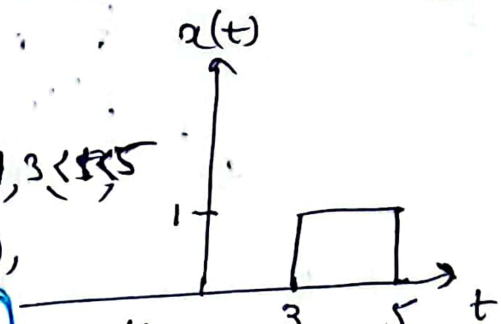
Practice sheet 3 solutions

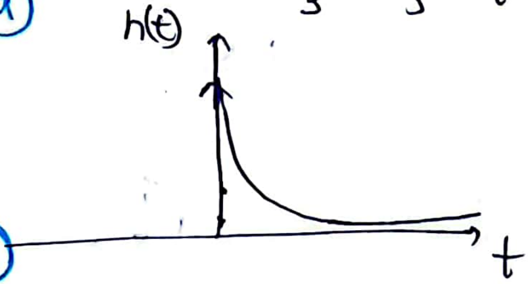
Q1- Given,

$$x(t) = u(t-3) - u(t-5) \begin{cases} 1, & 3<t<5 \\ 0, & \end{cases}$$



(a)

$$h(t) = e^{-3t}.u(t)$$



(b)

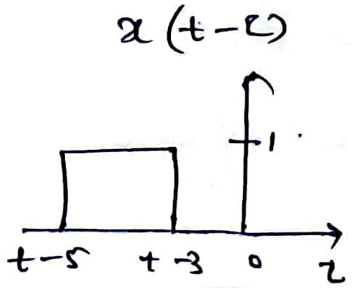
(a.) $y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau).d\tau.$$

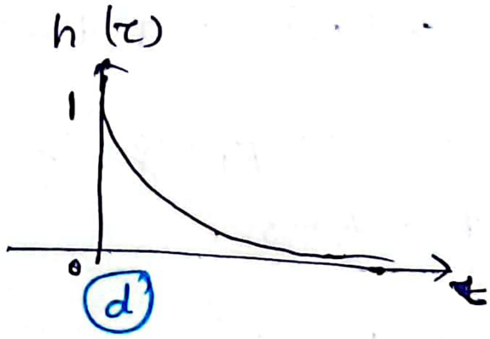
$$y(t) = \int_{-\infty}^{\infty} h(\tau).x(t-\tau).d\tau$$

The signal $x(t)$ & $h(t)$ is represented above.

Now.

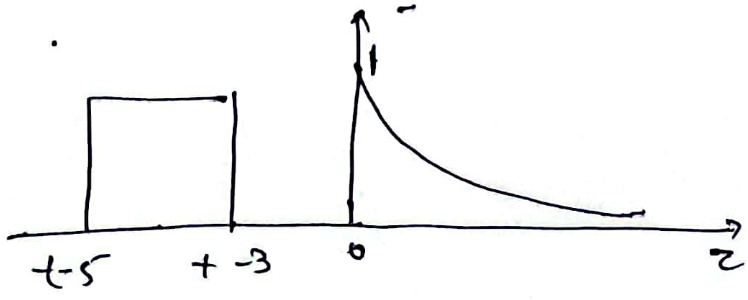


(c)



(d)

