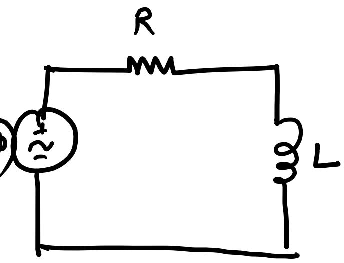
Lecture 7 / Chapter 18

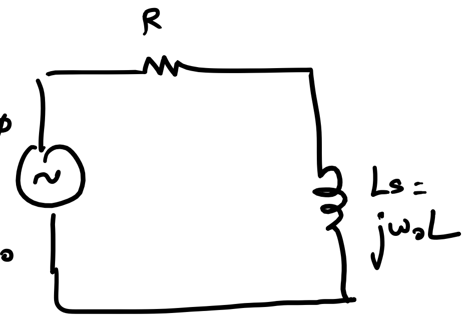
Fourier Circuit Analysis

Review

$v(t) = V_0 \cos(\omega_0 t + \phi)$



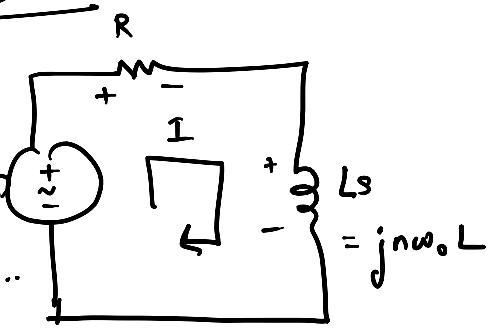
$\longrightarrow$ $V_0 e^{j\phi}$, $s = j\omega_0$



Instead

$V_s = \sum_{n=0}^{\infty} a_n \delta(\omega - n\omega_0)$

$\omega = 0, \omega_0, 2\omega_0, \cdots n\omega_0, \cdots$



$$+V_s - I_n \times R - I_n(jn\omega_0 L) = 0$$

$$I_n = \frac{V_s}{R + jn\omega_0 L} = \frac{a_n}{R + jn\omega_0 L}$$

$$I = \sum_{n=0}^{\infty} \frac{a_n}{R + nj\omega_0 L} \delta(\omega - n\omega_0) = \sum_{n=0}^{\infty} \frac{a_n \tan^{-1}\left(-\frac{n\omega_0 L}{R}\right)}{\sqrt{R^2 + (n\omega_0 L)^2}} \delta(\omega - n\omega_0)$$

$$\Rightarrow i(t) = \sum_{n=0}^{\infty} \frac{a_n}{\sqrt{R^2 + (n\omega_0 L)^2}} \cos\left(n\omega_0 t - \tan^{-1}\left(\frac{n\omega_0 L}{R}\right)\right)$$

Review

Source	Frequency Domain	Ckt
(1) $v(t) = V_0$	$V(s) = V_0$ $s = 0$	$R \to R$ $L \to Ls \to$ short $C \to \frac{1}{Cs} \to$ open
(2) $v(t) = V_0 \cos(\omega_0 t + \phi)$	$V(s) = V_0 \angle \phi$ $s = j\omega_0$	$R \to R$ $L \to j\omega_0 L$ $C \to \frac{1}{j\omega_0 C}$
(3) $v(t) = V_0 e^{-\sigma t} \cos(\omega_0 t + \phi)$	$V(s) = V_0 \angle \phi$ $s = -\sigma + j\omega_0$	$R \to R$ $L \to sL$ $C \to \frac{1}{sC}$

4 $v(t) = V_0 e^{-\sigma t}$

$V(s) = V_0$

$s = -\sigma$

$R \to R$

$L \to Ls$

$C \to \frac{1}{sC}$

5 $v(t) = f(t)\,u(t)$

$V(s)$ using $\mathcal{L}T$

$\left.\begin{array}{l} R \to R \\ L \to Ls \\ C \to \frac{1}{sC} \end{array}\right\}$ functions

6 $v(t)$ is periodic function with time period T, $\omega_0 = \frac{2\pi}{T}$

$V(s)$ using Fourier Series

$R \to R$

$\left.\begin{array}{l} L \to jn\omega_0 L \\ C \to \frac{1}{jn\omega_0 C} \end{array}\right\}$ n is an integer from 0 to ∞

