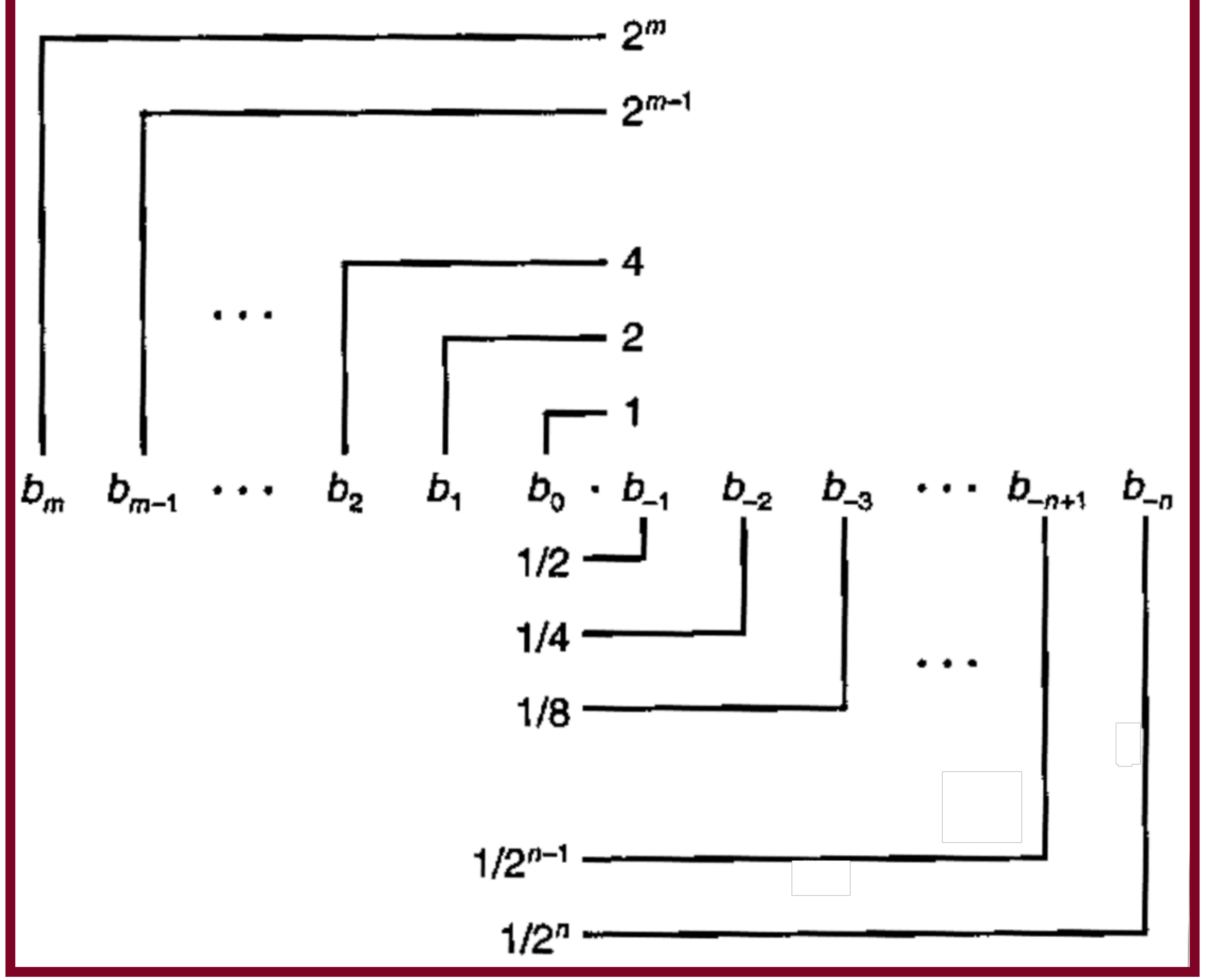
$$d_2 d_1 d_0 . d_{-1} d_{-2} = d_2 \times 10^2 + d_1 \times 10^1 + d_0 \times 10^0 + d_{-1} \times 10^{-1} + d_{-2} \times 10^{-2}$$

Digits after the decimal point are represented by negative powers of 10.

