Pricing Average Price Options

László Varga

MQA, Citi

Citi FinMath Course

Month Day, Year

Outline

1. Introduction

2. Commodity APOs and their pricing
   - Commodity APOs
   - Preliminary results and assumptions
   - Moment matching

3. Conclusions

Introduction – average price options

Average price options (APOs) or Asian options:

- Derivative contracts written on an average price
- Average price: arithmetic or geometric
- Usually European style
- First appearance: 1987 Tokyo
- Advantages:
  - smooths volatile market movements
  - excellent hedging tools when the market participants are exposed to average prices – popular in commodity markets
- Pricing methods:
  - exact calculation: not always possible or extremely computing intensive
  - **moment matching** – most popular
  - upper/lower price bounds
  - numerical solution of the pricing PDE
  - transformations (Laplace)
  - Monte Carlo simulation

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Introduction – APO types

Payoff of the different European Call APOs:

Averaging type	Continuously monitored	Discretely monitored
Geometric	$\left(\exp\left\{\frac{1}{T}\int\limits_0^T \log(S_t)\,dt\right\} - K\right)_+$	$\left(\left(\prod\limits_{i=1}^n S_{t_i}\right)^{1/n} - K\right)_+$
Arithmetic	$\left(\frac{1}{T}\int\limits_0^T S_t\,dt - K\right)_+$	$\left(\frac{1}{n}\sum\limits_{i=1}^n S_{t_i} - K\right)_+$

where

- $(S_t)_{t\in[0,T]}$: asset price process
- $\{t_1,\ldots,t_n\}$: fix observation times, $0 \le t_1 \le \ldots \le t_n \le T$
- T: exercise date
- K: strike

Outline

Commodity APOs

Payoff of the European APO for commodity underlyings:

$$\text{Payout}_{\text{APO}} = \left(\theta\left(\frac{1}{n}\sum_{i=1}^{n} F(t_i, T(t_i)) - K\right)\right)_+$$

where

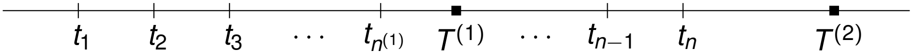
- θ: $+1$ for call, -1 for put options
- n: total number of averaging days
- $\{t_1, \dots, t_n\}$: averaging days, usually consecutive
- $T(\cdot)$: function, mapping the maturity of the front month contract to the time input
- $F(t_i, \tau)$: closing forward price on date t_i for a commodity contract maturing at τ

Commodity APOs

For most products the averaging period is 1 month that covers 2 adjacent futures contracts with maturities $T^{(1)}$ and $T^{(2)}$

$$\sum_{i=1}^{n} F(t_i, T(t_i)) = \sum_{i=1}^{n^{(1)}} F(t_i, T^{(1)}) + \sum_{i=n^{(1)}+1}^{n} F(t_i, T^{(2)}))$$

where $n^{(1)}$ is the rollover day – last day when the first contract is the front contract: $n^{(1)} = \max\{i \in \mathbb{Z} : T(t_i) = T^{(1)}\}$



Outline

Black '76 formula

Black-Scholes model in case the underlying is a forward contract

Notations:

t	valuation date
T	option expiry
T^*	forward contract expiry $\quad t \leq T \leq T^*$
r	interest rate
$D_T = e^{-r(T-t)}$	discount factor
$F_{t,T} = D_T S_t$	forward price

Forward price dynamics: $dF_{t,T} = F_{t,T}\sigma\, dW_t \quad \rightsquigarrow$ GBM

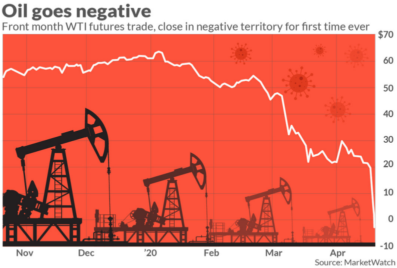
Price of the European call option at time t:

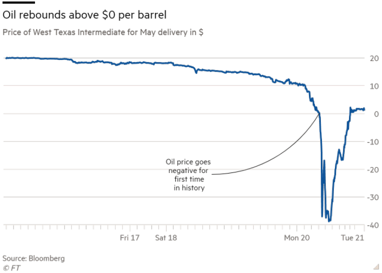
$$\text{Black}_{\text{Call}}(t, F_{t,T}, K, \sigma\sqrt{T}, D_{T^*}) := D_{T^*}\left[F_{t,T}\Phi(d_+) - K\Phi(d_-)\right], \text{where}$$

$$d_\pm = \frac{\log\left(\frac{F_{t,T}}{K}\right) \pm \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}}$$

Black '76 formula

What happens with the option price if $\frac{F_{t,T}}{K} < 0 \quad \Longleftrightarrow \quad F_{t,T} < 0$?





