Period-doubling and Chaos in Nonlinear Difference Equations: Applications to Models of Biological Populations

Introduction:

Chaos in mathematics refers to a random behaviour exhibited in a system, with sensitive dependence on initial conditions. Several dynamic systems exhibit chaotic behaviour for certain range of parameter values, such as the flow of water from a faucet, population growth over time and a swinging pendulum. In this module, we emphasize on the biological population models and particularly nonlinear difference equation to illustrate chaotic behaviour. We have investigated the harvesting model and the discrete logistic difference equation. However, it is a relatively abstract concept. We may find it challenging to understand it and investigate other models independently in the future. Thus, this essay will revisit and explore Ricker equation. We will first illustrate the numerical results with cobweb maps and orbit diagrams generated by RStudio. Then we will focus on finding periodic solutions of period 2. By combining mathematical and numerical analysis, we hope that students could better understand the concepts of period-doubling and chaos.

Ricker Equation Revisited:

The Ricker equation is a simple biological model for single species fisheries, which is given by:

$$x_{n+1} = x_n \exp\left(r\left(1 - \frac{x_n}{K}\right)\right),$$

where $r > 0$ and $K > 0$ are constants and x_n is the population density in year n. In Question 2 of Tutorial 7, we have explored the steady states of the equation and their stability. In this essay, we will focus on the case when $K = 1$, that is

$$x_{n+1} = x_n \exp\left(r(1 - x_n)\right). \tag{1}$$

We have worked out that the graph of (1) is a concave down parabola with a maximum value of $\frac{1}{r}\exp(r-1)$ at $x_n = \frac{1}{r}$. As shown in Figure 1, we have chosen r to be 1 and hence, there is a maxima of 1 at $x_n = 1$. In the tutorial, we showed that the steady states with their stability in terms of different ranges of r are:

- $x_* = 0$ is unstable for $r > 0$;
- $x_* = 1$ is monotonically asymptotically stable for $0 < r < 1$;
- $x_* = 1$ is oscillatory asymptotically stable for $1 < r < 2$;
- $x_* = 1$ is oscillatory unstable for $r > 2$.

In the following part, we will graphically represent it and explore long-term behaviour of (1).

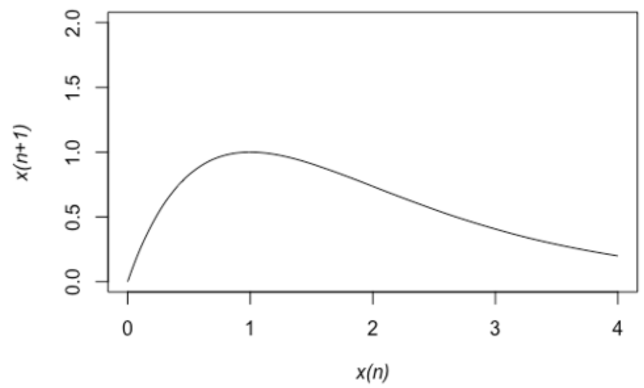

Figure 1: plot of the Ricker equation (1) (x_{n+1} vs. x_n) for $r = 1$

Ricker Equation of Numerical Analysis:

According to the steady state and stability analysis, we chose the initial condition $x_0 = 0.5$ and plotted the time series of x_n vs. n with corresponding cobweb maps for fixed values of r. To make the graph clearer, we used line segments; however, we need to remember that this is a single species, discrete time model. There are five separate cases considered:

Case 1: For $0 < r < 1$, the steady state $x_* = 1$ is monotonically asymptotically stable. That means that the population grows and eventually reaches to 1 (Figure 2).

