Elliptic curves cryptography

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Introduction

- What do we need to construct the group ?
- How to construct the group ?
- What are its application in cryptography ?
- Why it works ?
- What are the cons and pros ?

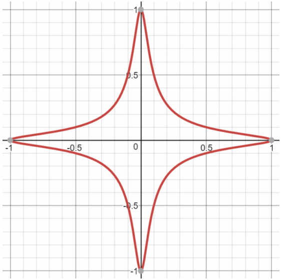


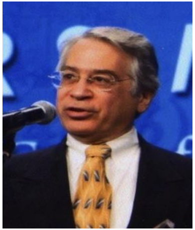
Figure 1: Edwards' Curve : $x^2 + y^2 = 1 + 300x^2y^2$

1. Introduction
    - Research papers
    - Tools
2. Projective Geometry
    - Projective plane
    - Affine slice
    - Projective plane and infinity point
3. Elliptic curves
    - Elliptic curves' definition
    - Rationals points
4. (E,+) group
    - Non associative binary operation
    - (E,+) abelian binary operation
5. Application
    - The discrete logarithm problem
    - Public key-sharing protocol
    - Elliptic curves' pros and cons
6. References

Research papers

- Neal Koblitz, *Elliptic curve cryptosystems*, 1985 [4]
- Victor S. Miller, *Use of elliptic curve in cryptography*, 1985 [6]

Neal Koblitz



Victor Saul Miller

The tools to build $(E, +)$

- Projective plane
- Projective lines
- Straight and tangent lines
- Rationals points