Steps

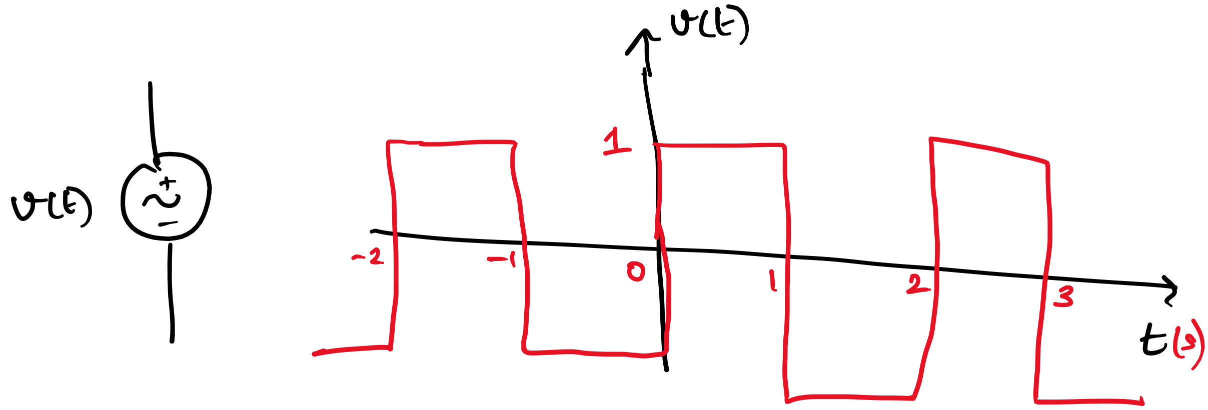
1. Find time period T & $\omega_0 = \frac{2\pi}{T}$ rad/s

2. Determine if function is odd or even

$f(-t) = f(t)$ (even function)

$f(-t) = -f(t)$ (odd function)

→



Soln :

$$T = 2s$$

$$\omega_0 = \frac{2\pi}{T} = \pi \quad rad/s$$

Using Fourier Series

$$v(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_0^T v(t)\, dt$$

$$a_n = \frac{2}{T}\int_0^T v(t) \cos(n\omega_0 t)\, dt$$

$$b_n = \frac{2}{T}\int_0^T v(t) \sin(n\omega_0 t)\, dt$$

In this case, v(t) is odd

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{+\frac{T}{2}} v(t)\, dt = \frac{1}{T}\left[\underbrace{\int_{-\frac{T}{2}}^{0} v(t)\, dt}_{I} + \int_0^{+T/2} v(t)\, dt\right]$$

$$u = -t \qquad t = -T/2, \; u = +T/2$$

$$du = -dt \qquad t = 0, \; u = 0$$

$$\Rightarrow \quad I = \int_{+\frac{T}{2}}^{0} v(-u)\,(-du) = \int_{0}^{+\frac{T}{2}} v(-u)\,du$$

Since it is odd, $\quad v(-u) = -v(u)$

$$\Rightarrow \quad I = -\int_{0}^{T/2} v(u)\,du = -\int_{0}^{T/2} v(t)\,dt$$

$$\Rightarrow \quad a_0 = -\int_{0}^{T/2} v(t)\,dt + \int_{0}^{T/2} v(t)\,dt = 0$$

$$\text{III}^{ly} \quad a_n = 0 \quad \forall\, n$$

$\left.\right\}$ odd function

$$b_n = \frac{2}{T}\int_{0}^{T} v(t)\sin(n\omega_0 t)\,dt$$

$$= \frac{2}{2}\int_{0}^{2} v(t)\sin(n\pi t)\,dt$$

$$= \int_{0}^{1} 1\sin(n\pi t)\,dt - \int_{1}^{2} \sin(n\pi t)\,dt$$

$$= \left[\frac{-\cos(n\pi t)}{n\pi}\right]_0^1 - \left[-\frac{\cos(n\pi t)}{n\pi}\right]_1^2$$

$$= \frac{1 - \cos(n\pi)}{n\pi} + \frac{\cos(2n\pi) - \cos(n\pi)}{n\pi}$$

$$= \frac{1 - 2\cos(n\pi) + \cos(2n\pi)}{n\pi} = \frac{2(1-\cos(n\pi))}{n\pi}$$

$$\Rightarrow b_n = \begin{cases} \dfrac{4}{n\pi}, & n \text{ is odd} \\ 0, & n \text{ is even} \end{cases}$$

