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# Pricing of European Spread Options

## 1 Introduction

Existence of various financial products is one of the features of the modern financial system. Many financial derivatives are actively traded on different exchanges throughout the world. Most of these derivatives are defined as exotic. Exotic options are considered to be the most common exotic products. In finance, an exotic option is an option which has features making it more complex than commonly traded vanilla options. Like the more general exotic derivatives they may have several triggers relating to determination of payoff. An exotic option may also include non-standard underlying instrument, developed for a particular client or for a particular market. Exotic options are more complex than options that trade on an exchange, and are generally traded over the counter (OTC).

A large group of exotic options is related to non-standard specifications of underlying assets. One of them is spread option whose payoff is based on the difference in price between two underlying assets. Spread options are generally traded over the counter, rather than on exchange.

In this paper we study the main features of the European spread call and put options and investigate the methods of pricing them. This report is organized in the following way. In the second section we give a description to the main characteristics of spread options and illustrate different scenarios of payoffs of a spread option depending on the asset price movements. We then analyze different approaches to pricing of spread options and their implementation. Section 4 provides the investigation of sensitivity parameters of spread options with respect to basis option characteristics. Additionally, a modification of standard spread option specification is introduced in the paper. This modification is based on the redefining of spread in prices of underlying assets. We show that most of sensitivity parameters change their functional forms compared to the standard setting. The final section concludes the discussion.

## 2 Description of the product

### 2.1 Characteristics of spread options

Consider a frictionless market with no arbitrage opportunities and with a constant riskless interest rate r. Assume two assets (asset 1 and asset 2) whose prices at the future date $T$ are

$$S_1(T) = F_1 e^{-\frac{1}{2}\sigma_1^2 T + \sigma_1 \sqrt{T}\epsilon_1} \tag{1}$$

$$S_2(T) = F_2 e^{-\frac{1}{2}\sigma_2^2 T + \sigma_2 \sqrt{T}\epsilon_2} \tag{2}$$

with respect to the Equivalent Martingale Measure (EMM), where $F_1$ and $F_2$ are the current forward prices for delivery at the future date $T$, $\sigma_1$ and $\sigma_2$ are volatilities and $\epsilon_1$ and $\epsilon_2$ are standard normal random variables with correlation $\rho$. It follows from above that the two assets prices are lognormal, and that the expected future price for each asset (with respect to the EMM) coincides with the current forward price.

Consider a European Call option on the **price spread** $(S_1(T) - S_2(T))$ and time to exercise $T$. The call option payoff at time $T$ is

$$C(T) = (S_1(T) - S_2(T) - K)^+ \tag{3}$$

where $()^+$ denotes the positive values. The present value of the future payoff can be represented by

$$C = e^{-rT} E_0[(S_1(T) - S_2(T) - K)^+] \tag{4}$$

with expectation taken with respect to the EMM and $r$ is the riskless interest rate.

From the Call-Put Parity, the value of the European Put option on the price spread with strike $K \geq 0$ and time to exercise $T$ is given by

$$P = C - e^{-rT}(F_1 - F_2 - K) = e^{-rT} E_0[(K - S_1(T) - S_2(T))^+] \tag{5}$$

As for American Call Spread option, according to Hull (2015), there is no reason to exercise the option before maturity date, so the result for the American call option is the same as for European call.

At maturity date, if the spread is greater than the strike price, the option holder exercises the option and gains the difference between the spread and the strike price. If the difference is less than 0, the option holder does not exercise the option, and the payoff is 0. Spread options are frequently traded in the energy market. Two possible examples are:

- *Crack spreads:* options on the spread between refined petroleum products and crude oil. The spread represents the refinement margin made by the oil refinery by "cracking" the crude oil into a refined petroleum product.

- *Spark spreads:* options on the spread between electricity and some type of fuel. The spread represents the margin of the power plant, which takes fuel to run its generator to produce electricity.

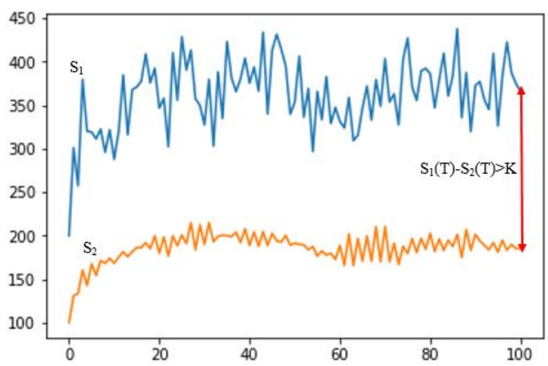
Besides the spread options on two commodities, one may also consider stocks, indices and exchange rates as the underlying of spread options.

As a result, the price of a spread option is a function of the following parameters: the current price of underlying assets ($S_1$ and $S_2$), correlation between them ($\rho$) and their volatilities ($\sigma_1$ and $\sigma_2$), strike ($K$), rates of returns on the assets ($q_1$ and $q_2$, if any), interest rate ($r$), time to maturity ($T$).

## 2.2 Payoff description

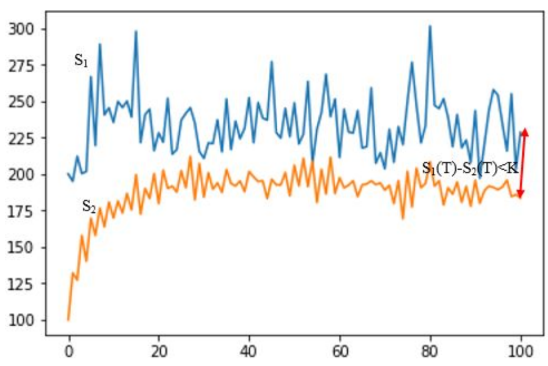
The previous section states the payoff function for both spread call and spread put options. Here we illustrate several possible scenarios based on the evolution of the asset prices.

Case 1: $S_1(T) - S_2(T) > K$



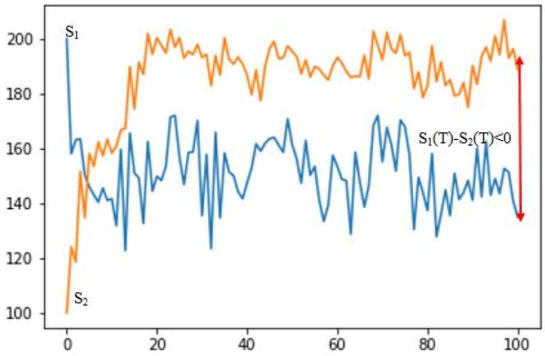

In this case the price of asset 1 at maturity exceeds the one of asset 2 for the amount greater than K (strike). Hence, payoff of a spread call is positive which means that call option is exercised. In contrast, spread put worth nothing in this case.

Case 2: $S_1(T) - S_2(T) < K$




Here the price of asset 1 at maturity exceeds the price of asset 2. However, the difference is less than strike. Hence, payoff of a spread call is zero which means that call option is not exercised. In contrast, spread put has positive value.

Case 3: $S_1(T) - S_2(T) < 0$




Here the price of asset 1 at maturity is smaller than the price of asset 2. Therefore, payoff of a spread call is zero which means that call option is not exercised, but spread put has positive value.

These examples demonstrate possible situations on the market and how they affect the decisions on option exercise.