$$y(t) = x(t) * h(t) \rightarrow$$



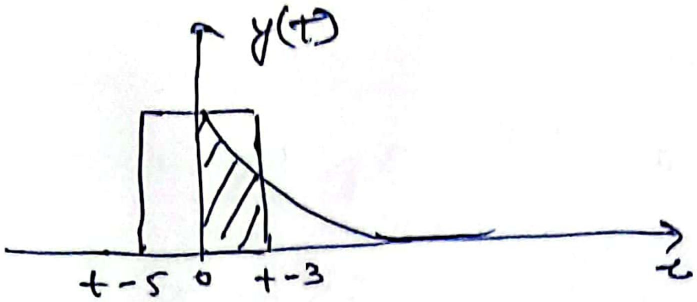
No overlapping of signal

$y(t) = 0$, if $t<3$

if $t-3<0 \Rightarrow t<3$

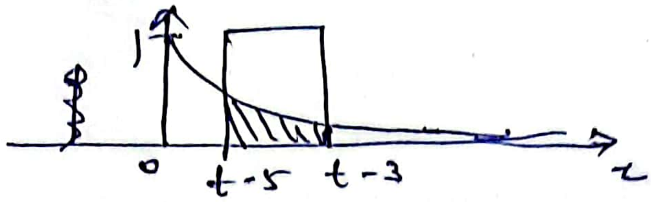
(e)

(f)



if $t-3 \geq 0$ and (2)
$t-5 < 0$
$3 \leq t < 5$

(g)



if $t-5 \geq 0$
$t \geq 5$

Now if $t-3 \geq 0 \Rightarrow t \geq 3$
$t-5 < 0 \Rightarrow t < 5$

there will be overlapping of signal. from
$\tau = 0$ to $\tau = t-3$. for $3 \leq t < 5$

$$y(t) = \int_0^{t-3} x(t-\tau)\cdot h(\tau)\cdot d\tau.$$

$$= \int_0^{t-3} (1)\cdot e^{-3\tau}\cdot d\tau = \frac{1-e^{-3(t-3)}}{3}$$

Now if $t-5 \geq 0 \Rightarrow t \geq 5$
$t-3 \leq \infty \Rightarrow t < \infty$
$\tau = t-5$ to $t = t-3$. for $5 \leq t \leq \infty$

$$y(t) = \int_{t-5}^{t-3} x(t-\tau)\cdot h(\tau)\cdot d\tau$$

$$y(t) = \int_{t-5}^{t-3} (1)\cdot e^{-3\tau} d\tau = \frac{(1-e^{-6})\, e^{-3(t-5)}}{3}$$

$$y(t) = \begin{cases} 0 & -\infty < t < 3 \\ \dfrac{1-e^{-3(t-3)}}{3} & 3 < t \leq 5 \\ \dfrac{(1-e^{-6})e^{-3(t-5)}}{3} & 5 < t \leq \infty \end{cases}$$

(b.) By differentiating $x(t)$ w.r.t time we get

$$\frac{dx(t)}{dt} = \delta(t-3) - \delta(t-5)$$

$$g(t) = \frac{dx(t)}{dt} * h(t)$$

$$= e^{-3(t-3)} \cdot u(t-3) - e^{-3(t-5)} u(t-5)$$

<u>Explanation</u>

$$g(t) = (\delta(t-3) * h(t) - \delta(t-5) * h(t))$$

$$g(t) = h(t-3) - h(t-5)$$ distributive property of convolution

$$= e^{-3(t-3)} \cdot u(t-3) - e^{-3(t-5)} \cdot u(t-5)$$

$$g(t) = \begin{cases} 0 & -\infty < t < 3 \\ e^{-3(t-3)} & 3 \leq t \leq 5 \\ e^{-3(t-3)} - e^{-3(t-5)}, & 5 \leq t < \infty \\ \text{or } (e^{-6} - 1)e^{-3(t-5)}, & \end{cases}$$

(c.) From the result of part (a) we can compute the derivative of $y(t)$.

$$\frac{d\,y(t)}{dt} = \begin{cases} 0 & -\infty < t < 3 \\ e^{-3(t-3)} & 3 < t \leq 5 \\ (e^{-6} - 1)\,e^{-3(t-5)} & 5 < t \leq \infty \end{cases}$$

We can say that

$$\boxed{g(t) = \frac{dy(t)}{dt}}$$

~~Q.1.~~

~~(a) Consider~~

Q

Q2. (a) $y(t) = t^2 x(t-1)$

(i)

$$x_1(t) \longrightarrow y_1(t) = t^2 x_1(t-1)$$
$$x_2(t) \longrightarrow y_2(t) = t^2 x_2(t-1)$$

$$x_3 = a x_1(t) + b x_2(t)$$

$$y_3(t) = t^2 x_3(t-1)$$
$$= t^2(a x_1(t-1) + b x_2(t-1))$$

$$y_3(t) = a y_1(t) + b y_2(t)$$

Therefore the system is linear.

