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Q1 (a) Let us say that we have a current-carrying loop. If the current 'I' is steady, there will be no change in the magnetic flux, 'ϕ' through the loop. If this current 'I' changes with time, it will create a change in the magnetic field B, and in turn produce a change in flux through the loop. According to Faraday's Law, an induced emf will arise to oppose the change in current, this emf is called "back-emf".

The property of the loop through which its own magnetic field opposes any change in current is called "self-inductance".

(b) Energy stored in a inductor

The formula for this energy is : $E = \frac{1}{2} Li^2$

where, L = inductance of the inductor (Henry)

i = current through the inductor (Ampere)

Energy stored in a magnetic Field.

Now, Flux ϕ through the inductor,

$$\phi = LI$$

Also, $\phi = \int \vec{B} \cdot d\vec{s}$

we have, $\vec{B} = \vec{\nabla} \times \vec{A}$ $\vec{A}$ = Magnetic vector potential.

$$\therefore \phi = \int (\vec{\nabla} \times \vec{A}) \cdot d\vec{s} = \oint \vec{A} \cdot d\vec{l}$$ [Stokes' Theorem]

$$\therefore LI = \oint \vec{A} \cdot d\vec{l}$$

Therefore,

$$E = \frac{1}{2} LI^2 = \frac{1}{2} I(LI) = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l}$$

$$= \frac{1}{2} \oint (\vec{A} \cdot \vec{I}) \, dl$$

$I \, d\vec{l} = \vec{I} \, dl$
(since the direction of current I is the same as $\hat{dl}$)

$\therefore E = \frac{1}{2} \oint (\vec{A} \cdot \vec{I})\, dl$

Introducing volume current $\vec{J}$, we have

$$E = \frac{1}{2} \int_V (\vec{A} \cdot \vec{J})\, dv$$

We have, $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$ [Ampere's Law]

$$\therefore E = \frac{1}{2} \int_V \frac{\vec{A} \cdot (\vec{\nabla} \times \vec{B})}{\mu_0} dv$$

$$= \frac{1}{2\mu_0} \int_V \vec{A} \cdot (\vec{\nabla} \times \vec{B})\, dv$$

We have, $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = -\vec{A} \cdot (\vec{\nabla} \times \vec{B}) + \vec{B} \cdot (\vec{\nabla} \times \vec{A})$

$$\therefore \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

$$\therefore E = \frac{1}{2\mu_0} \int_V \left\{ \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \right\} dv$$

$$\Rightarrow E = \frac{1}{2\mu_0} \int_V \left\{ \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \right\} dv$$

$$\Rightarrow E = \frac{1}{2\mu_0} \int_V B^2 dv - \frac{1}{2\mu_0} \int_V \vec{\nabla} \cdot (\vec{A} \times \vec{B})\, dv$$

$$\Rightarrow E = \frac{1}{2\mu_0} \int_V B^2 dv - \frac{1}{2\mu_0} \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{s}$$ [Divergence Theorem]

Let us now consider increasing the volume V ~~to all space~~ and hence the surface S also increases as well as the distance from the source of $\vec{B}$ and hence $\vec{A}$ as increases. Thus, the effect of $\vec{B}$ and $\vec{A}$ both decreases. Thus $\vec{A} \times \vec{B}$ decreases at a greater extent.

Hence, Energy stored in a magnetic field will be now

$$E = \frac{1}{2\mu_0} \int_V B^2 dV$$

Q2 Frequency dependent relative permittivity :

$$\epsilon_r(\omega) = 1 + \frac{Nq^2}{m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 + i\omega\gamma_j}$$

where, N = Number of molecules per unit volume

m = mass of one electron

q = charge of an electron

ω_j = natural vibration frequency of f_j number of electrons out of Z

γ_j = damping constant of f_j number of electrons out of Z

$\therefore \sum_j f_j = Z$

(a) Assuming all electrons have same natural frequency of oscillation and damping constant as well.

Let them be ω_Z and γ_Z respectively.

$$\sum_j f_j = Z$$

$$\therefore \epsilon_r(\omega) = 1 + \frac{Nq^2 Z}{m\epsilon_0(\omega_Z^2 - \omega^2 + i\omega\gamma_Z)}$$

(b) Under the assumption above,

considering just one molecule per unit volume, we have

N = 1 and hence,

$$\epsilon_r(\omega) = 1 + \frac{q^2 Z}{m\epsilon_0(\omega_Z^2 - \omega^2 + i\omega\gamma_Z)}$$

where, q = charge of an $\bar{e}$ = 1.6×10^{-19} C

m = mass of an $\bar{e}$ = 9.1×10^{-31} kg

$\epsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}

$$\therefore \frac{q^2}{m\epsilon_0} \simeq 3.2 \times 10^3 \text{ units (SI)} = K \text{ (let)}$$