# 3 Pricing of spread options

## 3.1 Model description

We consider a set of Brownian motions $B_1(t), B_2(t)$ with the correlation parameter $\rho$ on the probability space $(\omega, \mathcal{F}, \mathbb{P})$ endowed with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq T}$.

We look at the classical setting where besides a riskless bank account with constant interest rate r, our arbitrage-free market model comprises two assets whose prices at time $t$ are denoted by $S_1(t)$ and $S_2(t)$. We assume that their risk-neutral price dynamics are given by the following stochastic differential equations:

$$\frac{dS_1(t)}{S_1(t)} = \mu_1(t,T)dt + \sigma_1(t,T)dB_1(t)$$

$$\frac{dS_2(t)}{S_2(t)} = \mu_2(t,T)dt + \sigma_2(t,T)dB_2(t)$$

where the volatilities $\sigma_1$ and $\sigma_2$ are positive constants and $B_1$ and $B_2$ are two Brownian motions with correlation $\rho$.

In any case, the value of an spread option $V$ at time $t$ of the spread option with date of maturity $T$ and strike $K$ is given by the following risk-neutral expectation:

$$\begin{aligned} V(t,T) &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\{(S_1(T) - S_2(T) - K)^+\} \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\{(S_1(t)e^{(\mu_1-\sigma_1^2/2)T+\sigma_1 B_1(T)} - S_2(t)e^{(\mu_2-\sigma_2^2/2)T+\sigma_2 B_2(T)} - K)^+\} \\ &= e^{-rT}\int\int(S_1(t)e^{(\mu_1-\sigma_1^2/2)T+\sigma_1 B_1(T)} - S_2(t)e^{(\mu_2-\sigma_2^2/2)T+\sigma_2 B_2(T)} - K)^+\phi_T(B_1,B_2)dB_1 dB_2 \end{aligned}$$

which shows that the value of the spread options $V(t,T)$ is given by the double integral of a function of two variables with respect to a bivariate Gaussian distribution, namely the joint distribution of $B_1(T)$ and $B_2(T)$.

## 3.2 Pricing methods

There are several methods to price spread options. For instance, the following approaches may be considered: the closed form solution, e.g. Kirk's method (Kirk, 1995), finite differences approach, and Monte Carlo simulations to price spread options.

The advantages and disadvantages of each method are the following:

- Closed form solutions and approximations of partial differential equations (PDE) are advantageous because they are very fast, and extend well to computing sensitivities (namely, Greeks). However, closed form solutions are not always available, for example for American spread options.

- The finite difference method is a numerical procedure to solve PDEs by discretizing the price and time variables into a grid. It can handle cases where closed form solutions are not available. Moreover, finite difference extends well to calculating sensitivities because it outputs a grid of option prices for a range of underlying prices and times. Nevertheless, this approach performs slower than the closed form solutions.
- Monte Carlo simulation uses random sampling to simulate movements of the underlying asset prices. It handles cases where closed solutions do not exist. However, it usually suffers from increasing of computation time which is heavily affected by the precision requirements.

### 3.3 Kirk's approach

The most common theoretical result in the field of spread option analysis is the closed form solution proposed by Kirk (1995). The proposed formula for a European spread call option is presented below. Here we provide an extension of the formula suggested by Kirk by assuming the constant rate of return (e.g. dividends for stocks) on the assets 1 and 2, $q_1$ and $q_2$ respectively.

$$C_K(t, S_1(t), S_2(t), T) = S_1(t)e^{-q_1(T-t)}N(d_1) - (S_2(t)e^{-q_2(T-t)} + Ke^{-r(T-t)})N(d_2) \tag{6}$$

where

$$d_1 = \frac{ln\frac{S_1(t)e^{-q_1(T-t)}}{S_2(t)e^{-q_2(T-t)}+Ke^{-r(T-t)}} + \frac{\sigma_K^2}{2}(T-t)}{\sigma_K\sqrt{T-t}} \tag{7}$$

$$d_2 = \frac{ln\frac{S_1(t)e^{-q_1(T-t)}}{S_2(t)e^{-q_2(T-t)}+Ke^{-r(T-t)}} - \frac{\sigma_K^2}{2}(T-t)}{\sigma_K\sqrt{T-t}} = d_1 - \sigma_K\sqrt{T-t} \tag{8}$$

and

$$\sigma_K^2 = \sigma_1^2 + \sigma_2^2\Big(\frac{S_2(t)e^{-q_2(T-t)}}{S_2(t)e^{-q_2(T-t)} + Ke^{-r(T-t)}}\Big)^2 - 2\sigma_1\sigma_2\rho\frac{S_2(t)e^{-q_2(T-t)}}{S_2(t)e^{-q_2(T-t)} + Ke^{-r(T-t)}} \tag{9}$$

### 3.4 Monte Carlo simulation

Another way to compute an expectation considered here is to use Monte-Carlo approach. The idea is to simulate randomly values of the variables $S_1(t), S_2(t)$ for every $t \in [0, T]$. That generates multiple trajectories or sample paths, and for each of them we compute the value of the function of the path whose expectation we evaluate, and then to average these values over the sample paths. The principle of the method consists of computing the sample average that is quite straightforward, the only difficulty is to quantify and control the error.

If we assume that the coefficients of the stochastic differential equation that characterize the dynamics of the underlying assets are deterministic, then the situation becomes simpler. In such a case the joint distribution of the underlying assets at the terminal time can be computed explicitly, and there is no need to simulate entire sample paths: one can simulate samples from the terminal distribution directly. This distribution is well-known (joint log-normal distribution). The simulation of samples from the underlying asssets values at maturity is very easy. Indeed, if the coefficients are constant, the pair $(S_1(t), S_2(t))$ of prices at maturity can be written in the form:

$$S_1(T) = S_1(t)exp\Big[(\mu_1 - \frac{\sigma_1^2}{2})T + \sigma_1 dB_1(t)\Big] \tag{10}$$

$$S_2(T) = S_2(t)exp\Big[(\mu_2 - \frac{\sigma_2^2}{2})T + \sigma_2(\rho dB_1(t) + \sqrt{1-\rho^2}dB_2(t))\Big] \tag{11}$$

### 3.5 Implementation

Design and Implementation of different option pricing models was held in Python 3.6 package. As it was mentioned above, one can use current prices of underlying assets, correlation between them, rates of return on the assets and their volatilities, interest rate, strike and time to maturity as inputs for both Kirk's and Monte-Carlo methods. Additional parameter for the functions that authors of the paper suggest is type of the option (Call or Put).

Here and after we use the following values for inputs:

| Input | Variable | Value | |
|---|---|---|---|
| Time t | t | 0 | |
| Maturity time | T | 10 | |
| Initial price of asset 1 | S1_t | 150 | |
| Initial price of asset 2 | S2_t | 100 | |
| Strike | K | 50 | |
| Interest rate | r | 0.05 | |
| Dividend rate for asset 1 | q1 | 0.02 | |
| Dividend rate for asset 2 | q2 | 0.01 | |
| Volatility of price for asset 1 | sigma1 | 0.25 | |
| Volatility of price for asset 2 | sigma2 | 0.15 | |
| Correlation between assets | corr | 0.4 | |
| | | Call | Put |
| Type of the option | optType | 1 | -1 |

#### 3.5.1 Kirk's model

As previously mentioned, we consider the extension of Kirk's formula. To implement this algorithm, one may just make computation using the equations (6)-(9) and obtain the result.

