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Review

① $v(t) = 5e^{-3t}\cos(5t + 45^\circ)$ $\longrightarrow$ Don't use $\mathcal{L}T$

$V(s) = 5\angle 45^\circ = 5e^{j45^\circ} = 5(\cos 45^\circ + j\sin 45^\circ)$

$s = -3 \pm j5$

② $v(t) = 5e^{-3t}$ $\longrightarrow$ Don't use $\mathcal{L}T$

$V(s) = 5\angle 0^\circ = 5$

$s = -3$

③ $v(t) = 5e^{-3t}u(t)$ $\longrightarrow$ use $\mathcal{L}T$

Laplace Transform

Time domain: $f(t)$

Frequency domain (s) $\quad F(s) \overset{\mathcal{L}T}{=} \int_{-\infty}^{\infty} f(t)e^{-st}\,dt$ $\Rightarrow$ used a lot

$$f(t) \overset{I\mathcal{L}T}{=} \frac{1}{2\pi j}\int_{\sigma_0 - j\infty}^{\sigma_0 + j\infty} F(s)e^{+st}\,ds$$ $\Rightarrow$ not used

① $\delta(t)$ Impulse / Dirac delta function

$$\delta(t) = \begin{cases} 1, & t = 0 \\ 0, & t \neq 0 \end{cases} \qquad \delta(t - t_0) = \begin{cases} 1, & t = t_0 \\ 0, & t \neq t_0 \end{cases}$$

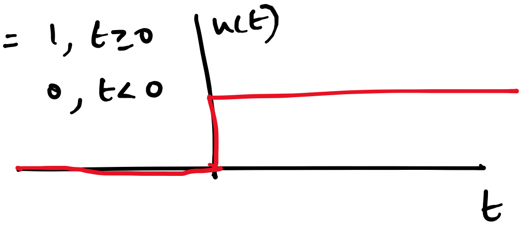
$$\delta(t)f(t) = \begin{cases} f(0), & t = 0 \\ 0, & t \neq 0 \end{cases}$$

$$LT\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-st}\, dt = e^{-s\times 0} = 1$$

$$LT\{5\,\delta(t)\} = 5$$

$$LT\{5\,\delta(t+3)\} = \int_{-\infty}^{\infty} 5\,\delta(t+3)\, e^{-st}\, dt = 5e^{-s(-3)} = 5e^{+3s}$$

(2) $f(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$



$$LT\{u(t)\} = \int_{-\infty}^{\infty} u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{-1}{s}\left[0 - 1\right] = \frac{1}{s}$$

$$LT\{100\, u(t)\} = \frac{100}{s}$$

→ ILT $\quad V(s) = 300\, e^{-3s} + \dfrac{200}{s}$

$$v(t) = 300\,\delta(t-3) + 200\, u(t)$$

(3) $f(t) = e^{-at}\, u(t)$ a is a real value

$$LT\{e^{-at}\, u(t)\} = \int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} e^{-(s+a)t}\, dt$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^{\infty} = \frac{1}{s+a}$$

(4) $f(t) = \cos(\omega_0 t)\, u(t)$ $\qquad$ ω_0 is a real value

$$= \left[\frac{e^{+j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] u(t)$$

$$F(s) = \mathcal{L}T\left\{\frac{1}{2} e^{j\omega_0 t} u(t)\right\} + \mathcal{L}T\left\{\frac{1}{2} e^{-j\omega_0 t} u(t)\right\}$$

$$= \frac{1}{2} \times \frac{1}{s - j\omega_0} + \frac{1}{2} \times \frac{1}{s + j\omega_0}$$

$$= \frac{1}{2}\left[\frac{s + j\omega_0 + s - j\omega_0}{(s - j\omega_0)(s + j\omega_0)}\right] = \frac{\cancel{2}s}{\cancel{2}(s^2 + \omega_0^2)} = \frac{s}{s^2 + \omega_0^2}$$

(5) $f(t) = \sin(\omega_0 t)\, u(t)$

$$F(s) = \mathcal{L}T\{f(t)\} = \frac{\omega_0}{s^2 + \omega_0^2}$$

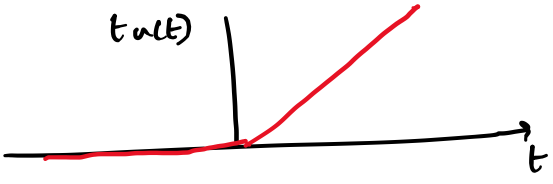
→ Find $I\mathcal{L}T$

a) $F(s) = 1.55 - \frac{2}{s}$ $\Rightarrow$ $f(t) = 1.55\, \delta(t) - 2u(t)$

b) $F(s) = \frac{1.5}{(s+9)} + \frac{18}{s^2 + 9}$ $\Rightarrow$ $f(t) = \left[1.5 e^{-9t} + 6 \sin(3t)\right] u(t)$

