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Homework #2

Exercise I.

parabola $y = -x^2 + 4.0$

line $y = 4x - 1.0$

$\boxed{x_0 = 1.5}$

we need to define a new function which indicates their intersection.

$-x^2 + 4.0 = 4x - 1.0$

$0 = x^2 + 4x - 5.0$

$f(x) = x^2 + 4x - 5.0$

a) Ten iterations of the Fixed Point Iterations Method

For Fixed Point Iterations Method, first we need an "initial guess" which is denoted as x_0.

x_0 = "initial guess"

$x_{i+1} = f(x_i) \quad \forall i = 0, 1, 2, ---$

We need to find the root of the $f(x)$

A real number r is a fixed point of $f(x)$ if $f(r) = r$

$f(x) = x^2 + 4x - 5.0$

$0 = x^2 + 4x - 5.0$

$-x^2 + 5.0 = 4x$

$\boxed{x = \frac{5.0 - x^2}{4} = p(x)} \longrightarrow x_{i+1} = P(x_i)$

I. $x_1 = P(x_0) = \frac{5.0 - (1.5)^2}{4} = 0,6875$

II. $x_2 = P(x_1) = \frac{5.0 - (0,6875)^2}{4} = 1,131835$

III. $x_3 = P(x_2) = \frac{5.0 - (1,131835)^2}{4} = 0,929737$

IV. $x_4 = P(x_3) = \frac{5.0 - (0,929737)^2}{4} = 1,033897$

V. $x_5 = P(x_4) = \frac{5.0 - (1,033897)^2}{4} = 0,982764$

VI. $x_6 = P(x_5) = \frac{5.0 - (0,982764)^2}{4} = 1,008543$

VII. $x_7 = P(x_6) = \frac{5.0 - (1,008543)^2}{4} = 0,995710$

VIII. $x_8 = P(x_7) = \frac{5.0 - (0,995710)^2}{4} = 1,002140$

IX. $x_9 = P(x_8) = \frac{5.0 - (1,002140)^2}{4} = 0,998928$

X. $x_{10} = P(x_9) = \frac{5.0 - (099\,8928)^2}{4} = 1,000535$

b) three iterations of the Newton's method.

For Newton-Raphson method we deal with derivatives.

we found the function as $f(x) = x^2+4x-5.0$ $\qquad f'(x) = 2x+4$

The process,

$$y - f(x_i) = f'(x_i)(x - x_i)$$

$$y_1 = 0 \Rightarrow x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{x_i^2+4x_i-5.0}{2x_i+4} = \frac{2x_i^2+4x_i - x_i^2 - 4x_i + 5.0}{2x_i+4}$$

$$x_{i+1} = \frac{x_i^2+5.0}{2x_i+4}$$ We know $x_0 = 1.5$

$$x_1 = \frac{x_0^2+5.0}{2x_0+4} = \frac{2,25+5}{3+4} = 1,035714$$

$$x_2 = \frac{x_1^2+5.0}{2x_1+4} = 1,000210$$

$$x_3 = \frac{x_2^2+5.0}{2x_2+4} = 1,000000$$

Exercise 2.

a) $$\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 8 & 5 & 4 \\ -1 & -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix}}_{b}$$

For finding LU decomposition, we use Gaussian Elimination

$$A = L.U$$

$$\begin{bmatrix} 4 & 2 & 1 \\ 8 & 5 & 4 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow[\text{from row 2}]{\text{Subtract row 1} \times 2} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow[\text{from row 3}]{\text{subt. } -1/4 \times \text{row 1}} \begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & -3/2 & -\frac{11}{4} \end{bmatrix} \xrightarrow[\text{from row 3}]{\text{subt. } -3/2 \times \text{row 2}} \underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1/4 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -0,25 & -1,5 & 1 \end{bmatrix}$$

