S&S MID SEMESTER EXAM – 2023
SOLUTIONS

SOL(1): Given that – $x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$

$$h(t) = \{u(t+1) - u(t-2)\}$$

$$y(t) = x(t) * h(t)$$

(a) $T = 2$

$$\therefore\ x(t) = \sum_{k=-\infty}^{\infty} \delta(t-2k)$$

$$\therefore\ y(t) = x(t) * h(t)$$

$$= \left\{\sum_{k=-\infty}^{\infty} \delta(t-2k)\right\} * \{u(t+1) - u(t-2)\}$$

$$= \{\cdots + \delta(t+4) + \delta(t+2) + \delta(t) + \delta(t-2) + \delta(t-4) + \cdots\}$$

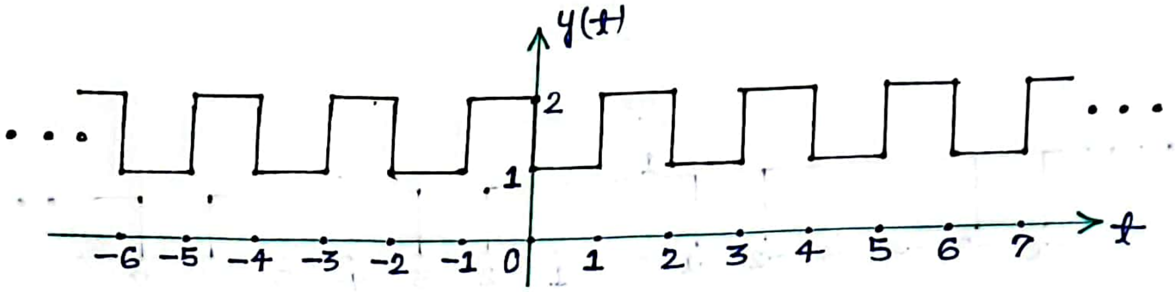
$$* \{u(t+1) - u(t-2)\}$$

$$= \int_{-\infty}^{\infty} \{\cdots + \delta(\tau+4) + \delta(\tau+2) + \delta(\tau) + \delta(\tau-2) + \delta(\tau-4) + \cdots\}$$
$$\times \{u(t-\tau+1) - u(t-\tau-2)\}\, d\tau$$

$$\because \int_{-\infty}^{\infty} x(t)\cdot\delta(t-t_1)\, dt = x(t_1)$$

$$= \cdots + \{u(t+5) - u(t+2)\} + \{u(t+3) - u(t)\} + \{u(t+1)$$
$$- u(t-2)\} + \{u(t-1) - u(t-4)\} + \{u(t-3) - u(t-6)\} + \cdots$$

→ (3 Points)



→ (1 Point)

$\therefore$ Time period for $y(t) = 2$ sec → (1 Point)

(b) $T = 4$

$$\therefore\ x(t) = \sum_{k=-\infty}^{\infty} \delta(t-4k)$$

$$\therefore\ y(t) = x(t) * h(t)$$

$$= \left\{\sum_{k=-\infty}^{\infty} \delta(t-4k)\right\} * \left\{u(t+1)-u(t-2)\right\}$$