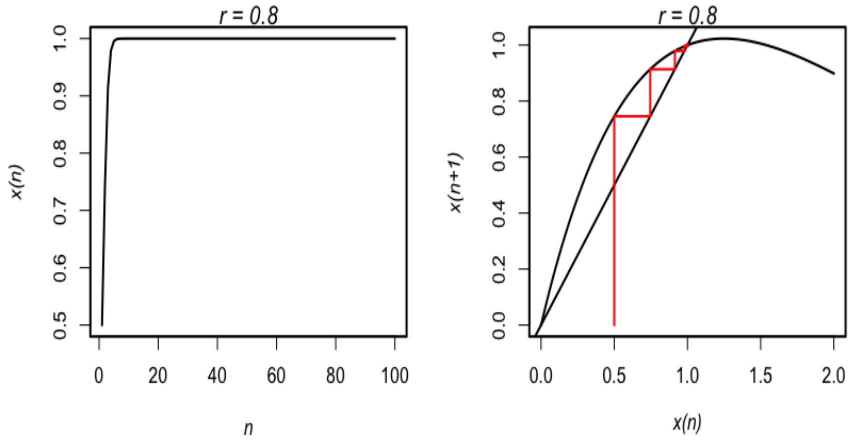

Figure 2: Cobweb maps (right) and solution behaviour (left) for the monotonically asymptotically stable case of (1) for $r = 0.8$

Case 2: For $1 < r < 2$, the steady state $x_* = 1$ is oscillatory asymptotically stable. That means that the population oscillates around the steady state and eventually reaches to it (Figure 3).

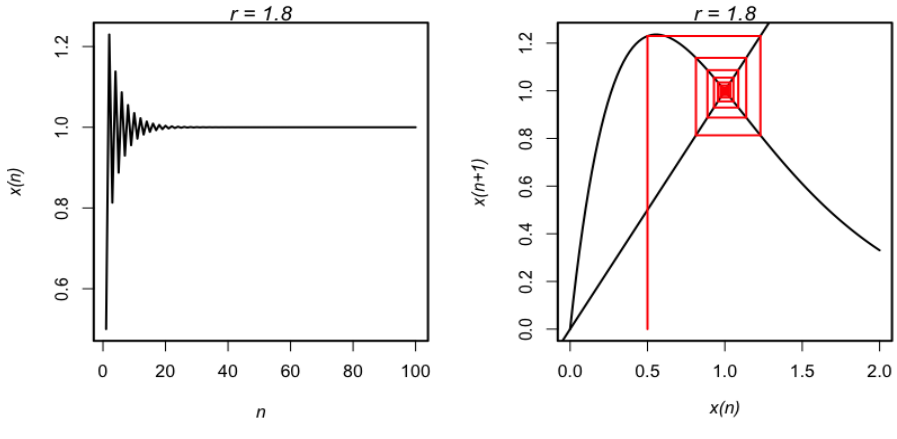

Figure 3: Cobweb maps (right) and solution behaviour (left) for the oscillatory asymptotically stable case of (1) for $r = 1.8$

Case 3: For slightly larger r ($r = 2.3$), there is a set of two fixed points. One fixed point represents a larger population in one generation and another one represents a smaller population in next generation. In this case, the population alternates between one and another. This is called a period-2 cycle (Figure 4).

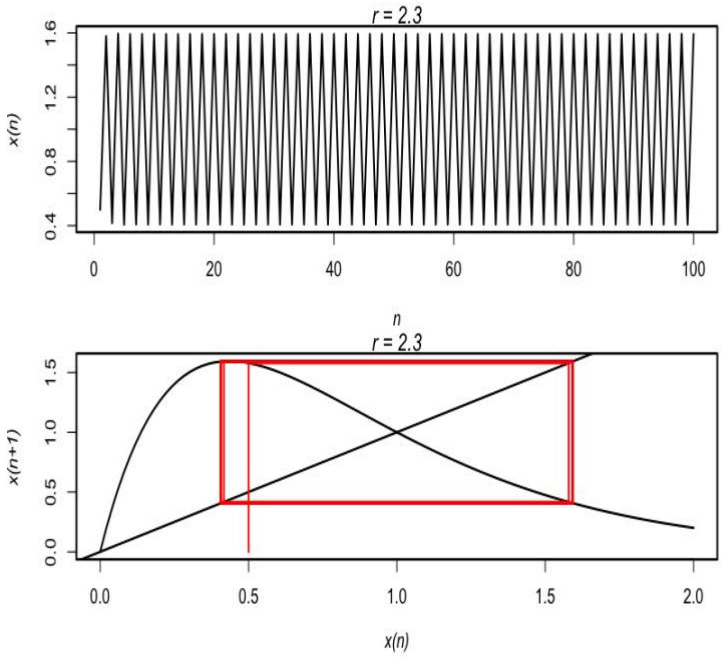

Figure 4: Cobweb maps (bottom) and solution behaviour (top) for the period-2 cycle case of (1) for $r = 2.3$

Case 4: For larger r ($r = 2.6$), there is a set of four repeating fixed points. x_n repeats every four iterations; and thus this is called a period-4 cycle (Figure 5).

