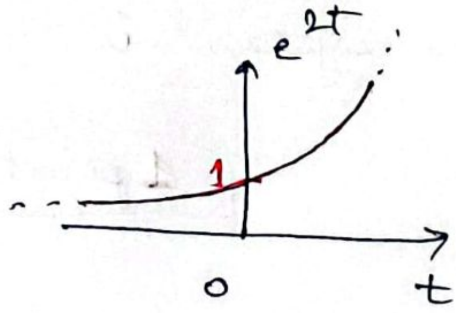
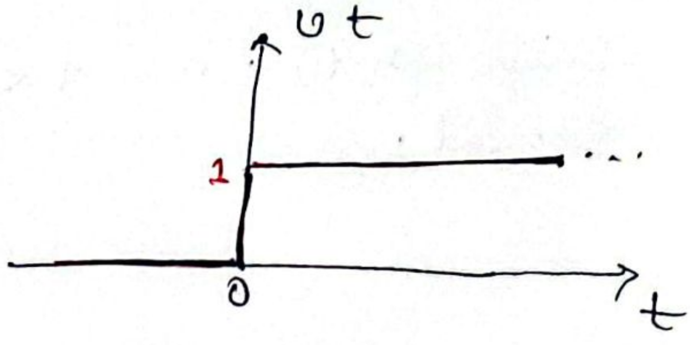
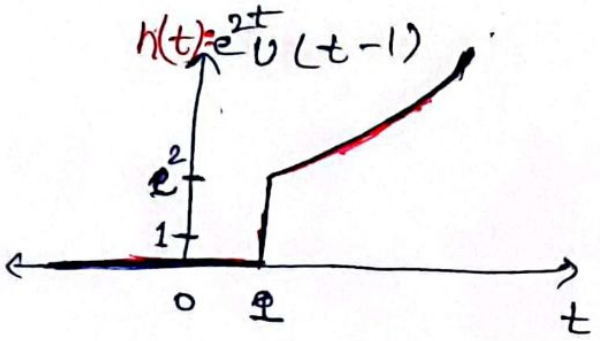
QUIZ 4 solution

Q1. Given that (4 MARKS)

$$h(t) = e^{2t} u(t-1)$$







→ $\boxed{\text{1 point}}$

(1) Stability Condition for LTI system

$$= \int_{-\infty}^{\infty} |h(t)|\, dt < \infty \quad \rightarrow \boxed{0.5\,pt}$$

$$S\{h(t)\} = \int_{1}^{\infty} |e^{2t}|\, dt$$

$$= \frac{e^{2t}}{2}\Big|_1^{\infty} = \frac{1}{2}\left[e^{2t}\right]_1^{\infty} = \frac{1}{2}\left[e^{\infty} - e^{2}\right] = \infty$$

→ $\boxed{0.5\,pt}$

Hence the system is unstable.

h(t) is not an energy signal → $\boxed{0.5\,pt}$

If the impulse response of LTI system is represented by an energy signal then the system will be stable.

ii) Condition for LTI system to be causal.

$$h(t) = 0 \quad \forall \quad t < 0$$ —(0.5 pt)

Since $h(t) = 0 \quad t < 0$ the system is causal.

1 point

Q 2. To prove that the rational number forms a field. We need to satisfy: (3 Marks)

(i) A set F

(ii) Two binary operation '+' and '.' such that

(a.) (F, +) is an Abelian group

(b.) $F^* = F - \{0\}$. $(F^*, \cdot)$ is abelian group

(c.) Multiplication operation distributive over addition.

- $x \cdot (y+z) = x \cdot y + x \cdot z \quad \forall\, x, y, z \in F$
- $(x+y) \cdot z = x \cdot z + y \cdot z \quad \forall\, x, y, z \in F.$

(1.) Closure under Addition and Multiplication 0.5 pt

. $a = p/q$, $b = r/s$. p, q, r and s ∈ Integer

$q \neq 0$ and $s \neq 0$

a, b in F

(A) $a + b = p/q + r/s = \frac{(ps + rq)}{qs}$ which is a rational number.

(B) $a \times b = \frac{p}{q} \times r/s = \frac{pr}{qs}$, which is a rational number.

Thus it is closure under addition and multiplication

2. Associativity and Commutativity 0.5 pt (4)

for all rational number a, b and c in F

For addition.

$$(a+b)+c = a+(b+c)$$

$a = p/q$, $b = r/s$ $c = u/v$ p, q, r, s, u, v are integers and $q \neq 0$, $s \neq 0$, and $v \neq 0$

Then, $(a+b)+c = \left(\frac{p}{q} + \frac{r}{s}\right) + \frac{u}{v}$

This is also a rational number.

$$= \left(\frac{ps+rq}{qs}\right) + \frac{u}{v} \Big\}$$ rational number.

For multiplication

$$(a \times b) \times c = a \times (b \times c)$$

$$\left(\frac{p}{q} \times \frac{r}{s}\right) \times \frac{u}{v}$$ } is a rational number.

Hence group of rational number doesn't affect the result.

