Next Steps for the Colorado Risk-Limiting Audit (CORLA) Program

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Abstract. Colorado CRS 1-7-515 requires risk-limiting tabulation audits (RLAs) starting in 2017. Most Colorado counties (comprising 98.2% of voters) have voting equipment amenable to ballot-level comparison audits, but some are only able to perform ballot-polling audits. Combining ballot-polling and ballot-level comparison audits to audit cross-jurisdictional contests was an unsolved problem. Moreover, Colorado's current audit software (RLATool) does not support audits of cross-jurisdictional contests, even contests entirely contained in counties that can conduct ballot-level comparison audits. This paper addresses both gaps, along the way introducing a simple, efficient method to use stratified sampling in RLAs. (Stratification makes it easier to combine ballot-polling and ballot-level comparisons, and also useful to reduce the required level of coordination among jurisdictions to audit cross-jurisdictional contests.) We present simple but inefficient methods, more efficient methods that combine ballot polling and ballot-level comparisons using stratified samples, and methods that combine ballot-level comparison and variable-size batch comparison audits without stratification, noting the changes to RLATool that each of these methods would require. We provide open-source reference implementations of the preferred methods in Jupyter notebooks.

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1 Introduction

A risk-limiting audit (RLA) of an election is a procedure that has a known, pre-specified minimum chance of correcting the electoral outcome if the outcome is incorrect—that is, if the reported outcome differs from the outcome that a full manual tabulation of the votes would find. RLAs require a durable, voter-verifiable record of voter intent, such as paper ballots, and they assume that this audit trail is sufficiently complete and accurate that a full hand tally would show the true electoral outcome. That assumption is not automatically satisfied: a *compliance audit* [10] is required.

Risk-limiting audits are generally (but not necessarily) incremental: they examine more ballots, or batches of ballots, until either (i) there is strong statistical evidence that a full hand tabulation would confirm the outcome, or (ii) the audit has led to a full hand tabulation, the result of which should become the official result.

RLAs have been piloted in California, Colorado, and Ohio, and a test of RLA procedures has been conducted in Arizona. RLA bills are being drafted or are already under consideration in California, Virginia, Washington, and other states. A number of laws have either allowed or mandated risk-limiting audits, including California AB 2023 (Saldaña), SB 360 (Padilla), and AB 44 (Mullin); Rhode Island SB 413A and HB 5704A; and Colorado Revised Statutes (CRS) 1-7-515. At the time of writing, California is considering another RLA bill, AB 2125.

CRS 1-7-515 requires Colorado to conduct risk-limiting audits beginning in 2017. (There are provisions to allow the Secretary of State to exempt some counties.) The first set of coordinated risk-limiting election audits across the state took place in Colorado in November, 2017.[4] Those audits only covered contests restricted to a single county, so counties could conduct audits independently. To audit statewide elections and contests that cross county lines, Colorado will need to implement new approaches and modify RLATool[5], the open-source audit software used for their 2017 audits.

Colorado's "uniform voting system" program[6] led many Colorado counties to purchase (or to plan to purchase) voting systems that are auditable at the ballot level: those systems export cast vote records (CVRs) for individual ballots in a manner that allows the corresponding paper ballot to be identified, and conversely, make it possible to find the CVR corresponding to any particular paper ballot. We call counties that have such systems "CVR" counties. It is estimated that by June, 2018, 98.2% of active Colorado voters will be in CVR counties. CVR counties can perform "ballot-level comparison audits," [3] which are currently the most efficient approach to risk-limiting audits in that they require examining fewer ballots than other methods do, when the outcome of the contest under audit is in fact correct.

Voting systems in other counties ("legacy" or "no-CVR" counties) do not allow auditors to check how the system interpreted voter intent for individual ballots. The election results can still be audited, provided the voting systems create a voter-verifiable paper trail (*e.g.*, voter-marked paper ballots) that is conserved to ensure that it remains accurate and intact, and organized well enough to permit ballots to be selected at random. Pilot audits in California

[4] See https://www.sos.state.co.us/pubs/elections/RLA/2017RLABackground.html

[5] https://github.com/FreeAndFair/ColoradoRLA/

[6] https://www.sos.state.co.us/pubs/elections/VotingSystems/UniformVotingSystem.html

suggest that the most efficient way to audit such systems is by "ballot-polling" [2,3] (in contrast to "batch-level comparisons," for example).

There is currently no literature on how to perform risk-limiting audits of contests that include CVR counties and no-CVR counties by combining ballot polling and ballot-level comparisons. Existing methods would either require all counties to use the lowest common denominator, ballot-polling (which does not take advantage of the CVRs, and thus is expected to require more auditing than a method that does), or would require no-CVR counties to perform batch-level comparisons, which were found in California to be (generally) less efficient than ballot-polling audits.[7] The audit software RLATool needs additional features to be able to audit contests that cross county lines.

First, the current version (1.1.0) of RLATool needs to be modified to recognize and group together contests that cross jurisdictional boundaries; currently, it treats every contest as if it were entirely contained in a single county. RLATool also does not allow the user to select the sample size, nor does it directly allow an unstratified random sample to be drawn across counties.

Second, to audit a contest that includes votes in no-CVR counties and votes in CVR counties, new statistical methods are needed to preserve the efficiency of ballot-level comparison audits in CVR counties.

