GR-Tensor Project

Haitham Ashraf

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1 Schwarzschild Metric and Solutions

The Schwarzschild metric is expressed through the line element equation as follows:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \tag{1}$$

This equation maintains spherical symmetry and encapsulates the spacetime configuration. The objective is to find the expressions for the functions $A(r)$ and $B(r)$.

In our Maple analysis, the solutions for $A(r)$ and $B(r)$ were derived as follows:

$$A(r) = c_1 + \frac{c_2}{r} \tag{2}$$

$$B(r) = \frac{c_1}{c_1 + \frac{c_2}{r}} \tag{3}$$

The solutions were obtained while adhering to specific boundary conditions, notably the Flatness condition:

$$\lim_{r\to\infty} A(r) = \lim_{r\to\infty} B(r) = 1 \tag{4}$$

Upon further refinement, the solutions were simplified to:

$$A(r) = 1 + \frac{c_2}{r} \tag{5}$$

$$B(r) = \frac{1}{1 + \frac{c_2}{r}} \tag{6}$$

Now to solve for C_2, We will study Schwarzschild equation at the Newtonian limit, at which the following conditions occour:

- Particles move with non-relativistic speeds (v$\ll$c)
- The gravitional field is static
- the gravitional field influence is weak

At this limit the metric tensor could be approximated to The Minkowski metric with addition to a perturbed term. In this case, the g_{00} takes the following form

$$g_{00} = \eta_{00} + h_{00} = -(1 + 2\phi) \tag{7}$$

Based on our assumption that the field is static, h_{00} took the following form $h_{00} = 2\phi(x^i)$.

By direct comparison of g_{00} component of the metric tensor in the Newtonian limit to following component in Schwarzschild metric. and since ϕ is the gravitional potential in Newton law. we find that: $g_{00} = -(1 - \frac{2GM}{r})$. Consquently, $c_2 = -2GM$ therefore, the Schwarzchild metric expressed in its line element finally becomes,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{8}$$

2 Verification of the Kerr Metric

In this report, we undertake the task of verifying that the Kerr metric, which characterizes the geometry of an axially symmetric spacetime configuration, satisfies the general relativity field equations. The Kerr metric is an essential extension of the Schwarzschild metric to incorporate the effects of rotation and is represented by the equation:

$$ds^2 = \left(\frac{r^2 + a^2 cos^2(\theta)}{r^2 - 2mr + a^2}\right) dr^2 + (r^2 + a^2 cos^2(\theta)) d\theta^2 + sin^2(\theta) \left(r^2 + a^2 + \frac{2mra^2 sin^2(\theta)}{r^2 + a^2 cos^2(\theta)}\right) d\phi^2 - \left(\frac{4mar sin^2(\theta)}{r^2 + a^2 cos^2(\theta)}\right) d\phi dt + \left(-1 + \frac{2mr}{r^2 + a^2 cos^2(\theta)}\right) dt^2$$

where a denotes the specific angular momentum of the rotating black hole.

Our primary objective is to validate that the Kerr metric adheres to the general relativity field equations. To achieve this, we must demonstrate that the Einstein tensor $G_{\mu\nu}$, derived from the Kerr metric, satisfies the equation:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Here, $T_{\mu\nu}$ represents the energy-momentum tensor accounting for matter and energy distribution and it equals zero in this case because we are solving in vaccum. By employing tensor calculations and utilizing tools like Maple, we have successfully established that the Einstein tensor for the Kerr metric evaluates to zero. This outcome serves as evidence that the left-hand side of the field equations becomes null, thereby confirming the Kerr metric's alignment with the fundamental principles of general relativity.

3 Scale Factor $a(t)$ Solutions for Different w Values

In this section, we explore the solutions for the scale factor $a(t)$ within the context of the Friedmann–Lemaître–Robertson–Walker (FLRW) metric for various values of the equation of state parameter w. Einstein's field equations are utilized to unveil how different energy components influence the universe's expansion.

The FLRW metric is given by:

$$ds^2 = -c^2 dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta \, d\phi^2 \right)$$

Here, $k = 0, \pm 1$ denotes the spatial curvature, and $a(t)$ signifies the time-dependent scale factor.

For varying w values, we uncover distinct expansion behaviors:

3.1 $w = 0$ - Matter-Dominated Universe

In a non-relativistic matter-dominated universe ($w = 0$), the energy density ρ decreases over time. The resulting $a(t)$ solution follows a power-law relation, signifying a decelerated expansion.

3.2 $w = \frac{1}{3}$ - Radiation-Dominated Universe

For a universe dominated by radiation ($w = \frac{1}{3}$), radiation energy density ρ decreases more rapidly than matter density. The $a(t)$ solution exhibits a power-law increase, indicating rapid early expansion.

3.3 $w = -1$ - Cosmological Constant

In the presence of a cosmological constant ($w = -1$), energy density remains constant. The $a(t)$ solution reflects exponential expansion, characteristic of cosmic inflation.