2022.06.06

设 $x_n = \left(1+\frac{1}{n^2}\right)\left(1+\frac{2}{n^2}\right)\cdots\left(1+\frac{n}{n^2}\right)$, 则极限 $\lim\limits_{n\to\infty} x_n$ = ____.

解 法一: 由于 $\frac{x}{1+x} < \ln(1+x) < x$, 故 $x - \ln(1+x) < \frac{x^2}{1+x}$. 因此

$$\left|\sum_{k=1}^{n} \ln\left(1+\frac{k}{n^2}\right) - \sum_{k=1}^{n}\frac{k}{n^2}\right| < \sum_{k=1}^{n}\frac{\left(\frac{k}{n^2}\right)^2}{1+\frac{k}{n^2}} < \sum_{k=1}^{n}\left(\frac{n}{n^2}\right)^2 = \frac{1}{n} \to 0(n\to\infty)$$

故由夹逼准则得:

$$\lim_{n\to\infty} \ln\left(1+\frac{k}{n^2}\right) = \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2} = \lim_{n\to\infty}\frac{n+1}{2n} = \frac{1}{2}.$$

故

$$\lim_{n\to\infty} x_n = \exp\left\{\lim_{n\to\infty}\ln\left(1+\frac{k}{n^2}\right)\right\} = \sqrt{e}.$$

法二: 因

$$x_n = \exp\left\{\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\right\} = \exp\left\{\sum_{k=1}^{n}\left[\frac{k}{n^2}+\mathcal{O}\left(\frac{k^2}{n^4}\right)\right]\right\} = \exp\left\{\frac{1}{2}+\mathcal{O}\left(\frac{1}{n}\right)\right\},$$

故 $\lim\limits_{n\to\infty} x_n = \sqrt{e}$.

2022.06.07

设 $x_0=1, x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}, n=1,2,\cdots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty} x_n$.

解 法一: $k=0$ 时, 有 $1\leqslant x_0<2$ 成立. 假设 $k=n\in\mathbb{N}$ 时, 有 $1\leqslant x_n<2$, 则 $k=n+1$ 时, 有

$$1\leqslant 1+\frac{x_n}{1+x_n}=x_{n+1}=2-\frac{1}{1+x_n}<2,$$

故由数学归纳法知 $1\leqslant x_n<2$. 又

$$x_{n+1}-x_n=\frac{1+2x_n}{1+x_n}-\frac{1+2x_{n-1}}{1+x_{n-1}}=\frac{x_n-x_{n-1}}{(1+x_n)(1+x_{n-1})}>0,$$

且 $x_1-x_0=\dfrac{1}{2}>0$, 知数列 $\{x_n\}$ 单调增加. 由单调有界准则得: 数列 $\{x_n\}$ 收敛, 令 $A=\lim\limits_{n\to\infty} x_n\geqslant 1$, 并对等式 $x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}$ 两边同时取极限, 得 $A=\dfrac{1+2A}{1+A}$ 即 $A^2-A-1=0$. 解得: $A=\dfrac{\sqrt{5}+1}{2}$ 或 $\dfrac{1-\sqrt{5}}{2}$ (舍去).

法二: 由于 $x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}$ 满足 $x_n=f(x_{n-1})$ 这种形式, 我们尝试令 $f(x)=\dfrac{1+2x}{1+x}$, 由于 $f'(x)=\dfrac{1}{(1+x)^2}>0$, 且 $f'(x)=\dfrac{1}{(1+x)^2}>0$, 知数列 $\{x_n\}$ 单调增加. 其他过程与法一类似.