Fractional Values in Binary

In binary, digits after the *binary* point are represented by negative powers of *two*:

$$b_2 b_1 b_0 . b_{-1} b_{-2} = b_2 \times 2^2 + b_1 \times 2^1 + b_0 \times 2^0 + b_{-1} \times 2^{-1} + b_{-2} \times 2^{-2}$$



Online binary to decimal converter:

http://web.stanford.edu/class/cs107/float/convert.html

Fractional Values in Binary

Example: 101.11 in binary:

$$1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 1 \times 2^{-2} = 4 + 0 + 1 + \frac{1}{2} + \frac{1}{4} = 5\frac{3}{4}$$

What happens to your number if you shift the binary point to the left by one?

Fractional Values in Binary

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The number is divided by two.

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What happens to your number if you shift the binary point to the right by one?

The number is multiplied by two.

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What happens to your number if you shift the binary point to the left by one?

The number is divided by two.

What happens to your number if you shift the binary point to the right by one?

The number is multiplied by two.

What is represented by 0.111111…1?

Fractional Values in Binary

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What happens to your number if you shift the binary point to the left by one?

The number is divided by two.

What happens to your number if you shift the binary point to the right by one?

The number is multiplied by two.

What is represented by 0.111111…1?

Numbers just below 1, e.g.: $0.111111_2 = \frac{63}{64}$ Shorthand: 1-ϵ

Fractional Values in Binary

Just like decimal with numbers like ⅓ and ⅙, binary cannot represent exactly any numbers like ⅓ and ⅕, nor even ⅒:

```
// testTenth.c
#include<stdio.h>
#include<stdlib.h>

int main()
{
    float f = 0.1;
    // print with 27 decimal places
    printf("%.27f\n",f);
    return 0;
}
```

```
$ ./testTenth
0.100000001490116119384765625
```

Fractional binary notation can only exactly represent numbers that can be written in the form:

$$x \times 2^y$$

IEEE Floating Point

When designing a number format, choices need to be made about the format specification. In the late 1970s, Intel sponsored William Kahan (from Berkeley…) to design a floating point standard, which formed the basis for the "IEEE Standard 754," or *IEEE Floating Point*, which almost all computers use today. The standard defines the bit pattern (32-bit, 64-bit, etc.) as a number in the form:

$$V = (-1)^s \times M \times 2^E$$

Where:

- The *sign s* is negative (s == 1) or positive (s == 0), with the sign for numerical value 0 as a special case.
- The *significand M* (sometimes called the *Mantissa*), is a fractional binary number that ranges *either* between 1 and 2-ϵ or between 0 and 1-ϵ.
- The *exponent E* weights the value by a (possibly negative) power of 2.

IEEE Floating Point Examples

$$V = (-1)^s \times M \times 2^E$$

Example: For s=0, M=1.5, E=9: $V = (-1)^0 \times 1.5 \times 2^9 = 768$

IEEE Floating Point

$$V = (-1)^s \times M \times 2^E$$

The bit representation of a floating point number is divided into three fields to encode these values:

- The single sign bit *s* directly encodes the sign *s*.
- The *k*-bit exponent field, $\mathrm{exp} = e_{k-1} \ldots e_1 e_0$ encodes the exponent *E*.
- The *n*-bit fraction field $\mathrm{frac} = f_{n-1} \ldots f_1 f_0$ encodes the significand *M*, but the value encoded also depends on whether or not the exponent field equals 0.

	31	30 — 23	22 — 0
Single precision (`float`)	s	exp	frac

Before We Continue

	31	30 … 23	22 … 0
Single precision (`float`)	s	exp	frac

Right now, you're saying to yourself, "Uhh…this is going to be complicated."

Before We Continue

Single precision (`float`)

31	30 … 23	22 … 0
s	exp	frac

Right now, you're saying to yourself, "Uhh…this is going to be complicated."