Projective plane

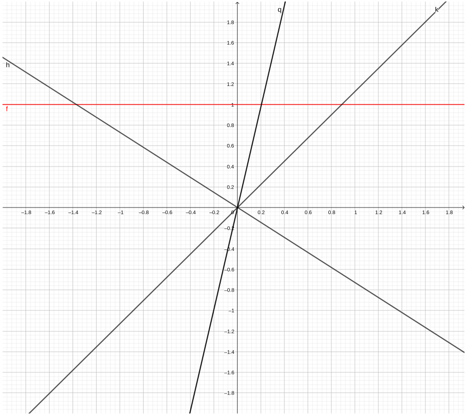


Figure 2: Projective line in red

Affine slice

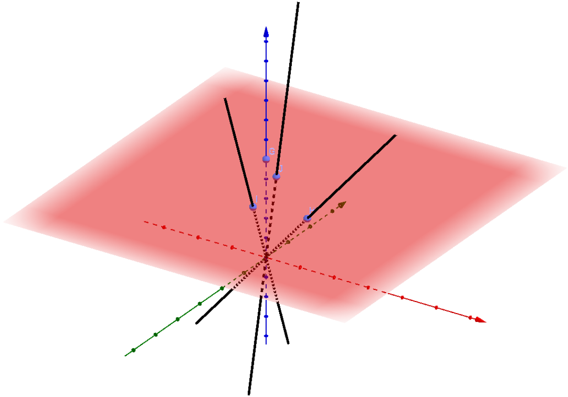


Figure 3: Affine slice of the projective plane in red

Affine slice of the projective plan and the infinity point

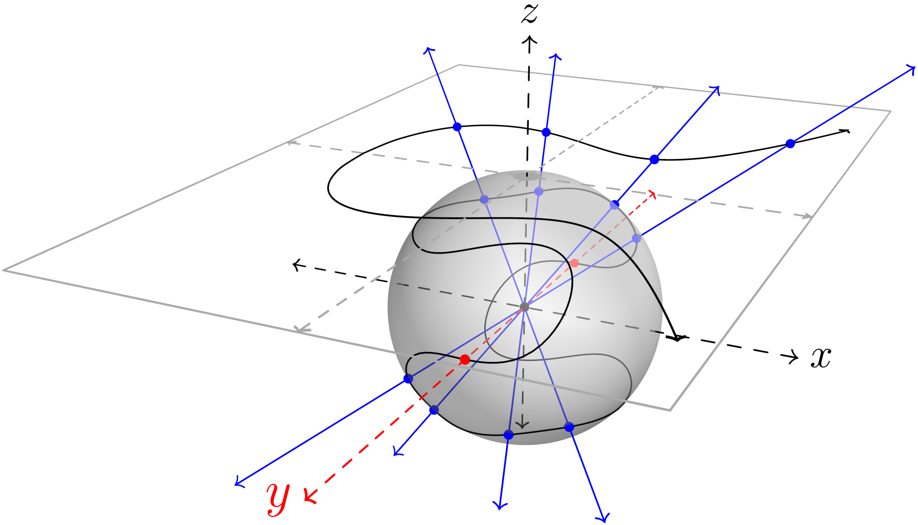


Figure 4: Elliptic curve of equation $y^2 = x^3 - x + 1$ on the projective plane

Elliptic curve

Definition

An elliptic curve define over a field K is a projective plane curve where its homogeneous equation is

