Solutions – Practice Sheet-5

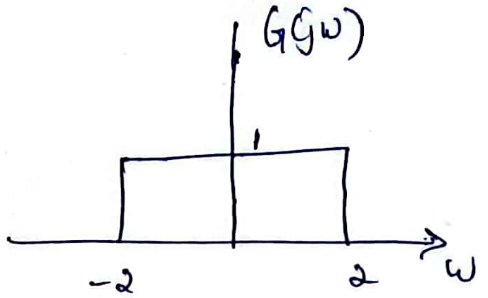
Q1.(a) Given $g(t) = x(t)\cos t$.

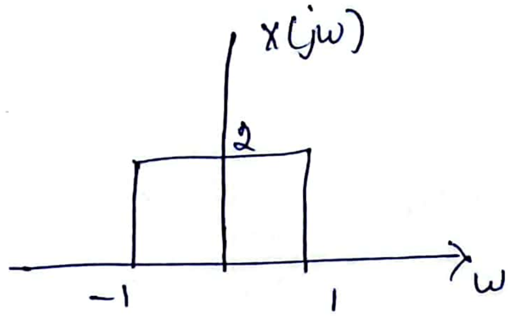
$$g(t) \xleftrightarrow{FT} G(j\omega).$$

Let $w(t) = \cos t \xleftrightarrow{FT} W(j\omega) = \pi[\delta(\omega-1) + \delta(\omega+1)]$

$$\therefore \quad G(j\omega) = \frac{1}{2\pi}[X(j\omega) * W(j\omega)]$$

$$= \frac{1}{2\pi}[X(j(\omega-1)) + X(j(\omega+1))]$$



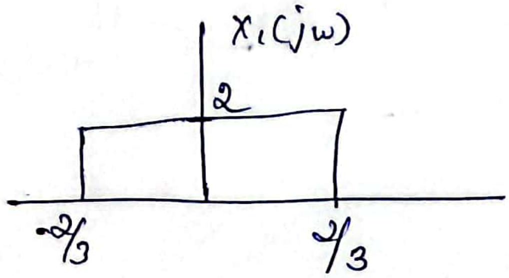


$$\therefore \quad x(t) = IFT(X(j\omega)) = \frac{2\sin t}{\pi t}$$

(b) $g(t) = x_1(t)\cos\left(\frac{2}{3}t\right)$

Here $w(t) = \cos\left(\frac{2}{3}t\right) \xleftrightarrow{FT} \pi\left[\delta\left(\omega - \frac{2}{3}\right) + \delta\left(\omega + \frac{2}{3}\right)\right]$

$\therefore$



Q2.

(a) $$H(j\omega) = \frac{j\omega+4}{6-\omega^2+5j\omega}$$

$$\therefore \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega+4}{6-\omega^2+5j\omega}$$

$$\therefore 6Y(j\omega) - \omega^2 Y(j\omega) + 5j\omega Y(j\omega) = j\omega X(j\omega) + 4X(j\omega).$$

Taking inverse FT on both sides.

$$\frac{\partial^2 y(t)}{\partial t^2} + 5\frac{\partial y(t)}{\partial t} + 6y(t) = \frac{\partial x(t)}{\partial t} + 4x(t).$$

(b) $$H(j\omega) = \frac{j\omega+4}{6-\omega^2+5j\omega}.$$

Splitting the denominator & finding the factors.

$$= \frac{2}{(2+j\omega)} - \frac{1}{(3+j\omega)}$$

Taking inverse FT,

$$h(t) = 2e^{-2t}u(t) - e^{-3t}u(t)$$

(c) $x(t) = e^{-4t}u(t) - t\,e^{-4t}u(t)$

$$X(j\omega) = \frac{1}{4+j\omega} - \frac{1}{(4+j\omega)^2}$$

$$\therefore \quad Y(j\omega) = X(j\omega)\,H(j\omega)$$

$$= \frac{3+j\omega}{(4+j\omega)^2} \times \frac{(4+j\omega)}{(2+j\omega)(3+j\omega)}$$

$$= \frac{1}{(4+j\omega)(2+j\omega)}$$

Taking IFT, $y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-4t}u(t)$

Q3.

