Approximation Final Work

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1 Theoretical Question

1.1 Background

Definition 1. *Let f be a continuous function on the interval $[a,b]$, and let p be a polynomial approximation. An alternating set on f, p is defined to be a sequence of points $x_0, \ldots, x_{n-1}$ such that:*

- $a \leq x_0 < \ldots < x_{n-1} \leq b$.
- $f(x_i) - p(x_i) = (-1)^i E$ *for* $i = 0, 1, \ldots, n-1$.

where either $E = ||f-p||$ or $E = -||f-p||$. The number n is the length of the alternating set.

Theorem 1.1. *Let $f \in C[a.b]$, and suppose that $p = p_n^*$ is the best approximation of f out of $P_n[x]$, that is*

$$||f-p||_\infty \leq ||f-q||_\infty \text{ for all } q \in P_n[x] \tag{1}$$

Then, there is an alternating set for $f-p$ consisting of at least $n+2$ points.

1.2 Question

prove that the best approximation polynomial is unique using Theorem 1.1.

Proof. The theorem is trivial if f is polynomial of degree $\leq n$, so we assume that is not true. to show the uniqueness, suppose that both p_n, q_n are polynomials of best approximation, and we will show that they are equal. Note that $\frac{(p_n+q_n)}{2}$ is also a polynomial of best approximation because:

$$\left\| f - \frac{(p_n+q_n)}{2} \right\| = \left\| \frac{(f-p_n)}{2} \frac{(f-q_n)}{2} \right\| \leq \frac{1}{2} \|f-p_n\| + \frac{1}{2} \|f-q_n\| = E$$

□