$$y^2z = x^3 + axz^2 + bz^3, \qquad (1)$$

where a and b are elements of K which verify the following condition

$$\Delta = -(4a^3 + 27b^2) \neq 0. \qquad (2)$$

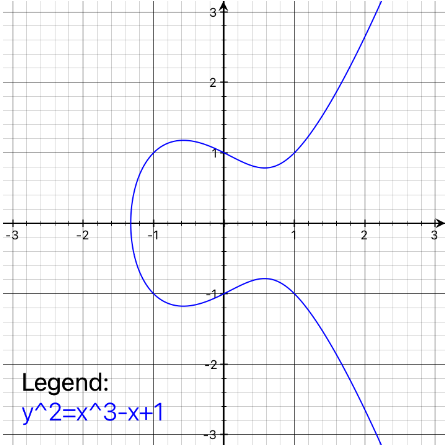


Figure 5: canonical curves with $\Delta < 0$

Weierstrass normal equation and rationals points

- **Weierstrass form**

$$y^2 = x^3 + ax + b,$$

- **rationals points set**

$$E(K) = \{P \in \mathbb{P}_2 \mid y^2 = x^3 + ax + b\} \cup \{\mathcal{O}\}$$

- **Infinity point**

$$\mathcal{O} = [0, 1, 0].$$

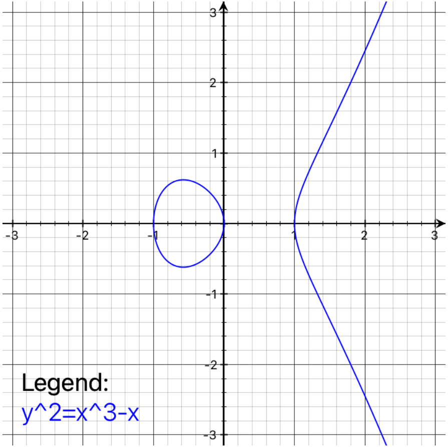


Figure 6: canonical curves with $\Delta > 0$

Non associative binary operation

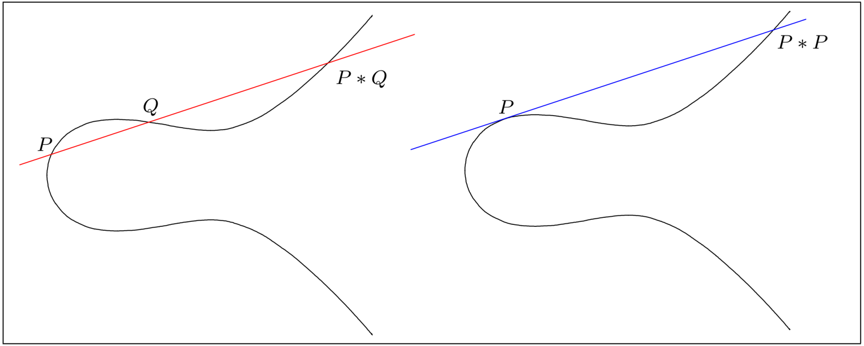


Figure 7: binary operation of chords and tangent of the curve

Abelian group

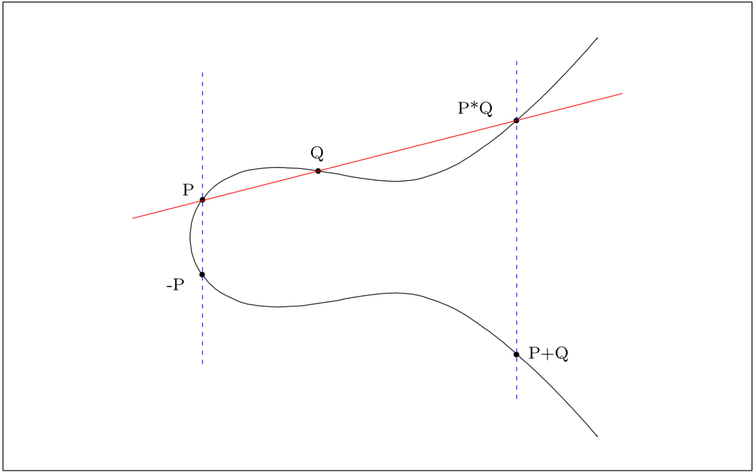


Figure 8: Geometric representation of rationals points' addition

Discrete logarithm

The problem of discrete logarithm

Let E be an elliptic curve define over a field K.
Let $Q \in E(K)$ be a rational point.
Given a rational point $P \in E(K)$, the discrete logarithm problem is to find $n \in \mathbb{N}$, if its exists, such as $P = nQ$.

Why it works:

- One-way functions exist.
- Unsolvability of the logarithm problem.
- No other way to solve Diffie-Hellman's problem.

Public key-sharing protocol

Diffie-Hellman protocol

Alice and Bob would like to share a secret key (i.e. know only by them) over a non-secure channel.

To do this they proceed the following way:

1) They choose and publish the triplet (K, E, P).
2) Alice choose $a > 0$ and compute $P_a = aP$, which she sends to Bob.
3) Bob $b > 0$ and compute $P_b = bP$, which she sends to Bob.
4) Alice and Bob compute aP_b and bP_a which give P_{ab}.

Pros and cons

Pros

- Abstract structure.
- Shorter secret key lengh.
- Low ressources usage.
- Hybrid cryptosystem compatibility.

Cons

- Many are patented
- Build-in trap doors? [3] [2]

The following is a statement of Serge Lang in his book *Elliptic curve: Diophantine analysis*, 1978 [5]:

"It is possible to write endlessly on elliptic curves. (This is not a threat)"

References

Pictures sources:

- Trustica.cz [7].
- Stackexchange forum [9].
- allaboutcircuits.com [1]
- Alchetron.com [8]

[1] *Dr. Neal Koblitz: Independent Co-creator of Elliptic Curve Cryptography - News*. URL: https://www.allaboutcircuits.com/news/dr.-neal-koblitz-independent-co-creator-of-elliptic-curve-cryptography/ (visited on 06/20/2022).

[2] Dan Goodin. *We don't enable backdoors in our crypto products, RSA tells customers*. Ars Technica. Sept. 20, 2013. URL: https://arstechnica.com/information-technology/2013/09/we-dont-enable-backdoors-in-our-crypto-products-rsa-tells-customers/ (visited on 06/21/2022).

[3] *How the NSA (may have) put a backdoor in RSA's cryptography: A technical primer*. The Cloudflare Blog. Jan. 6, 2014. URL: http://blog.cloudflare.com/how-the-nsa-may-have-put-a-backdoor-in-rsas-cryptography-a-technical-primer/ (visited on 06/21/2022).

[4] Neal Koblitz. *Elliptic curve cryptosystems*. 1985. URL: https://www.ams.org/journals/mcom/1987-48-177/S0025-5718-1987-0866109-5/S0025-5718-1987-0866109-5.pdf.

[5] Serge Lang (auth.) *Elliptic Curves: Diophantine Analysis*. 1st ed. Grundlehren der mathematischen Wissenschaften 231. Springer-Verlag Berlin Heidelberg, 1978. ISBN: 978-3-642-05717-5. (Visited on 06/20/2022).

[6] Victor S. Miller. *Use of elliptic curves in cryptography*. Lecture Notes in Computer Science. Vol. 218. Springer, 1985. URL: https://web.archive.org/web/20090206165338/http://dsns.csie.nctu.edu.tw/research/crypto/HTML/PDF/C85/417.PDF (visited on 06/16/2022).

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[9] Iñaki Viggers. *In Elliptic Curve, what does the point at infinity look like?* Cryptography Stack Exchange. May 13, 2019. URL: https://crypto.stackexchange.com/q/70507 (visited on 06/20/2022).