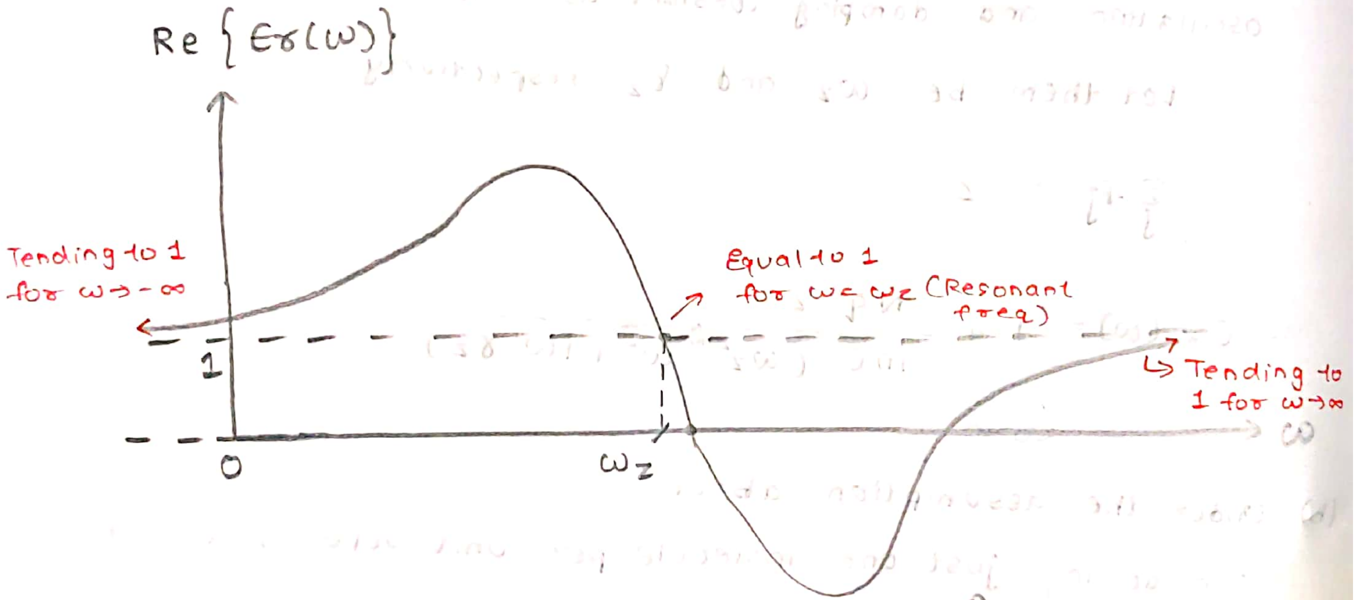
$$\therefore \quad \epsilon_r(\omega) = 1 + \frac{k_z}{\omega_z^2 - \omega^2 + i\omega\gamma_z}$$

$$\Rightarrow \epsilon_r(\omega) = 1 + \frac{k_1}{\omega_z^2 - \omega^2 + i\omega\gamma_z}$$

$$\Rightarrow \epsilon_r(\omega) = 1 + \frac{k_1(\omega_z^2 - \omega^2 - i\omega\gamma_z)}{(\omega_z^2 - \omega^2)^2 + (\omega\gamma_z)^2}$$

$$\text{Real}\{\epsilon_r(\omega)\} = 1 + \frac{k_1(\omega_z^2 - \omega^2)}{(\omega_z^2 - \omega^2)^2 + (\omega\gamma_z)^2}$$

$$\text{Imag}\{\epsilon_r(\omega)\} = \frac{-k_1\omega\gamma_z}{(\omega\gamma_z)^2 + (\omega_z^2 - \omega^2)^2}$$



$$\text{Re}\{\epsilon_r(\omega)\}\Big|_{\omega=\omega_z} = 1 + \frac{k_1\cancel{(\omega_z^2 - \omega_z^2)}^{\to 0}}{\cancel{(\omega_z^2-\omega_z^2)^2 + (\omega\gamma_z)^2}} = 1$$

$$\text{Re}\{\epsilon_r(\omega)\}\Big|_{\omega\to\infty} \cong 1 + \frac{k_1(\omega_z^2 - \omega^2)}{(\omega_z^2-\omega^2)^2 + (\omega\gamma_z)^2}$$

$$\cong 1 + \frac{k_1(-\omega^2)}{-\omega^4 + \omega^2\gamma_z^2} \cong 1 + \frac{k_1(-\omega^2)}{-\omega^4}$$

