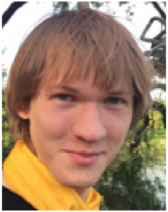
Shifting the thresold of phase transition in 2-SAT and random graphs

Sergey Dovgal[1,2] Vlady Ravelomanana[2]

[1]Université Paris-13, [2]Université Paris-Diderot



Acknowledgements: Élie de Panafieu, Fedor Petrov, `ipython+sympy+cpp`

June 13, 2017

Outline

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Phase transition in Erdős–Rényi random graphs

n vertices, m edges,

$$m = \frac{1}{2} n (1 + \mu n^{-1/3})$$

1. $\boxed{\text{“gas” } \mu \to -\infty}$: planar graph, trees and unicycles, max component size $O(\log n)$.
2. $\boxed{\text{“liquid” } |\mu| = O(1)}$: complex components appear, max component size $O(n^{2/3})$.
3. $\boxed{\text{“crystal” } \mu \to +\infty}$: non-planar, complex compontnes, max component size linear $O(n)$.

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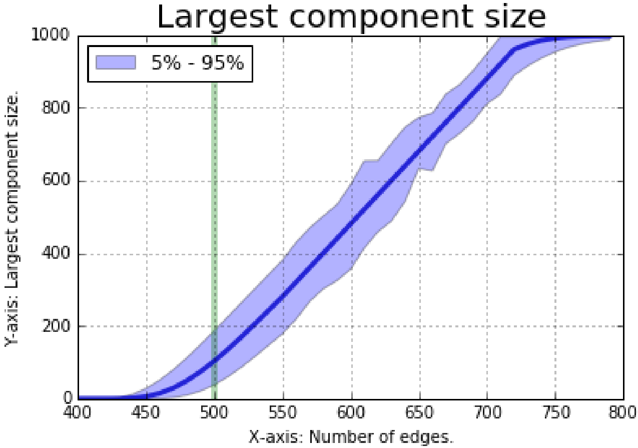
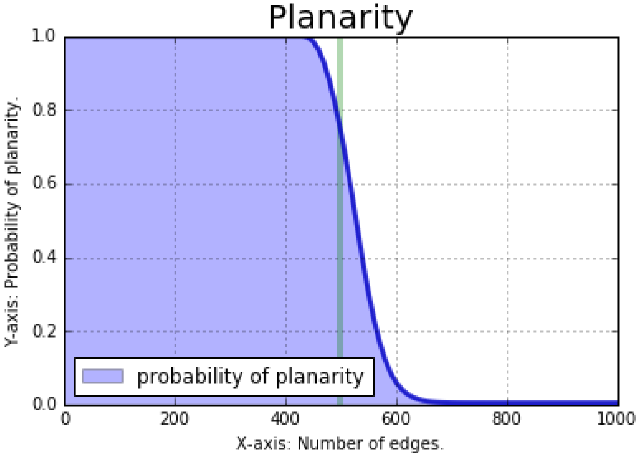
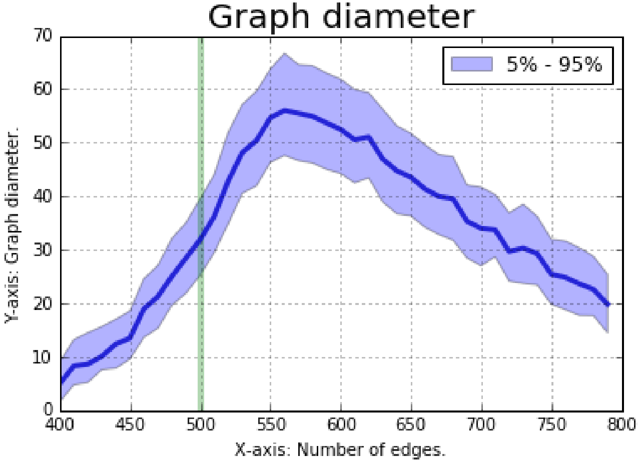
Phase transition :: largest component, $n = 1000$



Phase transition :: planarity, $n = 1000$



Phase transition :: diameter, $n = 1000$



Shifting the phase transition

$$m = \frac{1}{2} n(1 + \mu n^{-1/3}) \Rightarrow m = \alpha n(1 + \mu n^{-1/3})$$

1. Achlioptas percolation process
2. Degree sequence models
3. Degree set constraint :: current talk

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Example of graph with degree constraints

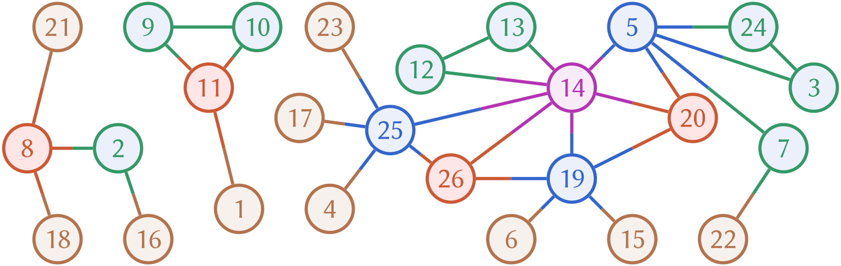

Figure: Random labeled graph from $\mathcal{G}_{26,30,\Delta}$ with the set of degree constraints $\Delta = \{1, 2, 3, 5, 7\}$.