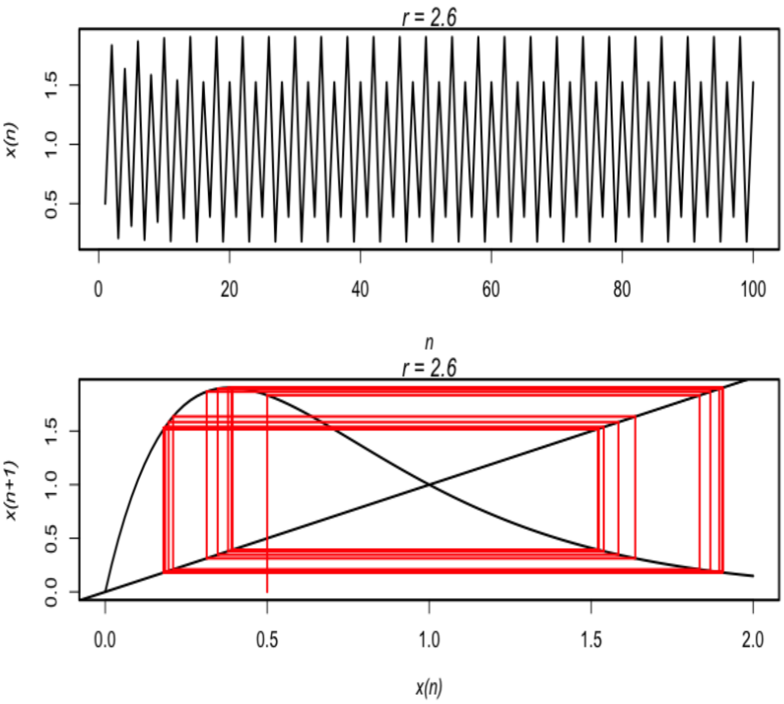

Figure 5: Cobweb maps (bottom) and solution behaviour (top) for the period-4 cycle case of (1) for $r = 2.6$

Case 5: For larger r ($r = 3$), there is no fixed point and periodic orbit. In other words, the trajectory does not repeat itself. Thus, the long-term behaviour of (1) is aperiodic. As shown in Figure 6, the corresponding cobweb maps are considerably complicated.

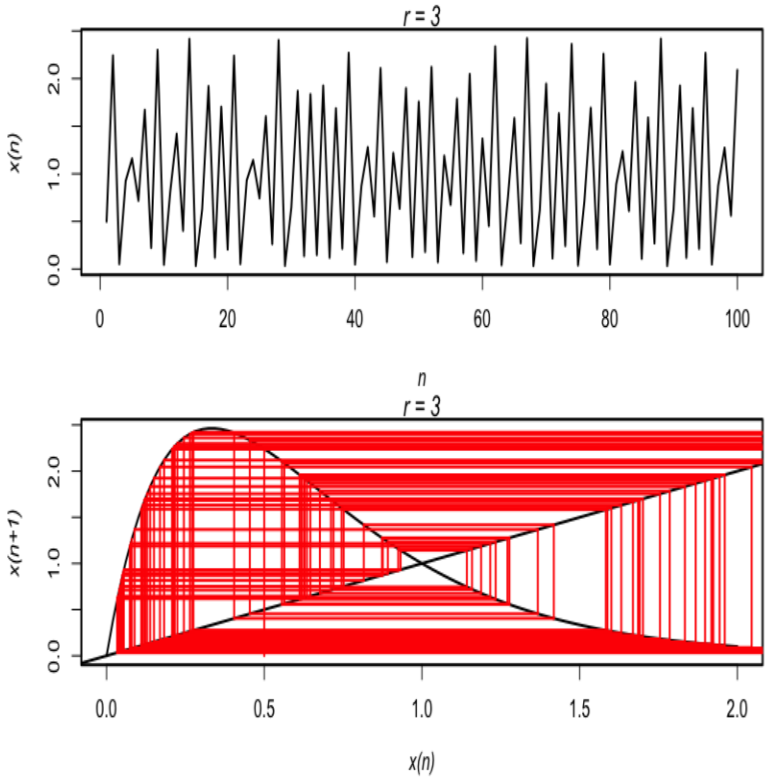

Figure 6: Cobweb maps (bottom) and solution behaviour (top) for the aperiodic case of (1) for $r = 3$