We focus on near-term requirements for risk-limiting audits in Colorado, incidentally developing a new method for using stratified samples in RLAs that is widely applicable. Section 3 presents crude but inefficient approaches that could be implemented easily. Section 4 presents an approach based on comparison audits with different batch sizes. This approach is statistically simple and relatively efficient, but might require changing how counties handle their ballots. Section 5 presents our recommended approach, which combines ballot-level comparisons in counties that can perform them with ballot-polling in the no-CVR counties. All the approaches require new software, including some changes to RLATool. We provide example software implementing the risk calculations for our recommended approach as a Python Jupyter notebook.[8] Sections 6 and 7 explain the recommended modifications to ballot-level comparison and ballot-polling audits, respectively. Section 8 summarizes our recommendations and considerations for implementation.

2 Preliminary notation

Here and generally throughout the paper, we discuss auditing a single plurality contest at a time, although the same sample can be used to audit more than one contest (and super-majority contests), and there are ways of combining audits of different contests into a single process [6,8]. We use terminology drawn from

[7] See [4] for a different (Bayesian) approach to auditing contests that include both CVR counties and no-CVR counties. Bayesian audits are not, in general, risk-limiting.

[8] See https://github.com/pbstark/CORLA18.

a number of papers; the key reference is Lindeman and Stark, 2012 [3]. An *overstatement error* is an error that caused the margin between *any* reported winner and *any* reported loser to appear larger than it really was. An *understatement error* is an error that caused the margin between *every* reported winner and *every* reported loser to appear to be smaller than it really was.

Throughout, we will refer to a contest between reported winner w and reported loser ℓ. The total number of reported votes for candidate w is denoted V_w and the total for candidate ℓ is denoted V_ℓ, so that $V_w > V_\ell$, since w is the reported winner.

We introduce additional notation below, as needed.

3 Simple approaches

3.1 Hand count the legacy counties

The simplest approach to combining legacy counties with CVR counties is to require every legacy county to do a full hand count, and to conduct a ballot-level comparison audit in CVR counties, based on contest margins adjusted for the results of the manual tallies in the CVR counties. For instance, imagine a contest with two candidates, reported winner w and reported loser ℓ. Suppose that a full manual tally of the votes in the legacy counties shows V'_w votes for w and V'_ℓ votes for ℓ. Suppose that a total of N ballots were cast in the CVR counties. Then the *diluted margin* for the comparison audit in the CVR counties is defined to be $[(V_w - V'_w) - (V_\ell - V'_\ell)]/N$. Requiring a full hand count in the legacy counties has obvious disadvantages, except perhaps in very close contests where ballot polling is not efficient. (But it does have the advantage of not forcing CVR counties to do additional auditing to compensate for the legacy counties.)

3.2 Treat legacy counties as if every ballot selected from them for audit has a two-vote overstatement

A third simple-but-pessimistic approach is to sample uniformly from all counties as if one were performing a ballot-level comparison audit everywhere, but to treat any ballot selected from a legacy county as a two-vote overstatement. This approach has the same disadvantages as the previous approach.

4 Variable batch sizes

Another approach is to perform a comparison audit across all counties, but to use batches consisting of more than one ballot (batch-level comparisons) in legacy counties and batches consisting of a single ballot (ballot-level comparisons) in CVR counties.[9] This requires that the no-CVR counties report vote subtotals

[9] For majority and plurality elections, including those in which voters can select more than one candidate, audits can be based on overstatement and understatement errors at the level of batches.

for physically identifiable batches. If a county's voting system can only report subtotals by precinct but the county does not sort paper ballots by precinct, this approach might require revising how the county handles its paper; we understand that this is the case in many Colorado counties.

That said, many California counties that do not sort vote-by-mail (VBM) ballots by precinct conduct the statutory 1% audits by manually retrieving the ballots for just those precincts selected for audit from whatever physical batches they happen to be in: the situation is identical to that in Colorado.

Another solution is the "Boulder-style" batch-level audit,[10] which requires generating vote subtotals after each physical batch is scanned, and exporting those subtotals in machine-readable form. That in turn may require using extra memory cards, repeatedly initializing and deleting tabulation databases, or other measures that add complexity and opportunity for error.

While those two approaches are laborious, they would provide a viable short-term solution, especially combined with information from SCORE to check that the reported batch-level results contain the correct number of ballots for each contest under audit. Moreover, it does not unduly increase the workload in CVR counties to compensate for legacy equipment.

Variable-batch-size comparison audits would require modifying or augmenting RLATool in several ways:

1. The CVR reporting tool would need to be modified to allow no-CVR counties to report batch-level results in a manner analogous to how CVR counties report ballot-level results, or an external tool would need to be provided.
2. The sampling algorithm would have to allow sampling batches with unequal probability. Efficient batch-level audits involve sampling batches with probability proportional to a bound on the possible overstatement error in the batch. It would also need to calculate the appropriate sampling probability for each batch. Again, this could be accommodated using an external tool to draw the sample from legacy counties.
3. The risk calculations would need to be modified. This, too, could be done with external software, with suitable provisions for capturing audit data from RLATool or directly from legacy counties.