Condition for the set Delta and random graph

n – number of vertices
m – number of edges

1. $1 \in \Delta$ $\longleftarrow$ Other cases $1 \notin \Delta$ remain open question
2. Period of Δ: $p \stackrel{def}{=} \gcd(d_1 - d_2 : d_1, d_2 \in \Delta)$,

   $p \mid 2m - n \cdot \min(\Delta)$ $\longleftarrow$ necessary, each degree $\in \Delta$
3. $2m/n \in$ fixed compact interval of $]\min(\Delta), \max(\Delta)[$

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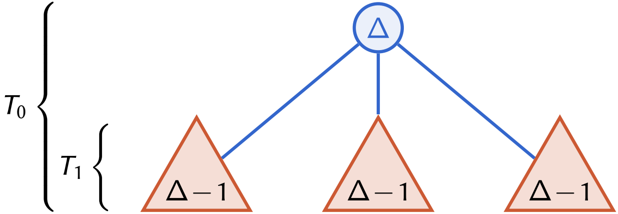
Trees with degree constraints

Rooted case



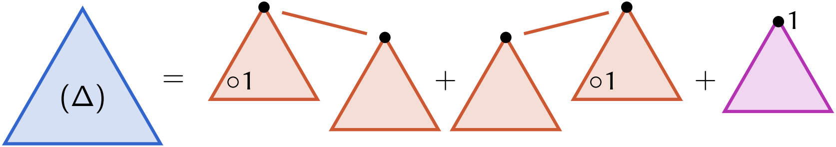
$$\Delta - k \stackrel{def}{=} \{d \colon d + k \in \Delta\}$$

$$\omega(z) = \sum_{d \in \Delta} \frac{z^d}{d!} = \frac{z^{d_1}}{d_1!} + \frac{z^{d_2}}{d_2!} + \dots, \qquad \begin{cases} T_0(z) = z\omega(T_1(z)), \\ T_1(z) = z\omega'(T_1(z)), \\ T_2(z) = z\omega''(T_1(z)). \end{cases}$$

Trees with degree constraints

Unrooted case Excercise

A variant of dissymmetry theorem:



Excercise: what is the EGF for unrooted trees?

Trees with degree constraints

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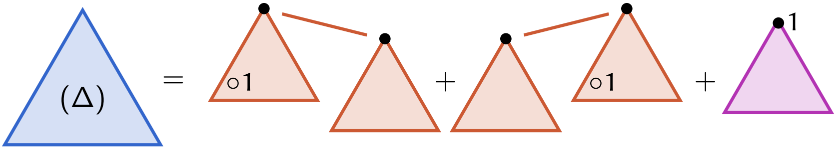
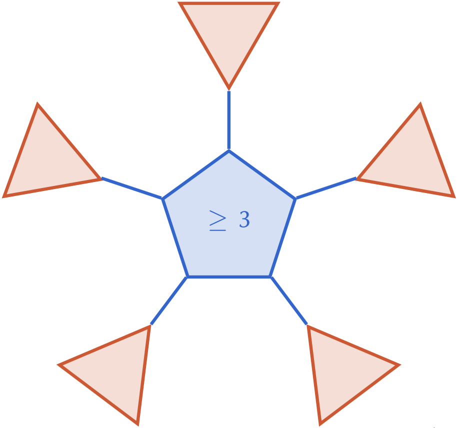


$$T_0(z) = \frac{T_1(z)^2}{2} + U(z) \quad \Leftrightarrow \quad U(z) = T_0(z) - \frac{T_1(z)^2}{2}$$

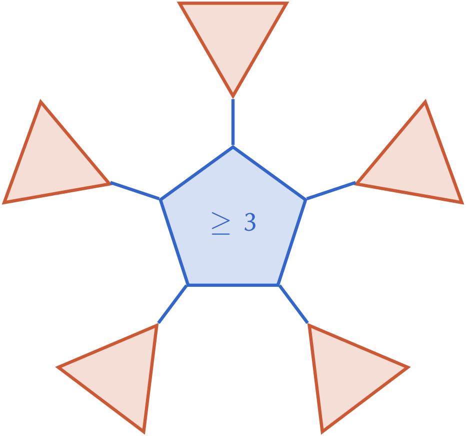
Unicycles with degree constraints

Excercise



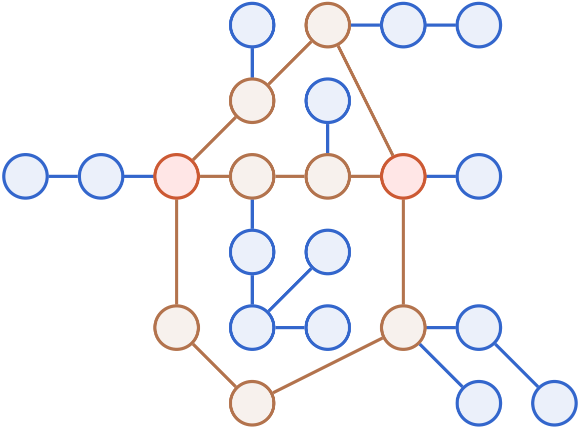
Excercise: what is the EGF for unicycles?

Unicycles with degree constraints



$$V(z) = \frac{1}{2}\left[\log\frac{1}{1-T_2(z)} - T_2(z) - \frac{T_2(z)^2}{2}\right]$$

2-core (the core) and 3-core (the kernel)



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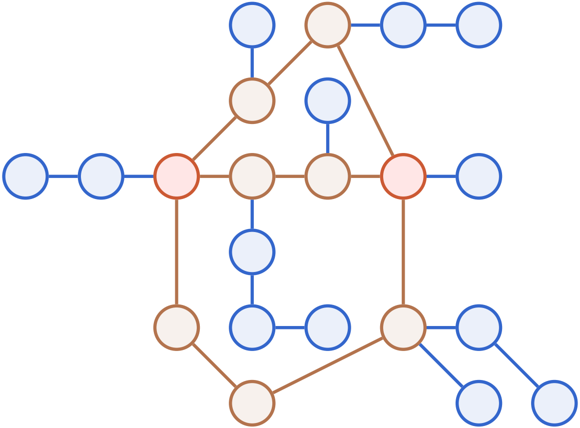