(ii) $y_1(t) = t^2 x_1(t-1)$

Now

$$x_2(t) = x_1(t-t_0)$$

$$y_2(t) = t^2 x_2(t-1) = t^2 x_1(t-1-t_0)$$

$$y_1(t-t_0) = (t-t_0)^2 x_1(t-1-t_0) \neq y_2(t)$$

System is not time invariant.

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Q2. (b) $y(t) = x(t^2)$

(i) $y_1(t) = x_1(t^2)$ —— (1)

(ii) $y_2(t) = x_2(t^2)$. —— (2)

$y_1(t) + y_2(t) = x_1(t^2) + x_2(t^2)$

$y_3(t) = a\,x_1(t^2) + b\,x_2(t^2)$

$y_3(t) = a\,y_1(t) + b\,y_2(t)$

The system is linear

(ii) $y_1(t) = x_1(t^2)$

$x_2(t) = x_1(t^2 - t_0)$ → delayed signal

$y_2(t) = x_2(t)$

$= x_1(t^2 - t_0)$

$y_1(t - t_0) = x_1((t - t_0)^2)$ → delayed output signal

∴ $y_1(t) \neq y_2(t)$

Hence system is time variant.

(c) $y[n] = x^2[n-2]$

(i) $y_1[n] = x_1^2[n-2]$

$y_2[n] = x_2^2[n-2]$

~~$y_3[n] = a x$~~ $x_3[n] = a\,x_1[n] + b\,x_2[n]$

$y_3[n] = x_3^2[n-2]$

$= \left(a\,x_1^2[n-2] + b\,x_2^2[n-2]\right)^2$

$y_3[n] \neq a\,y_1[n] + b\,y_2[n]$

Hence system is nonlinear

(ii) $y_1[n] = x_1^2[n-2]$

$x_2[n] = \cancel{x_2}[n] = x_1^2[n-2-n_0]$

$y_2[n] = x_2^2[n-2] = x_1^2[n-2-n_0]$

$y_1[n-n_0] = x_1^2[n-2-n_0]$

$y_2[n] = y_1[n-n_0]$

This implies the system is time invariant.

(d) $y[n] = x[n+1] - x[n-1]$

(i) $y_1[n] = x_1[n+1] - x_1[n-1]$

$y_2[n] = x_2[n+1] - x_2[n-1]$

$x_3[n] = a x_1[n] + b x_2[n] \Rightarrow x_3[n] = a x_1[n] + b x_2[n]$

$y_3[n] = x_3[n+1] - x_3[n-1]$

$= a\{x_1[n+1] - x_1[n-1]\} + b\{x_2[n+1] - x_2[n-1]\}$

$$\boxed{y_3[n] = a\, y_1[n] + b\, y_2[n]}$$

Hence, system is linear.

(ii) $y_1[n] = x_1[n+1] - x_1[n-1]$

$x_2[n] = x_1[n-n_0]$

$y_2[n] = x_2[n+1] - x_2[n-1]$

$= \cancel{x_1[n+ n_0]}$

$= x_1[n+1-n_0] - x_1[n-n_0-1]$

$y[n-n_0] = x_1[n+1-n_0] - x_1[n-1-n_0]$

$y_2[n] = y[n-n_0]$

Hence the system is time invariant.

3.