$$\Rightarrow v(t) = \sum_{n=1,3,5,\ldots} \frac{4}{n\pi} \sin(n\pi t)$$

$$V_0 \sin(n\pi t) = V_0 \cos(n\pi t - 90^\circ)$$

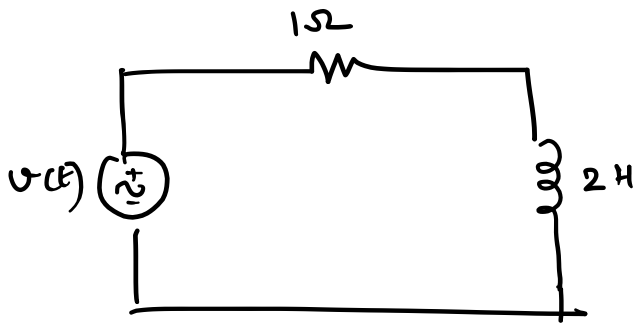
$$\Downarrow$$

$$V_0 \angle -90^\circ$$

$$V_0 [\cos(90^\circ) + j\sin(-90^\circ)]$$

$$= -jV_0$$

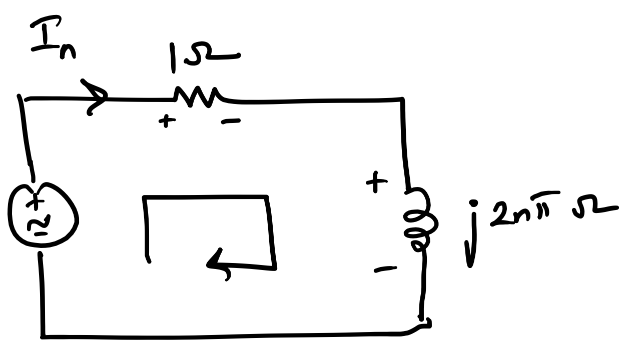
$$V(\omega) = \sum_{n=1,3,5,\ldots} -j \frac{4}{n\pi} \delta(\omega - n\pi)$$



Assume there are no initial conditions

$$\Downarrow$$

$$V(\omega) = \sum_{n=1,3,5,\ldots} -j\frac{4}{n\pi} \delta(\omega - n\pi)$$



$$+V(\omega) - I_n(1) - I_n(j2n\pi) = 0$$

$$\Rightarrow \frac{-j4}{n\pi} - I_n(1 + j2n\pi) = 0 \Rightarrow I_n = \frac{\frac{-j4}{n\pi}}{1 + j2n\pi}$$

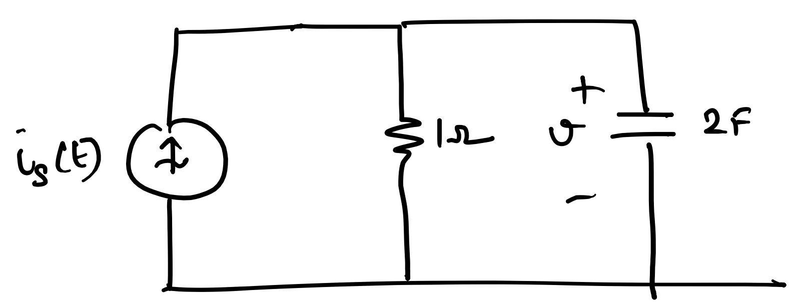
$$I_n = \frac{\frac{4}{n\pi}}{\sqrt{1 + 4n^2\pi^2}} \angle -90^\circ - \tan^{-1}(2n\pi)$$

$$\Rightarrow i_n(t) = \frac{4}{n\pi\sqrt{1+4n^2\pi^2}} \cos\left(n\pi t - 90^\circ - \tan^{-1}(2n\pi)\right)$$

