Approximation Final Work

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Contents

1 Theoretical Question

1.1 Background

Definition 1. *Let f be a continuous function on the interval $[a,b]$, and let p be a polynomial approximation. An alternating set on f,p is defined to be a sequence of points $x_0,\ldots,x_{n-1}$ such that:*

- $a \leq x_0 < \ldots < x_{n-1} \leq b$.
- $f(x_i) - p(x_i) = (-1)^i E$ *for* $i = 0,1,\ldots,n-1$.

where either $E = ||f-p||$ or $E = -||f-p||$. The number n is the length of the alternating set.

Theorem 1.1. *Let $f \in C[a.b]$, and suppose that $p = p_n^*$ is the best approximation of f out of $P_n[x]$, that is*

$$||f-p||_\infty \leq ||f-q||_\infty \text{ for all } q \in P_n[x] \tag{1}$$

Then, there is an alternating set for $f-p$ consisting of at least $n+2$ points.

1.2 Question

prove that the best approximation polynomial is unique using Theorem (1.1).

Proof. The theorem is trivial if f is polynomial of degree $\leq n$, so we assume that is not true. to show the uniqueness, suppose that both p_n, q_n are polynomials of best approximation, and we will show that they are equal. Note that $\frac{(p_n+q_n)}{2}$ is also a polynomial of best approximation because:

$$\left\| f - \frac{(p_n+q_n)}{2} \right\| = \left\| \frac{(f-p_n)}{2} \frac{(f-q_n)}{2} \right\| \leq \frac{1}{2} \|f-p_n\| + \frac{1}{2} \|f-q_n\| = \pm E$$

therefore, there are $n+2$ points at which:

$$\frac{f-p_n}{2} + \frac{f-q_n}{2} = \pm E$$

At each of those alternating points both $f-p_n$ and $f-q_n$ are equal to $\pm E$ then we can say that at each of the $n+2$ points:

$$(f(x_i) - p_n(x_i)) - (f(x_i) - q_n(x_i)) = 0 \text{ for each of the alternating point, } i = 1..n+2$$

Since both p_n, q_n are polynomials of degree$\leq$n then they must be identical, therefore p_n is unique. $\square$

2 Practical question

2.1 Background

Definition 2. *the **Chebyshev center**[1] of a bounded set $Q \subseteq X$ with non-empty interior is the center of the smallest ball that encloses the entire of the set Q,and the radius is:*

$$r = \inf\{\sup\{\|x - y\| : x \in Q\} : y \in X\}$$

The Chebyshev center is described as the solution to the following optimization problem:

$$\begin{aligned} \min_{\hat{x},r} \quad & r \\ \text{s.t.} \quad & \|\hat{x} - x\| \leq r \text{ for all } x \in Q \\ & r \geq 0 \end{aligned} \tag{2}$$

References

[1] Lieven Vandenberghe Stephen Boyd. *Convex Optimization*. Cambridge University Press, 2004. Chap. 8, pp. 416–418. ISBN: 978-0-521-83378-3. URL: https://web.stanford.edu/~boyd/cvxbook/bv_cvxbook.pdf#page=430.