Experiments in Physics

Physics 1292
General Physics II Lab

Columbia University

Department of Physics

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General Instructions

Purpose of the Laboratory

The laboratory experiments described in this manual are an important part of your physics course. Most of the experiments are designed to illustrate important concepts described in the lectures. Whenever possible, the material will have been discussed in lecture before you come to the laboratory. But some of the material, like the first experiment on measurement and errors, is not discussed at length in the lecture.

The sections headed Applications and Lab Preparation Examples, which are included in some of the manual sections, are *not* required reading unless your laboratory instructor specifically assigns some part. The Applications are intended to be motivational and so should indicate the importance of the laboratory material in medical and other applications. The Lab Preparation Examples are designed to help you prepare for the lab; you will not be required to answer all these questions (though you should be able to answer any of them by the end of the lab). The individual laboratory instructors may require you to prepare answers to a subset of these problems.

Preparation for the Laboratory

In order to keep the total time spent on laboratory work within reasonable bounds, the write-up for each experiment will be completed at the end of the lab and handed in *before the end of each laboratory period*. Therefore, it is imperative that you spend sufficient time preparing for the experiment *before* coming to laboratory. You should take advantage of the opportunity that the experiments are set up in the Lab Library (Room 506) and that TAs there are willing to discuss the procedure with you.

At each laboratory session, the instructor will take a few minutes at the beginning to go over the experiment and describe the equipment to be used and to outline the important issues. This does not substitute for careful preparation beforehand! You are expected to have studied the manual and appropriate references at home so that you are prepared when you arrive to perform the experiment. The instructor will be available primarily to answer questions, aid you in the use of the equipment, discuss the physics behind the experiment, and guide you in completing your analysis and write-up. Your instructor will describe his/her policy regarding expectations during the first lab meeting.

Some experiments and write-ups may be completed in less than the three-hour laboratory period, but under no circumstances will you be permitted to stay in the lab after the end of the period or to take your report home to complete it. If it appears that you will be unable to complete all parts of the experiment, the instructor will arrange with you to limit the experimental work so that you have enough time to write the report during the lab period.

Note: Laboratory equipment must be handled with care and each laboratory bench must be returned to a neat and orderly state before you leave the laboratory. In particular, you must turn off all sources of electricity, water, and gas.

Bring to Each Laboratory Session

- A pocket calculator (with basic arithmetic and trigonometric operations).
- A pad of 8.5×11 inch graph paper and a sharp pencil. (You will write your reports on this paper, including your graphs. Covers and staplers will be provided in the laboratory.)
- (optional) A ruler (at least 10 cm long).
- (optional) A personal laptop with Microsoft Excel for data analysis.

Graph Plotting

Frequently, a graph is the clearest way to represent the relationship between the quantities of interest. There are a number of conventions, which we include below.

- A graph indicates a relation between two quantities, x and y, when other variables or parameters have fixed values. Before plotting points on a graph, it may be useful to arrange the corresponding values of x and y in a table.
- Choose a convenient scale for each axis so that the plotted points will occupy a <u>substantial</u> part of the graph paper, but do <u>not</u> choose a scale which is difficult to plot and read, such as 3 or 3/4 units to a square. Graphs should usually be at least half a page in size.
- Label each axis to identify the variable being plotted and the units being used. Mark prominent divisions on each axis with appropriate numbers.
- Identify plotted *points* with appropriate symbols, such as crosses, and when necessary draw vertical or horizontal *error bars* through the points to indicate the range of uncertainty involved in these points.

- Often there will be a theory concerning the relationship between the two plotted variables. A linear relationship can be demonstrated if the data points fall along a single straight line. There are mathematical techniques for determining which straight line best fits the data, but for the purposes of this lab, we will be using Microsoft Excel's built-in fitting methods.

Error Analysis

All measurements, however carefully made, give a range of possible values referred to as an uncertainty or error. Since all of science depends on measurements, it is important to understand uncertainties and where they come from. Error analysis is the set of techniques for dealing with them.