Commutativity

for addition

$$a+b = b+a.$$

$a = p/q$, $b = r/s$ p, q, r, s are integer. $q \neq 0$, $s \neq 0$

$$a+b = \left(\frac{p}{q}\right)+\left(\frac{r}{s}\right) = \left(\frac{r}{s}\right)+\left(\frac{p}{q}\right) = b+a$$

Therefore, $a+b = b+a$ for all rational number.

Similarly, $a \times b = \left(\frac{p}{q}\right)\times\left(\frac{r}{s}\right) = \left(\frac{r}{s}\right)\times\left(\frac{p}{q}\right) = b \times a$

order of multiplication doesn't affect the result.

③ Existence of Additive identity (0) 0.5pt.

$$a+0 = (a \times 1)+(0 \times 1) = (a \times 1)+0 = a$$

Thus 0 is the additive identity for A.

i.e let $a = p/q$ p and q are integer $q \neq 0$

$$\left(\frac{p}{q}+0\right) = \left(\frac{p}{q}\times 1\right)+(0\times 1) = \left(\frac{p}{q}\times 1\right)+0 = \frac{p}{q}$$

④ Existence of Multiplicative Identity (1) 0.5pt

let a be arbitrary rational number,

$a \neq 0$ then $a \times 1 = a$

$\frac{p}{q} \neq 0$ then $\frac{p}{q} \times 1 = \frac{p}{q}$

Thus 1 is the multiplicative identity for R.

⑤ Existence of Additive Inverse: 0.5pt

$a+(-a) = 0$. $a = p/q$ $q \neq 0$.

$$\frac{p}{q}+\left(-\frac{p}{q}\right) = 0$$

Thus $-p/q$ is the additive inverse of p/q

(B.)

(5) Existence of Multiplicative Inverse. 0.5 pt

$a = p/q$ p and q are integer $q \neq 0$

$$p/q \times \left(1/p/q\right) = 1$$

Thus, $\left(1/p/q\right)$ is the multiplicative inverse of p/q.

We have proven that the set of rational number (R) forms a field since all the field properties are satisfied.

Q3 V → Set of periodic signals with the same time period T over the field $F = \mathbb{R}$ (3 MARKS)

Definition of periodic f^n

if f is periodic, then $\exists$, $T > 0$ There exist

s.t. $f(x+T) = f(x)$

Define addition of f^n & scalar multiplication

$$(f+g)(x) = f(x) + g(x)$$

$$\alpha f(x) = \alpha(f(x))$$

(1.) Closure under addition 0.5pt

let $f, g \in V$

$\Rightarrow f(x+T) = f(x)$

$g(x+T) = g(x)$

Now

$$(f+g)(x+T) = f(x+T) + g(x+T) \quad \text{By definition}$$

$$= f(x) + g(x)$$

$$(f+g)(x+T) = (f+g)(x)$$

$(f+g) \in V$ it holds Closure property.

(2.) Associative of addition 0.5 points

$(f+g)+h = f+(g+h)$

where $f, g, h \in V$

$$((f+g)+h)\cdot(x) = (f+g)(x) + h(x)$$
$$= (f(x) + g(x)) + h(x)$$
$$= f(x) + (g+h)(x)$$

$$\boxed{(f+g)+h = f+(g+h)}$$

(3.) Existence of additive Identity 0.25 pt

For every $f \in V$, $\exists\ 0 \in V$

s.t. $f+0 = 0+f = f$

(4.) Existence of additive inverse 0.25 pt

For Every $f \in V$, $\exists\ -f \in V$ s.t.

$f+(-f) = 0 = (-f)+f$ [since f is periodic, $-f$ is also periodic]

(5.) Commutative 0.5 pt

$(f+g)(x) = f(x) + g(x) = g(x) + f(x)$

$(f+g)(x) = (g+f)(x)$

$$f+g = g+f$$

V is an abelian group.

Now

⑥ Closure of scalar Multiplication 0.5pt

- for $f \in V$, $\alpha \in R = F$ (field)

$$(\alpha f)(x+T) = \alpha f(x+T)$$
$$= \alpha (f(x)$$
$$= (\alpha f)(x)$$

$$\alpha f \in V$$

→ It holds closure.

⑦ Compatibility of scalar multiplication with field multiplication. 0.5pt.

$$(\alpha \beta) f = \alpha (\beta f) \qquad \alpha, \beta \in F = R$$
$$f \in V$$

$$(\alpha \beta f)(x) = (\alpha \beta) f(x)$$
$$= \alpha (\beta f(x))$$
$$= \alpha (\beta f)(x).$$

$$(\alpha \beta) f = \alpha (\beta f)$$

- for $f, g \in V$, $\alpha \in F = R$

$$\alpha (f+g) = \alpha f + \alpha g$$

$$\alpha(f+g)(x) = \alpha\left[(f+g)(x)\right]$$
$$= \alpha\left[f(x) + g(x)\right]$$
$$= \alpha f(x) + \alpha g(x)$$
$$= (\alpha f)(x) + (\alpha g)(x)$$

$$(\alpha(f+g))(x) = (\alpha f + \alpha g)(x)$$

$$\Rightarrow \boxed{\alpha(f+g) = \alpha f + \alpha g}$$