For instance, one may run the code from Appendix B using earlier specified values of the variables and obtain the following values of options:

| Option type | Value |
|---|---|
| Call | 35.51 |
| Put | 33.51 |

#### 3.5.2 Monte-Carlo

For numerical optimization by Monte-Carlo method we need to generate multiple trajectories, and each of them might be generated by equations (10)-(11).

For this method an additional parameter should be indicated - number of simulation, which determines the number of different paths generated to obtain an expectations. In order to obtain sufficiently accurate evaluations we suggest the 10 000 simulations.

One may run the code from Appendix A using values of the variables specified earlier and obtain the following values of options:

| Option type | Value |
|---|---|
| Call | 35.82 |
| Put | 33.41 |

The results obtained are in line with the output of Kirk's approach.

# 4 Sensitivity analysis

## 4.1 Delta

The delta ($\Delta$) of an option is defined as the rate of change of its price with respect to the price of the underlying asset. It is the slope of the curve that relates the option price to the underlying asset price. In general,

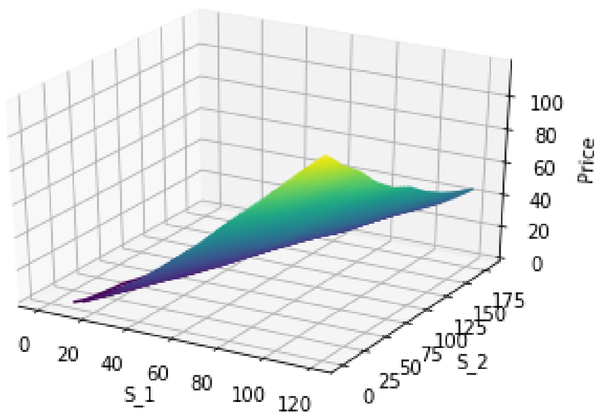
$$\Delta = \frac{\partial Price}{\partial S}$$

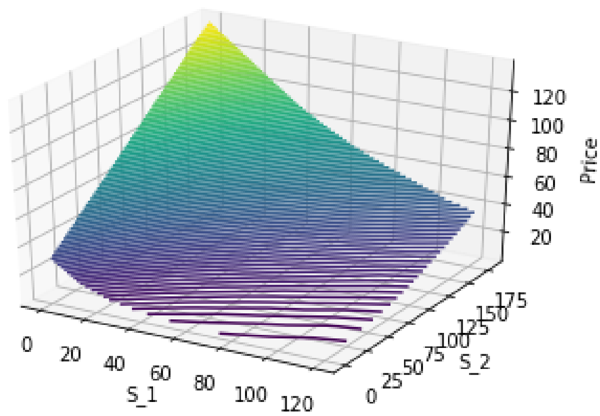
where S is the price of the underlying asset.

For the purposes of numerical computation, the partial derivatives are approximated with the finite differences:

$$\Delta = \frac{\partial Price}{\partial S} \approx \frac{\Delta Price}{\Delta S} = \frac{Price(S + \Delta S) - Price(S)}{\Delta S}$$

In our case we investigate the spread option whose price depends on two underlyings. Therefore it is quite important to study the sensitivity of the option price with respect to changes in prices of the both underlying assets ($S_1$ and $S_2$) as well as in the size of spread ($S_1 - S_2$). Thus, we have the following set of sensitivity parameters: $\Delta_{S_1} = \frac{\partial Price}{\partial S_1}$, $\Delta_{S_2} = \frac{\partial Price}{\partial S_2}$ and $\Delta_{spread} = \frac{\partial Price}{\partial (S_1 - S_2)}$.




| Parameter of Call | Value |
|---|---|
| $\Delta_{S_1}$ | 0.461 |
| $\Delta_{S_2}$ | -0.078 |
| $\Delta_{spread}$ | 0.463 |

| Parameter of Put | Value |
|---|---|
| $\Delta_{S_1}$ | -0.261 |
| $\Delta_{S_2}$ | 0.631 |
| $\Delta_{spread}$ | -0.393 |

The values of the parameters delta revealed the following connection between the prices of underlyings and the option prices:

- the price of a spread call option increases with the price of asset 1, whereas the relationship is opposite for a spread put option.
- an increase of the price of asset 2 lowers the call option price and enlarges the price of the put.
- the value of a spread between the prices of asset 1 and asset has positive impact on the spread call price and, on the contrary, decreases the price of put.

These conclusions are in line with the theoretical ones (e.g. derived from Kirk's formula).

## 4.2 Gamma

The gamma ($\Gamma$) of a portfolio of options on an underlying asset is the rate of change of the portfolio's delta with respect to the price of the underlying asset. It is the second partial derivative of the portfolio with respect to asset price:

$$\Gamma = \frac{\partial^2 Price}{\partial S^2}$$

Again, it is worth mentioning that the partial derivatives are approximated with the finite differences:

$$\Gamma = \frac{\partial^2 Price}{\partial S^2} \approx \frac{\Delta^2 Price}{(\Delta S)^2} = \frac{Price(S + \Delta S) + Price(S - \Delta S) - 2Price(S)}{(\Delta S)^2}$$

We study the second order sensitivity of the option price to changes in prices of the both underlying assets ($S_1$ and $S_2$) as well as in the size of spread ($S_1 - S_2$). Thus, we have the following set of gamma parameters: $\Gamma_{S_1} = \frac{\partial^2 Price}{\partial S_1^2}$, $\Gamma_{S_2} = \frac{\partial^2 Price}{\partial S_2^2}$ and $\Gamma_{spread} = \frac{\partial^2 Price}{\partial (S_1 - S_2)^2}$

| Parameter | Value for Call | Value for Put |
|---|---|---|
| $\Gamma_{S_1}$ | -0.02 | -0.0040 |
| $\Gamma_{S_2}$ | -0.067 | 0.005 |
| $\Gamma_{spread}$ | 0.013 | 0.010 |

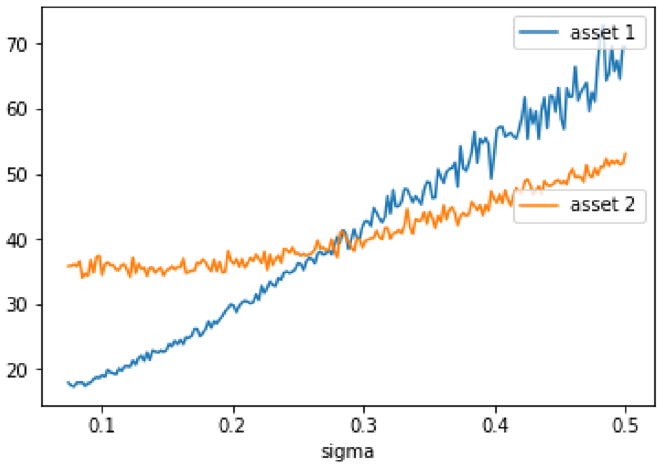
For the previous graphs, the parameter Gamma represents the curvature of the price curve of the option with respect to asset prices.

