

2022.06.06

设 $x_n = \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2}{n^2}\right) \cdots \left(1 + \frac{n}{n^2}\right)$, 则极限 $\lim_{n \rightarrow \infty} x_n = \underline{\hspace{2cm}}$.

解法一: 由于 $\frac{x}{1+x} < \ln(1+x) < x$, 故 $x - \ln(1+x) < \frac{x^2}{1+x}$. 因此

$$\left| \sum_{k=1}^n \ln \left(1 + \frac{k}{n^2}\right) - \sum_{k=1}^n \frac{k}{n^2} \right| < \sum_{k=1}^n \frac{\left(\frac{k}{n^2}\right)^2}{1 + \frac{k}{n^2}} < \sum_{k=1}^n \left(\frac{k}{n^2}\right)^2 = \frac{1}{n} \rightarrow 0 (n \rightarrow \infty)$$

故由夹逼准则得:

$$\lim_{n \rightarrow \infty} \ln \left(1 + \frac{k}{n^2}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2}.$$

故

$$\lim_{n \rightarrow \infty} x_n = \exp \left\{ \lim_{n \rightarrow \infty} \ln \left(1 + \frac{k}{n^2}\right) \right\} = \sqrt{e}.$$

法二: 因

$$x_n = \exp \left\{ \sum_{k=1}^n \ln \left(1 + \frac{k}{n^2}\right) \right\} = \exp \left\{ \sum_{k=1}^n \left[\frac{k}{n^2} + \mathcal{O} \left(\frac{k^2}{n^4} \right) \right] \right\} = \exp \left\{ \frac{1}{2} + \mathcal{O} \left(\frac{1}{n} \right) \right\},$$

故 $\lim_{n \rightarrow \infty} x_n = \sqrt{e}$.

2022.06.07

设 $x_0 = 1, x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}, n = 1, 2, \dots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim_{n \rightarrow \infty} x_n$.

解法一: $k=0$ 时, 有 $1 \leq x_0 < 2$ 成立. 假设 $k=n \in \mathbb{N}$ 时, 有 $1 \leq x_n < 2$, 则 $k=n+1$ 时, 有

$$1 \leq 1 + \frac{x_n}{1+x_n} = x_{n+1} = 2 - \frac{1}{1+x_n} < 2,$$

故由数学归纳法知 $1 \leq x_n < 2$. 又

$$x_{n+1} - x_n = \frac{1+2x_n}{1+x_n} - \frac{1+2x_{n-1}}{1+x_{n-1}} = \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})} > 0,$$

且 $x_1 - x_0 = \frac{1}{2} > 0$, 知数列 $\{x_n\}$ 单调增加. 由单调有界准则得: 数列 $\{x_n\}$ 收敛, 令 $A = \lim_{n \rightarrow \infty} x_n \geq 1$, 并对等式 $x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}$ 两边同时取极限, 得 $A = \frac{1+2A}{1+A}$ 即 $A^2 - A - 1 = 0$. 解

得: $A = \frac{\sqrt{5}+1}{2}$ 或 $\frac{1-\sqrt{5}}{2}$ (舍去).

法二: 由于 $x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}$ 满足 $x_n = f(x_{n-1})$ 这种形式, 我们尝试令 $f(x) = \frac{1+2x}{1+x}$, 由于 $f'(x) = \frac{1}{(1+x)^2} > 0$, 且 $f'(x) = \frac{1}{(1+x)^2} > 0$, 知数列 $\{x_n\}$ 单调增加. 其他过程与法一类似.

法三: 由于

$$\begin{aligned} \left| x_{n+1} - \frac{\sqrt{5}+1}{2} \right| &= \left| \frac{1+2x_n}{1+x_n} - \frac{\sqrt{5}+1}{2} \right| = \frac{|(3-\sqrt{5})x_n + 1 - \sqrt{5}|}{2(1+x_n)} \\ &= \frac{2}{(3+\sqrt{5})(1+x_n)} \left| x_n - \frac{\sqrt{5}+1}{2} \right| < \frac{2}{3+\sqrt{5}} \left| x_n - \frac{\sqrt{5}+1}{2} \right| < \left(\frac{2}{3+\sqrt{5}} \right)^2 \left| x_{n-1} - \frac{\sqrt{5}+1}{2} \right| \\ &< \dots < \left(\frac{2}{3+\sqrt{5}} \right)^{n+1} \left| x_0 - \frac{\sqrt{5}+1}{2} \right| \rightarrow 0 (n \rightarrow \infty), \text{ 故由夹逼准则得到: } \lim_{n \rightarrow \infty} x_n = \frac{\sqrt{5}+1}{2}. \end{aligned}$$