(a) $$y_2[n] = x_2[n-2] + \frac{1}{2} x_2[n-3]$$

$$= y_1[n-2] + \frac{1}{2} y_1[n-3]$$

$$= 2x_1[n-2] + 4x_1[n-3] +$$

$$\frac{1}{2}\left(2x_1[n-3] + 4x_1[n-4]\right)$$

$$= 2x_1[n-2] + 4x_1[n-3] + x_1[n-3] + 4x_1[n-4]$$

$$= 2x_1[n-2] + 5x_1[n-3] + 4x_1[n-4]$$

The input output relationship of S —

$$y[n] = 2x[n-2] + 5x[n-3] + 2x[n-4]$$

(b) Now lets reverse the S_1 & S_2.

$$\longrightarrow \boxed{S_2} \longrightarrow \boxed{S_1} \longrightarrow$$

S_2: $$y_1[n] = 2x_1[n] + 4x_1[n-1]$$

$$= 2y_2[n] + 4y_2[n-1]$$

$$= 2\left(x_2[n-2] + \frac{1}{2} x_2[n-3]\right) +$$

$$4\left(x_2[n-3] + \frac{1}{2} x_2[n-4]\right)$$

$$= 2x_2[n-2] + x_2[n-3] + 4x_2[n-3] + 2x_2[n-4]$$

$$= 2x_2[n-2] + 5x_2[n-3] + 2x_2[n-4]$$

we observe there is no change. The output remains same.

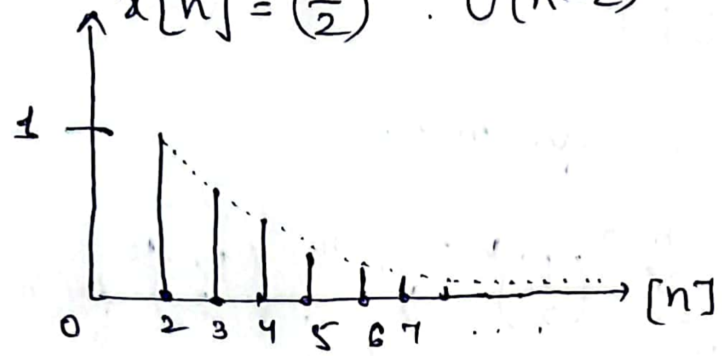
④ Given,

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2]$$

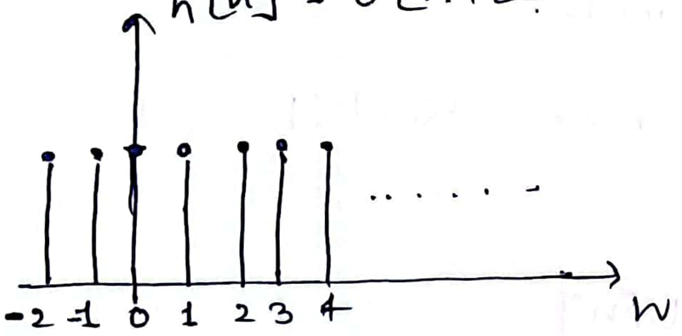
$$h[n] = u[n+2]$$

$$y[n] = x[n] * h[n] \quad ?$$

$x[n] = \left(\frac{1}{2}\right)^{n-2} \cdot u(n-2)$



$h[n] = u[n+2]$



We note that,

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[k-2]\, u[n+2-k]$$

Now by looking into $x[n]$ & $h[n]$ plot -

Conditions

$u[k-2] = 1$ when $k-2 \geq 0$ or $k \geq 2$

$= 0$ otherwise

$u[n+2-k] = 1$ $\quad n+2-k \geq 0$ or $k \leq n+2$

There are two conditions

(1.) $n+2 < 2 \rightarrow y[n] = 0$ for $n < 0$

(2.) $n+2 \geq 2 \rightarrow y[n] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} \Rightarrow \sum_{k=0}^{n} \left(\frac{1}{2}\right)^{k}$ for $n \geq 0$