Possible remediation: Option price = intrinsic value $= (F_{t,T} - K)_+$

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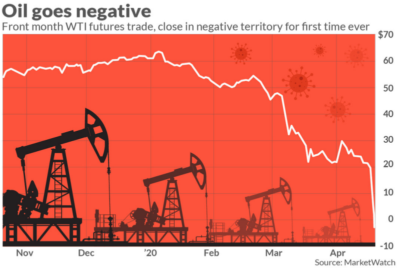

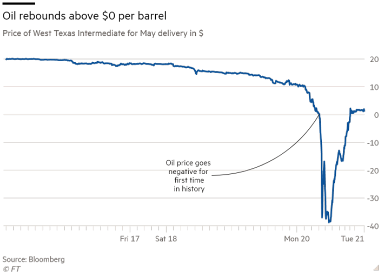

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Averaging method and conventions

Average price we concentrate on: $V := \sum_{i=1}^{n} \omega_i F(t_i, T_i)$, where

- ω_i: weights, for APOs $\omega_i = \frac{1}{n}$
- $T_i := T(t_i)$
- t: valuation date
  - usually $t < t_1 \leq \ldots \leq t_n$
  - if $t > t_1$, then some prices are already known and can be handled as deterministic

Conventions:

- Short duration APO (SD APO): averaging period is at most 1 month
- Long duration APO (LD APO): averaging period is longer than 1 month
- averaging days are weekdays

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Forward price dynamics

Multivariate lognormal dynamics:

$$dF(t, T_i) = \sigma(t, T_i)F(t, T_i))\, dW_t^{(i)} \qquad i = 1, \ldots, n$$
$$[W^{(i)}, W^{(j)}]_t = \rho_{i,j} t \qquad i, j = 1, \ldots, n$$

where $\sigma(t, T_i) = \sigma_i(t)$ is

- deterministic
- called *instantaneous volatility* of the contract maturing at T_i

Corollary:

- $F(t, T_i)$ has the martingale property
- If $t < t_i$, then

$$F(t_i, T_i) = F(t, T_i) \exp\left\{ -\frac{1}{2} \int_t^{t_i} \sigma_i^2(u) du + \int_t^{t_i} \sigma_i(u) dW^{(i)}(u) \right\}$$

Forward price dynamics

Multivariate lognormal dynamics:

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Models for the volatility

Instantaneous volatility models – strike dependence:

1. Flat model: $\sigma_{i,K}(u) = C_{i,K} \geq 0 \quad \forall u \in [t, T_i] \quad \forall i$
2. Samuelson model: $\sigma_{i,K}(u) = C_{i,K}(\sigma_L + e^{-\beta(T_i-u)}) \quad \forall u \in [t, T_i] \; \forall i$
    - σ_L and β are the Samuelson parameters
    - volatility increases if we get closer to the expiry date

Problem: the instantaneous volatility is not observable on the market, only *implied volatilites* can be obtained as average values:

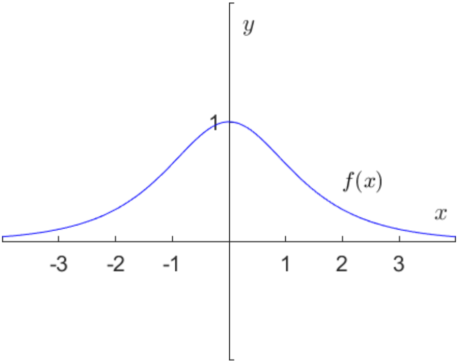
$$\widetilde{\sigma}_{i,K}(t) = \sqrt{\frac{1}{T_i - t}\int_t^{T_i} \sigma_{i,K}^2(u)du}$$

For the flat case it is easy to see that $\widetilde{\sigma}_{i,K}(t) = C_{i,K}$

We have 3 different feeder models for the volatility than incorporate volatility smile.