Yes, it does take some time to learn. We want you to appreciate a few things about the IEEE floating point format:

1. It is based on decisions and choices that were made, with good reason (we will discuss those reasons).
2. It is efficient, and attempts to eek out as much as it can from those 32 or 64 bits — computing is often about efficiency, and the people who came up with the standard really thought hard about it.
3. We don't want you to think "I could never come up with that!" — rather, we want you to appreciate it for what it is.

Normalized Floats

	31	30 – 23	22 – 0
Single precision (`float`)	s	exp	frac

- A float is considered to have a "normalized" value if the exponent is not all 0s and it is not all 1s, and is the most common case (e.g., bits 23-30 are not the value 0 or the value 255).
- The *exponent* is a *signed integer* in ***biased*** form. The exponent has a value exp - *bias*, where the "bias" is $2^{k-1} - 1$, and where *k* is the number of bits in the exponent (k=8 for floats, meaning that the bias is $2^7 - 1 = 127$). For floats, the exponent range is -126 to +127.
- The *fraction* is interpreted as having a fractional value *f*, where $0 \leq f < 1$, and having a binary representation of $0.f_{n-1} \cdots f_1 f_0$, with the binary point to the left of the most significant bit.
- The significand is defined to be $M = 1 + f$. This is **an implied leading 1 representation**, and a trick for getting an additional digit for free!

An extra bit of precision for free?

Single precision (`float`)

31	30 … 23	22 … 0
s	exponent (8 bits)	fraction (23 bits)

- Yes! We can always adjust the exponent so that the significand is in the range $1 \leq M < 2$ (assuming no overflow), so we don't need to explicitly represent the leading bit, because it is always 1 (very cool!)
- Remember, the designers of IEEE Floating Point wanted the best system, and this is a cool idea.

Example:

31	30 … 23	22 … 0
0	01111110	00000000000000000000000

- Sign: 0 (positive)
- Exponent: 01111110 = 126 (biased), so exponent of 2 will be 126 - 127 = -1
- Fraction: 0, which is assumed to be 1.0 (binary), which is 1.0 decimal
- Therefore, this number represents $+1.0 \times 2^{-1} = 0.5$ **(to the converter!)**

Let's try some more...

	31	30 ... 23	22 ... 0
Single precision (`float`)	s	exponent (8 bits)	fraction (23 bits)

- Example:

31	30 ... 23	22 ... 0
0	10000100	01010000000000000000000

- Sign: 0 (positive)
- Exponent: 10000100 = 132 (biased), so exponent of 2 will be 132 - 127 = 5
- Fraction: 0101, which is assumed to be 1.0101 (binary), which is:

$$1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} = 1.3125$$

- Therefore, this number represents +1.3125 x 2^5 = 42.0 **(to the converter!)**

https://www.h-schmidt.net/FloatConverter/IEEE754.html

http://web.stanford.edu/class/cs107/float/convert.html

Let's try some more...

	31	30 23	22 0
Single precision (`float`)	s	exponent (8 bits)	fraction (23 bits)

- Example:

31	30 23	22 0
0	10000100	01011000000000000000000

- Sign: 0 (positive)
- Exponent: 10000100 = 132 (biased), so exponent of 2 will be 132 - 127 = 5
- Fraction: 01011, which is assumed to be 1.01011 (binary), which is:

$$1 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} = 1.34375$$

- Therefore, this number represents +1.34375 x 2^5 = 43.0 **(to the converter!)**

Let's try some more...

	31	30 … 23	22 … 0
Single precision (`float`)	s	exponent (8 bits)	fraction (23 bits)

- Example:

31	30 … 23	22 … 0
1	10000101	10010000000000000000000

- Sign: 1 (negative)
- Exponent: 10000101 = 133 (biased), so exponent of 2 will be 133 - 127 = 6
- Fraction: 1001, which is assumed to be 1.1001 (binary), which is:

$$1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} = 1.5625$$

- Therefore, this number represents -1.5625 x 2^6 = -100.0 **<u>(to the converter!)</u>**

Let's try some more...