None of these changes is enormous; the mathematics and statistics are already worked out in published papers **PROVIDE CITES** that include exemplar code for calculating the batch-level error bounds, drawing the samples with probability proportional to an error bound, and calculating the attained risk from the sample results. Indeed, this is the method that was used in several of California's pilot audits, including the audit in Orange County. Section 6 derives a method for comparison audits with variable batch sizes.

[10] See http://bcn.boulder.co.us/~neal/elections/boulder-audit-10-11/.

5 Stratified "hybrid" audits

Other approaches involve *stratification*: partitioning the cast ballots into non-overlapping groups and sampling independently from those groups. One could stratify by county, but in general it is simpler and more efficient statistically (i.e., results in auditing fewer ballots) to minimize the number of strata. We consider methods that use two strata: CVR counties and no-CVR counties. Collectively, the ballots cast in CVR counties comprise one stratum and the ballots cast in legacy counties comprise a second stratum; every ballot cast in the contest is in exactly one of the two strata. We assume that the samples are drawn from the two strata independently.

Incidentally, the stratified method we present is applicable to other stratified audits, not just those with one stratum comprised of CVR counties and another stratum comprised of no-CVR counties.

5.1 Partitioning the total permissible overstatement into strata

The simplest approach to stratification involves partitioning the risk limit and the tolerable overstatement error of the tabulation into two pieces, one for the (pooled) CVR counties and one for the (pooled) no-CVR counties. Let $V_{w\ell} > 0$ denote the contest-wide margin (in votes) of reported winner w over reported loser ℓ. Let $V_{w\ell,s}$ denote the margin (in votes) of reported winner w over reported loser ℓ in stratum s. Note that $V_{w\ell,s}$ could be negative in one stratum. Let $A_{w\ell}$ denote the margin (in votes) of reported winner w over reported loser ℓ that a full hand count of the entire contest would show, that is, the *actual* margin rather than the *reported* margin. Reported winner w really beat reported loser ℓ if and only if $A_{w\ell} > 0$. Define $A_{w\ell,s}$ to be the actual margin (in votes) of w over ℓ in stratum s; this too may be negative.

Let $\omega_{w\ell,s} \equiv V_{w\ell,s} - A_{w\ell,s}$ be the *overstatement* of the margin of w over ℓ in stratum s. Reported winner w really beat reported loser ℓ if and only if $\omega_{w\ell} \equiv \omega_{w\ell,1} + \omega_{w\ell,2} < V_{w\ell}$.

Pick $\lambda_1 \in \Re$ and define $\lambda_2 = 1 - \lambda_1$. These values can be used to partition the total tolerable overstatement between the two strata: If $\omega_{w\ell,1} < \lambda_1 V_{w\ell}$ *and* $\omega_{w\ell,2} < \lambda_2 V_{w\ell}$, candidate w really received more votes than candidate ℓ. This condition is sufficient, but not necessary, to confirm that $\omega_{w\ell,1} + \omega_{w\ell,2} < V_{w\ell}$. Some (λ_1, λ_2) pairs can be ruled out *a priori*, because (for instance) $\omega_{w\ell,s} \in [-2N_s, 2N_s]$, where N_s is the number of ballots cast in stratum s.

For ballot-level comparison audits, auditing to ensure that $\omega_{w\ell,s} < \lambda_s V_{w\ell}$ is discussed in section 6. It is a minor modification of the method embodied in RLATool.

For ballot-polling audits, auditing to ensure that $\omega_{w\ell,s} < \lambda_s V_{w\ell}$ is discussed in section 7. Note that this requires a more substantial modification of the standard ballot-polling calculations, because the standard calculations consider only the fraction of ballots with a vote for either w or ℓ that contain a vote for w, while we need to make an inference about the difference between the number of votes for w and the number of votes for ℓ. This introduces an additional unknown nuisance parameter, the number of ballots with votes for either w or ℓ.

5.2 Constraining the total overstatement across strata

The goal is to constrain the *sum* of the overstatement errors in the two strata, rather than constrain the pieces separately. The null hypothesis $\omega_{w\ell,1} + \omega_{w\ell,2} \geq V_{w\ell}$ is true if and only if there exists *some* values of λ_1 and λ_2 such that $\omega_{w\ell,s} \geq \lambda_s V_{w\ell}$, $s = 1, 2$.[11] Thus, fixing λ_1 and λ_2 at single values and requiring that we reject both stratum-level null hypotheses would be unnecessarily conservative: there are many ways that the total overstatement could be less than $V_{w\ell}$ (i.e., the alternative hypothesis is true) without having the overstatement $\omega_{w\ell,s}$ in stratum s less than $\lambda_s V_{w\ell}$, $s = 1, 2$.

To that end, imagine *all* values ways of partitioning the error. If, for all (λ_1, λ_2) pairs, we can reject the hypothesis that the overstatement error in stratum 1 is greater than or equal to $\lambda_1 V_{w\ell}$ *and* the overstatement error in stratum 2 is greater than or equal to $\lambda_2 V_{w\ell}$, then we can conclude that the outcome is correct. This is efficient because it only requires rejecting one of the two stratum-wise null hypotheses, for all possible (λ_1, λ_2) pairs, rather than rejecting both null hypotheses for a particular pair.

