2022.06.06

设 $x_n=\left(1+\dfrac{1}{n^2}\right)\left(1+\dfrac{2}{n^2}\right)\cdots\left(1+\dfrac{n}{n^2}\right)$, 则极限 $\lim\limits_{n\to\infty}x_n$ = ____.

解 法一: 由于 $\dfrac{x}{1+x}<\ln(1+x)<x$, 故 $x-\ln(1+x)<\dfrac{x^2}{1+x}$. 因此

$$\left|\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)-\sum_{k=1}^{n}\frac{k}{n^2}\right|<\sum_{k=1}^{n}\frac{\left(\frac{k}{n^2}\right)^2}{1+\frac{k}{n^2}}<\sum_{k=1}^{n}\left(\frac{n}{n^2}\right)^2=\frac{1}{n}\to 0(n\to\infty)$$

故由夹逼准则得:

$$\lim_{n\to\infty}\ln\left(1+\frac{k}{n^2}\right)=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{n^2}=\lim_{n\to\infty}\frac{n+1}{2n}=\frac{1}{2}.$$

故

$$\lim_{n\to\infty}x_n=\exp\left\{\lim_{n\to\infty}\ln\left(1+\frac{k}{n^2}\right)\right\}=\sqrt{e}.$$

法二: 因

$$x_n=\exp\left\{\sum_{k=1}^{n}\ln\left(1+\frac{k}{n^2}\right)\right\}=\exp\left\{\sum_{k=1}^{n}\left[\frac{k}{n^2}+\mathcal{O}\left(\frac{k^2}{n^4}\right)\right]\right\}=\exp\left\{\frac{1}{2}+\mathcal{O}\left(\frac{1}{n}\right)\right\},$$

故 $\lim\limits_{n\to\infty}x_n=\sqrt{e}$.

2022.06.07

设 $x_0=1, x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}, n=1,2,\cdots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty} x_n$.

解 法一: $k=0$ 时, 有 $1\leqslant x_0<2$ 成立. 假设 $k=n\in\mathbb{N}$ 时, 有 $1\leqslant x_n<2$, 则 $k=n+1$ 时, 有

$$1\leqslant 1+\frac{x_n}{1+x_n}=x_{n+1}=2-\frac{1}{1+x_n}<2,$$

故由数学归纳法知 $1\leqslant x_n<2$. 又

$$x_{n+1}-x_n=\frac{1+2x_n}{1+x_n}-\frac{1+2x_{n-1}}{1+x_{n-1}}=\frac{x_n-x_{n-1}}{(1+x_n)(1+x_{n-1})}>0,$$

且 $x_1-x_0=\dfrac{1}{2}>0$, 知数列 $\{x_n\}$ 单调增加. 由单调有界准则得: 数列 $\{x_n\}$ 收敛, 令 $A=\lim\limits_{n\to\infty} x_n\geqslant 1$, 并对等式 $x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}$ 两边同时取极限, 得 $A=\dfrac{1+2A}{1+A}$ 即 $A^2-A-1=0$. 解得: $A=\dfrac{\sqrt{5}+1}{2}$ 或 $\dfrac{1-\sqrt{5}}{2}$ (舍去).

法二: 由于 $x_n=\dfrac{1+2x_{n-1}}{1+x_{n-1}}$ 满足 $x_n=f(x_{n-1})$ 这种形式, 我们尝试令 $f(x)=\dfrac{1+2x}{1+x}$, 由于 $f'(x)=\dfrac{1}{(1+x)^2}>0$, 且 $f'(x)=\dfrac{1}{(1+x)^2}>0$, 知数列 $\{x_n\}$ 单调增加. 其他过程与法一类似.

法三: 由于

$$\left|x_{n+1}-\frac{\sqrt{5}+1}{2}\right|=\left|\frac{1+2x_n}{1+x_n}-\frac{\sqrt{5}+1}{2}\right|=\frac{\left|\left(3-\sqrt{5}\right)x_n+1-\sqrt{5}\right|}{2\left(1+x_n\right)}$$

$$=\frac{2}{\left(3+\sqrt{5}\right)\left(1+x_n\right)}\left|x_n-\frac{\sqrt{5}+1}{2}\right|<\frac{2}{3+\sqrt{5}}\left|x_n-\frac{\sqrt{5}+1}{2}\right|<\left(\frac{2}{3+\sqrt{5}}\right)^2\left|x_{n-1}-\frac{\sqrt{5}+1}{2}\right|$$

$$<\cdots<\left(\frac{2}{3+\sqrt{5}}\right)^{n+1}\left|x_0-\frac{\sqrt{5}+1}{2}\right|\to 0(n\to\infty), \text{故由夹逼准则得到}: \lim_{n\to\infty}x_n=\frac{\sqrt{5}+1}{2}.$$

