Optional Assessment-2
Solutions

SOL(1): Given that—

$$x[n] = \begin{cases} \left(\frac{1}{3}\right)^n \cdot \cos\left(\frac{\pi}{4}n\right) & n \leqslant 0 \\ 0 & n > 0 \end{cases}$$

$$\therefore X[z] = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n \cdot \cos\left(\frac{\pi}{4}n\right) \cdot z^{-n}$$

$$= \left(\frac{1}{2}\right) \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n e^{j\frac{\pi}{4}n} \cdot z^{-n} + \left(\frac{1}{2}\right) \sum_{n=-\infty}^{0} \left(\frac{1}{3}\right)^n \cdot e^{-j\frac{\pi}{4}n} \cdot z^{-n}$$

$$= \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} \cdot e^{-j\frac{\pi}{4}n} \cdot z^{n} + \left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{-n} \cdot e^{j\frac{\pi}{4}n} \cdot z^{n}$$

$$= \left(\frac{1}{2}\right) \frac{1}{1 - 3 \cdot e^{-j\pi/4} \cdot z} + \left(\frac{1}{2}\right) \frac{1}{1 - 3 \cdot e^{j\pi/4} \cdot z}$$

→ (3 POINTS)

$\therefore$ ROC: $|z| < \frac{1}{3}$

→ (0.5 POINTS)

$\therefore$ The poles are at $z = \frac{1}{3} e^{-j\pi/4}$ & $z = \frac{1}{3} e^{j\pi/4}$

→ (0.5 POINTS)

SOL(2): Given that — $x[n] = 2^{n[n]}$

$$x[n] = 2^{n \cdot u[n]}$$

$$\therefore x[n] = \begin{cases} 1, & n < 0 \\ 2^n, & n \geq 0 \end{cases}$$

$$= u[-n-1] + 2^n u[n] \quad \text{——— (1)}$$

as we know that —

$$2^n \cdot u[n] \rightleftharpoons \frac{1}{1-2z^{-1}}, \quad ROC: |z| > 2$$

$$u[-n-1] \rightleftharpoons \frac{-1}{1-z^{-1}}, \quad ROC: |z| < 1$$

Now, by eq^n(1),

$$x[n] = \underbrace{u[-n-1]}_{ROC:\ |z|<1} + \underbrace{2^n u[n]}_{ROC:\ |z|>2} \rightarrow \text{(2 POINTS)}$$

There is no common ROC for $x[n]$. Hence $X(z)$ not exist here.

→ (2 POINTS)

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SOL(3): (a) Given causal LTI system 'S'—

$$\frac{d^3y(t)}{dt^3} + (1+\alpha)\frac{d^2y(t)}{dt^2} + \alpha(1+\alpha)\frac{dy(t)}{dt} + \alpha^2 y(t) = x(t)$$

Taking the Laplace transform of both sides of the given differential equation, we obtain—

$$s^3Y(s) + (1+\alpha)s^2Y(s) + \alpha(1+\alpha)sY(s) + \alpha^2Y(s) = X(s)$$

$$\left\{ \text{as we know—} \quad \frac{d^n f(t)}{dt^n} \rightleftharpoons s^n F(s) \quad \text{Bilateral Laplace Trans.} \right\}$$

$$[s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2]\,Y(s) = X(s)$$

Therefore, $$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2} \quad \text{—— (1)}$$

Given that — $$g(t) = \frac{d\,h(t)}{dt} + h(t)$$

Taking the Laplace transform of both sides of the given differential equation, we obtain—

$$G(s) = sH(s) + H(s)$$

$$G(s) = (s+1)\cdot H(s)$$

$$G(s) = \frac{(s+1)}{s^3 + (1+\alpha)s^2 + \alpha(1+\alpha)s + \alpha^2}$$

$$G(s) = \frac{(s+1)}{(s+1)(s^2+\alpha s+\alpha^2)}$$

$$G(s) = \frac{1}{(s^2+\alpha s+\alpha^2)}$$

$\therefore$ $G(s)$ has poles at $\alpha\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)$ & $\alpha\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$.

