# Approximation Theory 201020
# Final Projects

Aviv Gibali

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## 1 Instructions

**This final work contains two parts, theoretical and practical. each student must submit one exercise from each part and be ready to defend it.**

## 2 Theory

1. Let $f \in C[0,1]$, the Bernstein polynomials for $x \in C[0,1]$ are

$$\left(B_n(f)\right)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k \left(1-x\right)^k. \tag{2.1}$$

   Let $f_0(x) = 1$, $f_1(x) = x$ and $f_2(x) = x^2$. Prove that $B_n(f_0) = f_0$, $B_n(f_1) = f_1$ and

$$B_n(f_2) = \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1. \tag{2.2}$$

2. We saw in class the following theorem:

   **Theorem 2.1** *Let $f \in C[a,b]$, and suppose that $p = p_n^*$ is a best approximation to $f$ out of $P_n[x]$, that is*

$$\|f - p\|_\infty \leq \|f - q\|_\infty \text{ for all } q \in P_n[x]. \tag{2.3}$$

*Then, there is an alternating set for $f - p$ consisting of at least $n + 2$ points.*

Using this theorem show that the polynomial of best approximation to $f$ out of $P_n[x]$ is unique.

3. Using again the theorem from above show that for $f \in C[a, b]$, and $q \in P_n[x]$; If $f - p$ has an alternating set containing $n + 2$ (or more) points, then $p$ is the best approximation to $f$ out of $P_n[x]$.

4. Prove that $y = x - 1/8$ is the best linear approximation to $y = x^2$ in $[0, 1]$.

5. Let $T_n(x)$ be the $n - th$ Chebyshev polynomial. Prove that

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \text{ for all } n \geq 2. \tag{2.4}$$

6. Prove the convergence of Cimmino algorithm (also referred as *simultaneous projections method*) for a system of linear equations.

7. Provide an alternative proof to the ones given in class for the Weierstrass Approximation Theorem.

8. Prove the convergence of the Remez exchange algorithm.

9. Recall that for a given positive, Riemann integrable weight function $w(x)$ on $[a, b]$, the expression

$$\langle f, g \rangle := \int_a^b f(x)g(x)w(x)dx \tag{2.5}$$

defines an inner product on $C[a, b]$ and

$$\|f\|_2 := \sqrt{\int_a^b f^2(x)w(x)dx} \tag{2.6}$$

defines a (strictly convex) norm on $C[a, b]$. Prove that the Chebyshev polynomials are mutually orthogonal relative to the weight $w(x) = (1 - x^2)^{-1/2}$ on $[-1, 1]$, that is for $m \neq n \in \mathbb{N}$

$$\int_{-1}^1 T_n(x)T_m(x)w(x)dx = 0. \tag{2.7}$$

Calculate also $\|T_n\|_2$.

# 3 Practice

1. **Data fitting - COVID-19.** In the file "COVID-19-geographic-disbtribution-worldwide" (taken from the EU open data portal link) you can find the COVID-19 cases worldwide up-to-date. Choose one or several countries, plot their data as well as use any of the tools taught in Zoom for having plotting a continues function "close" to the data (best approximation, Padé approximation,...).

2. **Image Processing.** Given a matrix $K$ (usually called kernel) and a digital image $I$, the convolution $K$ and $I$, denoted by $K{*}I$ yields and output image. There are many kinds of kernels designed for different purposes such as edge detection, deblurring, de-noising and more. For more details you can check the recent paper presented in class: link.

   In this exercise you are asked to:

   (a) Rescale a given image using some kind of interpolation.

   (b) Write a Matlab code that receive a gray-scale image (or RGB transformed to gray-scaled). Now find a matrix $B$ such that:

   $$B * I = K * I. \tag{3.1}$$

   Write your own codes for the MATLAB codes "rgb2gray" and "conv2"). When this is achieved, apply **ANY** iterative method to reconstruct the original image.

3. **Information Retrieval.** Term-document matrices (link) are very large, on the other hand the number of topics that people talk about is small (in some sense, clothes, movies, politics, ...). The question is if we can we represent the term-document space by a lower dimensional latent space? This is just one of the applications of Low-Rank Matrix Approximation (link). For solving this problem you are asked to write a Matlab code for the Alternating projections algorithm. For further information read "Information Retrieval.ppt".

4. **Computed Tomography.** Use link to write a Matlab code for reconstruction from projections, either filtered back projection (FBP) or Algebraic Reconstruction Technique (ART, Kaczmarz). The Matlab commands "phantom" "radon" and "iradon" could be helpful. You can write a code to generate the sinogram.

5. **Signal Processing.** The Parks–McClellan algorithm (link and also link) is a variation of the Remez exchange algorithm which aims for finding the optimal Chebyshev finite impulse response (FIR) filter (optimal filter coefficients). The algorithm design filters with an optimal fit between the desired and actual frequency responses. The filters are optimal in the sense that the maximum error between the desired frequency response and the actual frequency response is minimized. In this exercise you are requested to write your own "firpm" Matlab function for this purpose.

6. **More practical problems.** Please check the additional file with more practical problems.