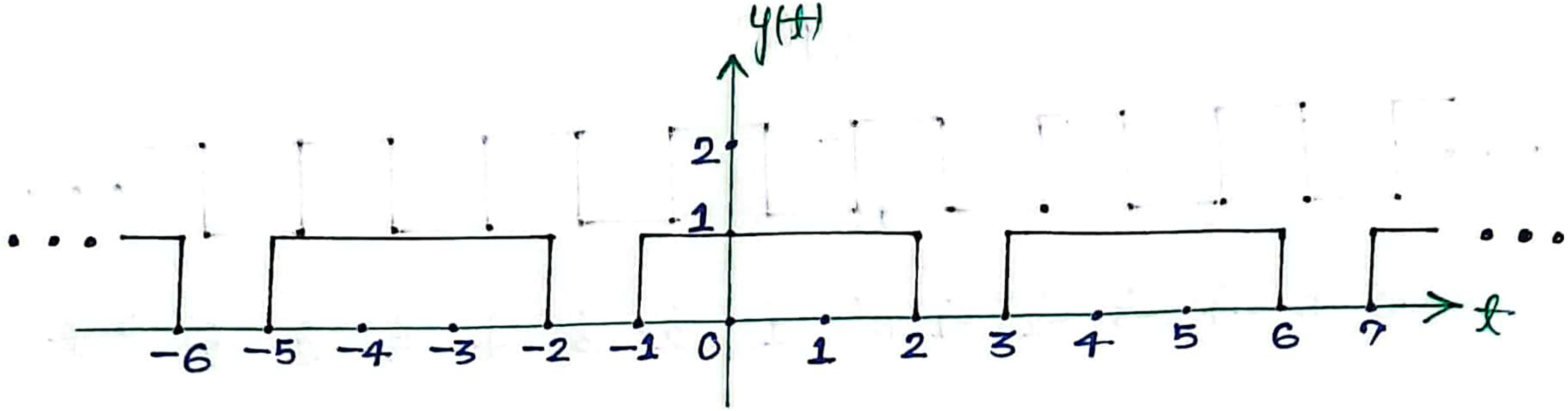
$$= \left\{\cdots + \delta(t+4) + \delta(t) + \delta(t-4) + \cdots\right\} * \left\{u(t+1)-u(t-2)\right\}$$

$$= \int_{-\infty}^{\infty} \left\{\cdots + \delta(\tau+4) + \delta(\tau) + \delta(\tau-4) + \cdots\right\}\left\{u(t-\tau+1)-u(t-\tau-2)\right\} d\tau$$

$$\because \int_{-\infty}^{\infty} x(t)\cdot\delta(t-t_1)\,dt = x(t_1)$$

$$= \cdots + \left\{u(t+5)-u(t+2)\right\} + \left\{u(t+1)-u(t-2)\right\}$$
$$+ \left\{u(t-3)-u(t-6)\right\} + \cdots$$

→ (3 Points)



→ (1 Point)

∴ Time period for $y(t) = 4$ sec → (1 Point)

Q2.

Given that,

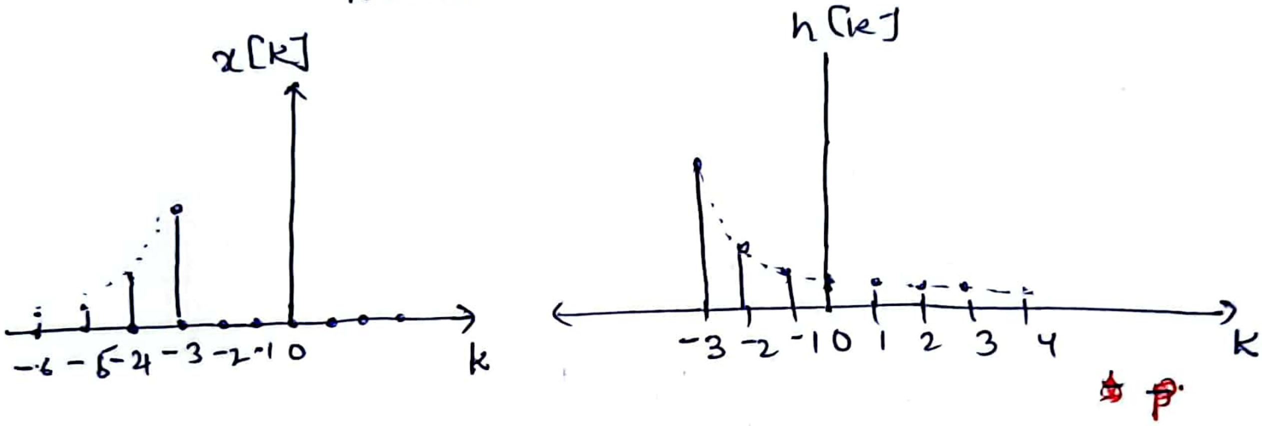
$$x[n] = 2^n u[-n-3]$$

$$h[n] = \left(\frac{1}{5}\right)^n u[n+3]$$

we know that,

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] \quad \text{or} \quad \sum_{k=-\infty}^{\infty} x[n-k] h[k]$$



