Introduction to Data Science Math 3900 / CS 5963
Lecture 16: Elections and Arrow's Impossibility Theorem
Revised: November 4, 2016

> "At a Lawn Tennis Tournament where I chanced, some while ago, to be a spectator, the present method of assigning prizes was brought to my notice by the lamentations of one of the Players, who had been beaten (and had thus lost all chance of a prize) early in the contest, and who had had the mortification of seeing the 2nd prize carried off by a Player whom he knew to be quite inferior to himself."
>
> —Charles Dodgson a.k.a Lewis Carroll (1883)

We define an *election procedure* as a method for generating a ranking (ordering) of a set of candidates from voter preferences. Note that an election does not just name the top candidate, but instead, ranks all of the candidates. (This slightly differs from everyday usage.) We refer to a procedure that only decides the top candidate as a *winner selection method.*

What is the relationship between these two problems? Clearly, one could recursively use a winner selection method to define an election procedure. (First choose a winner, then remove the winner from the ballot and use the winner selection method to choose the runner-up, *etc.*. . .) Conversely, the number one ranked alternative in an election procedure can be considered the winner, so every election procedure generates a winner selection method. So are these two problems—elections and winner selection—really equivalent? No, there are election procedures which cannot be derived from recursively selecting a winner. Thus, election procedures are more general and winner selection methods should be considered a subset of election procedures.

There are many different election systems and the argument for which one is the "best" has a rich history. Before we try to address this type of question, let's consider a few different election procedures and winner selection methods. We'll see that some of the methods greatly differ from one another and, for example, from the methods used in U. S. political elections.

First, note that if there are only two candidates, then the election and winner-selection methods are the same. In this case, any reasonable method will select the candidate for which the *majority* of the electorate favors. In the following discussion, we'll assume that there are three or more candidates. The interesting case arises when no single candidate receives a majority of the votes.

In the U. S., elections are generally decided by the plurality method or some variation thereof[1]. In the *plurality method*, each voter votes for a single candidate and the candidate with the most votes is declared the winner. It often occurs that the candidate does not receive a majority of the votes. For example, George W. Bush was elected president in 2000 with 47.87% of the votes and Bill Clinton was elected president in 1992 with 43.01% of the votes. In both cases, "third-party" candidates (Ralph Nader and Ross Perot respectively)

1. Presidential elections in the U.S. involve the electoral college, but I'm not going to discuss this here.

prevented the winner (or, possibly, the runner-up) from receiving a majority vote. Other voting methods may have resulted in alternative outcomes in these two elections. For example, in France, Russia, and Brazil, a *runoff election* is held, in which only the top two candidates appear on the ballot. This idea can be further generalized to an election procedure called the *elimination method*, where the candidate with the smallest number of votes is removed from the ballot in each round. This method is used by the International Olympic Committee to select the host for the Olympic games.

To formalize our discussion, we introduce some terminology. We write $a, b, c, \ldots$ to denote candidates and $i, j, k, \ldots$ to denote voters. Each voter has his/her own preferences. We write $a \succ_i b$ if candidate a is preferred to candidate b by voter i. If voter i has no preference over candidates a and b, we write $a =_i b$. Finally we write $a \succeq_i b$ to indicate that candidate b is not preferred to candidate a by voter i; that is, either: $a \succ_i b$ or $a =_i b$. We require the voter preferences to satisfy the following relationships:

1. For each pair of candidates, a and b, exactly one of the following holds: $a \succ_i b$, $b \succ_i a$, or $a =_i b$.

2. For all candidates a, $a =_i a$.

3. Each voter ranking should give a transitive relation: $a \succeq_i b$ and $b \succeq_i c$ implies that $a \succeq_i c$ with $a =_i c$ if and only if $a =_i b$ and $b =_i c$.

An *election procedure* is a method for generating a ranking of the candidates from the voter preferences that satisfy the above three properties. The relationships generated by the election procedure are denoted by $\succ$, $=$, and $\succeq$.

A *preference schedule* is a way to summarize individual voter preferences. Here is an example:

3	a	b	c
2	b	a	c
2	c	b	a

You should read this table as follows: 3 voters have preferences given by $a \succ b \succ c$, 2 voters have preferences give by $b \succ a \succ c$ and 2 voters have preferences given by $c \succ b \succ a$. To illustrate differences between election procedures, we consider two: the Borda count method and the Copeland method.