On applying finite sum formula.

$$\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} \quad \text{for } n \geq 0 \text{ \& } 0 < |a| < 1$$

$$y[n] = \frac{1-(1/2)^{n+1}}{1-1/2} u[n]$$

$$y[n] = 2\left[1-(1/2)^{n+1}\right] \cdot u[n]$$

5.

$$x[n] \longrightarrow \boxed{S_1} \xrightarrow{w[n]} \boxed{S_2[n]} \longrightarrow y[n]$$

$$w[n] = \frac{1}{2} w[n-1] + x[n] \quad \text{— (1)}$$

$$y[n] = \alpha y[n-1] + \beta w[n] \quad \text{— (2)}$$

$$w[n] = \frac{1}{\beta}\left\{y[n] - \alpha y[n-1]\right\} \text{ from eq}^n \text{ (2)}$$

— (a)

$$\frac{1}{2} w[n-1] = \frac{1}{2\beta}\left\{y[n-1] - \frac{\alpha}{2} y[n-2]\right\} \quad \text{— (b)}$$

from eq^n (1)

~~$w[n-1]$~~

$$w[n] - \frac{1}{2}w[n-1] = x[n]$$

from eq^n (a) & (b),

$$w[n] - \frac{1}{2}w[n-1] = \frac{1}{\beta} y[n] - \frac{\alpha}{\cancel{\beta}} y[n-1] - \frac{1}{2\beta} y[n-1] + \frac{\alpha}{2\beta} y[n-2]$$

$$w[n] - \frac{1}{2}w[n-1] = \frac{1}{\beta} y[n] - \left(\frac{\alpha}{\beta} + \frac{1}{2\beta}\right) y[n-1] + \frac{\alpha}{2\beta} y[n-2]$$

$$x[n] = \frac{1}{\beta} y[n] - \left(\frac{\alpha}{\beta} + \frac{1}{2\beta}\right) y[n-1] + \frac{\alpha}{2\beta} y[n-2] \quad \text{— (3)}$$

On Comparing $y[n] = -\frac{1}{8} y[n-2] + \frac{3}{4} y[n-1] + x[n]$

$$\Rightarrow x[n] = + y[n] + \frac{1}{8} y[n-2] - \frac{3}{4} y[n-1] \quad \text{— (4)}$$

On comparing eqn 3 & 4, we get

$$\frac{1}{\beta} = 1 \Rightarrow \boxed{\beta = 1}$$

$$\frac{\alpha}{2\beta} = \frac{1}{8} \Rightarrow \frac{\alpha}{2} = \frac{1}{8}$$

$$\boxed{\alpha = 1/4}$$

6

$$x(t) = \delta(t) + \delta(t-1) + \delta(t-2)$$

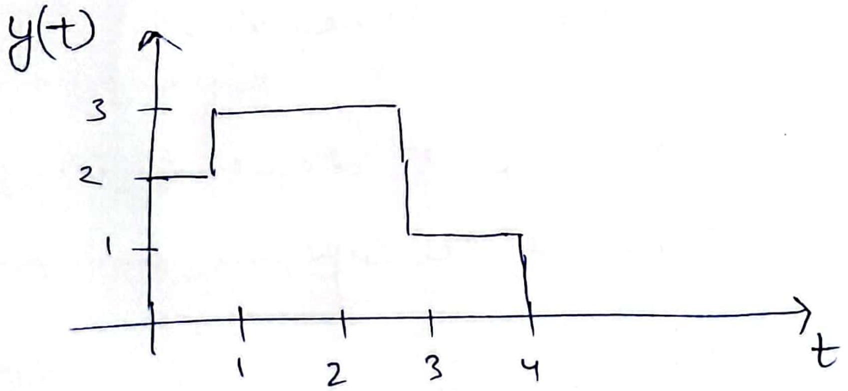
$$h(t) = 2u(t) - u(t-1) - u(t-2)$$

$$y(t) = x(t) * h(t)$$

$$= \{2u(t) - u(t-1) - u(t-2)\} * \{\delta(t) + \delta(t-1) + \delta(t-2)\}$$

$$= 2u(t) - u(t-1) - u(t-2) + 2u(t-1) - u(t-2) - u(t-3) + 2u(t-2) - u(t-3) - u(t-4)$$