31	30 … 23	22 … 0
s	exponent (8 bits)	fraction (23 bits)

Single precision (`float`)

- Example:

31	30 … 23	22 … 0
0	01111010	1001000000000000000000

- Sign: 0 (positive)
- Exponent: 01111010 = 122 (biased), so exponent of 2 will be 122 - 127 = -5
- Fraction: 1001, which is assumed to be 1.1001 (binary), which is:

$$1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 0 \times 2^{-3} + 1 \times 2^{-4} = 1.5625$$

- Therefore, this number represents +1.5625 x 2^{-5} = 0.048828125

(to the converter!)

3 minute break

When is a floating point value an integer?

	31	30 … 23	22 … 0
Single precision (`float`)	s	exponent (8 bits)	fraction (23 bits)

- Before we tackle that question, let's ask another: what is the exponent really doing to the significand?

- Remember:

$$V = (-1)^s \times M \times 2^E$$

- We are multiplying the significand (M) by a power of two…in other words, we are shifting it.
- So…if the un-biased exponent shifts the significand enough bits so that none of the fractional bits are still fractions, then we have an integer.

When is a floating point value an integer?

	31	30 … 23	22 … 0
Single precision (`float`)	s	exponent (8 bits)	fraction (23 bits)

- Example: Let's convert 1234567 decimal to floating point
- We need to divide by a power of two such that we get a fraction that is between 1 and 2.
- First, let's convert 1234567 to binary for the significand:

  100101101011010000111 (use this website for the conversion)
- Remove the leading 1 (it's free!), and add zeros at the end to get to 23 bits:

31	30 … 23	22 … 0
		00101101011010000111000

- Now determine the amount we need to shift **left** to get the binary representation in the form of 1.xxxx (in this case, 20), and add the bias: 20 + 127 = 147, and convert to binary (also, add the sign bit, 0):

31	30 … 23	22 … 0
0	10010011	00101101011010000111000

Denormalized Floats

Single precision (`float`)

31	30 – 23	22 – 0
s	00000000	fraction (23 bits)

- When the exponent is all zeros, this is called "denormalized" form. We interpret the exponent differently: the exponent is now 1-Bias (or, you can think of the bias now being 1 less, or 126 instead of 127 in the case of 32-bit floats). The significand value is simply the fraction, without a leading 1.
- Why do we do this?
  - We now have a way to represent zero (all 0s). Technically, all 0s is +0.0, and a 1 followed by all 0s is -0.0.
  - There is "gradual underflow," meaning that it allows us to extend the lower range of representable numbers, and to limit the amount of error with very small numbers. See here for more information than you may ever want: https://docs.oracle.com/cd/E19957-01/816-2464/ncg_math.html

Exceptional Floating Point Values

	31	30 ... 23	22 ... 0
Single precision (`float`)	s	11111111	fraction (23 bits)

- When the exponent is all ones, this is called "exceptional" form. These numbers are not real numbers in the sense that we can calculate with them (except in very certain circumstances).
- Exceptional numbers can denote the infinities:
  - `0 11111111 00000000000000000000000` is +infinity
  - `1 11111111 00000000000000000000000` is -infinity
- Exceptional numbers also define the "NaN" (Not a Number) numbers, which can have special purposes, but are largely ignored (and there are millions of them!)
- You can generate exceptional numbers in various ways:
  - The divisions 0/0 and $\pm\infty/\pm\infty$
  - The multiplications $0\times\pm\infty$ and $\pm\infty\times 0$.
  - The additions $\infty + (-\infty)$, $(-\infty) + \infty$ and equivalent subtractions.
  - The square root of a negative number.
- See https://en.wikipedia.org/wiki/NaN#Operations_generating_NaN for more details.

Arithmetic with Floating Point Numbers

On the first day of class, we looked at the following program:

```
#include<stdio.h>
#include<stdlib.h>

int main() {
    float a = 3.14;
    float b = 1e20;

    printf("(3.14 + 1e20) - 1e20 = %g\n", (a + b) - b);
    printf("3.14 + (1e20 - 1e20) = %g\n", a + (b - b));

    return 0;
}
```

```
$ ./floatMultTest
(3.14 + 1e20) - 1e20 = 0.000000
3.14 + (1e20 - 1e20) = 3.140000
```

we now have the tools to see why this happens!