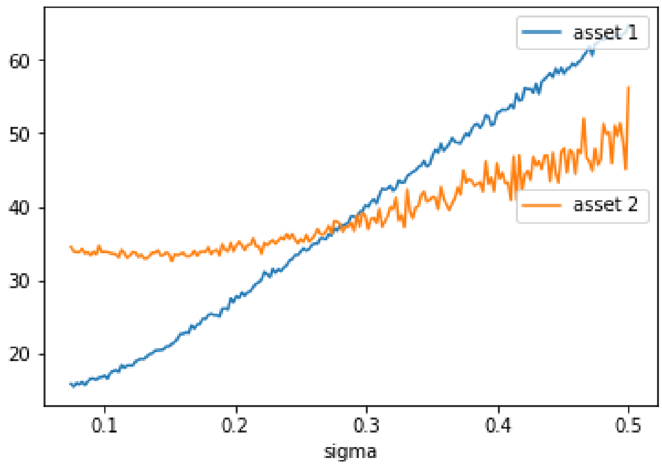
### 4.3 Vega

The vega ($\nu$) of a portfolio of derivatives is the rate of change of the value of the portfolio with respect to the volatility of the underlying asset.

$$\nu = \frac{\partial Price}{\partial \sigma} \approx \frac{\Delta Price}{\Delta \sigma} = \frac{Price(\sigma + \Delta\sigma) - Price(\sigma)}{\Delta \sigma}$$

In this study we analyze the effect of changes in volatilities on the price of a spread option. Thus, we have two following parameters: $\nu_1 = \frac{\partial Price}{\partial \sigma_1}$ and $\nu_2 = \frac{\partial Price}{\partial \sigma_2}$




| Parameter of Call | Value |
|---|---|
| $\nu_1$ | 65.899 |
| $\nu_2$ | 36.545 |

| Parameter of Put | Value |
|---|---|
| $\nu_1$ | 201.598 |
| $\nu_2$ | 26.258 |

Here one may notice that volatility of the price of asset 1 has more severe effect on the price of an option (both call and put) than volatility of the price of asset 2. This conclusion is based on the observation that vega of asset 1 is much larger than the one of asset 2.

The results obtained correspond to the theoretical arguments: volatility of an underlying asset (in our case the underlying assets 1 and 2) directly affects the price of an option meaning that the prices of both call and put options increase as volatility becomes larger.

### 4.4 Kappa

The kappa ($\kappa$) of a derivative is represents the rate of change of its value with respect to the changes in strike value:

$$\kappa = \frac{\partial Price}{\partial K} \approx \frac{\Delta Price}{\Delta K} = \frac{Price(K + \Delta K) - Price(K)}{\Delta K}$$

Here we obtain the following results:

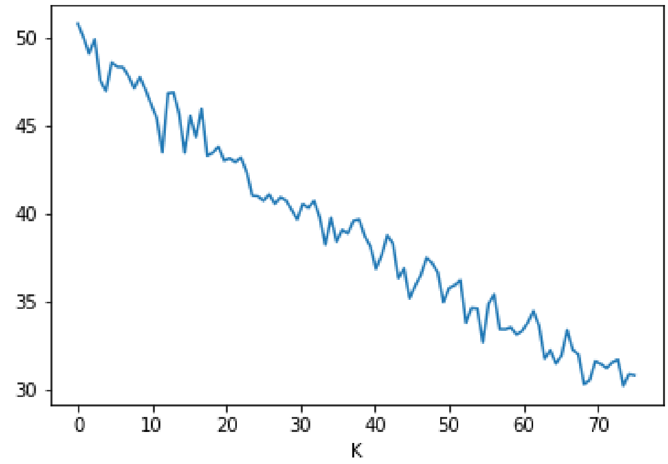
- the price of a spread call option decreases with the value of strike (K)
- the price of a spread put option increases with the value of strike
- this relation is linear

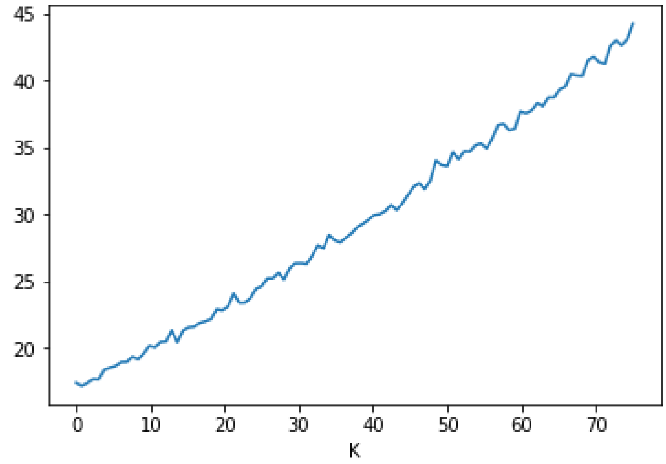
### 4.5 Theta

The theta ($\theta$) of an options is the rate of change of the value of the option with respect to the passage of time with all else remaining the same. Theta is sometimes referred to as the time decay of the portfolio. In general,

$$\theta = \frac{\partial Price}{\partial t} \approx \frac{\Delta Price}{\Delta t} = \frac{Price(t + \Delta t) - Price(t)}{\Delta t}$$

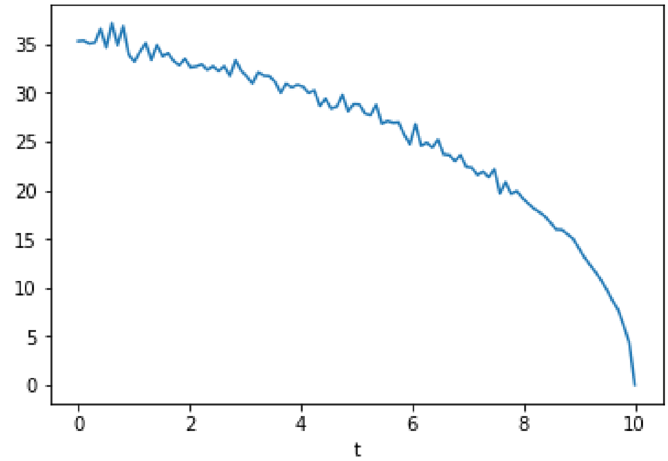
The graphs demonstrate the relation between the prices of both types of spread options and the time to maturity. In both cases the price decreases, meaning that the closer the maturity of an option is, the lower

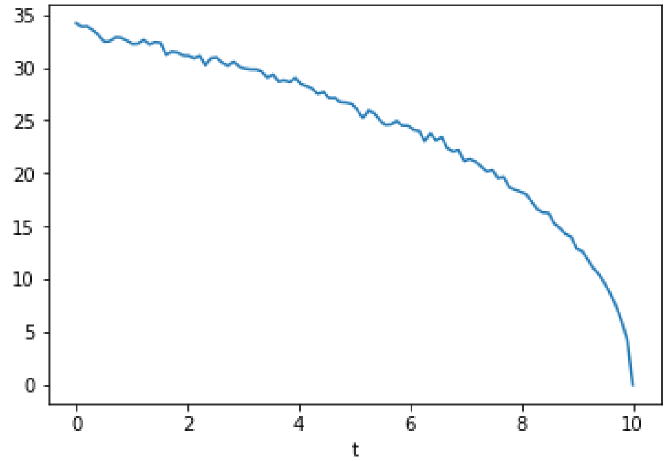



| Parameter of Call | Value |
|---|---|
| $\kappa$ | -2.308 |

| Parameter of Put | Value |
|---|---|
| $\kappa$ | 1.667 |




| Parameter of Call | Value |
|---|---|
| $\theta$ | -0.555 |

| Parameter of Put | Value |
|---|---|
| $\theta$ | -0.687 |

becomes its price which attains 0 exactly at maturity time (since forward price is exactly the spot price at maturity). Moreover, it is worth mentioning that the relationship between the option price and time to maturity is rather non-linear ($\sqrt{\cdot}$-kind of relationship).

