Quiz 1 Solution

F&W, ECE230, Winter 2022

Q1. Assume, a dielectric sphere of radius R and permittivity ϵ. The entire volume of the sphere is charged uniformly with a total charge Q. Calculate and plot the electric field as a function of r(r is the distance from the center of the sphere, $0 \leq r < \infty$.

7 points (4 for calculations, 3 for plot)

$\underline{r > R}$

$$\oint_S \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon}$$

$$\Rightarrow |\vec{E}| \cdot 4\pi r^2 = \frac{Q}{\epsilon}$$

$$\Rightarrow \vec{E} = \frac{Q}{4\pi\epsilon r^2} \hat{r}$$

$$= \frac{\rho \times \frac{4}{3}\pi R^3}{4\pi\epsilon \; r^2} \hat{r}$$

$$\therefore \boxed{\vec{E} = \frac{\rho R^3}{3\epsilon \; r^2} \hat{r}}$$

$\underline{r < R}$

$$\oint \vec{E} \cdot \vec{ds} = \frac{Q_{enc}}{\epsilon}$$

$$\Rightarrow E \cdot 4\pi r^2 = \frac{1}{\epsilon} \left[\frac{4}{3}\pi r^3 \times \rho\right]$$

where ρ = volume charge density

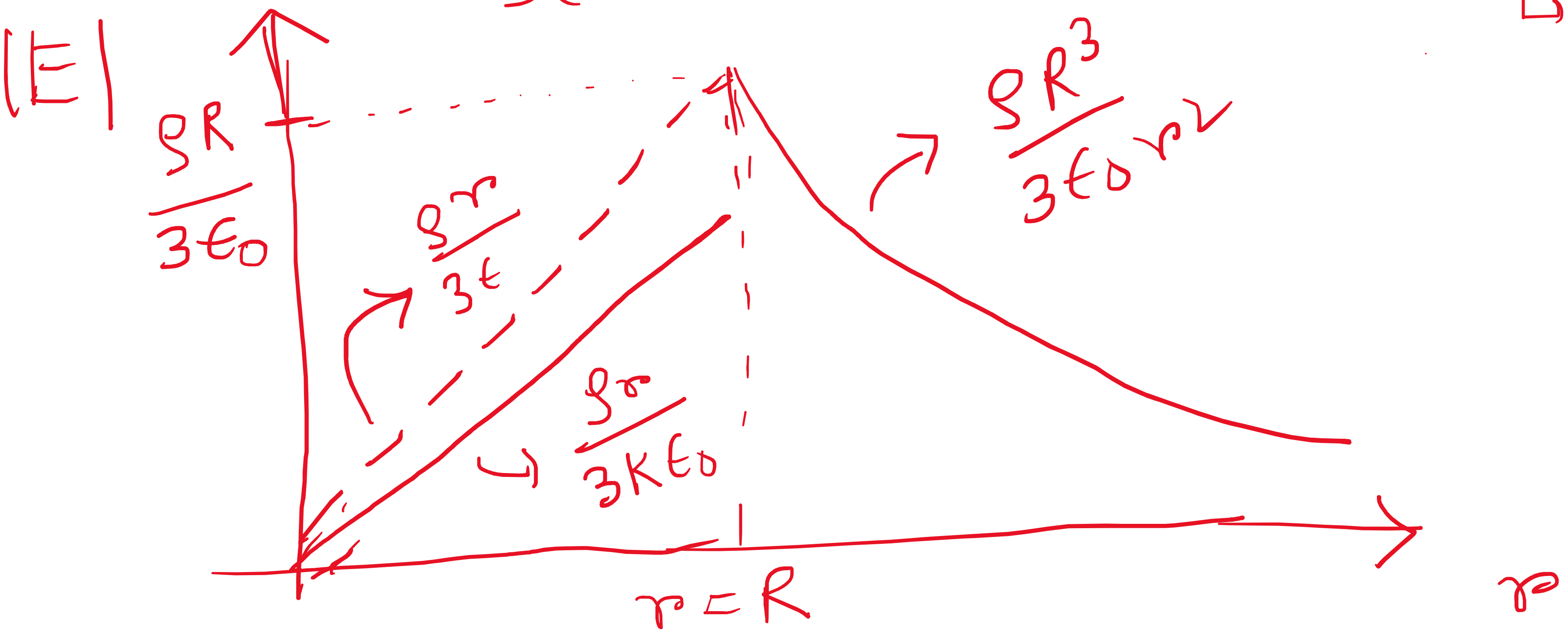
$$\Rightarrow \boxed{\vec{E} = \frac{\rho r}{3\epsilon} \hat{r}}$$

$$\therefore \ |E| = \frac{\rho R^3}{3\epsilon r^2} \quad \text{for } r > R$$

$$= \frac{\rho r}{3\epsilon} \quad \text{for } r < R$$

where $\epsilon = K\epsilon_0$

↳ dielectric constant



Q2. Prove that $\vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)=0$.
3 points