$$= \frac{1.5}{s+9} + 6 \times \left(\frac{3}{s^2 + 3^2}\right)$$

(6) $f(t) = t\, u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}$



$$F(s) = \mathcal{L}T\{t\, u(t)\} = \int_{-\infty}^{\infty} t\, u(t)\, e^{-st}\, dt = \int_{0}^{\infty} t\, e^{-st}\, dt$$

$$= \left[t \times \frac{e^{-st}}{-s} - \int \frac{e^{-st}}{-s} \times 1\, dt\right]_0^{\infty} = 0 + \frac{1}{s}\int_0^{\infty} e^{-st}\, dt = \frac{1}{s^2}$$

Table 14.1 (pg 559)

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$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$t\,u(t)$	$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\cos(\omega_0 t)\,u(t)$	$\frac{s}{s^2+\omega_0^2}$
$\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$
$e^{-at}\cos(\omega_0 t)\,u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$
$e^{-at}\sin(\omega_0 t)\,u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$

$$-\frac{d}{ds}\left(\frac{1}{s}\right) = -1 \times -\frac{1}{s^2} = \frac{1}{s^2}$$

(1) $g(t) = f(t)\,e^{-at}$

$$F(s) = LT\{f(t)\}$$

$$G(s) = LT\{g(t)\} = \int_{-\infty}^{\infty} f(t)\,e^{-at}\,e^{-st}\,dt$$

$$F(s+a) = \int_{-\infty}^{\infty} f(t)\,e^{-(s+a)t}\,dt$$

→ $f(t) = \sin(5t)\,e^{-3t}\,u(t)$

$$LT\{\sin(5t)\,u(t)\} = \frac{5}{s^2+5^2} = \frac{5}{s^2+25}$$

$$\Rightarrow LT\{\sin(5t)\,e^{-3t}\,u(t)\} = \frac{5}{(s+3)^2+25}$$

→ $f(t) = t\,\underbrace{e^{-5t}}\,u(t) \Rightarrow F(s) = \frac{1}{(s+5)^2}$

$$-\frac{d}{ds}\left(\frac{1}{s+5}\right) = \frac{1}{(s+5)^2}$$

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(2) $g(t) = t f(t)$

$G(s) = -\frac{d}{ds} F(s)$

(3) $g(t) = \frac{f(t)}{t}$

$G(s) = \int_0^\infty F(s)\, ds$

(4) $g(t) = \frac{df}{dt}$

$G(s) = sF(s) - f(0^-)$

(5) $g(t) = \int_{-\infty}^{t} f(t)\, dt$

$G(s) = \frac{F(s)}{s} + \frac{1}{s}\int_{-\infty}^{0^-} f(t)\, dt$

$\rightarrow$ $F(s) = \frac{s^2 + 4s + 4}{s} = s + 4 + \frac{4}{s}$

$f(t) = I\mathcal{L}T\left\{s + 4 + \frac{4}{s}\right\} = I\mathcal{L}T\{s \times 1\} + 4\delta(t) + 4u(t)$

$= \frac{d}{dt}(\delta(t)) + 4\delta(t) + 4u(t)$

(assuming initial conditions are zero)

$\rightarrow$ $F(s) = \frac{s+2}{s^2 + 2s + 4}$

$= \frac{s+2}{(s^2+2s+1)+3} = \frac{(s+1)+1}{(s+1)^2 + 3}$

$= \frac{(s+1)}{(s+1)^2 + (\sqrt{3})^2} + \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{(s+1)^2 + (\sqrt{3})^2}$

$f(t) = e^{-1t}\cos(\sqrt{3}t)u(t) + \frac{1}{\sqrt{3}} e^{-1t}\sin(\sqrt{3}t)u(t)$

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$\rightarrow \quad F(s) = \dfrac{8}{(s+3)^2} + \dfrac{4s}{2s+3}$

$\rightarrow \quad F(s) = -\dfrac{1}{2s} + \dfrac{1}{(0.5s)^2} + \dfrac{4}{(s+5)^2} + 2$

$\rightarrow \quad F(s) = 7 - \dfrac{s+\frac{1}{s}}{s^2+3s+1}$

$$\frac{4s}{2s+3} = \frac{\overset{2}{\cancel{4}}s}{\cancel{2}\left(s+\frac{3}{2}\right)} = \frac{2s}{s+\frac{3}{2}} = \frac{2\left(s+\frac{3}{2}-\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)}$$

$$= 2 - \frac{3}{\left(s+\frac{3}{2}\right)}$$