$Ax = b \rightarrow LUx = b$

I. define $c = Ux$ and solve $Lc = b$ for c.

II. Solve $Ux = c$ for x.

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -0,25 & -1,5 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} \rightarrow \left.\begin{array}{l} c_1 = 3 \\ 2c_1 + c_2 = 7 \rightarrow c_2 = 1 \\ -0,25c_1 - 1,5c_2 + c_3 = -1 \rightarrow c_3 = 1,25 \end{array}\right\} \underbrace{\begin{bmatrix} 3 \\ 1 \\ 1,25 \end{bmatrix}}_{\vec{c}}$$

$$\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0,25 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1,25 \end{bmatrix} \rightarrow \begin{array}{l} x_3 = 5 \\ x_2 + 2x_3 = 1 \rightarrow x_2 = -9 \\ 4x_1 + 2x_2 + x_3 = 3 \quad x_1 = 4 \end{array}$$

$$\vec{x} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$$

b) Use $PA=LU$ decomposition

$$\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 8 & 5 & 4 \\ -1 & -2 & -3 \end{bmatrix}}_{A} \qquad \text{set } P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow[\text{exchange } r_2 \text{ and } r_1]{} \begin{bmatrix} 8 & 5 & 4 \\ 4 & 2 & 1 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow[\text{from } r_2]{\text{subtract } \frac{1}{2} r_1} \begin{bmatrix} 8 & 5 & 4 \\ \frac{1}{2} & -1/2 & -1 \\ -1 & -2 & -3 \end{bmatrix} \xrightarrow[\text{from } r_3]{\text{subtract } -\frac{1}{8} r_1} \begin{bmatrix} 8 & 5 & 4 \\ \frac{1}{2} & -1/2 & -1 \\ \frac{-1}{8} & -11/8 & -5/2 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

exchange r_2 and r_3

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = P$$

$$\begin{bmatrix} 8 & 5 & 4 \\ \frac{-1}{8} & -11/8 & -5/2 \\ \frac{1}{2} & \frac{4}{11} & -1/11 \end{bmatrix} \xleftarrow[\text{from } r_3]{\text{Subtract } +\frac{4}{11} \times r_2} \begin{bmatrix} 8 & 5 & 4 \\ \frac{-1}{8} & -11/8 & -5/2 \\ \frac{1}{2} & -1/2 & -1 \end{bmatrix}$$

$P.A = L.U$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 8 & 5 & 4 \\ -1 & -2 & -3 \end{bmatrix}}_{A} = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/8 & 1 & 0 \\ \frac{1}{2} & 4/11 & 1 \end{bmatrix}}^{L} \cdot \overbrace{\begin{bmatrix} 8 & 5 & 4 \\ 0 & -11/8 & -5/2 \\ 0 & 0 & -1/11 \end{bmatrix}}^{U}$$

$Ax=b \rightarrow PAx = Pb \rightarrow$ first, solve $Lc = Pb$ for c
$LUx = Pb$ — Second, solve $Ux = c$ for x

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/8 & 1 & 0 \\ 1/2 & 4/11 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 7 \\ -1 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$$

$\boxed{c_1 = 7}$

$-\frac{c_1}{8} + c_2 = -1 \rightarrow \boxed{c_2 = -1/8}$

$\frac{c_1}{2} + \frac{4c_2}{11} + c_3 = 3 \rightarrow \frac{7}{2} - \frac{1}{22} + c_3 = 3 \rightarrow \boxed{c_3 = -5/11}$

$$\begin{bmatrix} 8 & 5 & 4 \\ 0 & -11/8 & -5/2 \\ 0 & 0 & -1/11 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1/8 \\ -5/11 \end{bmatrix}$$

$-1/11 \cdot x_3 = -5/11 \rightarrow \boxed{x_3 = 5}$

$-\frac{11}{8}x_2 - \frac{5}{2}x_3 = \frac{-1}{8} \Rightarrow -\frac{11x_2}{8} - \frac{100}{8} = -\frac{1}{8} \Rightarrow -11x_2 = 99 \ \boxed{x_2 = -9}$

$8x_1 + 5x_2 + 4x_3 = 7 \Rightarrow 8x_1 - 45 + 20 = 7$

$\Rightarrow \boxed{x_1 = 4}$

$$\vec{X} = \begin{bmatrix} 4 \\ -9 \\ 5 \end{bmatrix}$$

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c) $x_1 + x_2 = 2$ Approximate solution = [2, 1]