Now considering,

$$y[n] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k]$$

$$= \sum_{k=-\infty}^{\infty} 2^{n-k} u[-(n-k)-3] \left(\frac{1}{5}\right)^k u[k+3]$$

$$= 2^n \sum_{k=-\infty}^{\infty} \left(\frac{1}{10}\right)^k u[-n+k-3]\; u[k+3]$$

$$= 2^n \sum_{k=-3}^{\infty} \left(\frac{1}{10}\right)^k u[-n+k-3]$$

here we have,

$k = -3$ to ∞

In $u[-n+k-3]$

k runs from $k-3-n \geq 0$

$k = n+3$ to ∞

So, effectively, the range will run from,

$k = \max(-3, n+3)$ to ∞ 1 point

Case 1 : $n+3 \leq -3$

$\Rightarrow n \leq -6$

$k = -3$ to ∞

$$y[n] = 2^n \sum_{k=-3}^{\infty} \left(\frac{1}{10}\right)^k$$

$$= 2^n \left[10^3 + 10^2 + 10 + 1 + 10^{-1} + 10^{-2} \cdots\right]$$

$$= 2^n \times 10^3 \left[\frac{10}{9}\right]$$

$$\boxed{y[n] = 2^n \times \frac{10^4}{9}}$$

2 point

$\because a = 1$

$S = \frac{a}{1-r}$

$S_\infty = \left(\frac{1}{1 - 1/10}\right)$

$S_\infty = \frac{10}{9}$

Case 2: $n+3 > -3$

$\Rightarrow n > -6$

$k = n+3$ to ∞

$$y[n] = 2^n \left[\sum_{n+3}^{\infty} \left(\frac{1}{10}\right)^k\right]$$

$$= 2^n \left[\left(\frac{1}{10}\right)^{n+3} + \left(\frac{1}{10}\right)^{n+4} + \left(\frac{1}{10}\right)^{n+5} \dots\right]$$

$$= 2^n \left(\frac{1}{10}\right)^{n+3} \left[1 + \left(\frac{1}{10}\right) + \left(\frac{1}{10}\right)^2 \dots \infty\right]$$

→ forms a gp series

$$= 2^n \left(\frac{1}{10}\right)^{n+3} \times \left(\frac{1}{1 - 1/10}\right)$$

$$= 2^n \left(\frac{1}{10}\right)^{n+3} \cdot \left(\frac{10}{9}\right)$$

S: $a = 1$, $r = 1/10$, $S_\infty = \frac{a}{1-r}$

$$y[n] = \left(\frac{1}{5}\right)^n \times \frac{1}{900}$$

2 point

Hence, we have,

$$\boxed{\begin{array}{lll} y[n] = 2^n \times \frac{10^4}{9} & & n \le -6 \\ \quad\;\; = \left(\frac{1}{5}\right)^n \times \frac{1}{900} & & n > -6 \end{array}}$$

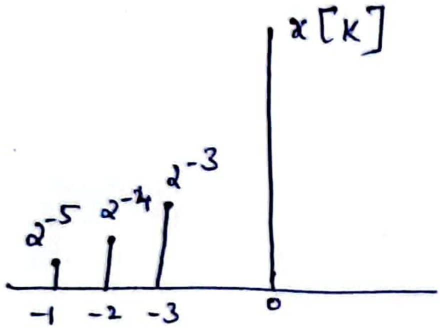
Q2. $x[n] = 2^n u[-n-3]$

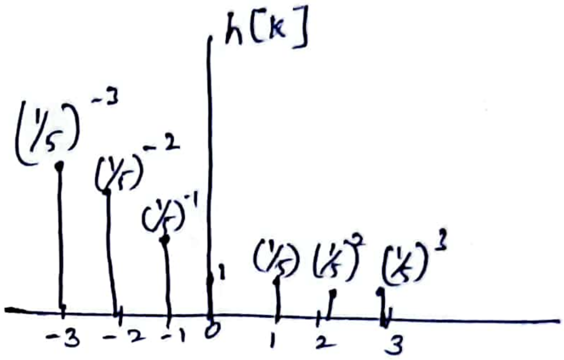
$h[n] = \left(\frac{1}{5}\right)^n u[n+3]$