Correlation structure - hyperbolic secant model

$$\text{sech}(x) = \frac{1}{\cosh(x)} = \frac{2}{e^x + e^{-x}}$$



Correlation between two forward contracts:

$$\rho_{i,j} = sech\left(\sqrt{2(1-\rho)}(T_i - T_j)\right)$$

where ρ is the so-called *nearby cross-contract correlation* parameter which is calibrated to asset classes

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Moment matching – Lévy approximation

- First proposed by Lévy for Asian options
- Payoff of the European call option: $(V - K)_+$, where V is unknown
- $M_i = EV^i$ is the ith moment of V
- We approximate V with lognormal distribution which is determined by its first two moments

$$V \approx F_T = F_0 e^{-\frac{\sigma^2 T}{2} + \sigma W_T}$$

Corollary:
$$EF_T = F_0 e^{-\frac{\sigma^2 T}{2}} E(e^{\sigma W_T}) = F_0 e^{-\frac{\sigma^2 T}{2}} e^{\frac{\sigma^2 T}{2}} = F_0$$
$$EF_T^2 = F_0^2 e^{-\sigma^2 T} E(e^{2\sigma W_T}) = F_0^2 e^{-\sigma^2 T} e^{\frac{4\sigma^2 T}{2}} = F_0^2 e^{\sigma^2 T}$$

Moment equations:
$$\begin{aligned} M_1 &= EF_T \\ M_2 &= EF_T^2 \end{aligned} \quad \Longrightarrow \quad \begin{aligned} \widehat{F_0} &= M_1 \\ \widehat{\sigma\sqrt{T}} &= \sqrt{\log\left(\frac{M_2}{(M_1)^2}\right)} \end{aligned}$$

Price of the option: $\text{Black}_{\text{Call}}(0, \widehat{F_0}, K, \widehat{\sigma\sqrt{T}}, D_{T^*})$

Moment matching – moments calculation

Turnbull and Wakeman used this moment matching methodology for the first time for pricing APOs on futures in 1991.

$$M_1 = E_t V = \sum_{i=1}^{n} \omega_i E_t F(t_i, T_i) = \sum_{i=1}^{n} \omega_i F(t, T_i)$$

$$M_2 = E_t V^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i \omega_j \underbrace{E_t[F(t_i, T_i) F(t_j, T_j)]}_{E_t[F(t_i, T_i) E_{t_i} F(t_j, T_j)]} =$$

$$= \sum_{i,j=1}^{n} \omega_i \omega_j E_t[F(t_i, T_i) F(t_i, T_j)]$$

Moment matching – calculation of M_2

$$M_2 = \sum_{i,j=1}^{n} \omega_i \omega_j F(t,T_i) F(t,T_j) \exp\left\{ -\frac{1}{2} \int_t^{t_i} (\sigma_i^2(u) + \sigma_j^2(u)) du \right\} \cdot$$

$$\cdot \underbrace{E_t \left[exp \left\{ \int_t^{t_i} \sigma_i(u) dW^{(i)}(u) + \int_t^{t_i} \sigma_j(u) dW^{(j)}(u) \right\} \right]}_{E_t\left(e^{X_i+X_j}\right) = e^{0+\frac{1}{2}D^2(X_i+X_j)} = e^{\frac{1}{2}\left(D^2X_i + D^2X_j + 2Cov(X_i,X_j)\right)}}$$

$X_i := \int_t^{t_i} \sigma_i(u) dW^{(i)}(u)$ is a stochastic integral with deterministic σ_i function, therefore (X_i, X_j) is jointly Gaussian with $EX_i = 0$,

$D^2 X_i = \int_t^{t_i} \sigma_i^2(u) du$ and $Cov(X_i, X_j) = \int_t^{t_i} \sigma_i(u) \sigma_j(u) \rho_{i,j} du$.

Moment matching – calculation of M_2

$$M_2 = \sum_{i,j=1}^{n} \omega_i \omega_j F(t, T_i) F(t, T_j) \exp\left\{ \rho_{i,j} \int_t^{t_i} \sigma_i(u) \sigma_j(u) du \right\}$$

For the flat volatility model we have

$$M_2 = \sum_{i,j=1}^{n} \omega_i \omega_j F(t, T_i) F(t, T_j) e^{\rho_{i,j}(t_i - t) C_{i,K} C_{j,K}}$$

For the Samuelson volatility model we have

$$M_2 = \sum_{i,j=1}^{n} \omega_i \omega_j F(t, T_i) F(t, T_j) \exp\left\{ \rho_{i,j} A_t^{i,j} (t_i - t) \tilde{\sigma}_{i,K}(t) \tilde{\sigma}_{j,K}(t) \right\}$$

where $A_t^{i,j}$ has an explicit, but very long form.

Conclusions

Experience with the moment matching technique

- Model in 10+ years' use
- The approximation is very good for SD APOs, somewhat worse for LD APOs
- The approximation gets worse the later the averaging period starts
- Fast calculations
- Independent Excel implementation: even the largest differences are only of 10^{-8} magnitude
- Risk are stable even under severely stressed conditions
- Benchmark with 4 moment matching – not much better
- Certain parameters are not enough frequently calibrated
- Risk sensitivities are not stable if prices go negative even with the intrinsic value usage

Thank you for the attention!

Questions / comments?

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