### 4.6 Rho

The rho ($\rho$) of a portfolio of options is the rate of change of the value of the portfolio with respect to the interest rate:

$$\rho = \frac{\partial Price}{\partial r} \approx \frac{\Delta Price}{\Delta r} = \frac{Price(r + \Delta r) - Price(r)}{\Delta r}$$

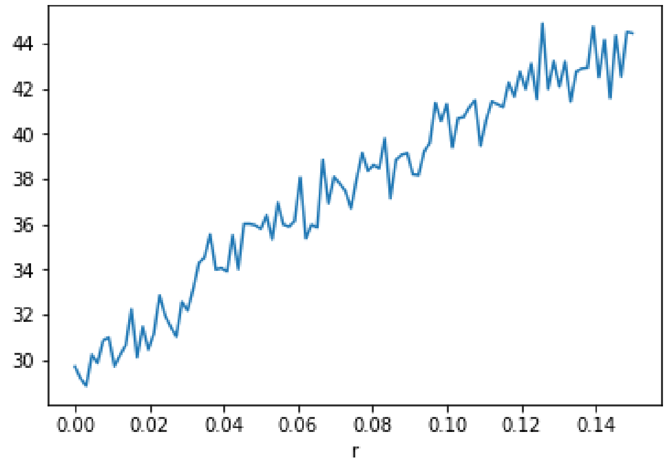
Here we see that the price of a spread call increases with the interest rate, whereas, in contrast, the price of a spread put decreases with this parameter. The relationship is close to linear.

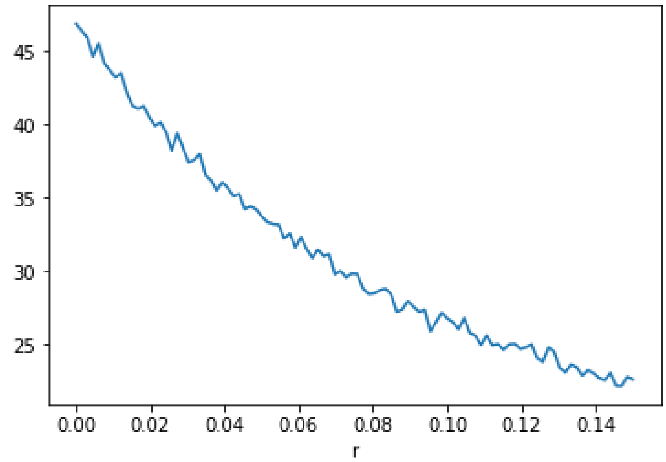
## 5 Modification of the spread option

### 5.1 Intuition

Various modifications of spread options may be suggested. Here we consider a modified spread value specification where the absolute difference in prices is taken into account rather than the amount that price of asset 1 exceeds the one of asset 2. The payoff function is then represented by

$$C(T) = (|S_1(T) - S_2(T)| - K)^+$$




| Parameter of Call | Value |
|---|---|
| $\rho$ | 98.287 |

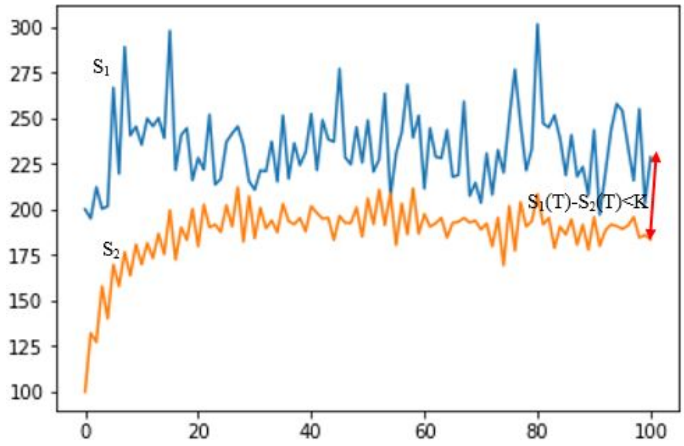
| Parameter of Put | Value |
|---|---|
| $\rho$ | -152.308 |

$$P(T) = (K - |S_1(T) - S_2(T)|)^+$$

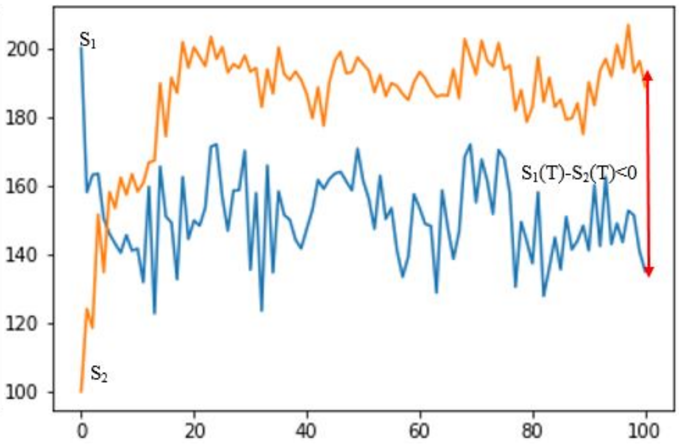
Evolution of the prices is still described by the equations discussed earlier.

This modification seems justified since an investor may be interested in hedging in situations when the difference of prices are large regardless of the direction of difference. In other words, by taking a long position in modified spread call an investor insures himself against the market movements when the price difference becomes significant.

This argument may be described by the illustrations from the previous chapter. Recall the two cases when the price difference is less than the strike value (namely, case 2 and case 3). In the standard setting, call option worth nothing in these cases as the price of asset 1 is less than $(S_2(T) + K)$. On the contrary, in the modified specification, call option is exercised even if the price of asset 1 drops below the price of asset 2 (case 3).




Case 2: $S_1(T) - S_2(T) < K$




Case 3: $S_2(T) - S_1(T) > K$ with K»0

## 5.2 Implementation

Here one may meet some difficulties. From the one hand, almost nothing has changed, small transformation was made - one may consider final model as vanilla option with the difference of two assets as underlying, but due to different specification of assets and correlation between them this approach to the problem gives totally incorrect result.

There are no closed form and finite difference solutions for this problem. Nevertheless, the Monte Carlo methodology can be applied in order to obtain the value of such option. One may transform the last step of numerical evaluation of standard spread option - evaluation of the expectation of the option value.