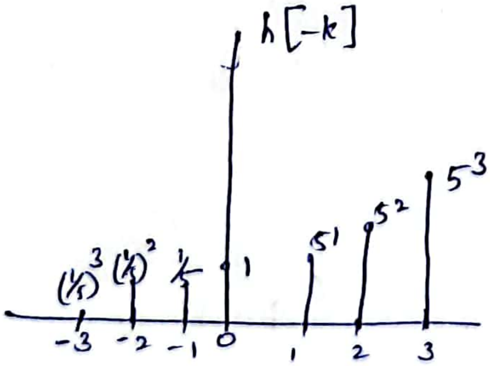
We know,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

Method 2







$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} 2^k u[-k-3] \left(\frac{1}{5}\right)^{n-k} u[n-k+3]$$

$$= \left(\frac{1}{5}\right)^n \sum_{k=-\infty}^{\infty} 2^k u[-k-3] \left(\frac{1}{5}\right)^{-k} u[n-k+3]$$

Case 1 $\quad n+3 > -3 \Rightarrow n > -6.$

$$y[n] = \sum_{k=-\infty}^{-3} 2^k \left(\tfrac{1}{5}\right)^{n-k} = \left(\tfrac{1}{5}\right)^n \sum_{k=-\infty}^{-3} 10^k.$$

$$= \tfrac{1}{5}^n \Big(\underbrace{\tfrac{1}{10^3} + \tfrac{1}{10^4} + \cdots}_{\text{GP Series}} \Big)$$

$$= \tfrac{1}{5}^n \left(\frac{1/10^3}{1 - 1/10} \right) = \tfrac{1}{5}^n \times \frac{1}{900}.$$

Case 2 $\quad n+3 \leq -3 \Rightarrow n \leq -6.$

$$y[n] = \sum_{k=-\infty}^{n+3} 2^k \left(\tfrac{1}{5}\right)^{n-k} = \left(\tfrac{1}{5}\right)^n \sum_{k=-(n+3)}^{\infty} 10^{-k}.$$

$$= \left(\tfrac{1}{5}\right)^n 10^{n+3} \Big(1 + \underbrace{\tfrac{1}{10} + \tfrac{1}{10^2} + \cdots}_{\text{GP series}} \Big)$$

$$= \left(\tfrac{1}{5}\right)^n 10^{n+3} \left(\frac{10}{9} \right)$$

$$= \left(\tfrac{1}{5}\right)^n \frac{10^{n+4}}{9} = 2^n \frac{10^4}{9}.$$

$\therefore$ Hence, we have,

$$y[n] = \begin{cases} 2^n \dfrac{10^4}{9} ; & n \leq -6 \\ \tfrac{1}{5}^n \dfrac{1}{900} & n > -6. \end{cases}$$

SOL(3):

(a) True

If $h(t)$ is periodic and non zero, then —

$$\int_{-\infty}^{\infty} |h(t)|\, dt = \infty$$

Therefore, $h(t)$ is unstable. → (1 Point)

(b) False

According to given condition,

Let assume, $h[n] = u[n]$

but $\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |u[n]| = \infty$

This is an unstable system. → (1 Point)

(C) True

Assuming that $h[n]$ is bounded and nonzero in the range $n_1 \leq n \leq n_2$,

$$\sum_{K=n_1}^{n_2} |h[K]| < \infty$$

This implies that the system is stable. → (1 Point)

(d) False

According to given condition,

Let assume, $h(t) = e^{t} u(t)$ is causal system

but it is not stable system.

→ (1 Point)

(e) False

For example, the cascade of a causal system with impulse response $h_1[n] = \delta[n-1]$ and a non-causal system with impulse response $h_2[n] = \delta[n+1]$ leads to a system with overall impulse response given by—

$$h[n] = h_1[n] * h_2[n] = \delta[n]$$

which is causal system. → (1 Point)

Ans 4.

(a) Let V be a F-vector space.

