Approximation Final Work

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1 Theoretical Question

1.1 Background

Definition 1. *Let f be a continuous function on the interval $[a,b]$, and let p be a polynomial approximation. An alternating set on f,p is defined to be a sequence of points $x_0,...,x_{n-1}$ such that:*

- $a \leq x_0 < ... < x_{n-1} \leq b$.
- $f(x_i) - p(x_i) = (-1)^i E$ *for* $i = 0,1,...,n-1$.

where either $E = ||f-p||$ or $E = -||f-p||$. The number n is the length of the alternating set.

Theorem 1.1. *Let $f \in C[a.b]$, and suppose that $p = p_n^*$ is the best approximation of f out of $P_n[x]$, that is*

$$||f-p||_\infty \leq ||f-q||_\infty \textit{ for all } q \in P_n[x] \tag{1}$$

Then, there is an alternating set for $f-p$ consisting of at least $n+2$ points.

1.2 Question

prove that the best approximation polynomial is unique using Theorem (1.1).

Proof. The theorem is trivial if f is polynomial of degree $\leq n$, so we assume that is not true. to show the uniqueness, suppose that both p_n, q_n are polynomials of best approximation, and we will show that they are equal. Note that $\frac{(p_n+q_n)}{2}$ is also a polynomial of best approximation because:

$$\left\| f - \frac{(p_n+q_n)}{2} \right\| = \left\| \frac{(f-p_n)}{2} \frac{(f-q_n)}{2} \right\| \leq \frac{1}{2} \|f-p_n\| + \frac{1}{2} \|f-q_n\| = \pm E$$

therefore, there are $n+2$ points at which:

$$\frac{f-p_n}{2} + \frac{f-q_n}{2} = \pm E$$

At each of those alternating points both $f-p_n$ and $f-q_n$ are equal to $\pm E$ then we can say that at each of the $n+2$ points:

$$(f(x_i) - p_n(x_i)) - (f(x_i) - q_n(x_i)) = 0 \text{ for each of the alternating point, } i = 1..n+2$$

Since both p_n, q_n are polynomials of degree≤n then they must be identical, therefore p_n is unique. □

2 Practical question

2.1 Background

Definition 2. *the **Chebyshev center**[3] of a bounded set $Q \subseteq X$ with non-empty interior is the center of the smallest ball that encloses the entire of the set Q,and the radius is:*

$$r = \inf\{\sup\{\|x - y\| : x \in Q\} : y \in X\}$$

The Chebyshev center is described as the solution to the following optimization problem:

$$\begin{aligned} \min_{\hat{x},r} \quad & r \\ \text{s.t.} \quad & \|\hat{x} - x\| \leq r \text{ for all } x \in Q \\ & r \geq 0 \end{aligned} \tag{2}$$

The optimization problem states that the radius r is the target function we would like to minimize is subjected to two types of constraints:

- All the points of $Q$are inside.
- r needs to be non-negative, as it is the radius.

Those it could also be described as the solution to the min-max problem.

$$\underset{\hat{x}}{\operatorname{argmin}} \max_{x \in Q} \|x - \hat{x}\| \tag{3}$$

which translate to the minimization of the distance of the furthest point from the center. Both of the optimization problems correspond to the first definition of Chebyshev centers.
Note: other books and papers[1] define the Chebyshev centers differently which is not equivalent to the definition (1).

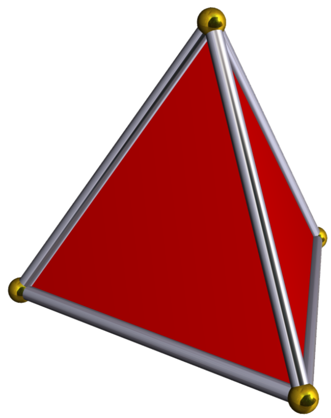
Definition 3. *The **Chebyshev center** of a bounded set $Q \subseteq X$ with non-empty interior is the center of the largest ball inscribed in Q, and the center is:*

$$\hat{x} = \underset{x \in X}{\operatorname{argmax}}\, d(x, Q) = \underset{x \in X}{\operatorname{argmax}} \min_{y \in Q} \|\hat{x} - x\| \tag{4}$$