Case 5 illustrates the chaotic behaviour and the system would become more and more chaotic as r increases. To visualize the long-term behaviour for all values of r, we plotted the bifurcation diagram. The graph is plotted as follows: for each value of parameter r incremented with 0.05 steps, we iterated x_n for 600 times and got 600 generations. We then discarded the first 300 generations for each r to allow the system settle down. Figure 7 has shown the population density in the last 300 generations and we discovered that:

- $0 < r < 2$ $\implies$ Stable Equilibrium
- $r \approx 2$ $\implies$ Bifurcate into a period-2 cycle
- $r \approx 2.526$ $\implies$ Bifurcate into a period-4 cycle
- As r increases, further period-doublings occur (period 8, 16, 32, etc)
- $r > 2.692$ $\implies$ Periodic behaviour ceases and we have chaotic behaviour. There are some regions where the periodic windows appear again and the system returns to a limit cycle, such as $r \approx 3.15$.

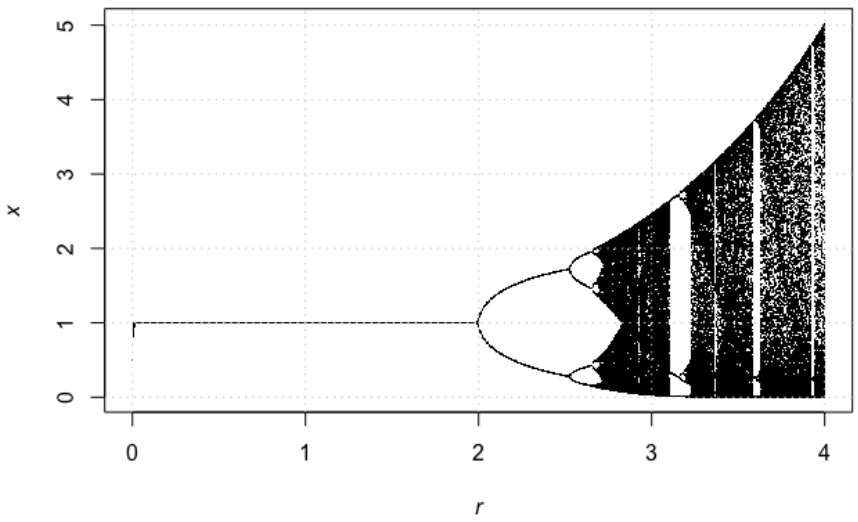

Figure 7: Asymptotic solution behaviour for the model (1) with parameter r

Ricker Equation of Mathematical Analysis:

Period-doubling and chaos have been graphically represented in pervious section. But, how can we show the system has a period-2 cycle which is asymptotically stable for $r > 2$ mathematically?

Recall that a periodic point with period k is such as

$$x_0 = f^k(x_0).$$

The point is asymptotically stable if

$$|f'(x_0)f'(x_1)\dots f'(x_{k-1})| < 1.$$

For a period-2 cycle, it holds

$$|f'(x_0)f'(x_1)| < 1.$$

We want to find a period-2 solution, which satisfies

$$x_0 = f(x_1) \text{ and } x_1 = f(x_0). \tag{2}$$

We have learned the notation that

$$f^2(x) = f(f(x))$$

in lectures. Combining with equation (2), we will get

$$x_0 = f\big(f(x_0)\big) = f^2(x_0) \quad \text{and} \quad x_1 = f\big(f(x_1)\big) = f^2(x_1).$$

Recall that x_* is a steady state of a map f if

$$x_* = f(x_*).$$

Hence x_0 and x_1 are steady states of the map f^2, and they satisfy (for computational simplicity, we now write x_0 and x_1 as x):

$$x = f(f(x)) = f\left(x \cdot exp(r(1-x))\right)$$
$$\Rightarrow \quad x = x \cdot \exp(r(1-x)) \exp\left(r\left(1 - xexp(r(1-x))\right)\right) \tag{3}$$