$$\cong 1$$

Similar for $\omega \to -\infty$

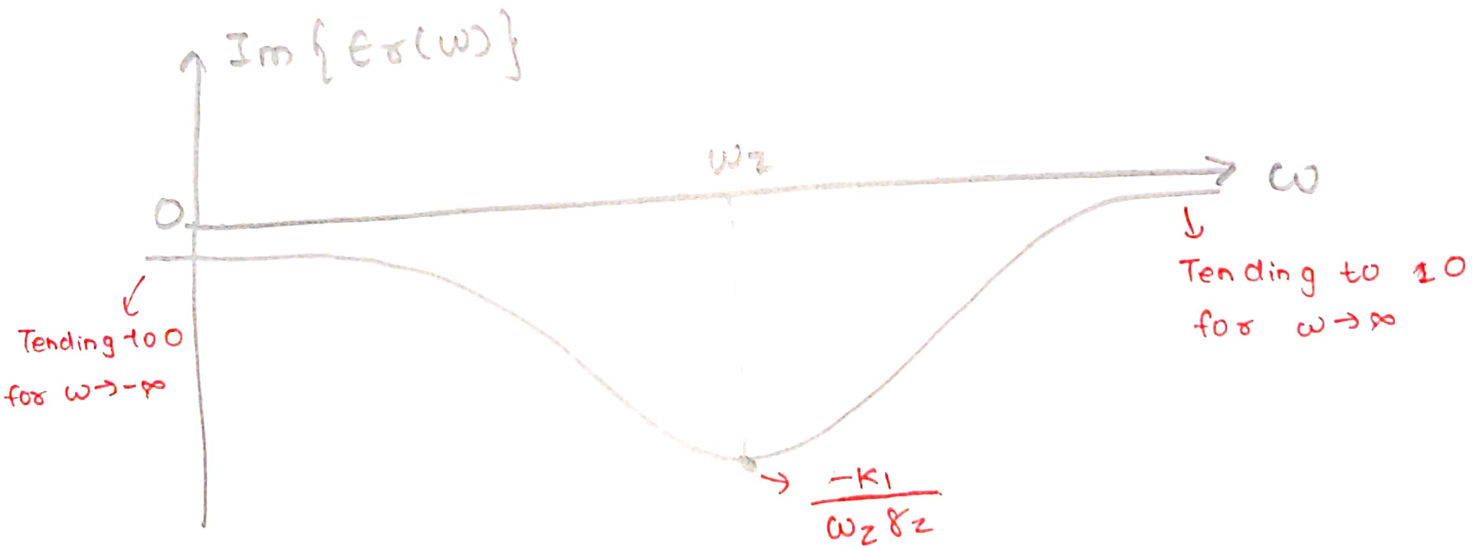
$$\mathrm{Im}\{\epsilon_r(\omega)\} = \frac{-k_1 \omega \gamma_z}{(\omega \gamma_z)^2 + (\omega_z^2 - \omega^2)^2}$$

Max. value for this will result if min. value of denominator is obtained. (i.e $\omega = \omega_z$)

$$\therefore \mathrm{Im}\{\epsilon_r(\omega)\}\Big|_{\omega=\omega_z} = \frac{-k_1 \omega_z \gamma_z}{\omega_z^2 \gamma_z^2 + 0} = \frac{-k_1}{\omega_z \gamma_z}$$

$$\mathrm{Im}\{\epsilon_r(\omega)\}\Big|_{\omega\to\infty} \cong \frac{-k_1 \omega \gamma_z}{\omega^2 \gamma_z^2 - \omega^4} \cong \frac{-k_1 \omega \gamma_z}{-\omega^4} \cong 0$$

Similar for $\omega \to -\infty$, $\mathrm{Im}\{\epsilon_r(\omega)\} = 0$



(c) Still carrying the assumption from part (a) and (b),

we have,

$$\epsilon_r(\omega) = 1 + \frac{k_1}{\omega_z^2 - \omega^2 + i\omega\gamma_z}$$

where $k_1 = \frac{Nq^2 z}{m\epsilon_0}$

Assumptions:

We are now considering metals, which have a lot of free electrons, hence the effect of damping force (or the binding force) for our oscillator model will be negligible. $\therefore \gamma_z \to 0$

Also, since there are a lot of free electrons, the resonant frequency ω_z will be less than usual.

Frequency of applied Electric field (ω) is such that $(\omega_z)^2$ is neglible infront of $(\omega)^2$.

$$\therefore \epsilon_r(\omega) = 1 + \frac{k_1}{-\omega^2}$$

$$\Rightarrow \epsilon_r(\omega) = 1 - \frac{k_1}{\omega^2}$$

Comparing with $\epsilon_r(\omega) = 1 - \frac{\omega_p^2}{\omega^2}$

we have, $\omega_p^2 = k_1$

$$\Rightarrow \omega_p = \sqrt{k_1}$$

$$= \sqrt{\frac{Nq^2 z}{m\epsilon_0}}$$

(d) Physical interpretation of ω_p (which is a constant for a particular metal) is that, it is called the plasma frequency.

For $\omega = \omega_p$ = plasma frequency

we have,

$$\epsilon_r = 1 - 1 = 0$$

∴ ~~The wave energy gets will get absorbed with little reflection.~~

This can be considered as the case where EM wave is traversed through a metal (PEC) and the wave is reflected back entirely and hence is called lossless material.

Plasma frequency is the frequency at which a charge displacement in an ideal plasma will naturally oscillate ~~lef~~ if left to itself.