To test the conjunction hypothesis (i.e., that both of those null hypotheses are true), we use Fisher's combining function. Let $p_s(\lambda_s)$ be the p-value of the hypothesis $\omega_{w\ell,s} \geq \lambda_s V_{w\ell}$. If the null hypothesis that $\omega_{w\ell,1} \geq \lambda_1 V_{w\ell}$ and $\omega_{w\ell,2} \geq \lambda_2 V_{w\ell}$ is true, then the combination

$$\chi(\lambda_1, \lambda_2) = -2 \sum_{s=1}^{2} \ln p_s(\lambda_s) \tag{1}$$

has a probability distribution that is dominated by the chi-square distribution with 4 degrees of freedom.[12] Fisher's combined statistic will be small when both null hypotheses are true and will be large when at least one null hypothesis is not true.

Hence, if, for all λ_1 and $\lambda_2 = 1 - \lambda_1$, the combined statistic $\chi(\lambda_1, \lambda_2)$ is greater than the $1 - \alpha$ quantile of the chi-square distribution with 4 degrees of freedom, the audit can stop.

This procedure involves maximizing Fisher's combined statistic over all pairs (λ_1, λ_2). For Colorado's audits, the calculation of $p_s(\lambda)$ uses the procedures discussed in sections 6 and 7. In general, $p_s(\lambda)$ could be the attained risk of any risk-limiting audit.

[11] Namely, letting $\lambda_1 = \frac{\omega_{w\ell,1}}{\omega_{w\ell,1} + \omega_{w\ell,2}}$ satisfies both inequalities.

[12] If the two tests had continuously distributed p-values, the distribution would be exactly chi-square with four degrees of freedom, but if either p-value has atoms when the null hypothesis is true, it is in general stochastically smaller. This follows from a coupling argument along the lines of Theorem 4.12.3 in [1].

6 Batch comparison audits of a tolerable overstatement in votes

In this section we expand previous comparison auditing work (already embodied in RLATool) to handle two new requirements. The first allows the specification of the λ parameters discussed in section 5. The second handles batch-level auditing.

The first requirement is to test whether the overstatement of any margin (in votes) exceeds some fraction λ of the overall margin $V_{w\ell}$ between reported winner w and reported loser ℓ. If the stratum contains all the ballots cast in the contest, then for $\lambda = 1$, this would confirm the election outcome. For stratified audits, we might want to test other values of λ, as described above.

In Colorado, comparison audits have been ballot-level (i.e., batches consisting of a single ballot). This section also addresses the second requirement by deriving a method for batches of arbitrary size, which might be useful for Colorado to audit contests that include CVR counties and legacy counties. We keep the *a priori* error bounds tighter than the "super-simple" method [8]. To keep the notation simpler, we consider only a single contest, but the MACRO test statistic [6,8] automatically extends the result to auditing $C > 1$ contests simultaneously. The derivation is for plurality contests, including "vote-for-k" plurality contests. Majority and super-majority contests are a minor modification [5].[13]

6.1 Notation

- $\mathcal{W}$: the set of reported winners of the contest
- $\mathcal{L}$: the set of reported losers of the contest
- N_s ballots were cast in all in the stratum. (The contest might not appear on all N_s ballots.)
- P "batches" of ballots are in stratum s. A batch contains one or more ballots. Every ballot in stratum s is in exactly one batch.
- n_p: number of ballots in batch p. $N_s = \sum_{p=1}^{P} n_p$.
- $v_{pi} \in \{0, 1\}$: the reported votes for candidate i in batch p
- $a_{pi} \in \{0, 1\}$: actual votes for candidate i in batch p. If the contest does not appear on any ballot in batch p, then $a_{pi} = 0$.
- $V_{w\ell,s} \equiv \sum_{p=1}^{P}(v_{pw} - v_{p\ell})$: Reported margin in stratum s of reported winner $w \in \mathcal{W}$ over reported loser $\ell \in \mathcal{L}$, in votes.
- $V_{w\ell}$: Overall reported margin of reported winner $w \in \mathcal{W}$ over reported loser $\ell \in \mathcal{L}$, in votes, for the entire contest (not just stratum s)
- V: smallest reported overall margin between any reported winner and reported loser: $V \equiv \min_{w\in\mathcal{W},\ell\in\mathcal{L}} V_{w\ell}$
- $A_{w\ell,s} \equiv \sum_{p=1}^{P}(a_{pw} - a_{p\ell})$: actual margin in the stratum of reported winner $w \in \mathcal{W}$ over reported loser $\ell \in \mathcal{L}$, in votes
- $A_{w\ell}$: actual margin of reported winner $w \in \mathcal{W}$ over reported loser $\ell \in \mathcal{L}$, in votes, for the entire contest (not just in stratum s)

[13] So are some forms of preferential and approval voting, such as Borda count, and proportional representation contests, such as D'Hondt [9]. Changes for IRV/STV are more complicated.

6.2 Reduction to maximum relative overstatement

If the contest is entirely contained in stratum s, then the reported winners of the contest are the actual winners if

$$\min_{w \in \mathcal{W}, \ell \in \mathcal{L}} A_{w\ell,s} > 0.$$

Here, we address the case that the contest may include a portion outside the stratum. To combine independent samples in different strata, it is convenient to be able to test whether the net overstatement error in a stratum exceeds a given threshold.