We suggest to add a specific binary variable, which indicates whether it is standard or modified version of the problem. The variable can take the value 0 in the case of the standard specification and 1 for the modified problem. One may run the code from Appendix A using values of the variables specified earlier (with the value of "modif"=1) and obtain the following values of options:

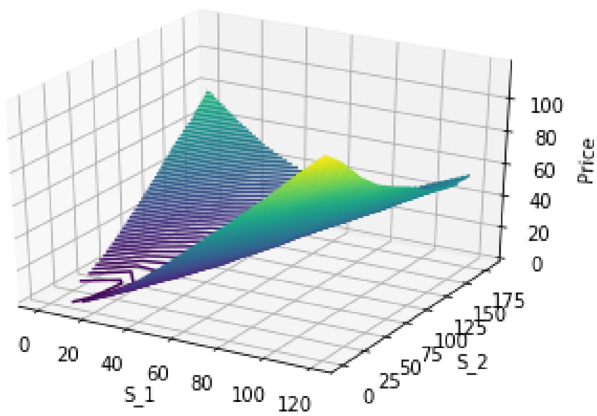
| Option type | Value |
|---|---|
| Call | 42.65 |
| Put | 6.33 |

### 5.3 Sensitivity of modified spread option

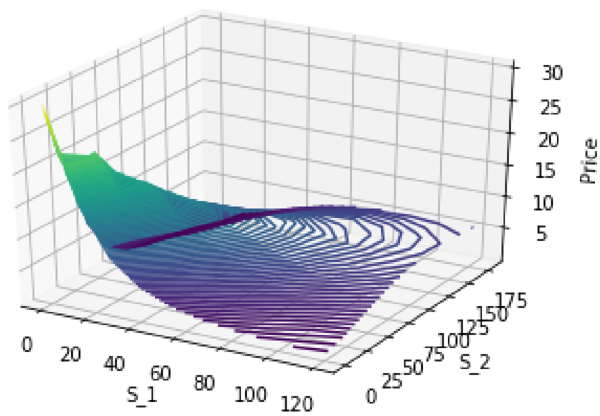
The set of sensitivity parameters is the same as in the previous chapter. We list their values in the table below.

| | Call option | | Put option | |
|---|---|---|---|---|
| Parameter | Value | Sign | Value | Sign |
| $\Delta_{S_1}$ | 0.339 | + | -0.019 | - |
| $\Delta_{S_2}$ | -0.118 | - | -0.066 | - |
| $\Delta_{spread}$ | 0.394 | + | 0.011 | + |
| $\Gamma_{S_1}$ | 0.030 | + | 0.008 | + |
| $\Gamma_{S_2}$ | 0.011 | + | 0.009 | + |
| $\Gamma_{spread}$ | -0.002 | - | 0.001 | + |
| $\nu_1$ | 174.236 | + | -3.422 | - |
| $\nu_2$ | 111.249 | + | 1.794 | + |
| $\kappa$ | 1.574 | + | 0.585 | + |
| $\theta$ | -0.593 | - | 0.817 | + |
| $\rho$ | 167.819 | + | -92.731 | - |

As one may notice, most of the parameters have the same sign as values of standard spread options. Below the graphs of relationships are provided.




Price of Call option with respect to underlying changes

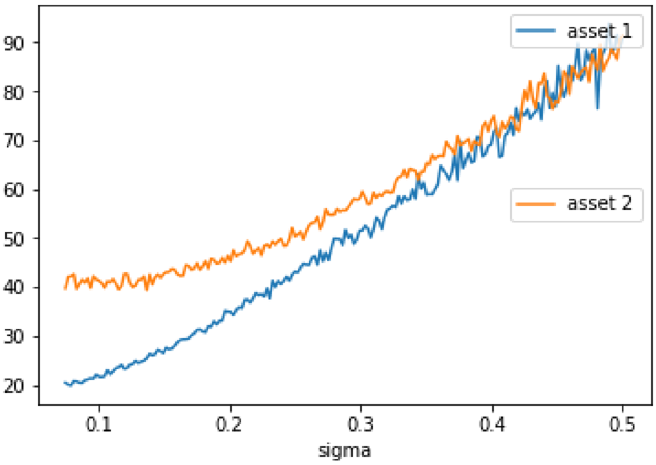



Price of Put option with respect to underlying changes

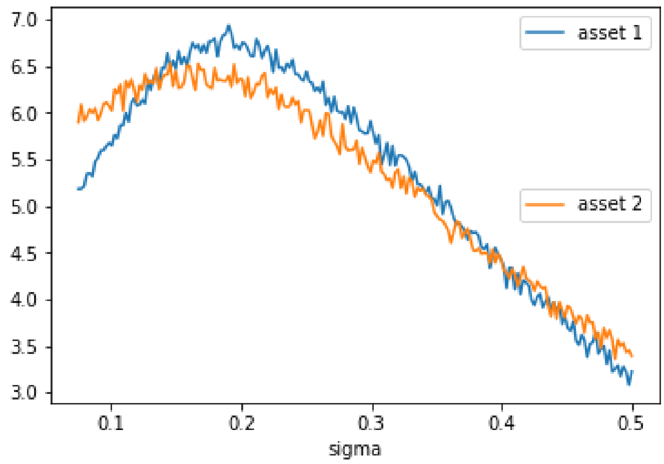
Here result meets expectations: in modified model due to transformation of assets price difference into its absolute value, the model takes not the linear form, as it did for standard model, but the quadratic one. Call option value changes in direct ratio with the difference between prices. As for Put option value, it increases when the sum of asset prices tends to 0.

On these graphs one can see the option values corresponding to particular volatility values of each asset. For example, price of Call option with volatility of asset 1 equal to 0.05 and volatility of asset 2 equal to 0.15 (initial value) is equal to 20. These graphs clearly represents the previous conclusion about quadratic form of relationship.

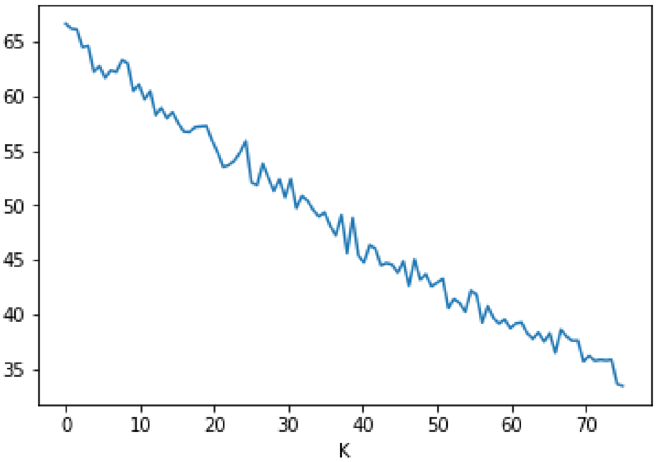
The relationship between the strike value and the option price is similar to the non-modified case: the price of call decreases with the value of strike, whereas the price of a modified put is increasing with K. Nevertheless, the functional form is quadratic, which differs from the non-modified specification.