The *Borda count* is an election procedure, which gives to each candidate, for each voter, a point for each lower-ranked candidate. The points then determine the ranking: the candidate with the most points is declared the winner, *etc.*. . . This method is named after Jean-Charles de Borda, although it has earlier roots[2]. For the example above, we would compute the Borda counts as follows. In the first row, candidate a gets $3 \times 2 = 6$ points since 3 voters prefer candidate a to candidates b and c. Also from the first row, candidate b receives $3 \times 1 = 3$ points. From the second row, candidate b receives $2 \times 2 = 4$ points and candidate a receives $2 \times 1 = 2$ points. From the third row, candidate c receives $2 \times 2 = 4$ points and candidate b receives $2 \times 1 = 2$ points. In total, candidate a receives $6 + 2 = 8$

2. It is often said that things are named after the *last* person to invent it.

points, candidate b receives $3 + 4 + 2 = 5$ points, and candidate c receives 4 points. The Borda count ranking is given by $b \succ a \succ c$ and b is declared the winner.

The *Copeland method* orders the candidates by the number of pairwise victories. (Ties count as half a victory.) The Copeland method is named after Arthur H. Copeland, but has earlier roots. In the example above, the Copeland method would work as follows. In the pairwise competition between candidates a and b, candidate b wins. In the pairwise competition competition between candidates a and c, candidate a wins. In the pairwise competition between candidates b and c, candidate b wins. Thus, the Copeland method assigns b two points, a one point, and c zero points. The pairwise comparison method ranking is $b \succ a \succ c$ and b is declared the winner.

In this example, the Borda count and Copeland methods rank the candidates the same. However, they differ for the following preference schedule (Exercise!).

3	a	b	c
2	b	c	a

So, which is "better": the Borda count or Copeland method? Well, it depends on what you mean by better. We illustrate this by defining the notion of a Condorcet candidate.

We say that a candidate is a *Condorcet candidate* if he beats every other candidate in head-to-head competition (that is, with all other candidates removed from the ballot). This concept is named after Nicolas de Condorcet, although (again) has earlier roots. It is not difficult to show that a Condorcet candidate does not necessarily exist, but if one does, it is unique. Intuitively it would seem desirable for a winner selection method to select the Condorcet candidate, if one exists. In the first example above, candidate b is a Condorcet candidate, while in the second example candidate a is the Condorcet candidate. Thus, the Copeland method selected the Condorcet candidate as winner, but the Borda count did not! In fact, one can show that the Copeland method always selects the Condorcet candidate as winner, provided one exists. The following example[3] shows that a Condorcet candidate does not always exist.

31	a	e	c	d	b
30	b	a	e		
29	c	d	b		
10	d	a	e		

Here, missing entries mean that the candidates were ranked equally and last, so the second row is $b \succ a \succ e \succ c = d$. Candidate a is the Copeland winner with 3 wins and 1 loss, but it is not the Condorcet candidate.

This might prompt us to write down a list of desirable properties and then see which election methods satisfy which properties. The following table[4] compares the properties of several different election methods. Each row of the table is an election method and each column is a desirable property. The entries of the table indicate whether or not the election method satisfies that property. Don't worry about how we define all of these methods or all of the properties. Note that Plurality, Runoff voting, Borda count, and Copeland are

3. from `http://en.wikipedia.org/wiki/Copeland\%27s_method`
4. from `http://en.wikipedia.org/wiki/Voting_system`

among the methods compared and that the Condorcet property is the third column in the table.

Looking at this table, we observe that none of the election methods satisfy all of the (desirable) properties. Of course, tomorrow someone could invent a new election method and we could then check to see which of these properties hold for that method. This leads us to the following question: is it possible that there exists a method that satisfies all (or possibly a chosen subset) of these properties?