$2x_1 + 2.01x_2 = 4$

$$\longrightarrow \begin{bmatrix} 1 & 1 \\ 2 & 2.01 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

1) Backward error $= \|r\|_\infty = \|b - A \cdot x_a\|$

$$= \left\| \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ 2 & 2.01 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\|_\infty$$

$$= \left\| \begin{bmatrix} 2 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 6.01 \end{bmatrix} \right\|_\infty = \left\| \begin{bmatrix} -1 \\ -2.01 \end{bmatrix} \right\|_\infty$$

$$\boxed{= \underline{2.01}}$$

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2) Forward Error $= \|x_a - x\|_\infty \longrightarrow \left\| \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\|_\infty$

$-2(x_1 + x_2 = 2)$

$+ \quad 2x_1 + 2.01x_2 = 4$

$0.01x_2 = 0$ $\boxed{x_1 = 2}$

$\boxed{x_2 = 0}$

$$= \left\| \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\|_\infty = 1$$

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3) Error magnification factor $= \dfrac{\text{Relative forward error}}{\text{Relative backward error}} = \dfrac{\|r\|_\infty}{\|b\|_\infty} \Big/ \dfrac{\|x - x_a\|_\infty}{\|x\|_\infty}$

$\|b\|_\infty = 4$

$\|x\|_\infty = 2$

$$= \frac{2.01}{4} \Big/ \frac{1}{2}$$

$$= \frac{2.01}{4} \cdot 2 = \frac{2.01}{2} = \underline{1,005}$$

Exercise 3.

$$S_{true} = 1 - (1 - S_{observed}) \cdot v^5$$

$$S_{observed} = 0.1 + 0.7\sqrt[4]{v}$$

We want to ensure that $S_{true} = 0.9$

a) Put $S_{observed}$ into S_{true}

$$S_{true} = 1 - (1 - \underbrace{(0.1 + 0.7\sqrt[4]{v})}_{S_{observed}}) \cdot v^5$$

$$\underbrace{1 - (0.1 + 0.7\sqrt[4]{v})}_{0.9 - 0.7\sqrt[4]{v}}$$

A single constraint in terms of v. $\longrightarrow$ $\boxed{S_{true} = 1 - (0.9 - 0.7\sqrt[4]{v}) \cdot v^5}$

b) Equate S_{true} to 0.9

$$0.9 = 1 - (0.9 - 0.7\sqrt[4]{v}) \cdot v^5$$

$$\boxed{f(v) = \frac{1}{9 - 7\sqrt[4]{v}} - v^5 = 0}$$

Now we can use bisection method on the interval $[0, 1]$

$f(0) = 1/9$ $\quad f(1) = -0.5$ $\longrightarrow$ Condition is satisfied, we can start iteration. $f(0) \cdot f(1) < 0$

$c = \frac{a+b}{2}$ → midpoint

	a	b	c	f(c)
1)	0.000000	1.000000	0.500000	0.289909
2)	0.500000	1.000000	0.750000	0.164986
3)	0.750000	1.000000	0.875000	-0.064443
4)	0.750000	0.875000	0.812500	0.070698
5)	0.812500	0.875000	0.843750	0.008841
6)	0.843750	0.875000	0.859375	-0.026291
7)	0.843750	0.859375	0.851563	-0.008358
8)	0.843750	0.851563	0.847656	0.000332
9)	0.847656	0.851563	0.849609	-0.003990
10)	0.847656	0.849609	0.848633	-0.001823
11)	0.847656	0.848633	0.848145	-0.000744
12)	0.847656	0.848145	0.847900	-0.000206
13)	0.847656	0.847900	0.847778	0.000063

$$\boxed{r_c = 0.847778}$$

c) We found that $r = 0.847778$. Plug it into the equation below,

$$S_{observed} = 0.1 + 0.7\sqrt[4]{V}$$

$$= 0.1 + 0.7 \times 0.9595$$

$$= \underline{\underline{0.77165}}$$