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right)\right]u[n]$$

Expanding $h[n]$

$$h[n] = \frac{1}{2}\left(\frac{1}{2}e^{j\pi/2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}e^{-j\pi/2}\right)^n u[n]$$

$$\therefore \quad H(j\omega) = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{j\pi/2}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{j\pi/2}e^{-j\omega}}$$

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(i) $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\therefore Y(e^{j\omega}) = \left[\frac{\frac{1}{2}}{1-\frac{1}{2}e^{j\pi/2}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\pi/2}e^{-j\omega}}\right]\left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right]$$

Rearranging the above,

$$= \frac{-j/[2(1-j)]}{1-(\frac{1}{2})e^{j\pi/2}e^{-j\omega}} + \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}} + \frac{1/[2(1+j)]}{1-\frac{1}{2}e^{-j\pi/2}e^{-j\omega}}$$

$$\therefore y[n] = \frac{-j}{2(1-j)}\left(\frac{j}{2}\right)^n u[n] + \frac{1}{2(1+j)}\left(\frac{-j}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{2}\right)^n u[n]$$

(ii) $x[n] = \cos\left(\frac{\pi n}{2}\right)$

$$y[n] = \frac{\cos\left(\frac{\pi n}{2}\right)}{3}\left[4 - \left(\frac{1}{2}\right)^n\right]u[n]$$

Q4. $y[n] + \frac{1}{2} y[n-1] = x[n]$

(a)

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1+\frac{1}{2}e^{-j\omega}}$$

(b)

(i) $x[n] = \left(\frac{1}{2}\right)^n u[n]$

$$X(e^{j\omega}) = \frac{1}{1-\frac{1}{2}e^{-j\omega}}$$

$$\therefore Y(e^{j\omega}) = \left[\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right]$$

$$= \frac{\frac{1}{2}}{1-\frac{1}{2}e^{-j\omega}} + \frac{\frac{1}{2}}{1+\frac{1}{2}e^{-j\omega}}$$

Taking the inverse fourier transform,

$$y[n] = \frac{1}{2}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(-\frac{1}{2}\right)^n u[n]$$

(ii) $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$

$$X(e^{j\omega}) = 1+\frac{1}{2}e^{-j\omega}$$

$\therefore Y(e^{j\omega}) = 1$ & $y[n] = \delta[n]$

(c)

(i) $X(e^{j\omega}) = \dfrac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$

$$Y(e^{j\omega}) = \left[\frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right]$$

$$= \frac{1}{(1+\frac{1}{2}e^{-j\omega})^2} - \frac{\frac{1}{4}e^{-j\omega}}{(1+\frac{1}{2}e^{-j\omega})^2}$$

Taking IFT,

$$y[n] = (n+1)\left(-\tfrac{1}{2}\right)^n u[n] - \tfrac{1}{4}\, n\left(-\tfrac{1}{2}\right)^{n-1} u[n-1]$$

(ii) $X(e^{j\omega}) = 1+2e^{-3j\omega}$

$$Y(e^{j\omega}) = \left[1+2e^{-3j\omega}\right]\left[\frac{1}{1+\frac{1}{2}e^{-j\omega}}\right]$$

$$= \frac{1}{1+\frac{1}{2}e^{-j\omega}} + \frac{2e^{-3j\omega}}{1+\frac{1}{2}e^{-j\omega}}$$

Taking IFT,

$$y[n] = \left(-\tfrac{1}{2}\right)^n u[n] + 2\left(-\tfrac{1}{2}\right)^{n-3} u[n-3].$$

Q5) (a) $x[n] = 2^n \sin\left(\frac{\pi}{4}n\right) u[n]$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{0} 2^n \sin\left(\frac{\pi}{4}n\right) e^{-j\omega n}$$

$$= -\sum_{n=0}^{\infty} 2^{-n} \sin(\pi n/4)\, e^{j\omega n}$$

$$= \frac{-1}{2j} \sum_{n=0}^{\infty} \left[(1/2)^n e^{j\pi n/4} e^{j\omega n} - \left((1/2)^n e^{-j\pi n/4} e^{j\omega n} \right) \right]$$

$$= \frac{-1}{2j} \left[\frac{1}{1-(1/2)e^{j\pi/4} e^{j\omega}} - \frac{1}{1-(1/2)e^{-j\pi/4} e^{j\omega}} \right]$$