sort by:																					
	Majority (MC)	Mutual MC	Condorcet	Cond. loser	Smith/ ISDA	LIIA	IIA	Cloneproof	Monotone	CC	PC	Reversal symmetry	Polytime/ Resolvable		Summable	ballot type	= ranks	>2 ranks	Later-no-harm/ Later-no-help		FBC:No favorite betrayal
Approval	Rated [nb 1]	No	No [nb 2] [nb 3]	No	No [nb 2]	Yes	Yes [nb 4]	Yes [nb 5]	Yes	Yes	Yes	Yes	$O(N)$	Yes	$O(N)$	approvals	Yes	No		[nb 6]	Yes
Borda count	No	No	No [nb 2]	Yes	No	No	No	No: teams	Yes	Yes	Yes	Yes	$O(N)$	Yes	$O(N)$	ranking	No	Yes	No	Yes	No
Copeland	Yes	Yes	Yes	Yes	Yes	No	No [nb 2]	teams, crowds	Yes	No [nb 2]	No [nb 2]	Yes	$O(N^2)$	No	$O(N^2)$	ranking	Yes	Yes	No [nb 2]	No	No [nb 2]
IRV (AV)	Yes	Yes	No [nb 2]	Yes	No [nb 2]	No	No	Yes	No	No	No	No	$O(N^2)$	Yes	$O(N!)$ [nb 7]	ranking	No	Yes	Yes	Yes	No
Kemeny-Young	Yes	Yes	Yes	Yes	Yes	Yes	No [nb 2]	No: spoilers	Yes	No [nb 2] [nb 8]	No [nb 2]	Yes	$O(N!)$	Yes	$O(N^2)$ [nb 9]	ranking	Yes	Yes	No [nb 2]	No	No [nb 2]
Majority Judgment [nb 10]	Rated [nb 11]	No [nb 12]	No [nb 2] [nb 3]	No	No [nb 2]	Yes	Yes [nb 4]	Yes	Yes	No [nb 13]	No [nb 14]	Depends [nb 15]	$O(N)$	Yes	$O(N)$ [nb 16]	scores [nb 17]	Yes	Yes	No [nb 18]	Yes	Yes
Minimax	Yes	No	Yes [nb 19]	No	No	No	No [nb 2]	No: spoilers	Yes	No [nb 2]	No [nb 2]	No	$O(N^2)$	Yes	$O(N^2)$	ranking	Yes	Yes	No [nb 2] [nb 19]	No	No [nb 2]
Plurality	Yes	No	No [nb 2]	No	No [nb 2]	No	No	No: spoilers	Yes	Yes	Yes	No	$O(N)$	Yes	$O(N)$	single mark	NA	No	NA [nb 20]	NA [nb 20]	No
Range voting	No	No	No [nb 2] [nb 3]	No	No [nb 2]	Yes	Yes [nb 4]	Yes	Yes	Yes	Yes	Yes	$O(N)$	Yes	$O(N)$	scores	Yes	Yes	No	Yes	Yes
Ranked pairs	Yes	Yes	Yes	Yes	Yes	Yes	No [nb 2]	Yes	Yes	No [nb 2]	No [nb 14] [nb 2]	Yes	$O(N^4)$	Yes	$O(N^2)$	ranking	Yes	Yes	No [nb 2]	No	No [nb 2]
Runoff voting	Yes	No	No [nb 2]	No	No [nb 2]	No	No	No: spoilers	No	No	No	No	$O(N)$ [nb 21]	Yes	$O(N)$ [nb 22]	single mark	NA	No [nb 23]	Yes	[nb 24]	No
Schulze	Yes	Yes	Yes	Yes	Yes	No	No [nb 2]	Yes	Yes	No [nb 2]	No [nb 14] [nb 2]	Yes	$O(N^3)$	Yes	$O(N^2)$	ranking	Yes	Yes	No [nb 2]	No	No [nb 2]
Random winner/ arbitrary winner [nb 25]	NA	NA	NA [nb 2]	NA	NA [nb 2]	Yes	Yes	No	NA	NA	NA	NA	$O(1)$	NA	$O(1)$	none	NA	NA	NA	NA	NA
Random ballot [nb 26]	No	No	No [nb 2]	No	No [nb 2]	Yes	Yes	Yes	Yes	No	Yes	Yes	$O(N)$	Yes	$O(N)$	single mark	NA	No	NA	NA	Yes

Arrow's Impossibility Theorem.

In the following, we define five sensible criteria which we expect a fair and reasonable election procedure should satisfy. Surprisingly, in 1963, Kenneth Arrow proved that there does not exist an election procedure which can satisfy all five criteria.

We say an election procedure is *fair* if it satisfies the following five criteria:

1. All conceivable voter rankings are allowed.

2. If $a \succeq_i b$ for all voters i, then $a \succeq b$ with equality if and only if $a =_i b$ for all voters i. We interpret this to mean that unanimous opinions are respected.