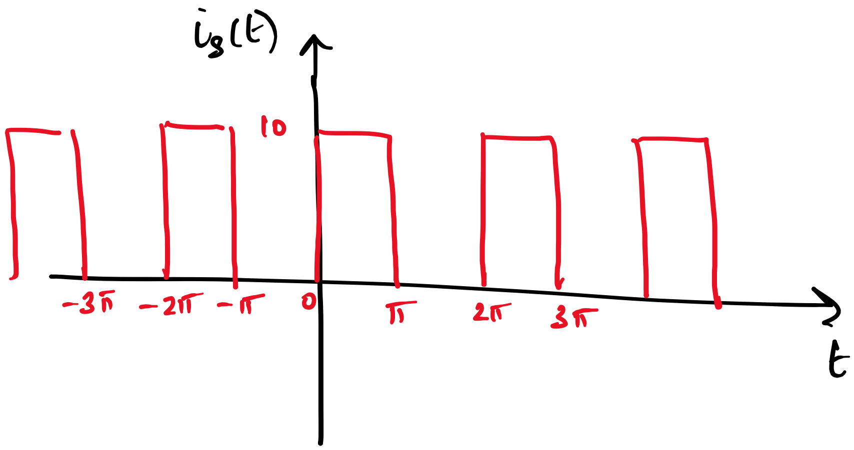
$$\Rightarrow i_n(t) = \frac{4}{n\pi\sqrt{1+4n^2\pi^2}} \sin\left(n\pi t - \tan^{-1}(2n\pi)\right)$$

$$i(t) = \sum_{n=1,3,5\ldots} \frac{4}{n\pi\sqrt{1+4n^2\pi^2}} \sin\left(n\pi t - \tan^{-1}(2n\pi)\right) \text{ A}$$

$\rightarrow$



Find $v(t)$ assuming no initial conditions



Soln :

$T = 2\pi$ seconds

$\omega_0 = \frac{2\pi}{T} = 1$ rad/s

$$i_s(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)$$

Neither odd nor even, so we need to find a_0, a_n + b_n

$$a_0 = \frac{1}{T}\int_0^T i_s(t)\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} i_s(t)\,dt = \frac{1}{2\pi}\left[\int_0^{\pi} 10\,dt + \int_{\pi}^{2\pi} 0\,dt\right]$$

$$= \frac{1}{2\pi}\times 10\pi = 5$$

$$a_n = \frac{2}{T}\int_0^T i_s(t)\cos(nt)\,dt$$

$$= \frac{2}{2\pi}\int_0^{\pi} 10\times\cos(nt)\,dt = \frac{10}{\pi}\left[\frac{\sin(nt)}{n}\right]_0^{\pi}$$

$$= \frac{10}{n\pi}\left[\sin(n\pi) - 0\right] = \frac{10\sin(n\pi)}{n\pi} = 0 \;\forall n$$

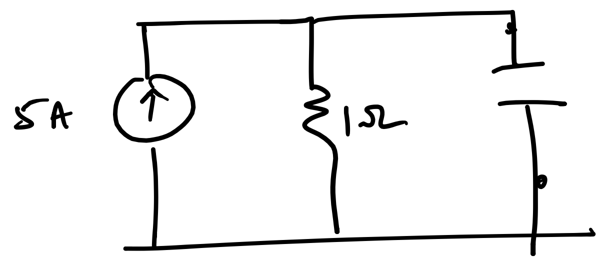
$$b_n = \frac{2}{T}\int_0^T i_s(t)\sin(nt)\,dt$$

$$= \begin{cases} 0, & n \text{ eren} \\ \frac{20}{n\pi}, & n \text{ odd} \end{cases}$$

$$i_s(t) = 5 + \sum_{n\text{ odd}} \frac{20}{n\pi}\sin(nt)$$

Redraw the ckt in frequency domain

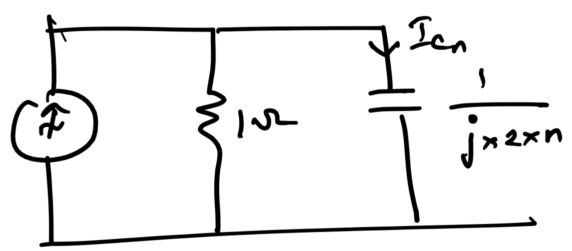
DC



$V = 5V$

n^{th} harmonic

$I_s = -j\frac{20}{n\pi}$ A



$$I_{cn} = \frac{I_s \times 1}{1 + \frac{1}{j2n}} = \frac{-j\frac{20}{n\pi}}{1 + \frac{1}{j2n}}$$

$$V_n = I_{cn} \times \frac{1}{j2n} = \left[\frac{-j\frac{20}{n\pi}}{1 + \frac{1}{j2n}}\right] \times \frac{\frac{1}{j2n}}{\frac{1}{j2n}}$$

$$= \frac{-\frac{10}{n^2\pi}}{\frac{1}{j2n} + \frac{-1}{4n^2}} = \frac{\frac{-10}{n^2\pi} \times j2n}{1 - \frac{1}{4n^2} \times j2n} = \frac{-j\frac{20}{n\pi}}{1 - \frac{j}{2n}}$$