法三: 由于

$$\left|x_{n+1}-\frac{\sqrt{5}+1}{2}\right|=\left|\frac{1+2x_n}{1+x_n}-\frac{\sqrt{5}+1}{2}\right|=\frac{\left|\left(3-\sqrt{5}\right)x_n+1-\sqrt{5}\right|}{2\left(1+x_n\right)}$$

$$=\frac{2}{\left(3+\sqrt{5}\right)\left(1+x_n\right)}\left|x_n-\frac{\sqrt{5}+1}{2}\right|<\frac{2}{3+\sqrt{5}}\left|x_n-\frac{\sqrt{5}+1}{2}\right|<\left(\frac{2}{3+\sqrt{5}}\right)^2\left|x_{n-1}-\frac{\sqrt{5}+1}{2}\right|$$

$$<\cdots<\left(\frac{2}{3+\sqrt{5}}\right)^{n+1}\left|x_0-\frac{\sqrt{5}+1}{2}\right|\to 0(n\to\infty), \text{故由夹逼准则得到}: \lim_{n\to\infty} x_n=\frac{\sqrt{5}+1}{2}.$$

法四: 由斐波那契数列 $\{f_n\}$ 与数学归纳法可证得 $x_n=\dfrac{f_{2n+2}}{f_{2n+1}}$. 结合数列 $\{f_n\}$ 的通项公式

$$f_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

得到:

$$x_n=\frac{f_{2n+2}}{f_{2n+1}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+2}}{\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}}=\frac{\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}-1}{2}\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}}{1+\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}}.$$

显然 $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}=0$, 等式两边同时取极限得到: $\lim\limits_{n\to\infty} x_n=\dfrac{\sqrt{5}+1}{2}$.

2022.06.08

设 $2x_1=1, 2x_n=1-x_n^2, n=1,2,\cdots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty} x_n$.

解 因 $0<x_1=\dfrac{1}{2}<1$, 假设 $k=n\in\mathbb{N}$ 时, $0<x_n<1$ 成立. 则 $0<x_{n+1}=\dfrac{1-x_n^2}{2}<1$. 由数学归纳法知 $0<x_n<1, n=1,2,\cdots$. 故

$$\begin{aligned}
\left|x_{n+1}-\left(\sqrt{2}-1\right)\right| &= \left|\frac{1-x_n^2}{2}-\sqrt{2}+1\right| = \frac{x_n^2+2\sqrt{2}-3}{2}\\
&= \frac{x_n+\sqrt{2}-1}{2}\left|x_n-\left(\sqrt{2}-1\right)\right|\\
&< \frac{1}{\sqrt{2}}\left|x_n-\left(\sqrt{2}-1\right)\right|\\
&< \left(\frac{1}{\sqrt{2}}\right)^2\left|x_{n-1}-\left(\sqrt{2}-1\right)\right| < \cdots\\
&< \left(\frac{1}{\sqrt{2}}\right)^n\left|x_1-\left(\sqrt{2}-1\right)\right| \to 0(n\to\infty).
\end{aligned}$$

故由夹逼准则得到: $\lim\limits_{n\to\infty} x_n=\sqrt{2}-1$.

2022.06.09

设 $f(x)=1-\cos x$, 则 $\lim\limits_{x\to 0}\dfrac{\left(1-\sqrt{\cos x}\right)\left(1-\sqrt[3]{\cos x}\right)\left(1-\sqrt[4]{\cos x}\right)\left(1-\sqrt[5]{\cos x}\right)}{f\left\{f\left[f\left(x\right)\right]\right\}}=$ ____.

解 $\dfrac{1}{15}$. 因

$$\begin{cases}
1-(\cos x)^a=1-[1-(1-\cos x)]^a\sim a(1-\cos x)\sim\dfrac{ax^2}{2},\\
f\left\{f\left[f\left(x\right)\right]\right\}\sim\dfrac{f^2\left[f\left(x\right)\right]}{2}\sim\dfrac{\left[\dfrac{f^2\left(x\right)}{2}\right]^2}{2}\sim\dfrac{1}{8}\left(\dfrac{x^2}{2}\right)^4=\dfrac{x^8}{128}.
\end{cases}$$