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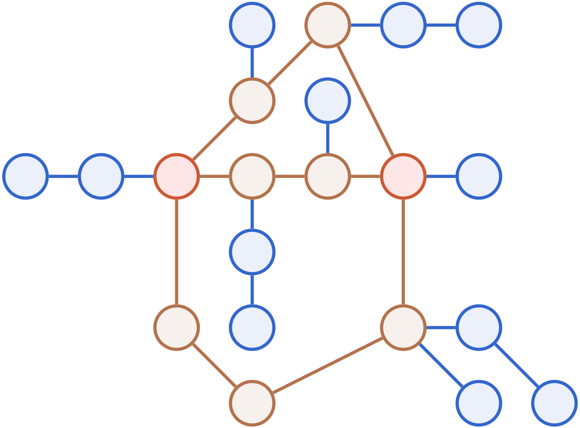

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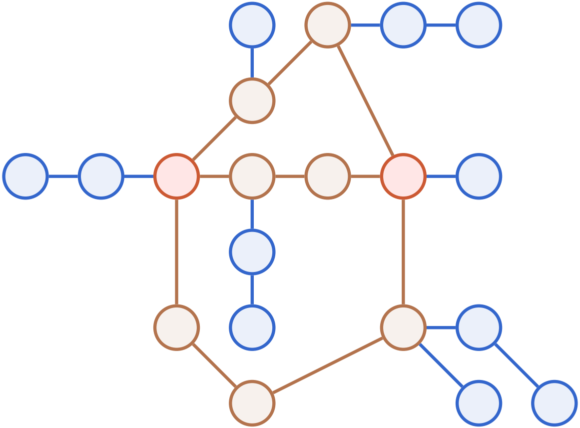

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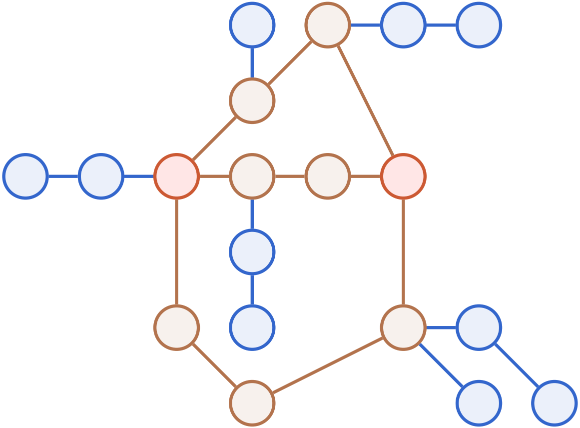

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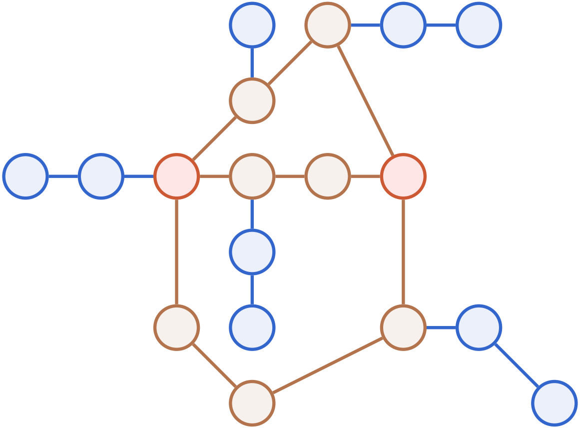

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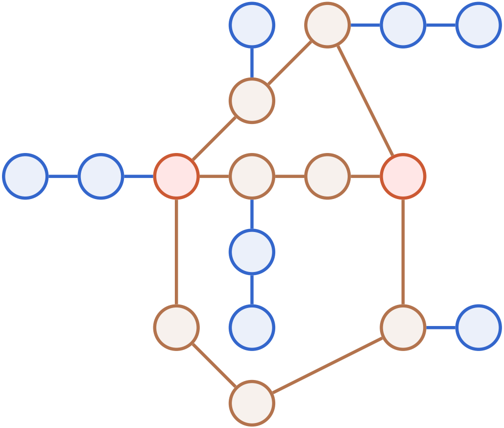

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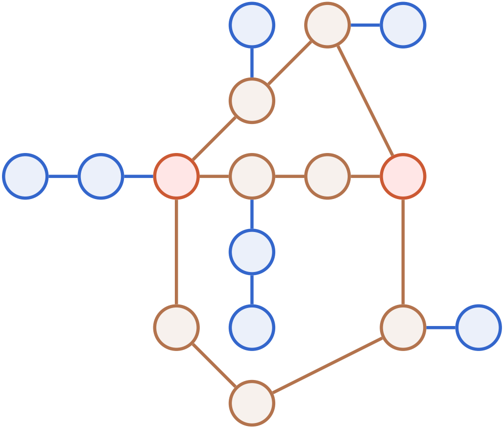

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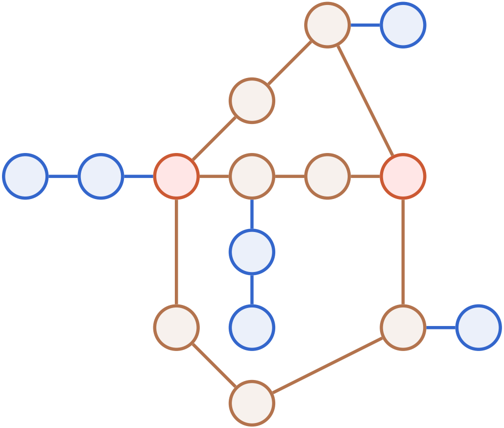

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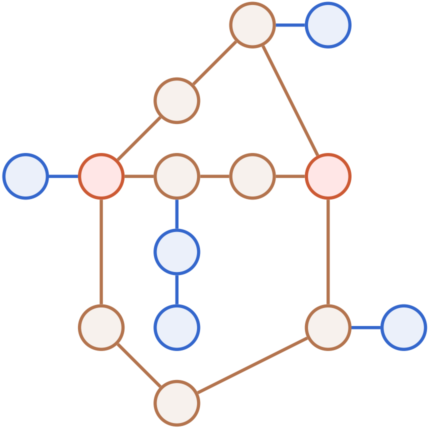