In science, the word "error" does not take the usual meaning of "mistake". Instead, we will use it interchangeably with "uncertainty" when talking about the result of a measurement. There are many aspects to error analysis and it will feature in some form in every lab throughout this course.

Inevitability of Experimental Error

In the first experiment of the semester, you will measure the length of a pendulum. Without a ruler, you might compare it to your own height and (after converting to meters) make an estimate of 1.5 m. Of course, this is only approximate. To quantify this, you might say that you are sure it is not less than 1.3 m and not more than 1.7 m. With a ruler, you measure 1.62 m. This is a much better estimate, but there is still uncertainty. You couldn't possibly say that the pendulum isn't 1.62001 m long. If you became obsessed with finding the exact length of the pendulum you could buy a fancy device using a laser, but even this will have an error associated with the wavelength of light.

Also, at this point you would come up against another problem. You would find that the string is slightly stretched when the weight is on it and the length even depends on the temperature or moisture in the room. So which length do you use? This is a problem of definition. During lab you might find another example. You might ask whether to measure from the bottom, top or middle of the weight. Sometimes one of the choices is preferable for some reason (in this case the middle because it is the center of mass). However, in general it is more important to be clear about what you mean by "the length of the pendulum" and consistent when taking more than one measurement. Note that the different lengths that you measure from the top, bottom or middle of the weight do not contribute to the error. *Error* refers to the range of values given by measurements of exactly the same quantity.

Importance of Errors

In daily life, we usually deal with errors intuitively. If someone says "I'll meet you at 9:00", there is an understanding of what range of times is OK. However, if you want to know how long it takes to get to JFK airport by train you might need to think about the range of possible values. You might say "It'll probably take an hour and a half, but I'll allow two hours". Usually it will take within about 10 minutes of this most probable time. Sometimes it will take a little less than 1hr20, sometimes a little more than 1hr40, but by allowing the most probable time plus three times this uncertainty of 10 minutes you are almost certain to make it. In more technical applications, for example air traffic control, more careful consideration of such uncertainties is essential.

In science, almost every time that a new theory overthrows an old one, a discussion or debate about relevant errors takes place. In this course, we will definitely not be able to overthrow established theories. Instead, we will verify them with the best accuracy allowed by our equipment. The first experiment involves measuring the gravitational acceleration g. While this fundamental parameter has clearly been measured with much greater accuracy elsewhere, the goal is to make the most accurate possible verification using very simple apparatus which can be a genuinely interesting exercise.

There are several techniques that we will use to deal with errors throughout the course. All of them are well explained, with more formal justifications, in "*An Introduction to Error Analysis*" by John Taylor, available in the Science and Engineering Library in the Northwest Corner Building.

Questions or Complaints

If you have a difficulty, you should attempt to work it through with your laboratory instructor. If you cannot resolve it, you may discuss such issues with:

- One of the laboratory Preceptors in Pupin Room 729;
- The Undergraduate Assistant in the Departmental Office – Pupin Room 704;
- The instructor in the lecture course, or the Director of Undergraduate Studies;
- Your undergraduate advisor.

As a general rule, it is a good idea to work downward through this list, though some issues may be more appropriate for one person than another.

Experiment 2-1
The Magnetic Field

Introduction

In this lab, we will determine the strength of the magnetic field in the gap of an electromagnet in two ways: first, by measuring the force applied on a current-carrying rod; and second, by measuring the effects of changing magnetic flux through a coil that is being inserted or removed from that gap. In doing so, we will also be verifying both Faraday's law and Lenz' law!

CAUTION:

- Always reduce the current through the electromagnet to zero before opening the circuit of the magnet coils.

- Remove wrist watches before placing hands near the magnet gaps.

Theory

Force on a Current-carrying Wire

Consider a rod of length L, held horizontally and normal to the direction of a uniform, horizontal magnetic field B. If a current i is passed through the wire, as indicated in Figure 1, then there will be a vertical force $F = iLB$ on the wire rod.