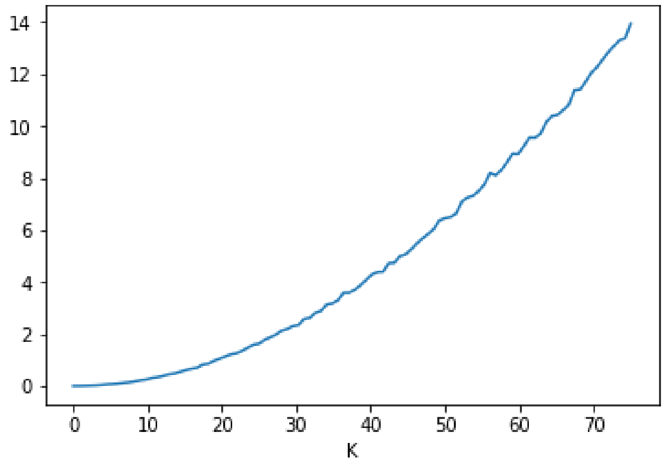
Price of Call option with respect to volatility



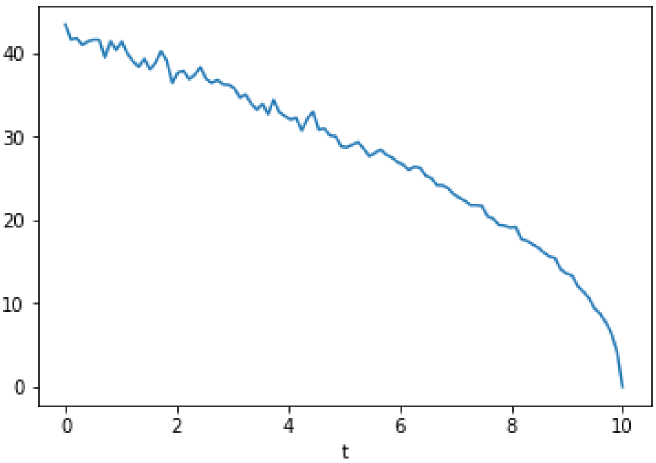
Price of Put option with respect to volatility



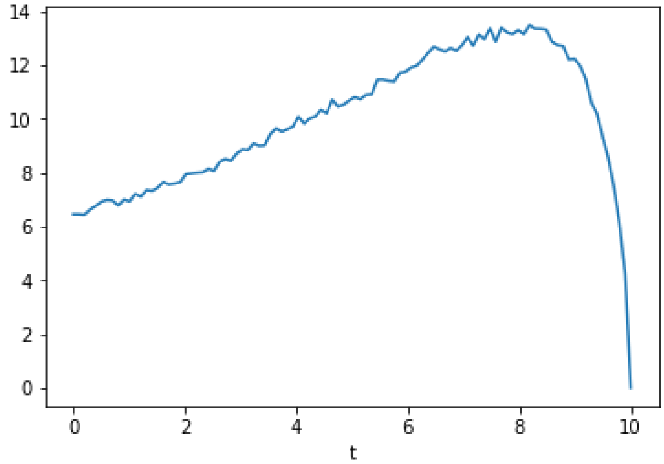
Price of Call option with respect to strike



Price of Put option with respect to strike



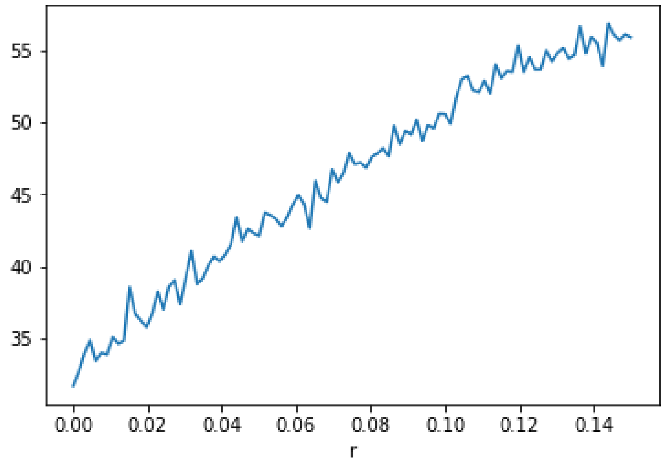
Price of Call option with respect to passage of time



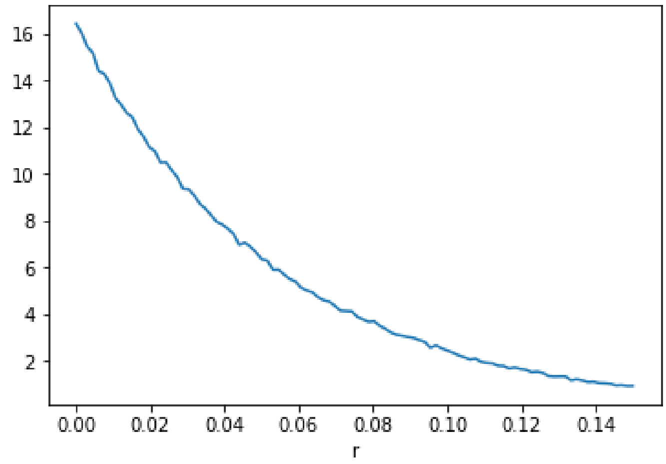
Price of Put option with respect to passage of time

For the modified call option the relationship between its price and time to maturity does not change. However, the relationship between the price of a modified put and time to maturity has a completely different functional form than in the non-modified case. Here it is rather quadratic with the increasing part corresponding to values of the time to maturity from 0 to 8.

Here the result is similar to one of the previous chapter when the standard spread option specification was considered. The price of a spread call increases with the interest rate, whereas, the price of a spread put decreases with this parameter.




Price of Call option with respect to interest rate




Price of Put option with respect to interest rate

## 6 Conclusion

We studied different approaches to the analysis of spread options. We showed that the main feature of spread options is that their values depend on two underlying assets whose prices are usually correlated. This connection between prices of different assets play an important role in the option pricing.

In addition, we provide the illustration of different scenarios of market price movements and how they affect the final payoffs of a spread option. A spread call option is exercised when the price of asset 1 sufficiently exceeds the price of asset 2. In contrast, a spread call is never exercised when these prices are close to each other or if the price of asset 2 is the largest. Nevertheless, a spread put is exercised in these situations.

We also discussed two most popular approaches in spread option pricing, namely Kirk's method and Monte Carlo simulation. Both techniques are proven to be justified and produce similar outcomes.

Moreover, the sensitivity analysis is conducted and the main characteristics of option price sensitivities are described. The conclusions obtained on sensitivity study are in line with the theoretical ones.

A modification of spread option is provided. The modification is based on reconsidering the definition of spread between asset prices. The sensitivity analysis revealed new properties which are different from ones of the model in a standard setting.

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# Appendix

## A. Monte Carlo simulation in Python

```
def PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif):
    """ Calculation of the price of a spread put/call european type option"""
    dt = T - t  # time to maturity
    drift1 = (r - q1 - 0.5*sigma1**2)*dt # drift term of asset 1
    vol1 = sigma1*sqrt(dt) # volatility term of asset 1
    drift2 = (r - q2 - 0.5*sigma2**2)*dt # drift term of asset 2
    vol2 = sigma2*sqrt(dt) # volatility term of asset 2
    mean = 0.0
    std = 0.0
    for i in range(numOfSim):
        dB1=gauss(0.0,1.0) # BM
        dB2=corr*dB1+sqrt(1-corr**2)*gauss(0.0,1.0) # another precess correlated to BM
        S1 = S1_t*exp(drift1 + vol1*dB1) # price of asset 1 corresponding to a particular path
        S2 = S2_t*exp(drift2 + vol2*dB2) # price of asset 2 corresponding to a particular path
        mean += max(optType*((S1-S2)*(1-modif)+abs(S1-S2)*modif-K),0) # recompute mean
        std += max(optType*((S1-S2)*(1-modif)+abs(S1-S2)*modif-K),0)*max(optType*((S1-S2)*(1-modif)
        +abs(S1-S2)*modif-K),0) # recompute variance
    mean *= (exp(-r*dt)/numOfSim) # mean price for numOfSim realizations
    std = sqrt((std/numOfSim - mean*mean)/numOfSim) # standard deviation for numOfSim realizations
    rslt = [mean,std]
    return rslt
Price=PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)
print ("Price by MC simulations is  " + str(round(Price[0],2)))
```