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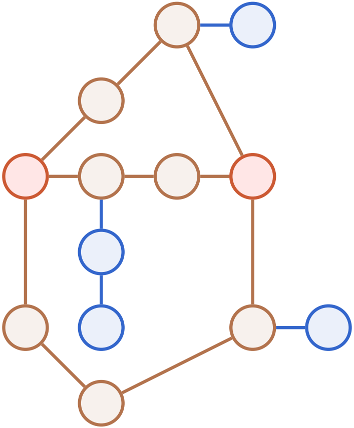

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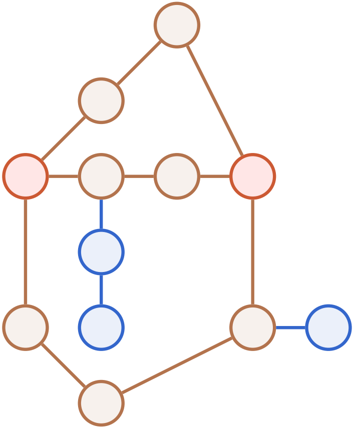

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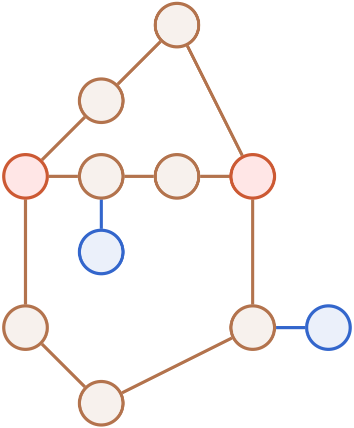

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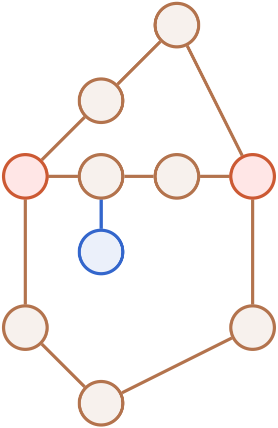

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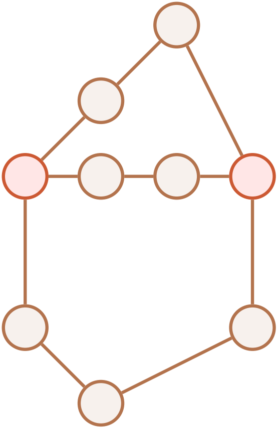

2-core of a graph

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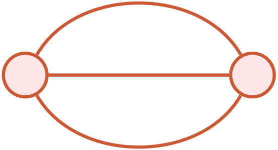
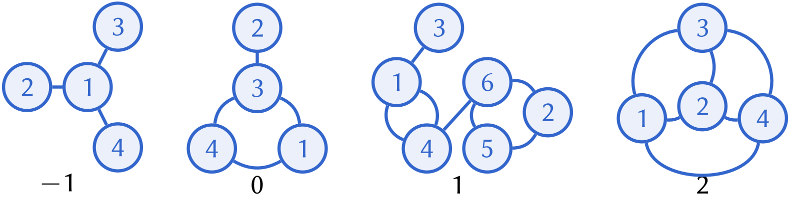


3-core of a graph

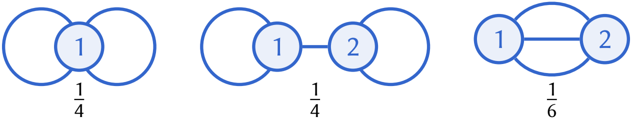
Notion of excess



$$\text{Excess} \overset{def}{=} \text{\# edges - \# vertices}$$

Kernel of a graph

Example: graphs with excess 1



All possible 3-core multigraphs and their compensation factors.
EGF for all connected bicyclic graphs ($\Delta = \mathbb{Z}_{\geq 0}$):

$$W(z) = \frac{1}{4}\frac{T(z)^5}{(1-T(z))^2} + \frac{1}{4}\frac{T(z)^6}{(1-T(z))^3} + \frac{1}{6}\underbrace{\frac{T(z)^2[3T(z)^2 - 2T^3(z)]}{(1-T(z))^3}}_{\text{inclusion-exclusion}}$$

$$W(z) \sim \frac{5}{24} \cdot \frac{1}{(1-T(z))^3} \text{ near } z = e^{-1}$$

Kernel of a graph

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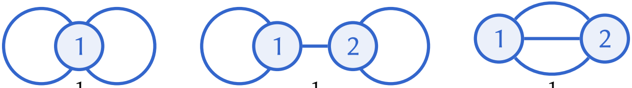

$\frac{1}{4}$ $\frac{1}{4}$ $\frac{1}{6}$

All possible 3-core multigraphs and their compensation factors.
EGF for all connected bicyclic graphs (arbitrary Δ):

$$W_\Delta(z) = \frac{1}{4}\frac{T_4(z)T_2(z)^4}{(1-T_2(z))^2} + \frac{1}{4}\frac{T_3(z)^2T_2(z)^4}{(1-T_2(z))^3} + \frac{1}{6}\underbrace{\frac{T_3(z)^2[3T_2(z)^2 - 2T_2(z)^3]}{(1-T_2(z))^3}}_{\text{inclusion-exclusion}}$$

$$W_\Delta(z) \sim (???) \cdot \frac{T_3(z)^2(???)}{(1-T_2(z))^3}$$

General symbolic lemma

EGF for connected graphs which reduce to given $\overline{\overline{M}}$ is:

$$W_{\Delta,\overline{\overline{M}}}(z) = \frac{\kappa(\overline{\overline{M}}) \prod_{v \in V} T_{\deg(v)}(z)}{n!} \cdot \frac{P(\overline{\overline{M}}, T_2(z))}{(1 - T_2(z))^{\mu}}$$

$$P(\overline{\overline{M}}, z) = \prod_{x=1}^{n} \left(z^{2\mu_{xx}} \prod_{y=x+1}^{n} z^{\mu_{xy}-1} (\mu_{xy} - (\mu_{xy} - 1)z) \right) ,$$

$$\begin{cases} T_k(z) & = z\omega^{(k)}(T_1(z)), \\ \mu_{xy} & = \#\text{ edges between nodes } x \text{ and } y \\ \kappa(\overline{\overline{M}}) & = \text{compensation factor} \\ \mu & = \#\text{ edges} \end{cases}$$