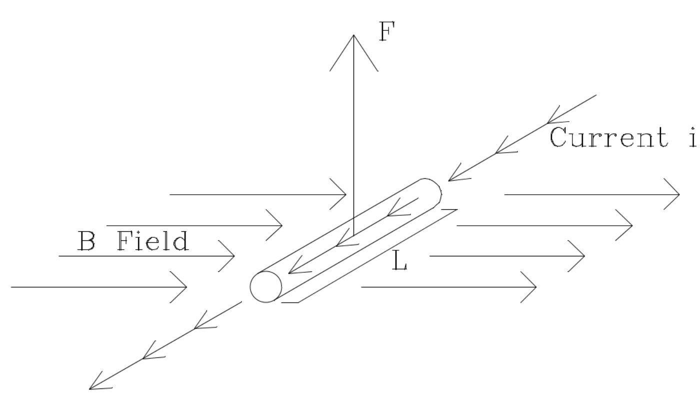

Figure 1: Force on a Current-carrying Wire

Induced EMF in a Coil

According to Faraday's Law of induction, a changing magnetic flux through a coil induces an EMF (electromagnetic force) ε given by

$$\varepsilon = N\frac{\Delta\Phi}{\Delta t} \tag{1}$$

where $\Phi = \int \vec{B} \cdot d\vec{A}$ is the flux of magnetic field B through a coil of area A and perpendicular to that area. N is the number of turns in the search coil. For this experiment, the area of the coil is constant and the magnetic field is assumed to be uniform, so the average EMF is given by

$$\varepsilon = -NA\frac{\Delta B}{\Delta t} \tag{2}$$

The negative sign in Faraday's Law comes from the fact that the EMF induced in the coil acts to oppose any change in the magnetic field. This is summarized as Lenz' Law. It is important to remember that EMF, despite being called a "force" is actually a potential and is measured in volts. Voltage will be induced as the coil enters and leaves magnetic field and its direction will be determined using Lenz' law.

Procedure

Force on a Current-carrying Wire

The experimental set-up is shown in Figure 2. The horizontal magnetic field B is produced in the air gap of a "C-shaped" iron electromagnet. The strength of B is determined by the current in the magnet coils I, which is supplied by an adjustable low-voltage power supply.

A more detailed drawing of the balance and electro-magnet arrangement is shown in Figure 3.

Note that the current i through the horizontal conductor is supplied by a separate power supply. The current balance is constructed out of conducting and insulating materials such that current can enter through one side of the knife-edge fulcrum, flow through one side of the balance arm to the horizontal conductor, and then flow back through the other side of the balance arm and out through the other knife-edge.

The current i in the balance is provided by the HP E3610A power supply, which can operate either in constant voltage or constant current mode. The voltage dial sets the *maximum voltage* the device will supply to the circuit; the current dial likewise sets the *maximum current*, and if the circuit tries to draw more current, the power