Arithmetic with Floating Point Numbers

You might be thinking: oh, this is just overflowing. But it is more subtle than that.

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %f\n", (a + b) - b);
printf("3.14 + (1e20 - 1e20) = %f\n", a + (b - b));
```

Let's look at the binary representations for 3.14 and 1e20:

	31	30 23	22 0
3.14:	0	10000000	10010001111010111000011

	31	30 23	22 0
1e20:	0	11000001	01011010111100011101100

How are we going to add these numbers?

Arithmetic with Floating Point Numbers

	31	30 – 23	22 – 0
3.14:	0	10000000	10010001111010111000011

	31	30 – 23	22 – 0
1e20:	0	11000001	01011010111100011101100

You cannot simply add the two significands together, you have to align their binary points. If we wanted to add the decimal values, it would look like this:

```
                        3.14
+  100000000000000000000.00
----------------------------
   100000000000000000003.14
```

Let's see what this looks like in 32-bit IEEE format…

Arithmetic with Floating Point Numbers

Let's see what this looks like in 32-bit IEEE format…

`100000000000000000003.14`

First: Convert to proper binary (http://web.stanford.edu/class/cs107/float/convert.html):

`1010110101111000111010111100010110101100011000100000000000000000011.001000111101011100001010001111010111…`

Second: Find the most significant 1 and take the next 23 digits after the 1 (we get the 1 for free!). We round up if the rest of the number contributes more than half (0.1b is 1/2):
`1` **`01011010111100011101011`** `1100`. (we round up to:
`01011010111100011101100`. This is the significand.

Third: Count how many places we need to shift **left** to put the number in 1.xxx format. In this case it is 66. We add 127 to this number, which gives us 127 + 66 = 193, which is our exponent (binary: **`11000001`**)

Fourth: if the sign is positive, the sign bit will be **`0`**, otherwise `1`.

Arithmetic with Floating Point Numbers

So, we are left with the following for 100000000000000000003.14 decimal:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

Let's compare this to 1e20 that we had before:

31	30 … 23	22 … 0
0	11000001	01011010111100011101100

Identical! We didn't have enough bits to differentiate between 1e20 and 100000000000000000003.14

Arithmetic with Floating Point Numbers

Back to our original example:

```
float a = 3.14;
float b = 1e20;
printf("(3.14 + 1e20) - 1e20 = %f\n", (a + b) - b);
printf("3.14 + (1e20 - 1e20) = %f\n", a + (b - b));
```

```
$ ./floatMultTest
(3.14 + 1e20) - 1e20 = 0.000000
3.14 + (1e20 - 1e20) = 3.140000
```

Clearly, `1e20 - 1e20` will produce `0` (no need to shift the binary points). What this really means is that **floating point arithmetic is not associative**. In other words, the order of operations matters.

Arithmetic with Floating Point Numbers

Here is another example:

```
int main()
{
    double a = 0.1;
    double b = 0.2;
    double c = 0.3;
    double d = a + b;
    printf("0.1 + 0.2 == 0.3 ? %s\n", a + b == c ? "true" : "false");
    return 0;
}
```

```
$ ./floatEquality
0.1 + 0.2 == 0.3 ? false
```

The rounding that happens during the calculation of 0.1 + 0.2 produces a different number than 0.3!

Arithmetic with Floating Point Numbers

```
int main()
{
    double a = 0.1;
    double b = 0.2;
    double c = 0.3;
    double d = a + b;
    printf("0.1 + 0.2 == 0.3 ? %s\n", a + b == c ? "true" : "false");
    printf("0.1:\t%.50g\n",a);
    printf("0.2:\t%.50g\n",b);
    printf("0.3:\t%.50g\n",c);
    printf("a + b:\t%.50g\n",d);
    return 0;
}
```