Role of cubic graphs

Demonstration on the blackboard

Let $\widehat{z}$ be the positive solution of $T_2(\widehat{z}) = 1$.
Then EGF for all (not necessary connected) complex multigraphs with excess r, has asymptotics near $\widehat{z}$, which comes from cubic graphs (degree of each vertex is equal to 3):

$$W_{\Delta,r}(z) \sim e_{r0} \frac{T_3(z)^{2r}}{(1 - T_2(z))^{3r}} \quad , \qquad e_{r0} = \frac{(6r)!}{2^{5r} 3^{2r} (3r)!(2r)!} \quad .$$

Local summary

1. EGF for unrooted trees with degree constraints
2. EGF for unicycles with degree constraints
3. EGF for graphs of fixed *excess* (main asymptotics)

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Outline

Expected results

n – number of vertices
m – number of edges

Framework: $m = \alpha n$, linear dependence.

1. $m = (1-\varepsilon)\alpha n \quad \longleftarrow$ only trees and unicycles
2. $m = \alpha n \quad \longleftarrow$ complex components with positive probability
3. $m = (1+\varepsilon)\alpha n \quad \longleftarrow$ probability of fixed excess is exponentially small

Expected results

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Framework: $m = \alpha n$, linear dependence.

1. $m = (1 - \varepsilon)\alpha n \quad \longleftarrow$ subcritical phase
2. $m = \alpha n \quad \longleftarrow$ cricital phase
3. $m = (1 + \varepsilon)\alpha n \quad \longleftarrow$ supercritical phase

Desired probability

Subcritical phase

$$\mathbb{P}\left(\text{graph } g \in \mathcal{G}(n, m, \Delta) \text{ consists only of trees and unicycles}\right)$$

$$= \frac{\#\text{ graphs from } \mathcal{G}(n, m, \Delta) \text{ whose components are trees and unicycles}}{\#\text{ graphs from } \mathcal{G}(n, m, \Delta)}$$

Number of graphs with degree constraints

1. $\Delta = \mathbb{Z}_{\geq 0}$. Stirling approximation:

$$\frac{n!}{(n-m)!\binom{\binom{n}{2}}{m}} \sim \sqrt{4\pi n\alpha} \cdot \frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}} \times \exp\Big(-n + \underbrace{\frac{m}{n} + \frac{m^2}{n^2}}_{3/4}\Big)$$

2. Arbitrary Δ ([de Panafieu, Ramos '16])

$$\frac{n!}{(n-m)!|\mathcal{G}_{n,m,\Delta}|} \sim \frac{\sqrt{4\pi n\alpha}}{p} \cdot \frac{2^m n^n m^m}{n^{2m}(n-m)^{n-m}} \times \exp\Bigg(-n\log\omega(\widehat{z}) + 2m\log\widehat{z} + \frac{1}{2}\phi_0(\widehat{z}) + \frac{1}{4}\phi_0^2(\widehat{z})\Bigg)$$

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Contour integrals for subcritical phase

Demonstration on the blackboard

$$\frac{n!}{|\mathcal{G}_{n,m,\Delta}|}\frac{1}{2\pi i}\oint \frac{U(z)^{n-m}}{(n-m)!}e^{V(z)}\frac{dz}{z^{n+1}} = 1 - O(\mu^{-3})$$

near the critical point $m = \alpha n$:

$$\begin{cases} 2\alpha & = \phi_0(\widehat{z}) \stackrel{def}{=} \widehat{z}\dfrac{\omega'(\widehat{z})}{\omega(\widehat{z})} \ , \\ 1 & = \phi_1(\widehat{z}) \stackrel{def}{=} \widehat{z}\dfrac{\omega''(\widehat{z})}{\omega'(\widehat{z})} \ . \end{cases}$$

Full range of densities

Theorem (Regime: $m = \alpha n(1 - \mu n^{-1/3})$)

1. *if* $\mu \to -\infty$, $|\mu| = O(n^{1/12})$, *then*

$$\mathbb{P}(G_{n,m,\Delta} \textit{ has only trees and unicycles}) = 1 - \Theta(|\mu|^{-3}) \ ;$$

2. *if* $|\mu| = O(1)$, *i.e.* μ *is fixed, then*

$$\mathbb{P}(G_{n,m,\Delta} \textit{ has only trees and unicycles}) \to \textit{constant} \in (0,1) \ ,$$

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Geometric statement

Demonstration on the blackboard

1. $h(z;r) \stackrel{def}{=} r\log\omega'(z) - r\log z + (1-r)\log(2\omega - z\omega')$
2. $\Phi(\theta;r) \stackrel{def}{=} \operatorname{Re} h(z_0 e^{i\theta};r)$,

$$\max_{\theta\in[0,2\pi]} \Phi(\theta;r) = \Phi(\theta;r)\Big|_{\theta_k=\frac{2\pi k}{p}}$$

Good old hypergeometric (Airy) function [FlJaKnŁuPi]

$$A(y,\mu) = \frac{e^{\mu^3/6}}{3^{(y+1)/3}} \sum_{k \geq 0} \frac{\left(\frac{1}{2} 3^{2/3} \mu\right)^k}{k! \Gamma((y+1-2k)/3)}$$