A map

$$\langle \cdot, \cdot \rangle : V \times V \to F$$

is called an inner product if it satisfies the following:

(i) $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0$ iff $v = 0 \ \forall v \in V$ [0.5 point]

(ii) $\langle v_1, v_2 \rangle = \overline{\langle v_2, v_1 \rangle} \quad \forall v_1, v_2 \in V$ [0.5 point]

(iii) It is linear in the first coordinate

$$\langle \alpha_1 v_1 + \alpha_2 v_2, w \rangle = \alpha_1 \langle v_1, w \rangle + \alpha_2 \langle v_2, w \rangle$$

$\forall v_1, v_2, w \in V$ and $\forall \alpha_1, \alpha_2 \in F$

[0.5 point]

(iv) It is conjugate linear in the second coordinate

[0.5 point]

(b) Given that,

$$\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) \cdot \overline{g(t)} \cdot dt$$

The vector space is defined over the Field $\mathbb{C}$. (the set of complex numbers).

(1) $\langle f(t), g(t) \rangle = \int_{-\infty}^{\infty} f(t) \cdot \overline{g(t)} \, dt$ [0.5 point]

$f(t) \in V$ for any $z \in C$ we know that

$$z \cdot \bar{z} = (z)^2$$

Hence.

$$= \int_{-\infty}^{\infty} |f(t)|^2 dt \geq 0$$

$$\langle f(t), f(t) \rangle = \int_{-\infty}^{\infty} |f(t)|^2 \cdot dt = 0$$

$$\text{iff } |f(t)|^2 = 0$$

$$f(t) = 0 \quad \forall \, t$$

2. Complex conjugate property: [0.5 point]

$$\langle f_1(t), f_2(t) \rangle = \int_{-\infty}^{\infty} f_1(t) \cdot \overline{f_2(t)} \cdot dt$$

$$= \int_{-\infty}^{\infty} \overline{\overline{f_1(t)}\, f_2(t)}\, dt$$

$f_1, f_2 \in V$

$\because z_1 \bar{z}_2 = \overline{\bar{z}_1 z_2}$

from conjugate property, we can write it as —

$$= \overline{\int_{-\infty}^{\infty} f_2(t) \cdot \overline{f_1(t)}\, dt}$$

$$= \overline{\langle f_2(t), f_1(t) \rangle}$$

Hence, we proved,

$$\boxed{\langle f_1(t), f_2(t) \rangle = \overline{\langle f_2(t), f_1(t) \rangle}}$$

3) It is linear in the first coordinate [0.5 point]

Let,

$$\langle \alpha f_1(t) + \beta f_2(t), g(t) \rangle$$

$$= \int_{-\infty}^{\infty} [\alpha f_1(t) + \beta f_2(t)]\, \overline{g(t)}\, dt$$

where $f_1, f_2, g \in V$, $\alpha, \beta \in C$

$$= \alpha \int_{-\infty}^{\infty} f_1(t)\,\overline{g(t)}\,dt + \beta \int_{-\infty}^{\infty} f_2(t)\cdot\overline{g(t)}\,dt$$

$$\boxed{= \alpha \langle f_1(t), g(t)\rangle + \beta \langle f_2(t), g(t)\rangle}$$

(4) Conjugate linear in the second coordinate [0.5 point]

$$\langle f(t), \alpha g_1(t) + \beta g_2(t)\rangle = \int_{-\infty}^{\infty} f(t)\,\overline{(\alpha g_1(t) + \beta g_2(t))}\,dt$$

$$f, g_1, g_2 \in V$$

$$\alpha, \beta \in \mathbb{C}$$

$$= \int_{-\infty}^{\infty} f(t)\,[\bar{\alpha}\,\overline{g_1(t)} + \bar{\beta}\,\overline{g_2(t)}]\,dt$$

$$= \bar{\alpha} \int_{-\infty}^{\infty} f(t)\cdot\overline{g_1(t)}\cdot dt + \bar{\beta} \int_{-\infty}^{\infty} f(t)\cdot\overline{g_2(t)}\cdot dt$$

$$\boxed{= \bar{\alpha} \langle f(t), g_1(t)\rangle + \bar{\beta} \langle f(t), g_2(t)\rangle}$$

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