Approximation Final Work

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Contents

1 Theoretical Question

1.1 Background

Definition 1. *Let f be a continuous function on the interval $[a,b]$, and let p be a polynomial approximation. An alternating set on f, p is defined to be a sequence of points $x_0, \ldots, x_{n-1}$ such that:*

- $a \leq x_0 < \ldots < x_{n-1} \leq b$.
- $f(x_i) - p(x_i) = (-1)^i E$ *for* $i = 0, 1, \ldots, n-1$.

where either $E = ||f-p||$ or $E = -||f-p||$. The number n is the length of the alternating set.

Theorem 1.1. *Let $f \in C[a.b]$, and suppose that $p = p_n^*$ is the best approximation of f out of $P_n[x]$, that is*

$$||f-p||_\infty \leq ||f-q||_\infty \text{ for all } q \in P_n[x] \tag{1}$$

Then, there is an alternating set for $f-p$ consisting of at least $n+2$ points.

1.2 Question

prove that the best approximation polynomial is unique using Theorem (1.1).

Proof. The theorem is trivial if f is polynomial of degree $\leq n$, so we assume that is not true. to show the uniqueness, suppose that both p_n, q_n are polynomials of best approximation, and we will show that they are equal. Note that $\frac{(p_n+q_n)}{2}$ is also a polynomial of best approximation because:

$$\left\| f - \frac{(p_n+q_n)}{2} \right\| = \left\| \frac{(f-p_n)}{2} \frac{(f-q_n)}{2} \right\| \leq \frac{1}{2} \|f-p_n\| + \frac{1}{2} \|f-q_n\| = \pm E$$

therefore, there are $n+2$ points at which:

$$\frac{f-p_n}{2} + \frac{f-q_n}{2} = \pm E$$

At each of those alternating points both $f-p_n$ and $f-q_n$ are equal to $\pm E$ then we can say that at each of the $n+2$ points:

$$(f(x_i) - p_n(x_i)) - (f(x_i) - q_n(x_i)) = 0 \text{ for each of the alternating point, } i = 1..n+2$$

Since both p_n, q_n are polynomials of degree$\leq$n then they must be identical, therefore p_n is unique. $\square$

2 Practical question

2.1 Background

Definition 2. *the **Chebyshev center**[2] of a bounded set $Q \subseteq X$ with non-empty interior is the center of the smallest ball that encloses the entire of the set Q,and the radius is:*

$$r = \inf\{\sup\{\|x-y\| : x \in Q\} : y \in X\}$$

The Chebyshev center is described as the solution to the following optimization problem:

$$\begin{aligned} \min_{\hat{x},r} \quad & r \\ \text{s.t.} \quad & \|\hat{x}-x\| \leq r \text{ for all } x \in Q \\ & r \geq 0 \end{aligned} \tag{2}$$

The optimization problem states that the radius r is the target function we would like to minimize is subjected to two types of constraints:

- All the points of $Q$are inside.
- r needs to be non-negative, as it is the radius.

Those it could also be described as the solution to the min-max problem.

$$\operatorname*{argmin}_{\hat{x}} \max_{x \in Q} \|x-\hat{x}\| \tag{3}$$

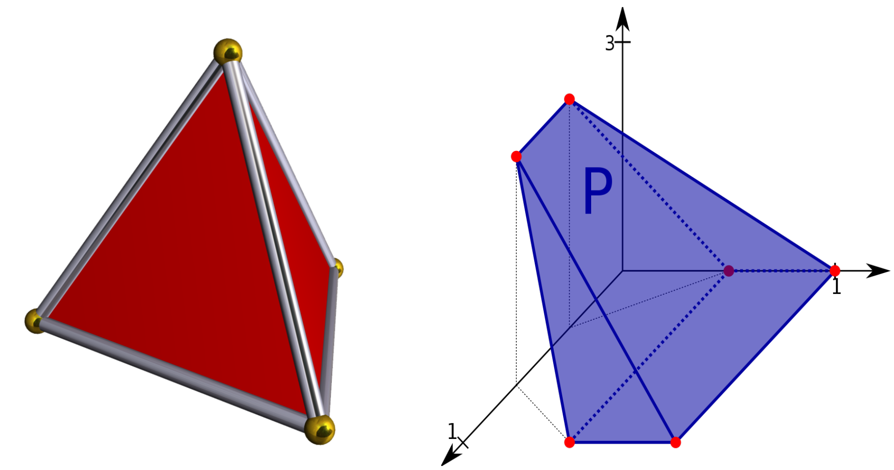
which translate to the minimization of the distance of the furthest point from the center. Both of the optimization problems correspond to the first definition of Chebyshev centers.
Note: other books and papers[1] define the Chebyshev centers differently which is not equivalent to the definition (1).

Definition 3. *The **Chebyshev center** of a bounded set $Q \subseteq X$ with non-empty interior is the center of the largest ball inscribed in Q, and the center is:*

$$\hat{x} = \operatorname*{argmax}_{x \in X} d(x,Q) = \operatorname*{argmax}_{x \in X} \min_{y \in Q} \|\hat{x}-x\| \tag{4}$$

The difference between the two definitions (1) and (3) is that definition (1) denote the center that encloses the set Q like in figure (???). However the second definition (3) states that the ball should be enclosed by the set Q, like in figure (???) and not the other way around.

Chebyshev center as a Linear programming problem
In contrast to the optimization problems above (2, 3) the definition used in order to contrast the linear problem is (3). In order to define the problem the set Q must be represented as an intersection of finitely many half-spaces. The intersection, if non-empty, creates a convex polytope, which is a geometrical object with flat faces created by the intersection of half-spaces as can be seen in figure 1.



(a) tetrahedron (b) 3-dimensional convex polytope

Figure 1: Two examples for convex polytopes.

References

[1] Dan Amir. *Best simultaneous approximation (Chebyshev centers)*. Vol. 11. International Series of Numerical Mathematics. Birkhäuser Basel, 1984. Chap. 2, pp. 19–35. ISBN: 9783034862554.

[2] Lieven Vandenberghe Stephen Boyd. *Convex Optimization*. Cambridge University Press, 2004. Chap. 8, pp. 416–418. ISBN: 978-0-521-83378-3. URL: https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf#page=430.