Instead of testing that condition directly, we will test a condition that is sufficient but not necessary for the inequality to hold, to get a computationally simple test that is still conservative (i.e., the risk is not larger than its nominal value).

For every winner, loser pair (w, ℓ), we want to test whether the overstatement error exceeds some threshold, generally one tied to the reported margin between w and ℓ. For instance, for stratified hybrid audit, we set the threshold to be $\lambda_s V_{w\ell}$.

We want to test whether

$$\sum_{p=1}^{P} (v_{pw} - a_{pw} - v_{p\ell} + a_{p\ell})/V_{w\ell} \geq \lambda_s.$$

The maximum of sums is not larger than the sum of the maxima; that is,

$$\max_{w \in \mathcal{W}, \ell \in \mathcal{L}} \sum_{p=1}^{P} (v_{pw} - a_{pw} - v_{p\ell} + a_{p\ell})/V_{w\ell} \leq \sum_{p=1}^{P} \max_{w \in \mathcal{W}, \ell \in \mathcal{L}} (v_{pw} - a_{pw} - v_{p\ell} + a_{p\ell})/V_{w\ell}.$$

Define

$$e_p \equiv \max_{w \in \mathcal{W} \ell \in \mathcal{L}} (v_{pw} - a_{pw} - v_{p\ell} + a_{p\ell})/V_{w\ell}.$$

Then no reported margin is overstated by a fraction λ_s or more if

$$E \equiv \sum_{p=1}^{P} e_p < \lambda_s.$$

Thus if we can reject the hypothesis $E \geq \lambda_s$, we can conclude that no pairwise margin was overstated by as much as a fraction λ_s.

Testing whether $E \geq \lambda_s$ would require a very large sample if we knew nothing at all about e_p without auditing batch p: a single large value of e_p could make E arbitrarily large. But there is an *a priori* upper bound for e_p. Whatever the reported votes v_{pi} are in batch p, we can find the potential values of the actual votes a_{pi} that would make the error e_p largest, because a_{pi} must be between 0 and n_p, the number of ballots in batch p:

$$\frac{v_{pw} - a_{pw} - v_{p\ell} + a_{p\ell}}{V_{w\ell}} \leq \frac{v_{pw} - 0 - v_{p\ell} + n_p}{V_{w\ell}}.$$

Hence,

$$e_p \le \max_{w\in\mathcal{W},\ell\in\mathcal{L}} \frac{v_{pw} - v_{p\ell} + n_p}{V_{w\ell}} \equiv u_p. \tag{2}$$

Knowing that $e_p \le u_p$ might let us conclude reliably that $E < \lambda_s$ by examining only a small number of batches—depending on the values $\{u_p\}_{p=1}^P$ and on the values of $\{e_p\}$ for the audited batches.

To make inferences about E, it is helpful to work with the *taint* $t_p \equiv \frac{e_p}{u_p} \le 1$. Define $U \equiv \sum_{p=1}^P u_p$. Suppose we draw batches at random with replacement, with probability u_p/U of drawing batch p in each draw, $p = 1, \ldots, P$. (Since $u_p \ge 0$, these are all positive numbers, and they sum to 1, so they define a probability distribution on the P batches.)

Let T_j be the value of t_p for the batch p selected in the jth draw. Then $\{T_j\}_{j=1}^n$ are IID, $\mathbb{P}\{T_j \le 1\} = 1$, and

$$\mathbb{E}T_1 = \sum_{p=1}^P \frac{u_p}{U} t_p = \frac{1}{U}\sum_{p=1}^P u_p \frac{e_p}{u_p} = \frac{1}{U}\sum_{p=1}^P e_p = E/U.$$

Thus $E = U\mathbb{E}T_1$. So, if we have strong evidence that $\mathbb{E}T_1 < \lambda_s/U$, we have strong evidence that $E < \lambda_s$.

This approach can be simplified even further by noting that u_p has a simple upper bound that does not depend on v_{pi}. At worst, the reported result for batch p shows n_p votes for the "least-winning" apparent winner of the contest with the smallest margin, but a hand interpretation would show that all n_p ballots in the batch had votes for the runner-up in that contest. Since $V_{w\ell} \ge V$ and $0 \le v_{pi} \le n_p$,

$$u_p = \max_{w\in\mathcal{W},\ell\in\mathcal{L}} \frac{v_{pw} - v_{p\ell} + n_p}{V_{w\ell}} \le \max_{w\in\mathcal{W},\ell\in\mathcal{L}} \frac{n_p - 0 + n_p}{V_{w\ell}} \le \frac{2n_p}{V}.$$

Thus if we use $2n_p/V$ in lieu of u_p, we still get conservative results. (We also need to re-define U to be the sum of those upper bounds.) An intermediate, still conservative approach would be to use this upper bound for batches that consist of a single ballot, but use the sharper bound (2) when $n_p > 1$. Regardless, for the new definition of u_p and U, $\{T_j\}_{j=1}^n$ are IID, $\mathbb{P}\{T_j \le 1\} = 1$, and

$$\mathbb{E}T_1 = \sum_{p=1}^P \frac{u_p}{U} t_p = \frac{1}{U}\sum_{p=1}^P u_p \frac{e_p}{u_p} = \frac{1}{U}\sum_{p=1}^P e_p = E/U.$$

So, if we have evidence that $\mathbb{E}T_1 < \lambda_s/U$, we have evidence that $E < \lambda_s$.

