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2022.06.06

设  $x_n = \left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2}{n^2}\right) \cdots \left(1 + \frac{n}{n^2}\right)$ , 则极限  $\lim_{n \rightarrow \infty} x_n = \underline{\hspace{2cm}}$ .

解法一: 由于  $\frac{x}{1+x} < \ln(1+x) < x$ , 故  $x - \ln(1+x) < \frac{x^2}{1+x}$ . 因此

$$\left| \sum_{k=1}^n \ln \left(1 + \frac{k}{n^2}\right) - \sum_{k=1}^n \frac{k}{n^2} \right| < \sum_{k=1}^n \frac{\left(\frac{k}{n^2}\right)^2}{1 + \frac{k}{n^2}} < \sum_{k=1}^n \left(\frac{n}{n^2}\right)^2 = \frac{1}{n} \rightarrow 0 (n \rightarrow \infty)$$

故由夹逼准则得:

$$\lim_{n \rightarrow \infty} \ln \left(1 + \frac{k}{n^2}\right) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{k}{n^2} = \lim_{n \rightarrow \infty} \frac{n+1}{2n} = \frac{1}{2}.$$

故

$$\lim_{n \rightarrow \infty} x_n = \exp \left\{ \lim_{n \rightarrow \infty} \ln \left(1 + \frac{k}{n^2}\right) \right\} = \sqrt{e}.$$

法二: 因

$$x_n = \exp \left\{ \sum_{k=1}^n \ln \left(1 + \frac{k}{n^2}\right) \right\} = \exp \left\{ \sum_{k=1}^n \left[ \frac{k}{n^2} + \mathcal{O} \left( \frac{k^2}{n^4} \right) \right] \right\} = \exp \left\{ \frac{1}{2} + \mathcal{O} \left( \frac{1}{n} \right) \right\},$$

故  $\lim_{n \rightarrow \infty} x_n = \sqrt{e}$ .

2022.06.07

设  $x_0 = 1, x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}, n = 1, 2, \dots$ . 证明数列  $\{x_n\}$  收敛, 并求极限  $\lim_{n \rightarrow \infty} x_n$ .

**解法一:**  $k=0$  时, 有  $1 \leq x_0 < 2$  成立. 假设  $k=n \in \mathbb{N}$  时, 有  $1 \leq x_n < 2$ , 则  $k=n+1$  时, 有

$$1 \leq 1 + \frac{x_n}{1+x_n} = x_{n+1} = 2 - \frac{1}{1+x_n} < 2,$$

故由数学归纳法知  $1 \leq x_n < 2$ . 又

$$x_{n+1} - x_n = \frac{1+2x_n}{1+x_n} - \frac{1+2x_{n-1}}{1+x_{n-1}} = \frac{x_n - x_{n-1}}{(1+x_n)(1+x_{n-1})} > 0,$$

且  $x_1 - x_0 = \frac{1}{2} > 0$ , 知数列  $\{x_n\}$  单调增加. 由单调有界准则得: 数列  $\{x_n\}$  收敛, 令  $A = \lim_{n \rightarrow \infty} x_n \geq 1$ , 并对等式  $x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}$  两边同时取极限, 得  $A = \frac{1+2A}{1+A}$  即  $A^2 - A - 1 = 0$ . 解

得:  $A = \frac{\sqrt{5}+1}{2}$  或  $\frac{1-\sqrt{5}}{2}$  (舍去).

**法二:** 由于  $x_n = \frac{1+2x_{n-1}}{1+x_{n-1}}$  满足  $x_n = f(x_{n-1})$  这种形式, 我们尝试令  $f(x) = \frac{1+2x}{1+x}$ , 由于  $f'(x) = \frac{1}{(1+x)^2} > 0$ , 且  $f'(x) = \frac{1}{(1+x)^2} > 0$ , 知数列  $\{x_n\}$  单调增加. 其他过程与法一类似.

**法三:** 由于

$$\begin{aligned} \left| x_{n+1} - \frac{\sqrt{5}+1}{2} \right| &= \left| \frac{1+2x_n}{1+x_n} - \frac{\sqrt{5}+1}{2} \right| = \frac{|(3-\sqrt{5})x_n + 1 - \sqrt{5}|}{2(1+x_n)} \\ &= \frac{2}{(3+\sqrt{5})(1+x_n)} \left| x_n - \frac{\sqrt{5}+1}{2} \right| < \frac{2}{3+\sqrt{5}} \left| x_n - \frac{\sqrt{5}+1}{2} \right| < \left( \frac{2}{3+\sqrt{5}} \right)^2 \left| x_{n-1} - \frac{\sqrt{5}+1}{2} \right| \\ &< \dots < \left( \frac{2}{3+\sqrt{5}} \right)^{n+1} \left| x_0 - \frac{\sqrt{5}+1}{2} \right| \rightarrow 0 (n \rightarrow \infty), \text{ 故由夹逼准则得到: } \lim_{n \rightarrow \infty} x_n = \frac{\sqrt{5}+1}{2}. \end{aligned}$$