```
$ ./floatEquality
0.1 + 0.2 == 0.3 ? false
0.1:    0.1000000000000000055511151231257827021181583404541
0.2:    0.200000000000000011102230246251565404236316680908 2
0.3:    0.2999999999999999888977697537484345957636833190918
a + b:  0.30000000000000004440892098500626161694526672363281
```

See extra slide for gdb run.

Arithmetic with Floating Point Numbers

Here is another example:

```
#include<stdio.h>
#include<stdlib.h>
int main()
{
    printf("16777224.0f == 16777225.0f ? %s\n",
            16777224.0f == 16777225.0f ? "true" : "false");
    return 0;
}
```

Arithmetic with Floating Point Numbers

Here is another example:

```
#include<stdio.h>
#include<stdlib.h>
int main()
{
    printf("16777224.0f == 16777225.0f ? %s\n",
           16777224.0f == 16777225.0f ? "true" : "false");
    return 0;
}
```

```
$ ./floatEquality2
16777224.0f == 16777225.0f ? true
```

It turns out that `16777225` is an integer that you cannot represent as a 32-bit float.

Floating Point Takeaways

- The IEEE Floating Point Standard was a carefully thought-out way to get the most out of a discrete set of bits. It may not be simple, but it is a great study in good engineering design.
- Floating point numbers represent a very large range, in a limited number of bits. A 32-bit float can only hold a bit over 4 billion numbers and has a range of **-3.4E+38 to +3.4E+38**. Not only is this literally an infinite number of reals that the format must try and represent, but that is a phenomenal range of numbers. The 64-bit double range is **-1.7E+308 to +1.7E+308** (!)
- Most numbers are, therefore, only represented approximately in float format, including many integers. Example:
  - 1 trillion = 1,000,000,000,000, and in 32-bit floating point, it is actually represented by the value 999,999,995,904, off by 4096!
  - You almost certainly *don't* want to use floats for currency!

Floating Point Takeaways

- You have to be very careful with your arithmetic when you are dealing with floats:
  - Associativity does not hold for numbers far apart in the range.
  - Many numbers are not exact (e.g., 0.1, 0.4, etc.)
  - Equality comparison operations are often unwise.
  -

References and Advanced Reading

- References:
  - IEEE 754: https://en.wikipedia.org/wiki/IEEE_754
  - IEEE Floating point: http://steve.hollasch.net/cgindex/coding/ieeefloat.html
  - Floating point arithmetic: https://en.wikipedia.org/wiki/Floating-point_arithmetic#Dealing_with_exceptional_cases
- Advanced Reading:
  - Comparing floats using equality: https://stackoverflow.com/questions/1088216/whats-wrong-with-using-to-compare-floats-in-java
  - Floating point converter: https://www.h-schmidt.net/FloatConverter/IEEE754.html
  - What Every Computer Scientist Should Know About Floating-Point Arithmetic
  - Floating point rounding errors: https://softwareengineering.stackexchange.com/questions/101163/what-causes-floating-point-rounding-errors
  - Why do we have a bias in floating point exponents?

Extra Slides

gdb run for 0.1 + 0.2 != 0.3

```
$ gdb floatEquality
The target architecture is assumed to be i386:x86-64
Reading symbols from floatEquality...done.
(gdb) break main
Breakpoint 1 at 0x400535: file floatEquality.c, line 5.
(gdb) run
Starting program: /afs/ir.stanford.edu/class/cs107/samples/lect10/floatEquality

Breakpoint 1, main () at floatEquality.c:5
5       double a = 0.1;
(gdb) n
6       double b = 0.2;
(gdb)
7       double c = 0.3;
(gdb)
8       double d = a + b;
(gdb)
9       printf("0.1 + 0.2 == 0.3 ? %s\n", a + b == c ? "true" : "false");
(gdb) x/gt &c
0x7fffffffe9e0: 0011111111010011001100110011001100110011001100110011001100110011
(gdb) x/gt &d
0x7fffffffe9e8: 0011111111010011001100110011001100110011001100110011001100110100
(gdb)
```