6.3 Testing $\mathbb{E}T_1 \ge \lambda_s/U$

To test whether $\mathbb{E}T_1 < \lambda_s/U$, there are a variety of methods available. One particularly "clean" sequential method is based on Wald's Sequential Probability Ratio Test (SPRT) ([11]). Harold Kaplan pointed out this method on a website that no longer exists. A derivation of this "Kaplan-Wald" method is given in Appendix A of [9]; to apply the method here, take $t = \lambda_s$ in their equation 18.

A different sequential method, the Kaplan-Markov method (also due to Harold Kaplan), is given in [7].

7 Ballot-polling audits of a tolerable overstatement in votes

In this section we develop a new method to conduct ballot-polling audits in legacy counties. This is a substantial change from existing ballot-polling methods [2], because specifying the λ parameters discussed in section 5complicates the statistical problem.

Existing ballot-polling methods consider only the fraction of ballots with a vote for either w or ℓ that contain a vote for w, making the statistical test one for a proportion. In this case, we need to make an inference about the *difference* between the number of votes for w and the number of votes for ℓ. This requires dealing with an unknown nuisance parameter not needed in the existing methods, the number of ballots with votes for either w or ℓ.

We address this nuisance parameter in two ways. First, our proposal explicitly takes into account ballots with no vote for w or for ℓ, including ballots for other candidates and invalid ballots, Second, the risk is maximized over all possible values of the nuisance parameter, ensuring that the test is conservative.

7.1 Conditional tri-hypergeometric test

We consider a single stratum s, containing N_s ballots. Of the N_s ballots, $A_{w,s}$ have a vote for w but not for ℓ, $A_{\ell,s}$ have a vote for ℓ but not for w, and $A_{u,s} = N_s - N_{w,s} - N_{\ell,s}$ have votes for both w and ℓ or neither w nor ℓ, including undervotes and invalid ballots. We might draw a simple random sample of n ballots (n fixed ahead of time), or we might draw sequentially without replacement, so the sample size B could be random. For instance, the rule for determining B could depend on the data.[14]

Regardless, we assume that, conditional on the attained sample size n, the ballots are a simple random sample of size n from the N_s ballots in the population. In the sample, B_w ballots contain a vote for w but not ℓ, with B_ℓ and B_u defined analogously. The conditional joint distribution of (B_w, B_ℓ, B_u) is tri-hypergeometric:

$$\mathbb{P}_{A_{w,s},A_{\ell,s}}\{B_w = i, B_\ell = j | B = n\} = \frac{\binom{A_{w,s}}{i}\binom{A_{\ell,s}}{j}\binom{N_s - A_{w,s} - A_{\ell,s}}{n-i-j}}{\binom{N_s}{n}}. \qquad (3)$$

[14] Sampling with replacement leads to simpler arithmetic, but is not as efficient.

Define the diluted sample margin, $D \equiv (B_w - B_\ell)/B$. We want to test the compound hypothesis $A_{w,s} - A_{\ell,s} \le c$. The value of c is inferred from the definition $\omega_{w\ell,s} \equiv V_{w\ell,s} - A_{w\ell,s} = V_{w,s} - V_{\ell,s} - (A_{w,s} - A_{\ell,s})$. Thus,

$$c = V_{w,s} - V_{\ell,s} - \omega_{w\ell,s} = V_{w\ell,s} - \lambda_s V_{w\ell}.$$

The alternative is the compound hypothesis $A_{w,s} - A_{\ell,s} > c$.[15] Hence, we will reject for large values of D. Conditional on $B = n$, the event $D = (B_w - B_\ell)/B = d$ is the same as $B_w - B_\ell = nd$.[16]

The p-value of the simple hypothesis that there are $A_{w,s}$ ballots with a vote for w but not for ℓ, $A_{\ell,s}$ ballots with a vote for ℓ but not for w, and $N - A_{w,s} - A_{\ell,s}$ ballots with votes for both w and ℓ or neither w nor ℓ (including undervotes and invalid ballots) is the probability that $B_w - B_\ell \ge nd$. Therefore,

$$\mathbb{P}_{A_{w,s},A_{\ell,s},N_s}\{D \ge d \mid B = n\} = \sum_{\substack{(i,j):i,j\ge 0\\ i-j\ge nd\\ i+j\le n}} \frac{\binom{A_{w,s}}{i}\binom{A_{\ell,s}}{j}\binom{N_s - A_{w,s} - A_{\ell,s}}{n-i-j}}{\binom{N_s}{n}}. \tag{4}$$

7.2 Maximizing the *p*-value over the null set

The composite null hypothesis does not specify $A_{w,s}$ or $A_{\ell,s}$ separately, only that $A_{w,s} - A_{\ell,s} \le c$ for some fixed, known c. Define $\mathcal{S}$ to be the set of pairs (i,j) such that $i, j \ge 0, i - j \ge nd$, and $i + j \le n$. The (conditional) p-value of this composite hypothesis for $D = d$ is the maximum p-value for all values $(A_{w,s}, A_{\ell,s})$ that are possible under the null hypothesis,

$$\max_{A_{w,s},A_{\ell,s}\in\{0,1,\ldots,N\}:A_{w,s}-A_{\ell,s}\le c, A_{w,s}+A_{\ell,s}\le N_s} \sum_{(i,j)\in\mathcal{S}} \frac{\binom{A_{w,s}}{i}\binom{A_{\ell,s}}{j}\binom{N_s - A_{w,s} - A_{\ell,s}}{n-i-j}}{\binom{N_s}{n}}, \tag{5}$$

wherever the summand is defined. (Equivalently, define $\binom{m}{k} \equiv 0$ if $k > m$, $k < 0$, or $m \le 0$.)