法四: 由斐波那契数列 $\{f_n\}$ 与数学归纳法可证得 $x_n=\dfrac{f_{2n+2}}{f_{2n+1}}$. 结合数列 $\{f_n\}$ 的通项公式

$$f_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

得到:

$$x_n=\frac{f_{2n+2}}{f_{2n+1}}=\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2n+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+2}}{\left(\frac{1+\sqrt{5}}{2}\right)^{2n+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{2n+1}}=\frac{\frac{1+\sqrt{5}}{2}-\frac{\sqrt{5}-1}{2}\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}}{1+\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}}.$$

显然 $\lim\limits_{n\to\infty}\left(\dfrac{\sqrt{5}-1}{\sqrt{5}+1}\right)^{2n+1}=0$, 等式两边同时取极限得到: $\lim\limits_{n\to\infty}x_n=\dfrac{\sqrt{5}+1}{2}$.

2022.06.08

设 $2x_1=1, 2x_n=1-x_n^2, n=1,2,\cdots$. 证明数列 $\{x_n\}$ 收敛, 并求极限 $\lim\limits_{n\to\infty} x_n$.

解 因 $0<x_1=\dfrac{1}{2}<1$, 假设 $k=n\in\mathbb{N}$ 时, $0<x_n<1$ 成立. 则 $0<x_{n+1}=\dfrac{1-x_n^2}{2}<1$. 由数学归纳法知 $0<x_n<1, n=1,2,\cdots$. 故

$$
\begin{aligned}
\left|x_{n+1}-\left(\sqrt{2}-1\right)\right| &= \left|\frac{1-x_n^2}{2}-\sqrt{2}+1\right| = \frac{x_n^2+2\sqrt{2}-3}{2} \\
&= \frac{x_n+\sqrt{2}-1}{2}\left|x_n-\left(\sqrt{2}-1\right)\right| \\
&< \frac{1}{\sqrt{2}}\left|x_n-\left(\sqrt{2}-1\right)\right| \\
&< \left(\frac{1}{\sqrt{2}}\right)^2\left|x_{n-1}-\left(\sqrt{2}-1\right)\right| < \cdots \\
&< \left(\frac{1}{\sqrt{2}}\right)^n\left|x_1-\left(\sqrt{2}-1\right)\right| \to 0(n\to\infty).
\end{aligned}
$$

故由夹逼准则得到: $\lim\limits_{n\to\infty} x_n=\sqrt{2}-1$.

2022.06.09

设 $f(x)=1-\cos x$, 则 $\lim\limits_{x\to0}\dfrac{\left(1-\sqrt{\cos x}\right)\left(1-\sqrt[3]{\cos x}\right)\left(1-\sqrt[4]{\cos x}\right)\left(1-\sqrt[5]{\cos x}\right)}{f\left\{f\left[f\left(x\right)\right]\right\}}=$ ____.

解 $\dfrac{1}{15}$. 因

$$\begin{cases} 1-(\cos x)^a=1-[1-(1-\cos x)]^a\sim a(1-\cos x)\sim\dfrac{ax^2}{2}, \\ f\left\{f\left[f\left(x\right)\right]\right\}\sim\dfrac{f^2\left[f\left(x\right)\right]}{2}\sim\dfrac{\left[\dfrac{f^2\left(x\right)}{2}\right]^2}{2}\sim\dfrac{1}{8}\left(\dfrac{x^2}{2}\right)^4=\dfrac{x^8}{128}. \end{cases}$$

故

$$\begin{aligned} &\lim_{x\to0}\frac{\left(1-\sqrt{\cos x}\right)\left(1-\sqrt[3]{\cos x}\right)\left(1-\sqrt[4]{\cos x}\right)\left(1-\sqrt[5]{\cos x}\right)}{f\left\{f\left[f\left(x\right)\right]\right\}} \\ =&\lim_{x\to0}\frac{\left(\dfrac{x^2}{4}\right)\left(\dfrac{x^2}{6}\right)\left(\dfrac{x^2}{8}\right)\left(\dfrac{x^2}{10}\right)}{\dfrac{x^8}{128}}=\frac{1}{15}. \end{aligned}$$

2022.06.10