故

$$\begin{aligned}
&\lim_{x\to 0}\frac{\left(1-\sqrt{\cos x}\right)\left(1-\sqrt[3]{\cos x}\right)\left(1-\sqrt[4]{\cos x}\right)\left(1-\sqrt[5]{\cos x}\right)}{f\left\{f\left[f\left(x\right)\right]\right\}}\\
=&\lim_{x\to 0}\frac{\left(\dfrac{x^2}{4}\right)\left(\dfrac{x^2}{6}\right)\left(\dfrac{x^2}{8}\right)\left(\dfrac{x^2}{10}\right)}{\dfrac{x^8}{128}}=\frac{1}{15}.
\end{aligned}$$

2022.06.10

$$\lim_{x\to 0}\frac{\tan(\sin x)-x}{\arctan x-\arcsin x}=\underline{\qquad}.$$

解 $-\frac{1}{3}$. $\displaystyle\lim_{x\to 0}\frac{\tan(\sin x)-x}{\arctan x-\arcsin x}=\lim_{x\to 0}\frac{(\tan(\sin x)-\sin x)-(x-\sin x)}{(\arctan x-x)+(x-\arcsin x)}=\lim_{x\to 0}\frac{(x^3/3)-(x^3/6)}{(-x^3/3)+(-x^3/6)}=-\frac{1}{3}$.

2022.06.11

$$\lim_{x\to 0}\frac{\sqrt[4]{1-\sqrt[3]{1-\sqrt{1-x}}}-1}{(1+x)^{1/\sqrt[3]{x^2}}-1}=\underline{\qquad}.$$

解 $-\dfrac{1}{4\sqrt[3]{2}}$.

$\because \sqrt[4]{1-\sqrt[3]{1-\sqrt{1-x}}}-1\sim-\dfrac{\sqrt[3]{1-\sqrt{1-x}}}{4}\sim-\dfrac{\sqrt[3]{x/2}}{4};\quad (1+x)^{1/\sqrt[3]{x^2}}-1\sim\sqrt[3]{x}\ (x\to 0).$

$\therefore \displaystyle\lim_{x\to 0}\frac{\sqrt[4]{1-\sqrt[3]{1-\sqrt{1-x}}}-1}{(1+x)^{1/\sqrt[3]{x^2}}-1}=\lim_{x\to 0}\frac{-\sqrt[3]{x/2}/4}{\sqrt[3]{x}}=-\frac{1}{4\sqrt[3]{2}}.$

2022.06.12

$$\lim_{x\to 0}\frac{(3+\sin x^2)^x-3^{\sin x}}{x^3}=\underline{\qquad}.$$

解 $\dfrac{1}{3}+\dfrac{\ln 3}{6}$.

$$\begin{aligned}\lim_{x\to 0}\frac{(3+\sin x^2)^x-3^{\sin x}}{x^3}&=\lim_{x\to 0}\frac{3^x\left\{\left[\left(1+\frac{\sin x^2}{3}\right)^x-1\right]+\left(3^{\sin x-x}-1\right)\right\}}{x^3}\\&=\lim_{x\to 0}\frac{\left(\frac{x\sin x^2}{3}\right)-[\ln 3\,(\sin x-x)]}{x^3}\\&=\lim_{x\to 0}\frac{\left(\frac{x^3}{3}\right)-\left[\ln 3\cdot\left(-\frac{x^3}{6}\right)\right]}{x^3}=\frac{1}{3}+\frac{\ln 3}{6}.\end{aligned}$$

2022.06.13

$$\lim_{x\to 1}\frac{x-x^x}{1-x+\ln x}=\underline{\qquad}.$$

解 2.

$$\lim_{x\to 1}\frac{x-x^x}{1-x+\ln x}=\lim_{x\to 0}\frac{e^x-e^{xe^x}}{1-e^x+x}=\lim_{x\to 0}\frac{e^{xe^x}x(1-e^x)}{-x^2/2}=2.$$

2022.06.14

$$\lim_{x\to 0}\frac{(1+x)^{2/x}-e^2\left[1-\ln(1+x)\right]}{x}=\underline{\qquad}.$$