法四: 由斐波那契数列 $\{f_n\}$ 与数学归纳法可证得 $x_n = \frac{f_{2n+2}}{f_{2n+1}}$. 结合数列 $\{f_n\}$ 的通项公式

$$f_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right]$$

得到:

$$x_n = \frac{f_{2n+2}}{f_{2n+1}} = \frac{\left(\frac{1+\sqrt{5}}{2} \right)^{2n+2} - \left(\frac{1-\sqrt{5}}{2} \right)^{2n+2}}{\left(\frac{1+\sqrt{5}}{2} \right)^{2n+1} - \left(\frac{1-\sqrt{5}}{2} \right)^{2n+1}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{\sqrt{5}-1}{2} \left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1}}{1 + \left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1}}.$$

显然 $\lim_{n \rightarrow \infty} \left(\frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1} = 0$, 等式两边同时取极限得到: $\lim_{n \rightarrow \infty} x_n = \frac{\sqrt{5}+1}{2}$.

2022.06.08

设 $2x_1 = 1, 2x_n = 1 - x_n^2, n = 1, 2, \dots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim_{n \rightarrow \infty} x_n$.

解 因 $0 < x_1 = \frac{1}{2} < 1$, 假设 $k = n \in \mathbb{N}$ 时, $0 < x_n < 1$ 成立. 则 $0 < x_{n+1} = \frac{1 - x_n^2}{2} < 1$. 由数学归纳法知 $0 < x_n < 1, n = 1, 2, \dots$. 故

$$\begin{aligned} \left| x_{n+1} - (\sqrt{2} - 1) \right| &= \left| \frac{1 - x_n^2}{2} - \sqrt{2} + 1 \right| = \frac{x_n^2 + 2\sqrt{2} - 3}{2} \\ &= \frac{x_n + \sqrt{2} - 1}{2} \left| x_n - (\sqrt{2} - 1) \right| \\ &< \frac{1}{\sqrt{2}} \left| x_n - (\sqrt{2} - 1) \right| \\ &< \left(\frac{1}{\sqrt{2}} \right)^2 \left| x_{n-1} - (\sqrt{2} - 1) \right| < \dots \\ &< \left(\frac{1}{\sqrt{2}} \right)^n \left| x_1 - (\sqrt{2} - 1) \right| \rightarrow 0 (n \rightarrow \infty). \end{aligned}$$

故由夹逼准则得到: $\lim_{n \rightarrow \infty} x_n = \sqrt{2} - 1$.

2022.06.09

设 $f(x) = 1 - \cos x$, 则 $\lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})(1 - \sqrt[3]{\cos x})(1 - \sqrt[4]{\cos x})(1 - \sqrt[5]{\cos x})}{f\{f[f(x)]\}} = \underline{\hspace{2cm}}$.

解 $\frac{1}{15}$. 因

$$\begin{cases} 1 - (\cos x)^a = 1 - [1 - (1 - \cos x)]^a \sim a(1 - \cos x) \sim \frac{ax^2}{2}, \\ f\{f[f(x)]\} \sim \frac{f^2[f(x)]}{2} \sim \frac{\left[\frac{f^2(x)}{2}\right]^2}{2} \sim \frac{1}{8} \left(\frac{x^2}{2}\right)^4 = \frac{x^8}{128}. \end{cases}$$

故

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})(1 - \sqrt[3]{\cos x})(1 - \sqrt[4]{\cos x})(1 - \sqrt[5]{\cos x})}{f\{f[f(x)]\}} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{x^2}{4}\right)\left(\frac{x^2}{6}\right)\left(\frac{x^2}{8}\right)\left(\frac{x^2}{10}\right)}{\frac{x^8}{128}} = \frac{1}{15}. \end{aligned}$$

2022.06.13

$$\lim_{x \rightarrow 1} \frac{x - x^x}{1 - x + \ln x} = \underline{\hspace{2cm}}.$$

解 2.

$$\lim_{x \rightarrow 1} \frac{x - x^x}{1 - x + \ln x} = \lim_{x \rightarrow 0} \frac{e^x - e^{xe^x}}{1 - e^x + x} = \lim_{x \rightarrow 0} \frac{e^{xe^x} x (1 - e^x)}{-x^2/2} = 2.$$

2022.06.14

$$\lim_{x \rightarrow 0} \frac{(1+x)^{2/x} - e^2 [1 - \ln(1+x)]}{x} = \underline{\hspace{2cm}}.$$

解 0.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1+x)^{2/x} - e^2 [1 - \ln(1+x)]}{x} \\ &= e^2 \lim_{x \rightarrow 0} \frac{\left(\frac{1+x}{e^x}\right)^{2/x} - 1}{x} + e^2 \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} \\ &= e^2 \lim_{x \rightarrow 0} \frac{1+x - e^x}{xe^x} \cdot \frac{2}{x} + e^2 = 2e^2 \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2}}{x^2} + e^2 = 0. \end{aligned}$$