Since $x_* = f(x_*) = f(f(x_*)) = f^2(x_*)$, x_* is also a steady state of f^2. In this case, the two steady states solve (3) and other solutions are periodic steady states of (1). We further simplified (3) and got

$$r = \frac{1}{1-x} \ln\left(\frac{2-x}{x}\right). \tag{4}$$

We treated LHS as function h (red line) and RHS as function g (purple line). To find the solution, we plotted them based on three scenarios ($r < 2$, $r = 2$, $r > 2$). Figure 8 further confirmed that period-2 solution occurs when $r > 2$. We then used the trick stated in the article by RM May. By writing

$$x_* = 1 + y,$$

equation (4) could be manipulated into the form

$$y = \tanh\left(\frac{1}{2}ry\right). \tag{5}$$

A graphic solution of equation (5) is shown in Figure 9. The graph confirmed that

- $r < 2$, showing one real solution, $y = 0$ ($x_0 = x_1 = 1$); hence stable equilibrium
- $r > 2$, showing two non-trivial solutions, $y = \pm y_0$ with $y_0 < 1$ ($x_{0,1} = 1 \pm y_0$), and one trivial solution.

We found out that the trivial solution is always unstable for $r > 2$, and non-trivial solutions are stable if

$$-1 < f'(x_0)f'(x_1) < 1$$

where

$$f'(x_{0,1}) = (1 - rx_{0,1}) \exp\left(r(1 - x_{0,1})\right).$$

Thus,

$$-1 < [1 - r(x_0 + x_1) + r^2 x_0 x_1] \cdot \exp(2r - r(x_0 + x_1)) < 1$$

where

$$x_0 + x_1 = 2 \text{ and } x_0 x_1 = 1 - {y_0}^2.$$

So,

$$0 < r[2 - r(1 - {y_0}^2)] < 2. \tag{6}$$

By equation (5), equation (6) eventually shows the constraint $r < 2.526$. Thus, there is a 2-cycle which is asymptotically stable for $2 < r < 2.526$.

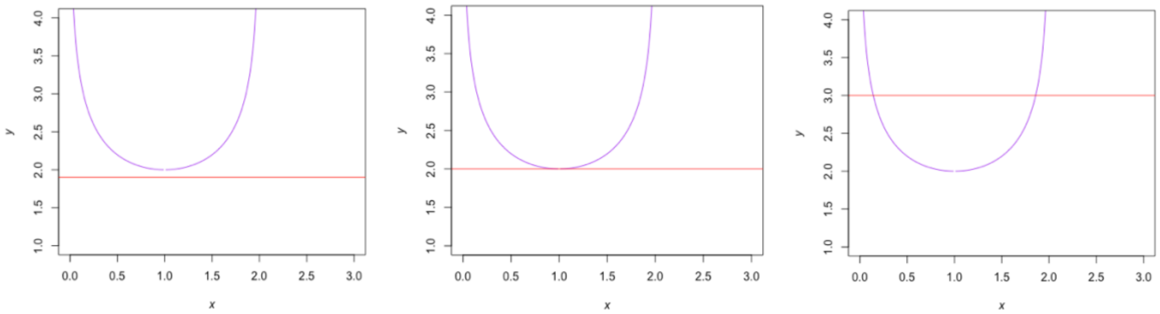

Figure 8: plot of function h and g for three cases $r = 1.9$ (left), $r = 2$ (middle), $r = 3$ (right)

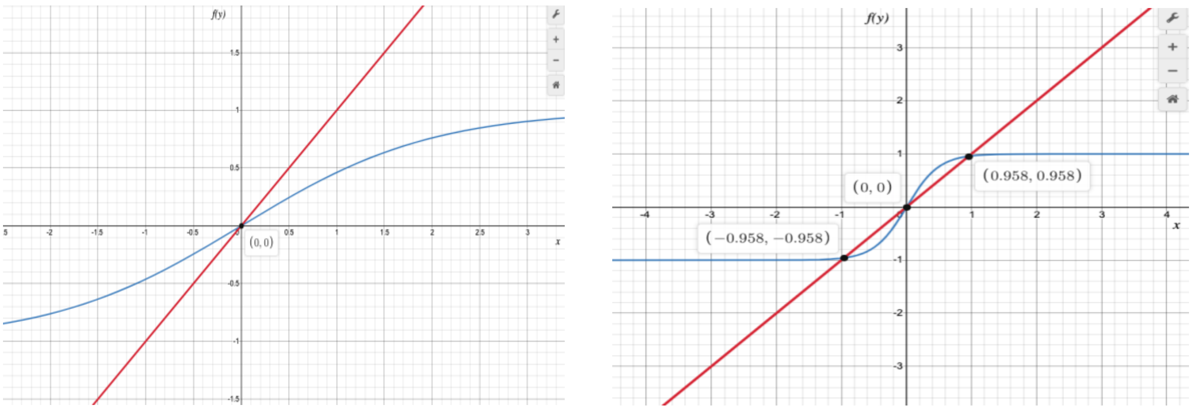

Figure 9: Graphical solutions of equation (5) where red line is $f(y) = y$ and blue line illustrates $\tanh\left(\frac{1}{2}ry\right)$ with $r = 1$ (left) and $r = 4$ (right)