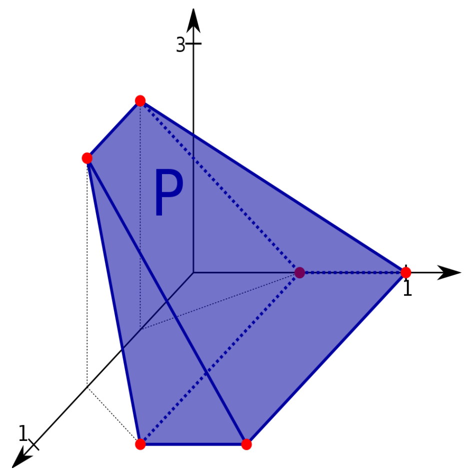
The difference between the two definitions (1) and (3) is that definition (1) denote the center that encloses the set Q like in figure (???). However the second definition (3) states that the ball should be enclosed by the set Q, like in figure (???) and not the other way around.

Chebyshev center as a Linear programming problem

In contrast to the optimization problems above (2, 3) the definition used in order to contrast the linear problem is (3). In order to define the problem the set Q must be represented as an intersection of finitely many half-spaces. The intersection, if non-empty, creates a convex polytope, which is a geometrical object with flat faces created by the intersection of half-spaces as can be seen in figure 1.



(a) tetrahedron



(b) 3-dimensional convex polytope

Figure 1: Two examples for convex polytopes.

The Chebyshev center could be found by solving the following problem:

$$Q = \{x \in \Re^n : Ax \le b\}$$

$$\begin{aligned} \min_{\hat{x},r} \quad & r \\ \text{s.t.} \quad & a_i\hat{x} + \|a_i\|\, r \le b_i, \text{ for every half-space of } Q \\ & r \ge 0 \end{aligned} \tag{5}$$

The optimization's target function is the same as before, but because the second definition (3) is used in this formulation, then instead of minimizing the radius of the ball into the set Q, we maximize the radius of the ball inside of it, while the constants require us that the ball and it's sphere should be inside of the convex polytope.

2.2 Experiments

The conducted Experiments included two steps: randomizing data points in the 2d plane, creating and solving the optimization problem in order to find the Chebyshev center and it's radius. Randomizing the data was relatively simple process, I decided to randomize 20 points in the 2d plane as a proof of concept. The coordinates for each of the points elements was in the range of $[-25, 25]$ in order to spread them in the plane, then I calculated the convex hull of the set of points, in order to create a set that fits both definition (1) and definition (3).

2.3 Experiment 1 - calculating the bounding circle of the convex hull

The experiments took the the set of points (including the boundaries of the convex hull) and solved the problem defined in (2) in order to find the enclosing circle of the set of points.

The non-linear solver

To solve the problem I used the function *fmincon* in matlab, which takes the objective function and the non-linear constraints and solves the problem using an algorithm called "**interior point**". The algorithm solves a sequence of approximated minimization problems that convert the inequalities into equality constraints because they are easier to solve (the exact technique that is used is called barrier function [2]). Then the algorithm uses one of two main types of steps at each iteration:

- A direction step, which attempts to solve the KKT equation via linear approximation, this is also called Newton step.
- A CG (conjugate gradient) step, using trust region.

At first the algorithm attempts to take a direction step, if it cannot The algorithm attempts a CG step.

2.3.1 Experiment 1 - definition 1 on randomized data

asd

References

[1] Dan Amir. *Best simultaneous approximation (Chebyshev centers)*. Vol. 11. International Series of Numerical Mathematics. Birkhäuser Basel, 1984. Chap. 2, pp. 19–35. ISBN: 9783034862554.

[2] J. Byrd R. Gilbert J. & Nocedal. *A Trust Region Method Based on Interior Point Techniques for Nonlinear Programming*. Vol. 89. Mathematical Programming 1. 2000, pp. 149–185.

[3] Lieven Vandenberghe Stephen Boyd. *Convex Optimization*. Cambridge University Press, 2004. Chap. 8, pp. 416–418. ISBN: 978-0-521-83378-3. URL: https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf#page=430.