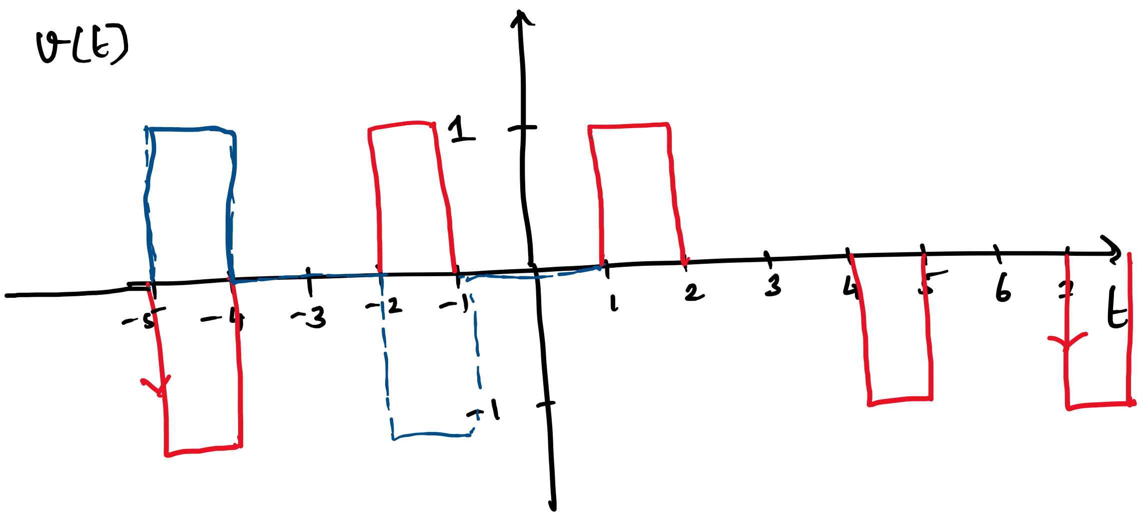
$$V_n = \frac{20}{n\pi\sqrt{1 + \frac{1}{4n^2}}} \angle -90^\circ - \tan^{-1}\left(\frac{-1}{2n}\right)$$

$$V_n = \frac{20}{\cancel{n}\pi} \times \frac{2\cancel{n}}{\sqrt{4n^2+1}} \;\angle -90^\circ + \tan^{-1}\left(\frac{1}{2n}\right)$$

$$v_n(t) = \frac{40/\pi}{\sqrt{4n^2+1}} \sin\left(nt + \tan^{-1}\left(\frac{1}{2n}\right)\right)$$

$$v(t) = 5 + \sum_{n=1,3,5\ldots} \frac{40/\pi}{\sqrt{1+4n^2}} \sin\left(nt + \tan^{-1}\left(\frac{1}{2n}\right)\right) \quad \checkmark$$

$\rightarrow$ $v(t)$



Soln:

$T = 12$ s

$$w_0 = \frac{2\pi}{T} = \frac{2\pi}{12} = \frac{\pi}{6} \text{ rad/s}$$

Even function

Half wave symmetric

$$f(t) = -f\left(t - \frac{T}{2}\right)$$

Table 18.1

$b_n = 0 \quad \forall \; n$

$$a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos(n w_0 t), \quad n \text{ odd}$$

$$0 \qquad n \text{ even}$$

$$a_n = \frac{8}{12} \int_0^3 v(t) \cos\left(\frac{n\pi t}{6}\right) dt$$

$$= \frac{2}{3} \int_1^2 \cos\left(\frac{n\pi t}{6}\right) dt = \frac{2}{3} \left[\frac{\sin\left(\frac{n\pi t}{6}\right)}{\frac{n\pi}{6}} \right]_1^2$$

$$= \frac{4}{n\pi} \left[\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{n\pi}{6}\right) \right] , \quad n \text{ odd}$$

$$v(t) = \sum_{n=1,3,5} \frac{4}{n\pi} \left[\sin\left(\frac{n\pi}{3}\right) - \sin\left(\frac{n\pi}{6}\right) \right] \cos\left(\frac{n\pi t}{6}\right)$$