解 0.

$$\begin{aligned}&\lim_{x\to 0}\frac{(1+x)^{2/x}-e^2\left[1-\ln(1+x)\right]}{x}\\&=e^2\lim_{x\to 0}\frac{\left(\dfrac{1+x}{e^x}\right)^{2/x}-1}{x}+e^2\lim_{x\to 0}\frac{\ln(1+x)}{x}\\&=e^2\lim_{x\to 0}\frac{1+x-e^x}{xe^x}\cdot\frac{2}{x}+e^2=2e^2\lim_{x\to 0}\frac{-\dfrac{x^2}{2}}{x^2}+e^2=0.\end{aligned}$$

2022.06.15

$$\lim_{x\to 0}\frac{(1+x)^{1/x}-(1+2x)^{1/(2x)}}{\sin x}=\underline{\qquad}.$$

解 $\dfrac{e}{2}$.

$$\begin{aligned}&\lim_{x\to 0}\frac{(1+x)^{1/x}-(1+2x)^{1/(2x)}}{\sin x}\\&=\lim_{x\to 0}(1+2x)^{1/(2x)}\lim_{x\to 0}\frac{1}{x}\left[\left(\frac{(1+x)^2}{1+2x}\right)^{1/(2x)}-1\right]\\&=e\lim_{x\to 0}\frac{1}{x}\cdot\frac{x^2}{(1+2x)\cdot(2x)}=\frac{e}{2}.\end{aligned}$$

2022.06.16

$$\lim_{x\to 0}\frac{1+\dfrac{x^2}{2}-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin x^2}=\underline{\qquad}.$$

解 $-\dfrac{1}{12}$.

$$\lim_{x\to 0}\frac{1+\dfrac{x^2}{2}-\sqrt{1+x^2}}{(\cos x-e^{x^2})\sin x^2}=\lim_{x\to 0}\frac{\left(1+\dfrac{x^2}{2}\right)^2-\left(\sqrt{1+x^2}\right)^2}{[(\cos x-1)-(e^{x^2}-1)]\cdot x^2\cdot\left(1+\dfrac{x^2}{2}+\sqrt{1+x^2}\right)}$$

$$=\lim_{x\to 0}\frac{x^4/4}{2x^2\left[(-x^2/2)-(x^2)\right]}=-\frac{1}{12}.$$

2022.06.17

$$\lim_{x\to 0}\frac{x\sin x^2-2(1-\cos x)\sin x}{x^5}=\underline{\qquad}.$$

解 $\dfrac{1}{4}$.

$$\lim_{x\to 0}\frac{x\sin x^2-2(1-\cos x)\sin x}{x^5}$$

$$=-\lim_{x\to 0}\frac{x(x^2-\sin x^2)}{x^5}+\lim_{x\to 0}\frac{x^3-2\sin x+\sin 2x}{x^5}$$

$$=0+\lim_{x\to 0}\frac{3x^2-2\cos x+2\cos 2x}{5x^4}=\lim_{x\to 0}\frac{6x+2\sin x-4\sin 2x}{20x^3}$$

$$=\lim_{x\to 0}\frac{6+2\cos x-8\cos 2x}{60x^2}=\lim_{x\to 0}\frac{-2\sin x+16\sin 2x}{120x}=\frac{1}{4}.$$

2022.06.18

$f(x)$ 在 $x=0$ 处二阶可导, 且 $f'(0)=0$, 求极限 $\displaystyle\lim_{x\to 0}\frac{f(x)-f(\ln(1+x))}{x^3}=\underline{\qquad}$.