$$y(t) = 2u(t) + u(t-1) - 2u(t-3) - u(t-4)$$



$$\boxed{y(2) = 3}$$

7. $y[n] = x^2[n] + \frac{1}{x^2[n-1]}$ d

a) let input $= x_1[n]$

$$y_1[n] = x_1^2[n] + \frac{1}{x_1^2[n-1]}$$

input $= x_2[n]$

$$y_2[n] = x_2^2[n] + \frac{1}{x_2^2[n-1]}$$

$$y[n] = [x_1(n) + x_2(n)]^2 + \frac{1}{(x_1^2[n-1] + x_2[n-1])^2}$$

$$y[n] \neq y_1[n] + y_2[n]$$

Hence system is not linear.

b) The present output depends on past & present values hence system is causal.

$$y[1] = x^2[1] + \frac{1}{x^2[0]}$$

$$y[0] = x[0] + \frac{1}{x^2[-1]}$$

c.) For delayed input $x(n-t)$

$$y_1(n) = x_1^2(n-t) + \frac{1}{x_1^2(n-t-1)}$$

for delayed output $y(n-t)$

$$y(n-t) = x_1^2(n-t) + \frac{1}{x_1^2(n-t-1)}$$

$y(n-t) = y_1[n]$

Hence system is time invariant system

8. Given,

(a) ~~$h[n] = a^n u[n]$~~ $h[n] = a^n u[n+2]$

Stability is determined by checking whether the impulse response is absolutely summable —

$$\sum_{k=-\infty}^{\infty} |h[n]| = \sum_{k=-2}^{\infty} |a^n| \qquad \text{——— (1)}$$

$$= a^{-2} + a^{-1} + \sum_{k=0}^{\infty} |a|^n$$

The infinite geometric sum in the equation (1) converges only if $|a| < 1$. Hence the system is stable and provide $0 < |a| \leq 1$. The system is not causal, since the impulse response $h[n]$ is nonzero for $n = -1, -2$, The system is not memoryless because $h[n]$ is nonzero for $n = -1, -2$.

The system is not memoryless because $h[n]$ is nonzero for some values $n \neq 0$.

(b) $h[n] = n\cos\left(\frac{\pi}{4}n\right) \cdot U[n]$

we know that

$$\sum_{k=-\infty}^{\infty} |h_1[k]| = \sum_{k=-\infty}^{\infty} \left|k\cos\left(\frac{\pi}{4}k\right) U[k]\right|$$

$$= \sum_{k=0}^{\infty} \left|k\cos\left(\frac{\pi}{4}k\right)\right|$$

This sum doesnot have a finite value. because function $\left|k\cos\left(\frac{\pi}{4}k\right)\right|$ increases as the value of k increases. Therefore, $h_1[n]$ cannot be impulse response of a stable LTI system.

9. Given,

$$h[n] = \left(\frac{1}{5}\right)^n U[n].$$

(a) $h[n] - A\,h[n-1] = \delta[n]$

$$\left(\frac{1}{5}\right)^n U[n] - A\left(\frac{1}{5}\right)^{(n-1)} U[n-1] = \delta[n]$$

Putting $n = 1$ and solving for A gives

$$\boxed{A = 1/5}$$

(b) from part (a)

$$h[n] - \frac{1}{5} h[n-1] = \delta[n]$$

$$h[n] * \delta[n] - \frac{1}{5} \delta[n-1] = \delta[n]$$

From the definition of inverse system, we can say —

$$g[n] = \delta[n] - \frac{1}{5} \delta[n-1].$$