Optimizing over the parameter *c* The following result enables us to only test hypotheses along the boundary of the null set.

Theorem 1 *Assume that* $n < A_{w,s} + A_{\ell,s}$. *Suppose the composite null hypothesis is* $N_w - N_\ell \le c$. *The p-value is maximized on the boundary of the null region, i.e. when* $N_w - N_\ell = c$.

[15] To use Wald's Sequential Probability Ratio Test, we might pick a simple alternative instead, e.g., $A_{w,s} = V_{w,s}$ and $A_{\ell,s} = V_{\ell,s}$, the reported values, provided $V_{w,s} - V_{\ell,s} > c$.

[16] In contrast, the BRAVO ballot-polling method [2] conditions only on $B_w + B_\ell = m$.

Proof. Without loss of generality, let $c = 0$ and assume that $A_{u,s} = N_s - A_{w,s} - A_{\ell,s}$ is fixed. Let $N_{w\ell,s} \equiv A_{w,s} + A_{\ell,s}$ be the fixed, unknown number of ballots for w or for ℓ in stratum s. The p-value p_0 for the simple hypothesis that $c = 0$ is

$$p_0 = \sum_{(i,j)\in\mathcal{S}} \frac{\binom{N_{w\ell,s}/2}{i}\binom{N_{w\ell,s}/2}{j}\binom{A_{u,s}}{n-i-j}}{\binom{N_s}{n}} = \sum_{(i,j)\in\mathcal{S}} T_{ij}, \tag{6}$$

where T_{ij} is defined as the (i, j) term in the summand and $T_{ij} \equiv 0$ for pairs (i, j) that don't appear in the summation.

Assume that $c > 0$ is given. The p-value p_c for this simple hypothesis is

$$\begin{aligned} p_c &= \sum_{(i,j)\in\mathcal{S}} \frac{\binom{(N_{w\ell,s}+c)/2}{i}\binom{(N_{w\ell,s}-c)/2}{j}\binom{A_{u,s}}{n-i-j}}{\binom{N_s}{n}} \\ &= \sum_{(i,j)\in\mathcal{S}} T_{ij} \frac{\frac{N_{w\ell,s}+c}{2}(\frac{N_{w\ell,s}+c}{2}-1)\cdots(\frac{N_{w\ell,s}}{2}+1)(\frac{N_{w\ell,s}-c}{2}-j)\cdots(\frac{N_{w\ell,s}}{2}-1-j)}{(\frac{N_{w\ell,s}+c}{2}-i)\cdots(\frac{N_{w\ell,s}}{2}+1-i)(\frac{N_{w\ell,s}-c}{2})\cdots(\frac{N_{w\ell,s}}{2}-1)}. \end{aligned}$$

Terms in the fraction can be simplified: choose the corresponding pairs in the numerator and denominator. Fractions of the form $\frac{\frac{N_{w\ell,s}}{2}+a}{\frac{N_{w\ell,s}}{2}+a-i}$ can be expressed as $1+\frac{i}{\frac{N_{w\ell,s}}{2}+a-i}$. Fractions of the form $\frac{\frac{N_{w\ell,s}}{2}-a-j}{\frac{N_{w\ell,s}}{2}-a}$ can be expressed as $1-\frac{j}{\frac{N_{w\ell,s}}{2}-a}$. Thus, the p-value can be written as

$$\begin{aligned} p_c &= \sum_{(i,j)\in\mathcal{S}} T_{ij} \prod_{a=1}^{c/2} \left(1+\frac{i}{\frac{N_{w\ell,s}}{2}+a-i}\right)\left(1-\frac{j}{\frac{N_{w\ell,s}}{2}-a}\right) \\ &> \sum_{(i,j)\in\mathcal{S}} T_{ij} \left[\left(1+\frac{i}{\frac{N_{w\ell,s}+c}{2}-i}\right)\left(1-\frac{j}{\frac{N_{w\ell,s}}{2}+1}\right)\right]^{c/2} \\ &= \sum_{(i,j)\in\mathcal{S}} T_{ij} \left[1+\frac{\frac{N_{w\ell,s}+c}{2}j+\frac{N_{w\ell,s}}{2}i+i}{(\frac{N_{w\ell,s}+c}{2}-i)(\frac{N_{w\ell,s}}{2}+1)}\right]^{c/2} \\ &> \sum_{(i,j)\in\mathcal{S}} T_{ij} \\ &= p_0 \end{aligned}$$

The last inequality follows from the fact that i and j are nonnegative, and that $i < \frac{N_{w\ell,s}+c}{2}$ – it is a possible outcome under the null hypothesis.