解 $\dfrac{f''(0)}{2}$.

$$\lim_{x\to 0}\frac{f(x)-f(\ln(1+x))}{x^3}=\lim_{x\to 0}\frac{f'(\xi)(x-\ln(1+x))}{x^3}\quad(\ln(1+x)<\xi<x)$$

$$=\lim_{x\to 0}\frac{f'(\xi)-f'(0)}{\xi-0}\lim_{x\to 0}\frac{\xi}{x}\lim_{x\to 0}\frac{x-\ln(1+x)}{x^2}=\frac{f''(0)}{2}.\quad\left(1\leftarrow\frac{\ln(1+x)}{x}<\frac{\xi}{x}<1\right)$$

2022.06.19

$$\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\left(\sqrt{a}\arctan\sqrt{\frac{x}{a}}-\sqrt{b}\arctan\sqrt{\frac{x}{b}}\right)=\underline{\qquad}.$$

解 $\dfrac{b-a}{3ab}$.

$$\begin{aligned}
&\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\left(\sqrt{a}\arctan\sqrt{\frac{x}{a}}-\sqrt{b}\arctan\sqrt{\frac{x}{b}}\right)\\
=&\sqrt{a}\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\left(\arctan\sqrt{\frac{x}{a}}-\sqrt{\frac{x}{a}}\right)-\sqrt{b}\lim_{x\to 0^+}\left(\arctan\sqrt{\frac{x}{b}}-\sqrt{\frac{x}{b}}\right)\\
=&\sqrt{a}\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\cdot\frac{x\sqrt{x}}{3a\sqrt{a}}-\sqrt{b}\lim_{x\to 0^+}\frac{1}{x\sqrt{x}}\cdot\frac{x\sqrt{x}}{3b\sqrt{b}}=\frac{1}{3a}-\frac{1}{3b}=\frac{b-a}{3ab}
\end{aligned}$$

1. $\left(1+\dfrac{1}{n}\right)^n < \left(1+\dfrac{1}{n+1}\right)^{n+1}$.

解 由基本不等式可得:

$$\left(1+\frac{1}{n}\right)^n = 1\cdot\left(1+\frac{1}{n}\right)^n < \left[\frac{1}{n+1}\left(1+\sum_{k=1}^{n}\left(1+\frac{1}{n}\right)\right)\right]^{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1}$$

2. $\left(1+\dfrac{1}{n}\right)^{n+1} > \left(1+\dfrac{1}{n+1}\right)^{n+2}$.

解 由基本不等式可得:

$$\left(1+\frac{1}{n}\right)^{n+1} = \frac{1}{1\cdot\left(\dfrac{n}{n+1}\right)^{n+1}} > \frac{1}{\left[\dfrac{1}{n+2}\left(1+\displaystyle\sum_{k=1}^{n+1}\left(\dfrac{n}{n+1}\right)\right)\right]^{n+2}} = \left(1+\frac{1}{n+1}\right)^{n+2}$$

3. 设 Ω 是由曲面 $z=\sqrt{x^2+y^2}$ 与 $z=\sqrt{1-x^2-y^2}$ 所围成的区域, 则 $I=\displaystyle\iiint\limits_{\Omega}(x+z)\,\mathrm{d}v$.

解 $\dfrac{\pi}{8}$.

$$\begin{aligned}
I &= \iiint\limits_{\Omega}(x+z)\,\mathrm{d}v = \iiint\limits_{\Omega} z\,\mathrm{d}v \\
&= \iint\limits_{D}\mathrm{d}\sigma\int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} z\,\mathrm{d}z \\
&= \frac{1}{2}\iint\limits_{D} z^2\Big|_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}}\,\mathrm{d}\sigma \\
&= \iint\limits_{D}\left(\frac{1}{2}-x^2-y^2\right)\mathrm{d}\sigma \\
&= \int_0^{2\pi}\mathrm{d}\theta\int_0^{1/\sqrt{2}}\left(\frac{1}{2}-r^2\right)r\,\mathrm{d}r \\
&= 2\pi\left(\frac{r^2}{4}-\frac{r^4}{4}\right)\Big|_0^{1/\sqrt{2}} \\
&= \frac{\pi}{8}.
\end{aligned}$$