Q1

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

We know that, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \longrightarrow (1)$ [Ampere Circuital Law]

also, $\vec{\nabla} . \vec{B} = 0$ ($\vec{B}$ ≡ solenoidal field)

Hence, $\vec{B}$ can be written as curl of any vector field

Since divergence of curl of a vector field is zero.

$\therefore \vec{B} = \vec{\nabla} \times \vec{A}$ ($\vec{A}$ ≡ magnetic vector potential)

Now, $\vec{A}$ can be written as $\vec{A}_0 + \vec{\nabla} \lambda$ [Gauge Freedom]

{ Here, $\vec{A}_0$ be our original vector potential where $\vec{B} = \vec{\nabla} \times \vec{A}_0$ }

Since, $\vec{\nabla} \times \vec{A}$

$= \vec{\nabla} \times (\vec{A}_0 + \vec{\nabla} \lambda)$

$= \vec{\nabla} \times \vec{A}_0 + \vec{\nabla} \times (\vec{\nabla} \lambda)$

$= \vec{\nabla} \times \vec{A}_0 + 0$

$= \vec{B}$

$\therefore \vec{B} = \vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_0$

For $\vec{\nabla} . \vec{A} = 0$

$\Rightarrow \vec{\nabla} . \vec{A}_0 + \nabla^2 \lambda = 0$

$\Rightarrow \vec{\nabla} . \vec{A}_0 = -\nabla^2 \lambda$

Find λ by this, and hence we have $\vec{A}$

Now, we have the liberty to define divergence of $\vec{A}$ as per our convenience. This is known as coulomb gauge.

Let, $\vec{\nabla} . \vec{A} = 0$ (For simplicity in calculation)

$\therefore$ using (1) $\Rightarrow \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$

$\left[\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla} . \vec{v}) - \nabla^2 \vec{v} \right]$

$\Rightarrow \vec{\nabla} (\vec{\nabla} . \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

$\Rightarrow \vec{\nabla} (0) - \nabla^2 \vec{A} = \mu_0 \vec{J}$

$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$

Hence Proved

Note: For this Ampere Circuital Law to be defined, our magnetic field should be constant / not time varying.

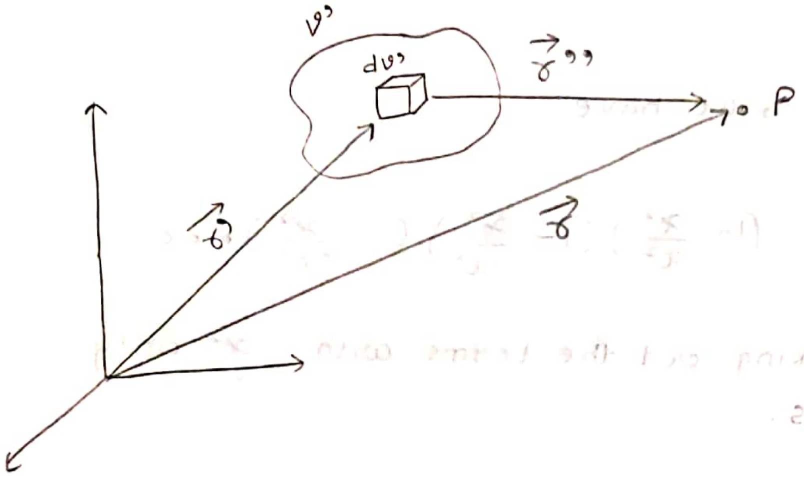
Q2

Vector Potential for a single magnetic dipole $\vec{m}$:

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}''}{r''^2} \qquad \left[\vec{r}'' = \vec{r} - \vec{r}' \right]$$

For a distribution of magnetic dipoles

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \left[\int_{V'} \frac{\vec{J}_b(r')}{\vec{r}''} dv' + \oint_S \frac{\vec{K}_b(r')}{\vec{r}''} ds' \right]$$



To find the vector potential $\vec{A}(r)$ due to an accumulation of dipoles, we simply take a volume integral of a small element from a material which contains many many number of dipoles as shown in the above figure.

Let the Magnetic dipole per unit volume be $\vec{M}(r')$

$\therefore$ For small dv', $d\vec{m} = \vec{M} dv'$

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \int_{V'} \frac{d\vec{m} \times \hat{r}''}{(r'')^2}$$

$$\Rightarrow \vec{A}(r) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{M}(r') \times \hat{r}''}{(r'')^2} dv'$$

$$\Rightarrow \vec{A}(r) = \frac{\mu_0}{4\pi} \int_{V'} \vec{M}(r') \times \left(\frac{\hat{r}''}{r''^2} \right) dv' = \frac{\mu_0}{4\pi} \int_{V'} \vec{M}(r') \times \left(\vec{\nabla}' \frac{1}{r''} \right) dv'$$

Note: $\vec{\nabla} \equiv -\vec{\nabla}'$

(Shifting the origin to $\vec{r}'$)

(Refer to class notes)