1. As $\mu \to -\infty$,

$$A(y,\mu) = \frac{1}{\sqrt{2\pi} |\mu|^{y-1/2}} \left(1 - \frac{3y^2+3y-1}{6|\mu|^3} + O(\mu^{-6})\right)$$

2. As $\mu \to +\infty$,

$$A(y,\mu) = \frac{e^{-\mu^3/6}}{2^{y/2} |\mu|^{1-y/2}} \left(\frac{1}{\Gamma(y/2)} + \frac{4\mu^{-3/2}}{3\sqrt{2}\Gamma(y/2-3/2)} + O(\mu^{-2})\right)$$

In-class excercise

Excercise

Trees and unicycles: $y = 1/2$. Complex component: $y \geq 1 + 1/2$.
As $\mu \to -\infty$,

$$A(y, \mu) = \frac{1}{\sqrt{2\pi}|\mu|^{y-1/2}} \left(1 - \frac{3y^2 + 3y - 1}{6|\mu|^3} + O(\mu^{-6})\right)$$

What is the asymptotics of $A(y, \mu)$?

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Asymptotics of $A(y, \mu)$:

$$A(1/2, \mu) \sim \frac{1}{\sqrt{2\pi}} \left(1 - \frac{5}{24|\mu|^3}\right)$$

$$A(3/2, \mu) \sim \frac{1}{\sqrt{2\pi}|\mu|} \left(1 + O(|\mu|^{-3})\right)$$

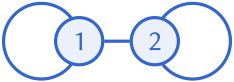
More than just probability: Analytic lemma

Excercise

As $m = \alpha n(1 - \mu n^{-1/3})$,

$$\frac{n!}{(n-m)!|\mathcal{G}_{n,m,\Delta}|}[z^n]\frac{U(z)^{n-m}}{(1-T_2(z))^y} \sim \sqrt{2\pi}C_y \cdot A(y, \widetilde{C}\mu)n^{y/3-1/6}$$

Excercise: $\mathbb{P}$ of 1 bicycle inside critical phase. Asymptotics?



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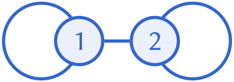

2-core contains 3 paths.

More than just probability: Analytic lemma

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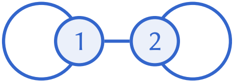

2-core contains 3 paths. “Mnemonic rule” $y = -\frac{1}{2} \Rightarrow y = 3 - \frac{1}{2}$.

More than just probability: Analytic lemma

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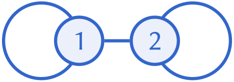

$$C\frac{n^{(y+3)/3-1/6}}{n^{y/3-1/6}} \cdot \frac{1}{n-m+1} = O(1)$$

Local summary

1. Airy function from [FlJaKnŁuPi]
2. Contour integrals for $m = \text{linear}(n) \lesssim \alpha n$
3. Core technical statement (due to Petrov): global max of real part complex function
4. Analytic lemma will be used after.
   We gain additional $n^{1/3}$ for each additional $\dfrac{1}{1 - T_2(z)}$.

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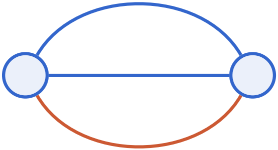
Outline

Length of a 2-path

Excercise

q — excess (condition and then sum over q)



$\longleftarrow$ marked 2-path inside complex component of some graph

$$\mathbb{E}[u^{P_n}] \propto [z^n] \frac{U(z)^{n-m+q}}{(n-m+q)!} e^{V(z)} \frac{1 - T_2(z)}{1 - uT_2(z)}$$

Question: asymptotics of length of 2-path? Hint: analytic lemma.

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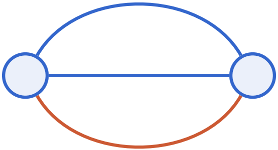

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$$\mathbb{E}P_n \Leftarrow \frac{d}{du}(*)\Big|_{u=1} \underset{\text{Analytic lemma}}{\Leftarrow} \mathbb{E}P_n \sim C \cdot n^{1/3}$$

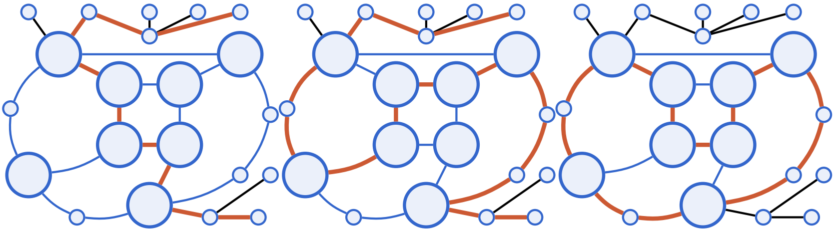
Bivariate EGF for tree height

[Flajolet, Odlyzko '82]

$$F(z,u) = \sum_{n\geq 0} \frac{z^n}{n!} \sum_{h=0}^{n} \underset{\text{\# trees height } h}{\underset{\uparrow}{A_n^{[h]}}} \cdot u^h$$

1. $\displaystyle \frac{d}{du}F(z,u)\bigg|_{u=1} \sim C_1 \log \sqrt{1-\frac{z}{\rho}}\ ,$
2. $\displaystyle \frac{d^2}{du^2}F(z,u)\bigg|_{u=1} \sim C_2 \left(1-\frac{z}{\rho}\right)^{-1/2}$
3. Modify analytic lemma for $\log(1-\phi_1(z))$.

