Module 6 Problem 8.8

8.8 Find Participation Factors where $\dot{x} = Ax$

1) $\begin{bmatrix} 3 & 8 \\ 2 & 3 \end{bmatrix}$ 2) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$

1) $\det A = |A - \lambda I| = \begin{bmatrix} 3-\lambda & 8 \\ 2 & 3-\lambda \end{bmatrix}$ $(3-\lambda)^2 - 16$

Char eqn: $\lambda^2 - 6\lambda - 7 = 0$

$\lambda_{1,2} = \{-1, 7\}$

REV @ $\lambda_1 = -1$

$AV_1 = -V_1$ $\quad V_1 = \begin{bmatrix} V_{11} \\ V_{21} \end{bmatrix}$ ~~V_{21}~~

$3V_{11} + 8V_{21} = -V_{11} \Rightarrow -4V_{11} = 8V_{21}$

$2V_{11} + 3V_{21} = -V_{21} \Rightarrow 2V_{11} = -4V_{21}$

$\Big\} V_{11} = -2V_{21}$ $\quad V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

REV @ $\lambda_2 = 7$

$V_2 = \begin{bmatrix} V_{12} \\ V_{22} \end{bmatrix}$ $\quad AV_2 = 7V_2$

$3V_{12} + 8V_{22} = 7V_{12} \Rightarrow 8V_{22} = 4V_{12}$

$2V_{12} + 3V_{22} = 7V_{22} \Rightarrow 2V_{12} = 4V_{22}$

$\Big\} V_{12} = 2V_{22}$ $\quad V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

LEV @ $\lambda_1 = -1$

$W_1^t = \begin{bmatrix} W_{11}^t & W_{21}^t \end{bmatrix}$

$3W_{11}^t + 2W_{21}^t = -W_{11}^t \Rightarrow 4W_{11}^t = -2W_{21}^t$

$8W_{11}^t + 3W_{21}^t = -W_{21}^t \Rightarrow 8W_{11}^t = -4W_{21}^t$

$\Big\} 2W_{11}^t = -W_{21}^t$ $\quad W_1^t = \begin{bmatrix} 2 & -1 \end{bmatrix}$

LEV @ $\lambda_2 = 7$

$W_2^t = \begin{bmatrix} W_{12}^t & W_{22}^t \end{bmatrix}$

$3W_{12}^t + 2W_{22}^t = 7W_{12}^t \Rightarrow 2W_{22}^t = 4W_{12}^t$

$8W_{12}^t + 3W_{22}^t = 7W_{22}^t \Rightarrow 4W_{22}^t = 8W_{12}^t$

$\Big\} W_{22}^t = 2W_{12}^t$ $\quad W_2^t = \begin{bmatrix} 2 & 1 \end{bmatrix}$

$V_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $V = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix}$

$W_1^t = [2 \ -1]$, $W_2^t = [2 \ 1]$, $W^T = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$

Verify $W_2^t V_1 = W_1^t V_2 = 0$, $[2 \ 1]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$, $[2 \ -1]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$ ✓

Verify $W_1^t V_1 = W_2^t V_2$, $[2 \ -1]\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 4$, $[2 \ 1]\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 4$ ✓

$\lambda = -1$ $P_{11} = \frac{W_{11} V_{11}}{W_1^t V_1} = \frac{2}{4} = \frac{1}{2}$, $P_{21} = \frac{W_{21} V_{21}}{W_2^t V_2} = \frac{-4}{4} = -1$

$\lambda = 1$ $P_{12} = \frac{W_{12} V_{12}}{W_2^t V_2} = -\frac{1}{4}$, $P_{22} = \frac{W_{22}^t V_{22}}{W_2^t V_2} = \frac{2}{4} = \frac{1}{2}$

$$P = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} \\ -1 & \frac{1}{2} \end{bmatrix}$$

2| $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ $\det A = |A - \lambda I| = \begin{bmatrix} 1-\lambda & 2 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 5 & -1-\lambda \end{bmatrix}$