## B. Kirk's model in Python

```
"""Kirk's approach - for the case (modif=0) only!"""

sig=sqrt(sigma1**2+(sigma2*S2_t*exp(-q2*(T-t))/(S2_t*exp(-q2*(T-t))+K*exp(-r*(T-t))))**2-2*corr*sigma1*
               sigma2*(S2_t*exp(-q2*(T-t))/(S2_t*exp(-q2*(T-t))+K*exp(-r*(T-t))))) # volatility by Kirk
d1=(log(S1_t*exp(-q1*(T-t))/(S2_t*exp(-q2*(T-t))+K*exp(-r*(T-t))))+0.5*(sig**2)*(T-t))/(sig*sqrt(T-t))
d2=d1-sig*sqrt(T-t)
P_kirk=optType*(S1_t*exp(-q1*(T-t))*norm.cdf(optType*d1)-(S2_t*exp(-q2*(T-t))+K*exp(-r*(T-t)))*
                                               norm.cdf(optType*d2)) # price by Kirk's approach
print ("Price by Kirk's formula is  " + str(round(P_kirk,2)))
```

## C. Libraries used and set of parameters

```
import numpy as np
from random import gauss
from math import exp, sqrt, log
import matplotlib.pyplot as plt
from scipy.stats import norm
from mpl_toolkits.mplot3d import axes3d, Axes3D

"""Set the parameters"""

t=0 # time t
T=10 # maturity
S1_t=150 # price of asset 1 at time t
```

```
S2_t=100 # price of asset 2 at time t
K=50 # strike
r=0.05 # interest rate
q1=0.02 # dividend rate of asset 1 (=0 if assset does not yields any dividends)
q2=0.01 # dividend rate of asset 1 (=0 if assset does not yields any dividends)
sigma1=0.25 # volatility of the price of asset 1
sigma2=0.15 # volatility of the price of asset 2
corr=0.4 # correlation between the assets
optType=1 # type of the option: 1 if call, -1 if put
modif=0 # the model specification: 0 if standard spread option, 1 if modified

numOfSim=10000 # number of simulations
```

## D. Sensitivity analysis

```
""" Sensitivity analysis"""

"""Price sensitivity with respect to the asset prices - delta"""

n=10 # grid size
P=np.empty([n, n])
S1 = np.linspace(0,S1_t*1.2,n) # grid 1
S2 = np.linspace(0,S2_t*1.2,n) # grid 2
for i in range(n):
    for j in range(n):
        P[i][j]=PriceByMC(t,T,S1[i],S2[j],K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]
y,x = np.meshgrid(S1,S2)
fig = plt.figure()
ax = plt.axes(projection='3d')
ax.contour3D(x, y, P, 100, cmap='viridis')
ax.set_xlabel('S_1')
ax.set_ylabel('S_2')
ax.set_zlabel('Price')
fig

dS1=S1_t*0.05 # price1 change
dS2=-S2_t*0.05 # price2 change
delta_1=(PriceByMC(t,T,S1_t+dS1,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(dS1)
                                                                                        # delta(S1) - delta of asset 1
delta_2=(PriceByMC(t,T,S1_t,S2_t+dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(dS2)
                                                                                        # delta(S2) - delta of asset 2
delta_spread=(PriceByMC(t,T,S1_t+dS1,S2_t+dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]
                                                            -Price[0])/(dS1-dS2) # delta(S1-S2) - delta of spread

"""Further price sensitivity with respect to the asset prices - gamma"""

gamma_1=(PriceByMC(t,T,S1_t+dS1,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]+
          PriceByMC(t,T,S1_t-dS1,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-
          2*Price[0])/(dS1**2) # gamma(S1) - gamma of asset 1
gamma_2=(PriceByMC(t,T,S1_t,S2_t+dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]+
          PriceByMC(t,T,S1_t,S2_t-dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-
          2*Price[0])/(dS2**2) # gamma(S2) - gamma of asset 2
gamma_spread=(PriceByMC(t,T,S1_t+dS1,S2_t+dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]+
                PriceByMC(t,T,S1_t-dS1,S2_t-dS2,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-
                2*Price[0])/((dS1-dS2)**2) # gamma(S1-S2) - gamma of spread

```

```
"""Price sensitivity with respect to strike - kappa"""

m=100 # grid size
kgrid = np.linspace(0,K*1.5,m) # grid
P_k=[PriceByMC(t,T,S1_t,S2_t,kgrid[i],r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]
     for i in range(len(kgrid))]
plt.plot(kgrid.tolist(),P_k)
plt.xlabel('K')
plt.show()
dK=K*0.01 # strike change
kappa=(PriceByMC(t,T,S1_t,S2_t,K+dK,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(dK)
                                                                                  # kappa of the option



"""Price sensitivity with respect to time - theta"""

l=100 # grid size
tgrid = np.linspace(0,T,l) # grid
P_t=[PriceByMC(tgrid[i],T,S1_t,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]
     for i in range(len(tgrid))]

plt.plot(tgrid.tolist(),P_t)
plt.xlabel('t')
plt.show()

d_t=1 # time change
theta=(PriceByMC(t+d_t,T,S1_t,S2_t,K,r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(d_t)
                                                                                  # theta of the option



"""Price sensitivity with respect to interest rate - rho"""


p=100 # grid size
rgrid = np.linspace(0,r*3,p) # grid
P_r=[PriceByMC(t,T,S1_t,S2_t,K,rgrid[i],q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]
     for i in range(len(rgrid))]

plt.plot(rgrid.tolist(),P_r)
plt.xlabel('r')
plt.show()

d_r=r*0.5 # interest rate change
rho=(PriceByMC(t,T,S1_t,S2_t,K,r+d_r,q1,q2,sigma1,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(d_r)
                                                                                    # rho of the option

"""Price sensitivity with respect to volatility - vega"""

u=200 # grid size
sgrid = np.linspace(min(sigma1,sigma2)*0.5,max(sigma1,sigma2)*2,u) # grid
P_s1=[PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sgrid[i],sigma2,corr,numOfSim,optType,modif)[0] for i in range(len(sgrid))]
P_s2=[PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sigma1,sgrid[i],corr,numOfSim,optType,modif)[0] for i in range(len(sgrid))]

line1,=plt.plot(sgrid.tolist(),P_s1, label="asset 1")
line2,=plt.plot(sgrid.tolist(),P_s2, label="asset 2")
plt.xlabel('sigma')
```

```
first_legend = plt.legend(handles=[line1], loc=1)
ax = plt.gca().add_artist(first_legend)
plt.legend(handles=[line2], loc=5)
plt.show()

d_s=min(sigma1,sigma2)*0.1 # volatility change
vega1=(PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sigma1+d_s,sigma2,corr,numOfSim,optType,modif)[0]-Price[0])/(d_s)
                                                                                  # ega of asset 1
vega2=(PriceByMC(t,T,S1_t,S2_t,K,r,q1,q2,sigma1,sigma2+d_s,corr,numOfSim,optType,modif)[0]-Price[0])/(d_s)
                                                                                  # vega of asset 2
```