Diameter, circumference and longest path of complex component



All of order $\Theta(n^{1/3})$

Planarity

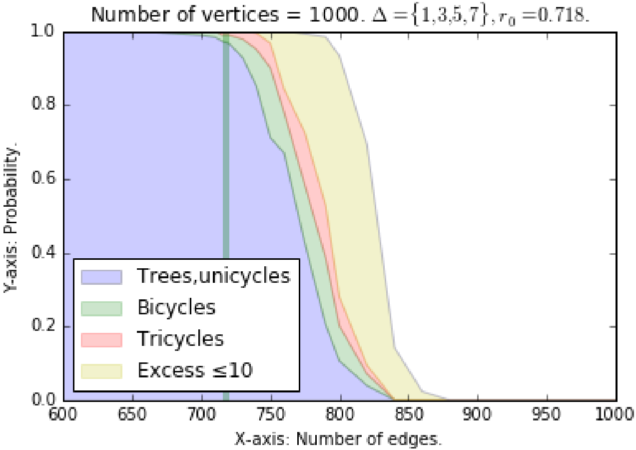
Demonstration on the blackboard

Let $p(\mu)$ be the probability that $G_{n,m,\Delta}$ is planar.

1. $p(\mu) = 1 - \Theta(|\mu|^{-3})$, as $\mu \to -\infty$;
2. $p(\mu) \to$ constant $\in (0, 1)$, as $|\mu| = O(1)$, and $p(\mu)$ is computable;
3. $p(\mu) \to 0$, as $\mu \to +\infty$.

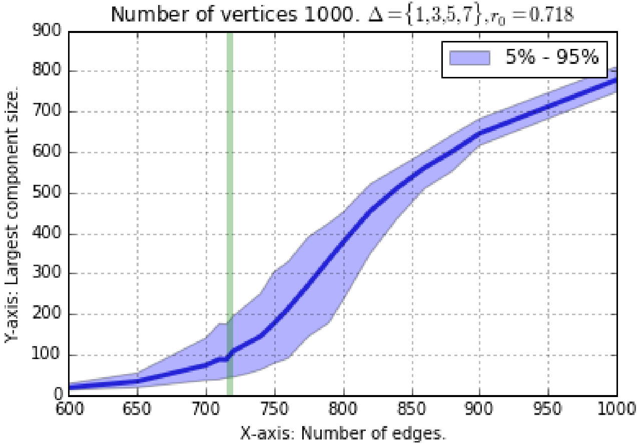
Experimental results

(1/3)



Experimental results

(2/3)



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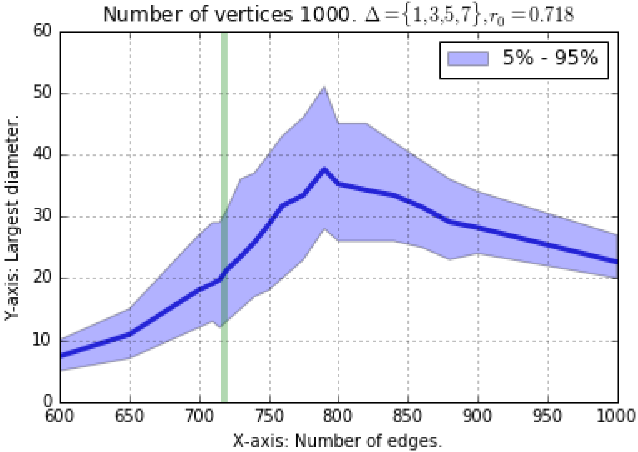
(3/3)



Local summary

1. Phase transition of diameter, circumference, longest path.
2. Phase transition for planarity.
3. Diameter of trees and unicycles — open question?
4. Largest component — open question?
5. Stay tuned for 2-SAT!

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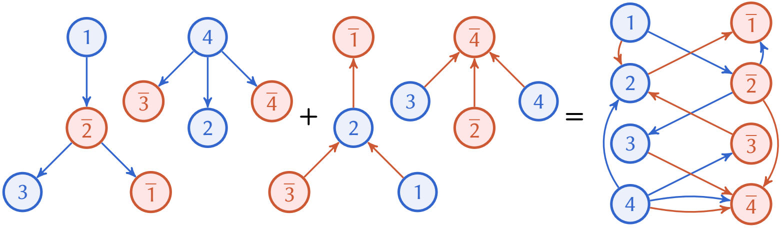
Outline

2-CNF formula and digraph model

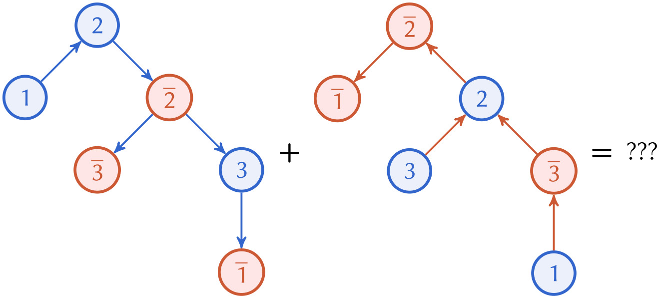
Digraph representation and sum-representation of a 2-SAT formula

$$(\overline{x}_1 \vee \overline{x}_2)(x_2 \vee x_3)(x_2 \vee \overline{x}_1)(\overline{x}_4 \vee \overline{x}_3)(\overline{x}_4 \vee x_2)(\overline{x}_4 \vee \overline{x}_4)$$



Excercise on 2-CNF representation construction

Excercise



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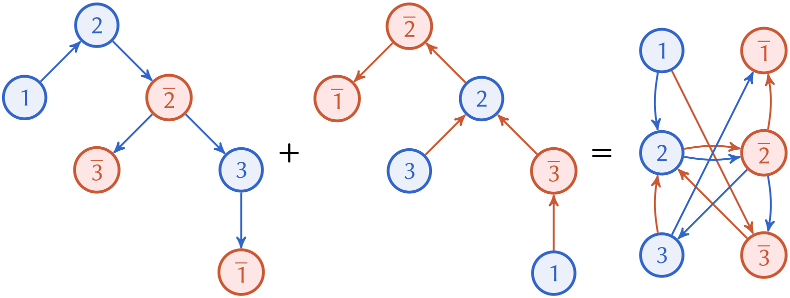

Digraph model from graph with degree constraint

Lemma: CNF is UNSAT iff it contains a circuit $x \rightsquigarrow \overline{x}$ and $\overline{x} \rightsquigarrow x$.

