Homework 6

May 5^{th}, 2024

[Problem 1]

Solution:

Let $a(y) = I_{(a\le y\le b)}$, $T(F) = \int y \, dF(y)$, thus $L_F(x) = a(x) - T(F) = I_{(a\le x\le b)} - (F(b) - F(a))$

$$\hat{\tau}^2 = \frac{1}{n}\sum_{i=1}^{n} L_{\hat{F}}(x_i) = \frac{1}{n}\sum_{i=1}^{n}\left(I_{(a\le x_i\le b)} - T(\hat{F}_n)\right)^2 = \frac{1}{n}\sum_{i=1}^{n}\left(I_{(a\le x_i\le b)} - \hat{\theta}\right)^2$$

$$\hat{se} = \frac{1}{\sqrt{n}}\hat{\tau} = \frac{1}{n\sqrt{n}}\sum_{i=1}^{n}\left(I_{(a\le x_i\le b)} - \hat{\theta}\right)^2 = \frac{1}{n\sqrt{n}}\sum_{i=1}^{n} I_{(a\le x_i\le b)} - \frac{1}{\sqrt{n}}\hat{\theta}^2$$

$\rightarrow$ $(1-\alpha)$ C.I. for θ: $[\hat{\theta} - z_{\alpha/2}\cdot\hat{se},\ \hat{\theta} + z_{\alpha/2}\cdot\hat{se}]$

[Problem 2]

Solution:

F strictly increasing, and with the positive density f.

$T(F) = F^{-1}(p)$, let $F(\varepsilon, y) = (1-\varepsilon)F(y) + \varepsilon\delta_x(y)$, $p = F(\varepsilon, T(F(\varepsilon, y)))$

$$\frac{d}{d\varepsilon}F(\varepsilon, T(F(\varepsilon,y))) = F_1 + F_2\cdot\frac{d}{d\varepsilon}T(F(\varepsilon,y))$$

$$= -F(T(F(\varepsilon,y))) + \delta_x(T(F(\varepsilon,y))) + [(1-\varepsilon)f(T(F(\varepsilon,y))) + \varepsilon\delta_x'(T(F(\varepsilon,y)))]\frac{d}{d\varepsilon}T(F(\varepsilon,y))$$

For the equation $F(\varepsilon, T(F(\varepsilon,y))) = p$, take derivative w.r.t. ε, then take $\varepsilon = 0$.

$$-p + \delta_x(F^{-1}(p)) + f(F^{-1}(p))\cdot L(x) = 0$$

When $\begin{cases} \theta = F^{-1}(p) \ge x, & \delta_x(F^{-1}(p)) = 1,\ L(x) = \dfrac{p-1}{f(\theta)} \\ \theta < x, & \delta_x(F^{-1}(p)) = 0,\ L(x) = \dfrac{p}{f(\theta)} \end{cases}$

$$\rightarrow L(x) = \begin{cases} \dfrac{p-1}{f(\theta)} & x \le \theta \\ \dfrac{p}{f(\theta)} & x > \theta \end{cases}$$