Char eqn: $(1-\lambda)[(3-\lambda)(-1-\lambda) - 5] - 2[0(-1-\lambda) - 0(1)] - 1[0(5) - 0(3-\lambda)]$

$(1-\lambda)[\lambda^2 - 2\lambda - 8]$

$\lambda = \{+1, -2, 4\}$

REV

@ $\lambda_1 = -2$, $V_1 = \begin{bmatrix} V_{11} \\ V_{21} \\ V_{31} \end{bmatrix}$

$V_{11} + 2V_{21} + V_{31} = -2V_{11} \Rightarrow V_{21} = V_{11}$

$3V_{21} + V_{31} = -2V_{21} \Rightarrow 5V_{21} = -V_{31}$

$5V_{21} - V_{31} = -2V_{31} \Rightarrow 5V_{21} = -V_{31}$

$V_1 = \begin{bmatrix} 1/5 \\ 1/5 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \end{bmatrix}$

@ $\lambda_2 = 4$, $V_2 = \begin{bmatrix} V_{12} \\ V_{22} \\ V_{32} \end{bmatrix} \Rightarrow V_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$2V_{22} + V_{32} = 3V_{12} \Rightarrow 2V_{22} + V_{22} = 3V_{12}$

$V_{32} = V_{22}$

$5V_{22} = 5V_{32}$

$V_{12} = V_{22} = V_{32}$

REV

@ $\lambda_3 = 1$ $\quad V_3 = \begin{bmatrix} V_{13} \\ V_{23} \\ V_{33} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$V_{13} + 2V_{23} + V_{33} = V_{13} \Rightarrow 2V_{23} = -V_{33}$

$3V_{23} + V_{33} = V_{23} \Rightarrow 2V_{23} = -V_{33}$

$5V_{32} - V_{33} = V_{33} \Rightarrow 5V_{32} = 2V_{33}$

LEV

@ $\lambda_1 = -2$ $\quad W_1^T = \begin{bmatrix} W_{11} & W_{21} & W_{31} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$

$W_{11} = W_{11} \Rightarrow W_{11} = 0$

$2W_{11} + 3W_{21} + 5W_{31} = -2W_{21} \Rightarrow 2W_{11} + 5W_{21} = -5W_{31} \Rightarrow W_{21} = -W_{31}$

$W_{11} + W_{21} - W_{31} = -2W_{31} \Rightarrow W_{11} + W_{21} = -W_{31}$

@ $\lambda_2 = 4$ $\quad W_2^T = \begin{bmatrix} W_{12} & W_{22} & W_{32} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 5 \end{bmatrix}$

$W_{12} = W_{12} = 0 \Rightarrow W_{12} = 0$

$3W_{22} + 5W_{32} = 4W_{22} \Rightarrow 5W_{32} = W_{22}$

$W_{12} + W_{22} - W_{32} = 4W_{32} \Rightarrow 5W_{32} = W_{22}$

@ $\lambda_3 = 1$ $\quad W_3^T = \begin{bmatrix} W_{13} & W_{23} & W_{33} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$

$W_{13} = 0$

$3W_{23} + 5W_{32} = W_{23} \Rightarrow 2W_{23} = -5W_{32}$

$W_{23} - W_{33} = W_{33} \Rightarrow W_{22} = 2W_{33}$

$$V = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -5 & 1 & 0 \end{bmatrix}, \quad W^T = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$$

$W_1^T V_1 = 0$

$W_2^T V_2 = 6$

$W_3^T V_3 = 0$

$$P_{11} = \frac{W_{11} V_{11}}{W_1^T V_1} = 0, \quad P_{21} = \frac{W_{21} V_{21}}{W_2^T V_2} = 0 \quad P_{31} = 0 = P_{32} = P_{33}$$

$$P_{12} = \frac{W_{12} V_{12}}{W_2^T V_2} = \frac{1}{6} \quad P_{22} = \frac{W_{22} V_{22}}{W_2^T V_2} = \frac{1}{6}$$

$$P_{13} = \frac{W_{13} V_{13}}{W_3^T V_3} = 0 \quad P_{23} = 0$$

$$P = \begin{bmatrix} 0 & \frac{1}{6} & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$