Random CNF: $\mathcal{G}(n, m, \Delta) \oplus \overline{\mathcal{G}(n, m, \Delta)}$

How to control the probability of circuit $1 \rightsquigarrow \overline{1} \rightsquigarrow 1$?
Idea: probability the same as for any other circuit.

Complicated excercise on cycles

Excercise

Fix nodes x, y (say, $x = 1$ and $y = 2$). Condider random directed graphs with vertex degrees from Δ, subcritical phase, condition on trees and unicycles.

$$\sum_{\ell \geq 1} 2^{\ell} \mathbb{P}(x, y \in \underline{\text{circuit}} \text{ of length } \ell) = ?$$

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$$\sum_{\ell \geq 1} 2^{\ell} \mathbb{P}(x, y \in \underline{\text{circuit}} \text{ of length } \ell)$$

Step 1. Mark the circuit + expectation of 2^L.

$$\vec{V}^{\bullet}(z) = \underbrace{\left(z\frac{d}{dz}\right)\left(z\frac{d}{dz}\right)}_{\text{2 markings}} \mathrm{CIRCUIT}_{>2}(uz)\Bigg|_{z \mapsto \vec{T}_2(z), u=2}$$

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$$\sum_{\ell \geq 1} 2^{\ell} \mathbb{P}(x, y \in \underline{\text{circuit}} \text{ of length } \ell)$$

Step 2. Count the coefficient by analytic lemma.

$$\propto \frac{1}{n(n-1)} [z^n] \vec{U}(z)^{n-m} \exp\left(\vec{V}(z)\right) \vec{V}^{\bullet}(z) \quad ,$$

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$$\sum_{\ell \geq 1} 2^{\ell} \mathbb{P}(x, y \in \underline{\text{circuit}} \text{ of length } \ell)$$

Step 3. Final expression. 2 markings $\mapsto n^{2/3}$ (subcritical $\times |\mu|^{-2}$):

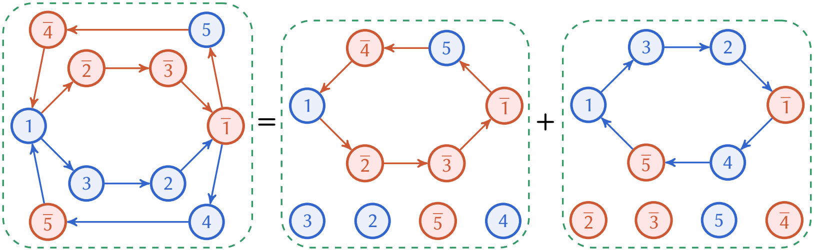
$$\sim \frac{n^{2/3}|\mu|^{-2}}{n(n-1)} \sim O(n^{-4/3}|\mu|^{-2})$$

Case without complex component

Demonstration on the blackboard

Subcritical case: $n = \alpha m(1 - \mu n^{-1/3})$, $\mu \to -\infty$.

$$\mathbb{P}(F_{n,m} \text{ is SAT}) \geq 1 - \frac{5}{24|\mu|^3} + O(|\mu|^{-6})$$



Full statement of the theorem

1. $\mathbb{P}(F_{n,m,\Delta} \text{ is } \textsc{sat}) \geq 1 - O(|\mu|^{-3})$ as $\mu \to -\infty$,
2. $\mathbb{P}(F_{n,m,\Delta} \text{ is } \textsc{sat}) \geq \Theta(1)$ as $|\mu| = O(1)$,
3. $\mathbb{P}(F_{n,m,\Delta} \text{ is } \textsc{sat}) \geq \exp(-\Theta(\mu^3))$ as $\mu \to +\infty$.

Local summary

1. No complex component, contradictory circuit.
   Done by marking + analytic lemma.
2. Correction for non-uniformity?
3. Upper bound — what happens inside complex components?

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Proportion of vertices of certain degree

Excercise

Given $\Delta = \{d_1, d_2, \ldots\}$, $1 \in \Delta$.
How many vertices of degree d?

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Given $\Delta = \{d_1, d_2, \ldots\}$, $1 \in \Delta$.
How many vertices of degree d?
Idea.

$$T(z,u) = z \cdot \left(\omega + (u-1)\frac{z^d}{d!}\right) \circ T(z,u)$$

Distribution is easily obtained through marking method.

Distribution of parameters from Hatami-Molloy

Sequence of degrees $\mathcal{D} = (d_v)_{v \in G}$.

$$Q := Q(\mathcal{D}) := \frac{\sum_{v \in G} d_v^2}{2|E|} - 2, \quad R := R(\mathcal{D}) := \frac{\sum_{v \in G} d_v (d_v - 2)^2}{2|E|}$$

Mark the vertex degree $\mapsto G(z, u)$.

$$\mathbb{E} d_1^2 = \frac{[z^n] \left(u \frac{d}{du} \right)^2 G(z, u) \Big|_{u=1}}{[z^n] G(z, 1)}$$

$$\mathbb{E} Q(\mathcal{D}) = \frac{4n\alpha - O(n^{2/3})}{2m} - 2 = O(n^{-1/3})$$

Results and Methods

1. Analytic description of phase transition in model with degree constraints
2. Fedor Petrov: help in the proof of geometric statement at mathoverflow
3. Study of distribution of parameters.
4. Height of sprouting tree – new result?
5. Lower bound for 2-SAT + improvement of bounds by Bollobas.
6. Degree constraints ≈ degree sequence in certain small window.

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Open problems

1. The case $1 \notin \Delta$.
2. Upper bound for 2-SAT.
3. Size of the largest component.
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That's all!