**法四:** 由斐波那契数列  $\{f_n\}$  与数学归纳法可证得  $x_n = \frac{f_{2n+2}}{f_{2n+1}}$ . 结合数列  $\{f_n\}$  的通项公式

$$f_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{1-\sqrt{5}}{2} \right)^n \right]$$

得到:

$$x_n = \frac{f_{2n+2}}{f_{2n+1}} = \frac{\left( \frac{1+\sqrt{5}}{2} \right)^{2n+2} - \left( \frac{1-\sqrt{5}}{2} \right)^{2n+2}}{\left( \frac{1+\sqrt{5}}{2} \right)^{2n+1} - \left( \frac{1-\sqrt{5}}{2} \right)^{2n+1}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{\sqrt{5}-1}{2} \left( \frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1}}{1 + \left( \frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1}}.$$

显然  $\lim_{n \rightarrow \infty} \left( \frac{\sqrt{5}-1}{\sqrt{5}+1} \right)^{2n+1} = 0$ , 等式两边同时取极限得到:  $\lim_{n \rightarrow \infty} x_n = \frac{\sqrt{5}+1}{2}$ .

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2022.06.08

设  $2x_1 = 1, 2x_n = 1 - x_n^2, n = 1, 2, \dots$ . 证明数列  $\{x_n\}$  收敛, 并求极限  $\lim_{n \rightarrow \infty} x_n$ .

**解** 因  $0 < x_1 = \frac{1}{2} < 1$ , 假设  $k = n \in \mathbb{N}$  时,  $0 < x_n < 1$  成立. 则  $0 < x_{n+1} = \frac{1 - x_n^2}{2} < 1$ . 由数学归纳法知  $0 < x_n < 1, n = 1, 2, \dots$ . 故

$$\begin{aligned} \left| x_{n+1} - (\sqrt{2} - 1) \right| &= \left| \frac{1 - x_n^2}{2} - \sqrt{2} + 1 \right| = \frac{x_n^2 + 2\sqrt{2} - 3}{2} \\ &= \frac{x_n + \sqrt{2} - 1}{2} \left| x_n - (\sqrt{2} - 1) \right| \\ &< \frac{1}{\sqrt{2}} \left| x_n - (\sqrt{2} - 1) \right| \\ &< \left( \frac{1}{\sqrt{2}} \right)^2 \left| x_{n-1} - (\sqrt{2} - 1) \right| < \dots \\ &< \left( \frac{1}{\sqrt{2}} \right)^n \left| x_1 - (\sqrt{2} - 1) \right| \rightarrow 0 (n \rightarrow \infty). \end{aligned}$$

故由夹逼准则得到:  $\lim_{n \rightarrow \infty} x_n = \sqrt{2} - 1$ .

2022.06.09

设  $f(x) = 1 - \cos x$ , 则  $\lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})(1 - \sqrt[3]{\cos x})(1 - \sqrt[4]{\cos x})(1 - \sqrt[5]{\cos x})}{f\{f[f(x)]\}} = \underline{\hspace{2cm}}$ .

解  $\frac{1}{15}$ . 因

$$\begin{cases} 1 - (\cos x)^a = 1 - [1 - (1 - \cos x)]^a \sim a(1 - \cos x) \sim \frac{ax^2}{2}, \\ f\{f[f(x)]\} \sim \frac{f^2[f(x)]}{2} \sim \frac{\left[\frac{f^2(x)}{2}\right]^2}{2} \sim \frac{1}{8} \left(\frac{x^2}{2}\right)^4 = \frac{x^8}{128}. \end{cases}$$

故

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{(1 - \sqrt{\cos x})(1 - \sqrt[3]{\cos x})(1 - \sqrt[4]{\cos x})(1 - \sqrt[5]{\cos x})}{f\{f[f(x)]\}} \\ &= \lim_{x \rightarrow 0} \frac{\left(\frac{x^2}{4}\right)\left(\frac{x^2}{6}\right)\left(\frac{x^2}{8}\right)\left(\frac{x^2}{10}\right)}{\frac{x^8}{128}} = \frac{1}{15}. \end{aligned}$$

2022.06.13

$\lim_{x \rightarrow 1} \frac{x - x^x}{1 - x + \ln x} = \underline{\hspace{2cm}}$ .

解 2.

$$\lim_{x \rightarrow 1} \frac{x - x^x}{1 - x + \ln x} = \lim_{x \rightarrow 0} \frac{e^x - e^{xe^x}}{1 - e^x + x} = \lim_{x \rightarrow 0} \frac{e^{xe^x}x(1 - e^x)}{-x^2/2} = 2.$$