Optimizing over the parameter $A_{w,s}$ We have shown empirically (but do not prove) that this tail probability, as a function of $A_{w,s}$, has a unique maximum at one of the endpoints when $A_{w,s}$ is either as small or as large as possible, given N, c, and the observed sample values B_w and B_ℓ. If the empirical result is true, then finding the maximum is trivial; otherwise, it is a trivial one-dimensional optimization problem to compute the unconditional p-value.

7.3 Conditional testing

If the conditional tests are always conducted at significance level α or less, i.e., so that $\mathbb{P}\{\text{Type I error}|B = n\} \leq \alpha$, then the overall procedure has significance level α or less:

$$\begin{aligned}\mathbb{P}\{\text{Type I error}\} &= \sum_{n=0}^{N} \{\text{Type I error}|B = n\}\mathbb{P}\{B = n\} \\ &\leq \sum_{n=0}^{N} \alpha \mathbb{P}\{B = n\} = \alpha. \qquad (7)\end{aligned}$$

In particular, this implies that our conditional hypergeometric test will have the correct risk limit unconditionally.

8 Recommendations

We have outlined several methods Colorado might use to audit cross-jurisdictional contests that include CVR counties and no-CVR counties. We expect that stratified "hybrid" audits will be the most palatable, given the constraints on time for software development and the logistics of the audit itself, because the workflow for counties would be the same as it was in November, 2017. Additionally, we expect that using this approach would tend to minimize the workload, compared to the other methods.

What would change is the risk calculation "behind the scene," including the algorithms used to decide when the audit can stop. Those algorithms could be implemented in software external to RLATool. The minimal modification to RLATool that would be required to conduct a hybrid audit is to allow the sample size from each county to be controlled externally, e.g. by uploading a parameter file once per round, rather than using a formula that is based on the margin within that county alone. The parameter file would be generated by external software that does the audit calculations described here based on the detailed audit progress and discrepancy data available from RLATools' rla_export command.

8.1 Software and examples

Examples of stratified hybrid audits are in Jupyter notebooks available at https://www.github.com/pbstark/CORLA18. The first two examples are contained in a single notebook, "hybrid-audit-example-1". The first example is a hypothetical medium-sized election with a total of $110,000$ votes and a diluted margin of 1.8%. 9.1% of the ballots come from no-CVR counties. The risk limit is 10%. In 93 of 100 simulations, a stratified "hybrid" audit with samples of 400 ballots in the CVR stratum and 600 ballots in the no-CVR stratum and risk limit 10% would have sufficed to confirm that the total overstatement error did not exceed the overall margin. This combined method would require a total of 1,000 ballots to be examined.

If a CVR were available for all counties and we could have run a ballot-level comparison audit for the entire contest, rather than stratifying, an audit with risk limit 10% that found no errors would have concluded after examining just 263 ballots. The efficiency gained comes from two sources. First, ballot-level comparison audits are substantially more efficient than ballot-polling audits. If no CVR were available in any county and we had run a ballot-polling audit for the entire contest, the expected sample size needed to confirm the audit with a risk limit of 10% would have been 13,988. The ballot-polling audit would require checking over 10% of ballots cast, an infeasible number in practice. Second, the hybrid audit divides the margin and risk limit between two strata under the hood. To attain a Fisher's combined p-value less than the risk limit, only one stratum-level risk needs to be small; the risk in the other stratum may be large. For instance, if the stratum-level risks were 90% and 2%, then the Fisher's combined p-value would be less than 10%, but if the stratum-level risks were 45% and 5%, it would not. Sample sizes must be increased to ensure that one or both of the stratum-level risks is small enough. The workload is more than a pure ballot-comparison audit, but substantially less than a pure ballot-polling audit.

Another method discussed in Section 3.2 is to perform a ballot-level comparison audit statewide, but to treat any ballot sampled from the no-CVR county as showing a two-vote overstatement. In this example, this worst-case method would lead to a full hand count. However, the situation may be more optimistic for Colorado: if only 1.2% of ballots came from the no-CVR stratum and the overall margin were in fact 10,000 votes, then this method would require checking 430 ballots.

The second example is a hypothetical large statewide election with a total of 2 million ballots and a diluted margin of nearly 20%. The risk limit is 5%. If a CVR were available for all counties and we could have run a ballot-level comparison audit for the entire contest, rather than stratifying, an audit with risk limit 5% that found no errors would have concluded after examining just 31 ballots. Using Fisher's method to conduct a hybrid audit does not increase the workload substantially: In 100% of 10,000 simulations, an audit with samples of 50 ballots in the CVR stratum and 20 ballots in the no-CVR stratum would have sufficed to confirm that the total overstatement error did not exceed the overall margin at the 5% risk limit.

A second notebook, "hybrid-audit-example-2," illustrates the workflow for conducting a hybrid audit of this kind. The example election has a total of 2 million ballots. The reported margin is just over 1%, but in reality the vote totals for the reported winner and reported loser are identical in both strata. The risk limit is 5%. We use Fisher's method to combine the audits in the CVR stratum and no-CVR stratum. The sample sizes in each stratum are large, but the maximum Fisher's combined p-value is over 20%, so the audit cannot be terminated. In each case, the audit leads to a full recount of all the ballots.

These notebooks can be modified and run with different contest sizes, margins, and risk limits, in order to estimate the workload of different scenarios.

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