(b) $x[n] = x[n-6]$ and $x[n] = u[n] - u[n-5]$ for $0 \le n \le 5$

$x[n]$ is periodic with $N = 6$

$\Rightarrow$ For a signal $x[n]$ with Fourier series representation $x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$

the Fourier transform is given by

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - 2\pi k/N)$$

$$\Rightarrow \quad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-j\omega_0 k n}$$

$$\Rightarrow \quad a_k = \frac{1}{6} \sum_{n=0}^{5} x[n]\, e^{-j(2\pi/6)n.k}$$

$$= \frac{1}{6} \sum_{n=0}^{4} e^{-j(2\pi/6)n.k}$$

$$= \frac{1}{6} \left[\frac{1 - e^{-j5\pi k/3}}{1 - e^{-j\pi k/3}} \right]$$

$$\Rightarrow X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi \left(\frac{1}{6} \left[\frac{1-e^{-j5\pi k/3}}{1-e^{-j\pi k/3}} \right] \right) \delta(\omega - \pi/3 - 2\pi l)$$

$$= \sum_{l=-\infty}^{\infty} (\pi/3) \left(\frac{1-e^{-j5\pi k/3}}{1-e^{-j\pi k/3}} \right) \delta(\omega - \pi/3 - 2\pi l)$$

(c) $x[n] = \left(\frac{1}{2}\right)^{|n|} \cos\left(\frac{\pi}{8}(n-1)\right)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} (1/2)^{|n|} \cos(\pi(n-1)/8)\, e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} (1/2)^{|n|} \frac{\left(e^{j\pi(n-1)/8} + e^{-j\pi(n-1)/8}\right)}{2} e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left((1/2)^{|n|+1} e^{j\pi(n+1)/8} e^{-j\omega n} + (1/2)^{|n|+1} e^{-j\pi(n+1)/8} e^{-j\omega n} \right)$$

$$= \sum_{n=-\infty}^{\infty} \left((1/2)^{|n|+1} e^{j(\pi(n+1)/8 - \omega n)} + (1/2)^{|n|+1} e^{-j(\pi(n+1)/8 + \omega n)} \right)$$

$$= \frac{1}{2}\left[\frac{e^{-j\pi/8}}{1-(1/2)e^{j(\pi/8-\omega)}} + \frac{e^{j\pi/8}}{1-(1/2)e^{-j(\pi/8+\omega)}} \right] +$$

$$\frac{1}{4}\left[\frac{e^{j(\pi/4+\omega)}}{1-(1/2)e^{j(\pi/8+\omega)}} + \frac{e^{-j(\pi/4-\omega)}}{1-(1/2)e^{-j(\pi/8-\omega)}} \right]$$

Q6) Consider $g(t) = (y_1(t) \cdot y_2(t)) * y_3(t)$

where $y_1(t) = \cos^2(t) = \dfrac{1+\cos(2t)}{2}$

$y_2(t) = x(t)$

$y_3(t) = \dfrac{\sin t}{\pi t}$

$$\Rightarrow \quad Y_1(jw) = \pi\delta(w) + \frac{\pi}{2}\left(\delta(w-2) + \delta(w+2)\right)$$

$$\begin{aligned} F.T.\{y_1(t)\cdot y_2(t)\} &= \left(Y_1(jw) * Y_2(jw)\right)/2\pi \\ &= \left(Y_1(jw) * X(jw)\right)/2\pi \\ &= \frac{X(jw)}{2} + \frac{X(j(w-2))}{4} + \frac{X(j(w+2))}{4} \end{aligned}$$

$$Y_3(jw) = \begin{cases} 1 & |w| < 1 \\ 0 & |w| \geq 1 \end{cases}$$

$\Rightarrow$ For $g(t) = (y_1(t)\, y_2(t)) * y_3(t)$,

$$\begin{aligned} G(jw) &= F.T.\{y_1(t)\, y_2(t)\}\cdot Y_3(jw) \\ &= \left(\frac{X(jw)}{2} + \frac{X(j(w-2))}{4} + \frac{X(j(w+2))}{4}\right) Y_3(jw) \end{aligned}$$

Since $X(jw) = 0$ for $|w| \geq 1$

$\Rightarrow$ $X(j(w-2)) = 0$ for $|w-2| \geq 1$ or $w \geq 3$ & $w \leq 1$

$X(j(w+2)) = 0$ for $|w+2| \geq 1$ or $w \leq -3$ & $w \geq -1$

$\Rightarrow$ F.T. $\{y_1(t)\, y_2(t)\}$ only exists between -3 to $+3$

where:

$$F.T.\{y_1(t)\, y_2(t)\} = \begin{cases} X(jw)/2 & -1 \leq w \leq 1 \\ X(j(w-2))/4 & 1 \leq w \leq 3 \\ X(j(w+2))/4 & -3 \leq w \leq -1 \end{cases}$$

$\Rightarrow$ Multiplying by $Y_3(jw)$ which only exists between -1 & 1:

$$G(jw) = \frac{1}{2}X(jw) \quad \Rightarrow \text{LTI system w/ } h(t) = \frac{1}{2}\delta(t)$$