2022.06.15

$$\lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - (1+2x)^{1/(2x)}}{\sin x} = \underline{\hspace{2cm}}.$$

解 $\frac{e}{2}$.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1+x)^{1/x} - (1+2x)^{1/(2x)}}{\sin x} \\ &= \lim_{x \rightarrow 0} (1+2x)^{1/(2x)} \lim_{x \rightarrow 0} \frac{1}{x} \left[\left(\frac{(1+x)^2}{1+2x} \right)^{1/(2x)} - 1 \right] \\ &= e \lim_{x \rightarrow 0} \frac{1}{x} \cdot \frac{x^2}{(1+2x) \cdot (2x)} = \frac{e}{2}. \end{aligned}$$

2022.06.16

$$\lim_{x \rightarrow 0} \frac{1 + \frac{x^2}{2} - \sqrt{1+x^2}}{(\cos x - e^{x^2}) \sin x^2} = \underline{\hspace{2cm}}.$$

解 $-\frac{1}{12}$.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{1 + \frac{x^2}{2} - \sqrt{1+x^2}}{(\cos x - e^{x^2}) \sin x^2} \\ &= \lim_{x \rightarrow 0} \frac{\left(1 + \frac{x^2}{2}\right)^2 - (\sqrt{1+x^2})^2}{[(\cos x - 1) - (e^{x^2} - 1)] \cdot x^2 \cdot \left(1 + \frac{x^2}{2} + \sqrt{1+x^2}\right)} \\ &= \lim_{x \rightarrow 0} \frac{x^4/4}{2x^2 [(-x^2/2) - (x^2)]} = -\frac{1}{12}. \end{aligned}$$

2022.06.17

$$\lim_{x \rightarrow 0} \frac{x \sin x^2 - 2(1 - \cos x) \sin x}{x^5} = \underline{\hspace{2cm}}.$$

解 $\frac{1}{4}$.

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{x \sin x^2 - 2(1 - \cos x) \sin x}{x^5} \\ &= -\lim_{x \rightarrow 0} \frac{x(x^2 - \sin x^2)}{x^5} + \lim_{x \rightarrow 0} \frac{x^3 - 2 \sin x + \sin 2x}{x^5} \\ &= 0 + \lim_{x \rightarrow 0} \frac{3x^2 - 2 \cos x + 2 \cos 2x}{5x^4} = \lim_{x \rightarrow 0} \frac{6x + 2 \sin x - 4 \sin 2x}{20x^3} \\ &= \lim_{x \rightarrow 0} \frac{6 + 2 \cos x - 8 \cos 2x}{60x^2} = \lim_{x \rightarrow 0} \frac{-2 \sin x + 16 \sin 2x}{120x} = \frac{1}{4}. \end{aligned}$$

$$1. \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1}.$$

解 由基本不等式可得:

$$\left(1 + \frac{1}{n}\right)^n = 1 \cdot \left(1 + \frac{1}{n}\right)^n < \left[\frac{1}{n+1} \left(1 + \sum_{k=1}^n \left(1 + \frac{1}{n}\right)\right) \right]^{n+1} = \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$$2. \left(1 + \frac{1}{n}\right)^{n+1} > \left(1 + \frac{1}{n+1}\right)^{n+2}.$$

解 由基本不等式可得:

$$\left(1 + \frac{1}{n}\right)^{n+1} = \frac{1}{1 \cdot \left(\frac{n}{n+1}\right)^{n+1}} > \frac{1}{\left[\frac{1}{n+2} \left(1 + \sum_{k=1}^{n+1} \left(\frac{n}{n+1}\right)\right) \right]^{n+2}} = \left(1 + \frac{1}{n+1}\right)^{n+2}$$

$$a_N := \begin{cases} \frac{1}{N}, & \text{当 } N = \text{整数的平方时}, \\ \frac{1}{N^2}, & \text{其他}, \end{cases}$$

$$b_N := \begin{cases} \frac{1}{N} + \frac{1}{N^2}, & \text{当 } N = \text{整数的平方时}, \\ \frac{1}{N^2}, & \text{其他}, \end{cases} = \frac{1}{N^2} + \begin{cases} \frac{1}{N}, & \text{当 } N = \text{整数的平方时}, \\ 0, & \text{其他}. \end{cases}$$

$$\text{此时 } S_N = \sum_{k=1}^N a_k \leq \sum_{k=1}^N b_k = \sum_{k=1}^N \frac{1}{k^2} + \sum_{k^2 \leq N} \frac{1}{(k^2)^1}$$

本题来自高等数学辅导讲义严选题第 5 页第 39 题:

39. 求极限 $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1}} + \cdots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right).$