Therefore, $G(s)$ has 2 Poles. → (1.5 POINTS)

(b) By eq^n(1),

$$H(s) = \frac{1}{(s+1)(s^2+\alpha s+\alpha^2)}$$

Therefore, H(s) has poles at -1, $\alpha\left(-\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)$ and $\alpha\left(-\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)$.

The given system is causal LTI system. For stability of causal system, poles of Transfer function H(s) should lies in Left Hand Side of s-plane. (For stability, ROC includes imaginary axis in s-plane).
For this to be true —

$Re\{s\} < 0$ → (1 POINT)

$$\left(-\frac{\alpha}{2}\right) < 0$$

$\therefore \alpha > 0$ → (1.5 POINTS)

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SOL(4): Given that − $x[n] = \left(\frac{1}{2}\right)^{|n|} \cdot \cos\left[\frac{\pi}{8}(n-1)\right] \rightleftharpoons X(e^{j\omega})$

$$\therefore X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|n|} \cdot \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} \cdot \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n} + \cos\frac{\pi}{8} - \cos\frac{\pi}{8}$$

$$+ \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}$$

$$= \underbrace{\sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}}_{X_1(e^{j\omega})} + \underbrace{\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}}_{X_2(e^{j\omega})} - \cos\frac{\pi}{8} \quad \text{--- (1)}$$

(1 POINT)

$$\therefore X_1(e^{j\omega}) = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^{-n} \cdot \cos\left[\frac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot \cos\left[\frac{\pi}{8}(-n-1)\right] \cdot e^{j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot \frac{e^{-j\frac{\pi}{8}(n+1)} + e^{j\frac{\pi}{8}(n+1)}}{2} \cdot e^{j\omega n}$$

$$= \frac{1}{2}\left\{ \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot e^{-j\frac{\pi}{8}(n+1)} \cdot e^{j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} \cdot e^{j\frac{\pi}{8}(n+1)} \cdot e^{j\omega n} \right\}$$

$$= \frac{1}{2}\left\{ \frac{e^{-j\pi/8}}{1-\left(\frac{1}{2}\right)\cdot e^{-j\pi/8} \cdot e^{j\omega}} + \frac{e^{j\pi/8}}{1-\left(\frac{1}{2}\right)\cdot e^{j\pi/8} \cdot e^{j\omega}} \right\} \quad \text{--- (2)}$$

(1 POINT)

$$\therefore X_2(e^{j\omega}) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \cos\left[\tfrac{\pi}{8}(n-1)\right] \cdot e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \cdot \frac{e^{j\pi/8(n-1)} + e^{-j\pi/8(n-1)}}{2} \cdot e^{-j\omega n}$$

$$= \frac{1}{2}\left\{ \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n e^{j\frac{\pi}{8}(n-1)} \cdot e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n \cdot e^{-j\frac{\pi}{8}(n-1)} \cdot e^{-j\omega n} \right\}$$

$$= \frac{1}{2}\left\{ \frac{e^{-j\pi/8}}{1 - \left(\frac{1}{2}\right) \cdot e^{j\pi/8} \cdot e^{-j\omega}} + \frac{e^{j\pi/8}}{1 - \left(\frac{1}{2}\right) \cdot e^{-j\pi/8} \cdot e^{-j\omega}} \right\}$$

— (3)

(1 POINT)

By eq^n(1), eq^n(2) & eq^n(3), we get —

$$X(e^{j\omega}) = \frac{1}{2}\left\{ \frac{e^{-j\pi/8}}{1 - \left(\frac{1}{2}\right) \cdot e^{-j\pi/8} \cdot e^{j\omega}} + \frac{e^{j\pi/8}}{1 - \left(\frac{1}{2}\right) \cdot e^{j\pi/8} \cdot e^{j\omega}} \right\}$$

$$+ \frac{1}{2}\left\{ \frac{e^{-j\pi/8}}{1 - \left(\frac{1}{2}\right) e^{j\pi/8} \cdot e^{-j\omega}} + \frac{e^{j\pi/8}}{1 - \left(\frac{1}{2}\right) \cdot e^{-j\pi/8} \cdot e^{-j\omega}} \right\}$$

$$- \cos(\pi/8)$$

(1 POINT)