Conclusion:

To conclude, we revisited Ricker equation and briefly stated the steady states with their linear stability. In numerical analysis section, we considered five cases with different values of parameter r. We generated cobweb maps and orbit diagram to graphically present the behaviour of the dynamic system (Ricker equation). We found that after $r > 2$, period doublings occur and the system exhibits the chaotic behaviour quickly. In mathematical analysis section, we showed the steps of finding periodic solutions of period 2. We also illustrated that there is a 2-cycle which is asymptotically stable for $2 < r < 2.526$. We think that period-doubling and chaos in nonlinear difference equations could be difficult to understand if only the mathematical equations are provided, and graphically representation of the dynamic system is considerably useful in mathematical biological modelling. In this case, by providing the R code of generating cobweb graphs and bifurcation diagram (see Appendix), we hope that students could be able to investigate other models independently in the future.

Appendix (R code):

Cobweb maps and the corresponding solution behaviour (1):

```
q_map <- function(r = 1, x_o = runif(1, 0, 1), N = 100, burn_in = 0,...) {
  par(mfrow = c(2, 1),mar = c(4, 4, 1, 2), lwd = 2)
  x <- array(dim = N)
  x[1] <- x_o
  for(i in 2:N)
    x[i] <- x[i-1] * exp(r * (1 - x[i-1]))
  plot(x[(burn_in + 1):N], type = 'l', xlab = 'n',ylab = 'x(n)',...)

  x <- seq(from = 0,to = 2,length.out = 100)
  x_np1 <- array(dim = 100)
  for(i in 1:length(x))
    x_np1[i] <- x[i] * exp(r * (1 - x[i]))
  plot(x, x_np1, type = 'l',xlab = "x(n)", ylab = "x(n+1)")
  abline(0,1)
  start = x_o
  vert = FALSE
  lines(x = c(start, start), y = c(0, start * exp(r * (1 - start))), col = "red")
  for(i in 1:(2*N)) {
    if(vert) {
      lines(x = c(start, start),y = c(start, start * exp(r * (1 - start))), col = "red")
      vert = FALSE
    } else {
      lines(x = c(start,
                  start * exp(r * (1 - start))),
            y = c(start * exp(r * (1 - start)),
                  start * exp(r * (1 - start))) , col = "red")
      vert = TRUE
      start = start * exp(r * (1 - start))
    }
  }
}
```

Bifurcation Diagram (2):

```
# r: bifurcation parameter
# x: initial value
# N: Number of iteration
# M: Number of iteration points to be returned
my.map <- function(r, x, N, M){
  z <- 1:N
  z[1] <- x
  for(i in c(1:(N-1))){
    z[i+1] <- z[i] * exp(r * (1 - z[i]))
  }
  # Return the last M iterations
  z[c((N-M):N)]
}


## Set scanning range for bifurcation parameter r
my.r <- seq(0, 4, by = 0.005)

Orbit <- sapply(my.r, my.map, x = 0.5, N = 600, M = 300)
Orbit <- as.vector(Orbit)
M = 300
r <- sort(rep(my.r, (M + 1)))

plot(Orbit ~ r, pch=".", xlab=expression(italic("r")), ylab=expression(italic("x")))
grid()
```

Reference of the code above:

(1) Chiver C. *Dynamical systems: Mapping chaos with R.* Available from: https://www.r-bloggers.com/2012/07/dynamical-systems-mapping-chaos-with-r/ [Accessed 22nd April 2021].

(2) Gesmann M. Logistic map: Feigenbaum diagram in R. *Dynamical Systems*. Available from: https://magesblog.com/post/2012-03-17-logistic-map-feigenbaum-diagram/ [Accessed 22nd April 2021].