Using Product Rule 7, we get (explained below #)

$$\vec{A}(r) = \frac{\mu_0}{4\pi} \int_{V'} \left\{ \frac{1}{r''} \left(\vec{\nabla}' \times \vec{M}(r') \right) - \left(\vec{\nabla}' \times \left(\frac{\vec{M}(r')}{r''} \right) \right) \right\} dV'$$

$$= \frac{\mu_0}{4\pi} \int_{V'} \frac{1}{r''} \left(\vec{\nabla}' \times \vec{M}(r') \right) dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{1}{r''} \left[\vec{M}(r') \times \vec{ds}' \right]$$

$$\left[\because \int_V (\vec{\nabla} \times \vec{A})\, dv = - \oint_S \vec{A} \times d\vec{s} \right]$$

Comparing with our question

$$\vec{J}_b(r') = \vec{\nabla}' \times \vec{M}(r')$$

$$\vec{K}_b(r') = \vec{M}(r') \times \hat{n} \quad \{ \hat{n} \text{ is unit normal vector} \}$$

$\vec{J}_b(r')$ is the potential of volume current.

$\vec{K}_b(r')$ is the potential of surface current.

(#) $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times(\vec{\nabla} f)$

$\Rightarrow \vec{A}\times(\vec{\nabla} f) = f(\vec{\nabla}\times\vec{A}) - \vec{\nabla}\times(f\vec{A})$

Comparing with our equation,

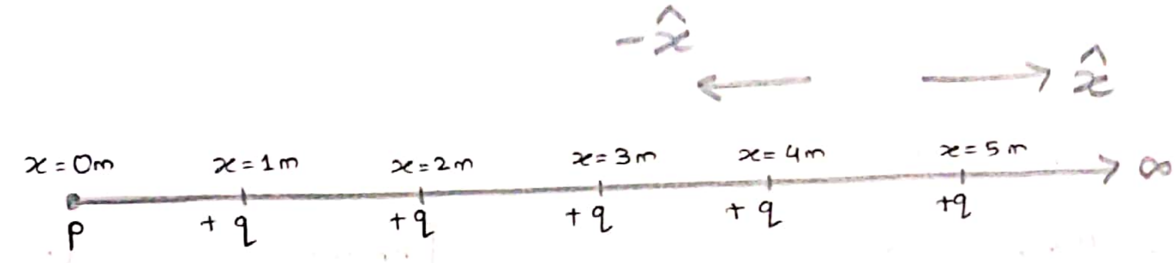
$\vec{A} = \vec{M}(r')$

$\vec{\nabla} = \vec{\nabla}'$

$f = \dfrac{1}{r''}$

$\therefore \vec{M}(r') \times \left(\vec{\nabla}' \dfrac{1}{r''}\right) = \dfrac{1}{r''}\left[\vec{\nabla}' \times \vec{M}(r')\right] - \vec{\nabla}' \times \left[\dfrac{\vec{M}(r')}{r''}\right]$

Q3



We know, Electric field due to a point charge q.

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

$\vec{E}$ due to q at $x = 1m$,

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{1^2}(-\hat{x})$$

Similarly, $$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{q}{2^2}(-\hat{x})$$

$$\vec{E}_3 = \frac{1}{4\pi\varepsilon_0}\frac{q}{3^2}(-\hat{x})$$

$\vdots$

$$\vec{E}_n = \frac{1}{4\pi\varepsilon_0}\frac{q}{n^2}(-\hat{x})$$

We have,

$$\vec{E}_P = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \dots + \vec{E}_n \text{, where } n \to \infty$$

$$\Rightarrow \vec{E}_P = \frac{1}{4\pi\varepsilon_0} q \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2}\right)(-\hat{x})$$

$$\Rightarrow \vec{E}_P = \frac{-q}{4\pi\varepsilon_0}\left\{\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)\right\}\hat{x} \longrightarrow (*)$$

Let us calculate $\sum_{n=1}^{\infty}\frac{1}{n^2}$

We have, $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\Rightarrow \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots \longrightarrow \text{(I)}$$

Now, we have,

Factors of $\sin x$ are $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

$$\therefore \sin x = (x-0)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots$$

or

$$\sin x = x\left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots$$

$$\Rightarrow \sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots \longrightarrow (2)$$

Using (1) and (2), we have

$$1-\frac{x^2}{3!}+\frac{x^4}{5!}-\ldots = \left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\ldots$$

Solving RHS and taking out the terms with x^2 only. same for LHS.

$$-\frac{x^2}{3!} = -\frac{x^2}{\pi^2}-\frac{x^2}{4\pi^2}-\frac{x^2}{9\pi^2}-\ldots$$

$$\Rightarrow \frac{x^2}{3!} = \frac{x^2}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right)$$

$$\Rightarrow \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots = \frac{\pi^2}{3!} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6} \longrightarrow (*)(*)$$

$$(*) \Rightarrow \vec{E_p} = \frac{-q}{4\cancel{\pi}\epsilon_0}\,\frac{\pi^{\cancel{2}}}{6}\,\hat{x} \quad (\text{using } (*)(*))$$

$$\boxed{\therefore \vec{E_p}\Big|_{q=1} = \frac{-\pi\,\hat{x}}{24\,\epsilon_0}\ \cancel{C/m^2}\ Nm/C}$$