3. If in two different elections, each voter ranks candidates a and b the same, then the election outcomes between a and b are the same in the two elections. That is, how candidates are ranked relative to each other in an election depends only on how the voters rank them relative to each other and not how they are ranked relative to other

candidates. Denoting preferences in the second election by the symbol $\supseteq$, we express this property in symbols:

$$\text{If } \forall i,\ a \succeq_i b \iff a \supseteq_i b, \quad \text{then } a \succeq b \iff a \supseteq b.$$

4. If there are two elections such that $a \succeq_i b$ implies $a \supseteq_i b$ for all i, and if also $a \succeq b$, then $a \supseteq b$. In other words, if a does at least as well as b in a later ranking by the voters as he did in the present ranking, and if he beat b in the present election, he'll beat b in the later election.

5. There is no voter i such that $a \succ b$ if and only if $a \succ_i b$. In other words, there is no dictator.

Theorem 1 (Arrow's Impossibility Theorem.) *No election procedure for three or more candidates is fair.*

A set of voters V is called *decisive* for a against b if when all voters in V agree on ranking a at least equal to b, then $a \succeq b$ regardless of how the remaining voters rank a and b. Furthermore, if $a = b$, then $a =_i b$ for all i in V. At least one decisive set exists for all a and b, since all the voters are decisive by (1)-(2). By (4), we can check if V is decisive by looking at an election with $a \succeq_i b$ for all i in V and $b \succ_i a$ for all i not in V.

Proof We will show that (1) through (4) imply the existence of a dictator.

Pick two candidates a and b. We will show that there is a single voter who is decisive for a, b. Suppose that is not the case and let V be the smallest decisive set for a, b. Then V has at least two voters in it, so we can split it into two nonempty, disjoint sets of voters V_1, V_2 so that $V = V_1 \cup V_2$, $V_1, V_2 \neq \emptyset$, $V_1 \cap V_2 = \emptyset$. Let c be another candidate and consider the election procedure in which

$$\begin{aligned} a \succeq_i b \succeq_i c \quad & \forall i \in V_1 \\ c \succeq_i a \succeq_i b \quad & \forall i \in V_2 \\ b \succ_i c \succ_i a \quad & \forall i \notin V \end{aligned}$$

The idea is to show that either V_1 is decisive for a, c or V_2 is decisive for b, c. Both statements contradict the minimality of V (Exercise). The general argument goes as follows. If $a \succeq c$, then V_1 is decisive for a and c, contradicting the minimality of V. Thus $c \succ a$. Since V is decisive for a, b, it follows from the above election that $a \succeq b$. Thus, by transitivity, $c \succ b$ and V_2 is decisive for c, b, contradicting the minimality of V. As an exercise, you can check the cases when equality occurs. Therefore, V contains a single voter i.

So far, we have that if $a \succeq_i b$, then $a \succeq b$. Let c be a third candidate and suppose that $a \succeq_i b \succeq_i c$. Consider the election procedure when $b \succ_j c \succ_j a$ for all $j \neq i$. By (2), $b \succ c$ and by decisiveness $a \succ b$. Thus $a \succ c$. By (3), we can ignore b and see that, if $a \succeq_i c$ and $c \succ_j a$ for all $j \neq i$, then $a \succeq c$. Hence i is decisive for a and c. Repeat this argument for d a candidate distinct from a, c. Therefore i is decisive for every pair and i is a dictator. ■

It can be shown that Criteria (3) follows from the other four. In practice, since criteria (1), (2), and (5) are pretty clear, its axiom (4) that is typically violated—the entry of a new candidate can affect the outcome of the rankings.

To illustrate the absurdity that the presence of the third option should not influence the voters preference between the other two, we consider the following anecdote attributed to Sidney Morgenbesser.

> After finishing his meal, a diner decides to order dessert from a restaurant. The waitress tells him he has two choices: apple pie and blueberry pie. The diner orders the apple pie. After a few minutes the waitress returns and says that they also have cherry pie at which point the diner says "In that case I'll have the blueberry pie."

In practice, one has to be very careful when considering the method used for an election. No method is fair and, at worse, the method can be exploited by a coalition of the voters.

Notes and further discussion.

See (Bender, 1978, pp. 124–126). Börgers (2010) contains a more extensive discussion of the mathematics of voting. Arrow's Impossibility theorem was proven in Arrow (1963).

References

K. J. Arrow. Social choice and individual values. Cowles Foundation Monograph 12, Wiley, 1963.

E. A. Bender. *An Introduction to Mathematical Modeling*. Dover, 1978.

C. Börgers. *Mathematics of Social Choice*. SIAM, 2010.