Homework 4

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Problem 1

```r
mood.stats <- function(X,Y){
  X <- sort(X); Y <- sort(Y)
  XY <- sort(c(X,Y))
  rank.X <- match(X, XY)
  return(sum((rank.X-length(XY)+1)/2)^2)
}


moods.pvalue <- function(X,Y,nrep){
  # Calculate the Mood test statistics
  n <- length(X)
  stats <- mood.stats(X, Y)
  # Simulate under the null
  mood.sim <- c()
  for (i in 1:nrep) {
    mood.sim[i] <- mood.stats(rnorm(n,0,1), rnorm(n,0,1))
    }

  return(2*min(mean(mood.sim < stats), mean(mood.sim> stats)))
  }


Sukhatme.stats <- function(X,Y){
  data.pos <- merge(X[which(X>0)],Y[which(Y>0)])
  data.neg <- merge(X[which(X<0)],Y[which(Y<0)])
  return(sum(data.pos[,1]<data.pos[,2])+sum(data.neg[,1]>data.neg[,2]))
  }


Sukhatme.pvalue <- function(X,Y,nrep){
  # Calculate the Sukhatme test statistics
  n <- length(X)
  stats <- Sukhatme.stats(X, Y)
  # Simulate under the null
  Sukhatme.sim <- c()
  for (i in 1:nrep) {
    Sukhatme.sim[i] <- Sukhatme.stats(rnorm(n,0,1), rnorm(n,0,1))
```

```
    }
  return(return(2*min(mean(Sukhatme.sim < stats), mean(Sukhatme.sim> stats))))
  }
```

Problem 1.1

```
# Calculate the p-value of the statistics under the problem setting
set.seed(243)
X = rnorm(100); Y = rnorm(100,0,1.2); nrep = 500
moods.pvalue(X,Y,nrep)
```

```
## [1] 0.624
```

```
Sukhatme.pvalue(X,Y,nrep)
```

```
## [1] 0.052
```

Conclusion:
Both of the test fail to reject the null hypothesis under the setting of Problem 1.1 with respectively p-value of 0.624 and 0.052.

Problem 1.2

```
set.seed(217)
# Case 1: False rejection when median is not 0
X = rnorm(100,1,1); Y = rnorm(100,1,1); nrep = 500
moods.pvalue(X,Y,nrep)
```

```
## [1] 0.372
```

```
Sukhatme.pvalue(X,Y,nrep)
```

```
## [1] 0
```

```
# Case 2: n is small
set.seed(217)
X = rnorm(10,0,1); Y = rnorm(10,0,5); nrep = 500
# Case 3: n is extremely large, Sukhatme brings larger computation cost
# and become less powerful than the Mood test
moods.pvalue(X,Y,nrep)
```

```
## [1] 0.768
```

```r
Sukhatme.pvalue(X,Y,nrep)
```

```
## [1] 0.036
```

Conclusion:
The first scenario is that, the false rejection when median is not 0; the second scenario is n is small; and the third scenario is that n is extremely large, Sukhatme test brings larger computation cost and become less powerful than the Mood test.

Problem 1.3

```r
# Case 1: Skewed distribution
set.seed(217)
X = rnorm(50,1,0.5); Y = rgamma(50,0.5,0.5); nrep = 500
moods.pvalue(X,Y,nrep)
```

```
## [1] 0
```

```r
Sukhatme.pvalue(X,Y,nrep)
```

```
## [1] 0.036
```

Conclusion:
For skewed distribution, Sukhatme test is more powerful than the Mood test.

Problem 2

Proof:
$P\{(Y<X<0)\cup(0<X<Y)\} = P\{Y<X<0\}+P\{0<X<Y\}$
$= \frac{1}{2}P\{(Y<X<0)\cup(X<Y<0)\}+\frac{1}{2}P\{(0<X<Y)\cup(0<Y<X)\}$
$= \frac{1}{2}(P\{X,Y<0\}+P\{X,Y>0\})$
$= \frac{1}{2}((\frac{1}{2})^2+(\frac{1}{2})^2)$
$= \frac{1}{4}$

Problem 3

3.1

```r
Wilcoxon<-function(x,y){
  z<-sort(c(x,y))
  z[match(x,z)]=match(x,z);z[match(y,z)]=0
  return(sum(z))
}

Wilcoxon_test<-function(n,N=10000){
  i=1;W=rep(0,N)
  while(i<=N){
    x<-sort(rnorm(n));y=sort(rnorm(n))
    W[i]<-Wilcoxon(x,y)
    i<-i+1
  }

  return(sort(W))
}

f_3<-Wilcoxon_test(100)
sprintf("The 0.95 Confidence Interval is [%d,%d]",f_3[250],f_3[9750])
```

```
## [1] "The 0.95 Confidence Interval is [9238,10821]"
```

```r
set.seed(10)
Wilcoxon(sort(rnorm(100)),sort(rnorm(100,0.1,1)))
```

```
## [1] 9571
```

Conclusion of the Wilcoxcon Rank-Sum Test:
The sample result falls into the 0.95 confidence interval, hence we fail to reject the null hypothesis, that is we believe the two samples have same distribution.

```r
TH_test<-function(x,y){
  z<-sort(c(x,y))
  z[match(x,z)]=1
  z[match(y,z)]=0
  z<-z*qnorm(ppoints(length(z)))
  return(sum(z))
}
  TH_sim<-function(n,N=10000){
    i=1
    TH=rep(0,N)
    while(i<=N){
      x<-sort(rnorm(n))
      y<-sort(rnorm(n))
      TH[i]=TH_test(x,y)
      i<-i+1
    }
    return(sort(TH))
  }
  TH_set=TH_sim(100)
  sprintf("The 0.95 Confidence Interval is [%f,%f]",TH_set[250],TH_set[9570])
```

```
## [1] "The 0.95 Confidence Interval is [-13.825180,12.050896]"
```

```r
set.seed(10)
TH_test(sort(rnorm(100)),sort(rnorm(100,0.1,1)))
```

```
## [1] -7.947415
```

Conclusion of the the Terry-Hoeffding Test:
The sample result falls into the 0.95 confidence interval, hence we fail to reject the null hypothesis, that is we believe the two samples have same distribution.

3.2

Wilcoxon Rank-Sum test's linear rank weight is the rank itself, so when the sample conforms to some heavy-tailed distribution(for instance, Pareto Distribution), the rank which is closer to the median would contain more information, and in turn the Wilcoxon Rank-Sum test would be more powerful and preferable.

3.3

The linear rank weight of Terry-Hoeffding test is the normal quantile, making the test more sensitive to the extreme values among the ranks. When the distribution's left and right tail contains more information (such as the uniform distribution or light-tail distribution with sudden fluctuations in its numerical value), Terry-Hoeffding test would be more powerful.