Let, $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$$\therefore \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Now, $\vec{\nabla}.\left(\vec{\nabla}\times\vec{A}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \frac{\partial^2 A_z}{\partial x\,\partial y} - \frac{\partial^2 A_y}{\partial x\,\partial z} - \frac{\partial^2 A_z}{\partial y\,\partial x} + \frac{\partial^2 A_x}{\partial y\,\partial z} + \frac{\partial^2 A_y}{\partial z\,\partial x} - \frac{\partial^2 A_x}{\partial z\,\partial y}$$

$$= 0$$

Q3. The electrostatic potential of some charge-distribution is given by:

$$V(r) = A\frac{e^{-\lambda r}}{r}$$

where, A and λ are constants. Find the electric field $\vec{E}(r)$ and the charge density $\rho(r)$. [3 + 7 = 10 points]

$$E = -\nabla V = -A \frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{r}$$

$$= -A\left\{\frac{r.(-\lambda).e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\}\hat{r}$$

$$\boxed{\vec{E} = \frac{A e^{-\lambda r}(r\lambda + 1)}{r^2}\hat{r}}$$

Now,

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \Rightarrow \rho = \epsilon_0(\vec{\nabla}\cdot\vec{E})$$

$$= \epsilon_0 A\left[\nabla\cdot \frac{e^{-\lambda r}(r\lambda+1)}{r^2}\hat{r}\right] = \epsilon_0 A\left[e^{-\lambda r}(r\lambda+1)\,\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2}\,\nabla\left(e^{-\lambda r}(r\lambda+1)\right)\right] \quad \text{—— (i)}$$

We know that, $\nabla\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\,\delta^3(r)$

Therefore, $\rho = \epsilon_0 A \left[e^{-\lambda r}(r\lambda+1) \, \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) + \frac{\hat{r}}{r^2} \nabla\left(e^{-\lambda r}(r\lambda+1)\right)\right] = \epsilon_0 A \left[e^{-\lambda r}(r\lambda+1) 4\pi \delta^3(r) + \frac{\hat{r}}{r^2} \cdot \nabla\left(e^{-\lambda r}(r\lambda+1)\right)\right]$

Now,

$$\nabla\left(e^{-\lambda r}(r\lambda+1)\right) = \hat{r}\frac{\partial}{\partial r}\left(e^{-\lambda r}(r\lambda+1)\right)$$

$$= \hat{r}\left[e^{-\lambda r}\lambda + (r\lambda+1)\cdot(-\lambda)\, e^{-\lambda r}\right]$$

$$= \hat{r}\left[\lambda e^{-\lambda r} - r\lambda^2 e^{-\lambda r} - \lambda e^{-\lambda r}\right]$$

$$= \hat{r}\left[-r\lambda^2 e^{-\lambda r}\right]$$

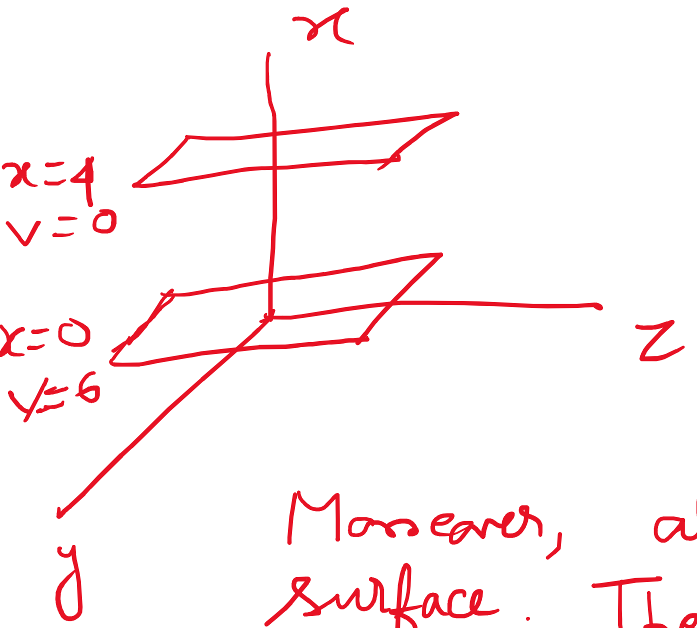
Again, $e^{-\lambda r}(r\lambda+1)4\pi\delta^3(r) = 4\pi\delta^3(r)$

$$\therefore \rho = \epsilon_0 A\left[4\pi\delta^3(r) + \frac{1}{r^2}\left(-r\lambda^2 e^{-\lambda r}\right)\right]$$

$$\boxed{\rho = \epsilon_0 A\left[4\pi\delta^3(r) - \frac{\lambda^2}{r} e^{-\lambda r}\right]}$$

Q4. Consider two perfectly conducting plates, infinite along the yz plane are placed at $x = 0$ and $x = 4$. The potential at first plate is $6V$ and the second plate is grounded. Find an expression for potential distribution in the region between the plates. [Hint: recall Poisson's and Laplace's equations. You might want to consider this as a 1-dimensional case]. Can you explain why this is 1 dimensional problem? This explanation must be solid so that I can award you the 1 point as indicated in the marks breakup.

$4 + 1 = 5$ points



we know that, from Laplace equation

$$\nabla^2 V = 0$$

$$\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = 0 \quad \text{— (1)}$$

Here, V is only varying along x direction.

Moreover, at a fixed value of x, we will get equipotential surface. Therefore we can write equation (1) as

$$\frac{\partial^2 V}{\partial x^2} = 0$$

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$\Rightarrow \quad V = Ax + B$

at $\quad x = 0, \quad V = 6 \quad \Rightarrow \quad 6 = A \cdot 0 + B$

$\Rightarrow \quad \boxed{B = 6}$

at $\quad x = 4, \quad V = 0 \quad \Rightarrow \quad 0 = A \cdot 4 + B$

$\Rightarrow \quad 0 = 4A + 6$

$\Rightarrow \quad A = -\frac{6}{4} = -1.5$

$\therefore \quad \boxed{V = -1.5x + 6}$