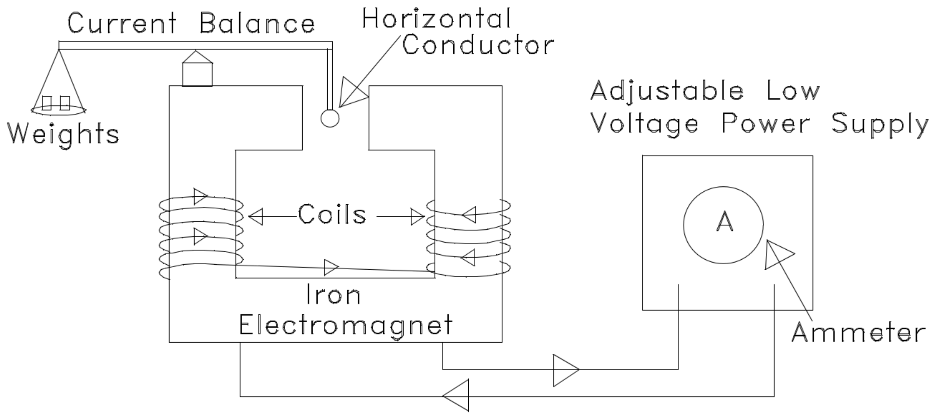

Figure 2: Experimental Set-up

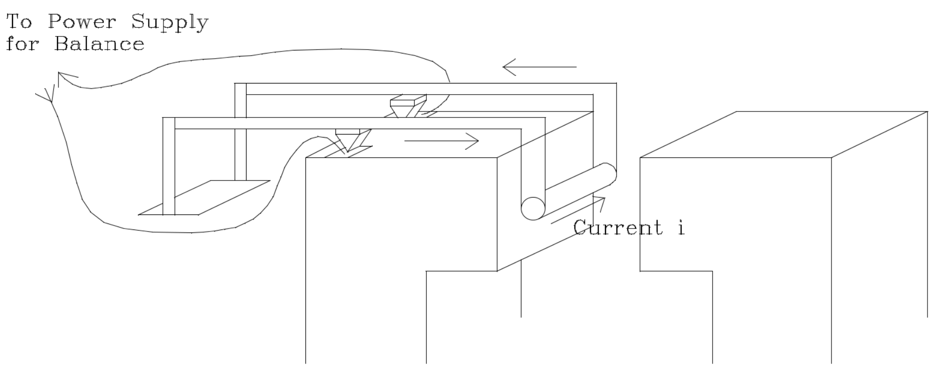

Figure 3: Setup of the Wire and Balance for Force Measurement

supply will reduce the voltage until it reaches whatever value is needed to maintain the maximum current (by $V = IR$).

To use the power supply in constant current mode, begin with the current dial turned all the way down (counter-clockwise) and the voltage dial turned all the way up (clockwise). Set the range to 3 Amps, and connect the leads to the + and − terminals. You can now set the current to the desired level. Note that the digital meters on the power supply show the *actual* voltage and current being supplied, so you will not normally see any current unless the leads are connected to a complete circuit. (If you want to set the current level without closing the circuit, you can hold in the CC Set button while you turn the current dial.) Once you close the circuit, the voltage adjusts automatically to maintain the constant current level, and the CC (Constant Current) indicator light should be on.

1. Set the current I through the electromagnet at 5 amperes. Place a small number of weights on the balance and determine the value off i necessary to reach equilibrium. Repeat for at least five different weights.

2. With Microsoft Excel, plot the weight used to balance the scale vs. the balance current i. Include error bars.

3. Draw a line of best fit and determine the slope with error using LINEST.

4. From your slope, determine the magnetic field strength B with error.

5. Repeat the above steps (steps 1-4) for two other magnet currents I (for a total of three current data sets). You do not need to do error analysis for these measurements, but plot each of your results on the same graph.

6. Draw a diagram similar to Figure 3 and indicate the directions of i, F and B for your setup.

7. Discuss potential sources of error.

Induced EMF in a Coil

Experimental Apparatus

A charge integrator (the Magnetic Field Module shown in Figure **??**) is used to measure the ΔQ produced by the EMF induced in the search coil. A capacitor in the module stores the charge ΔQ, and the voltage across this capacitor (read on the external voltmeter shown) is proportional to ΔQ. Therefore

$$V = K\Delta Q = K\frac{N}{R}\Delta\Phi \tag{3}$$

where K is a constant that depends on the capacitance and gain of the integrator circuit. Instead of trying to calculate a value of KN/R in terms of the components, it is more direct to calibrate the combination of the search coil and the integrator circuit by measuring V for a known $\Delta\Phi$. This known magnetic flux can be created by passing a measured current I_{sol} through a long air-core solenoid of n turns per meter and with a cross-sectional area of A_{sol} so that

$$\Delta\Phi_{\mathrm{sol}} = B_{\mathrm{sol}}A_{\mathrm{sol}} = \mu_0 n I_{\mathrm{sol}} A_{\mathrm{sol}} \tag{4}$$

where μ_0 is the permeability of free space ($\mu_0 = 4\pi \times 10^{-7}\,\mathrm{T}\cdot\mathrm{m/A}$).

Procedure

Connect the apparatus as shown in Figure **??**. Turn on the power supply. Before any measurements are made, depress the shorting switch, then release the shorting switch and turn the drift adjust control to minimize the drift in the output voltage as observed on your meter. The shorting switch must be used to discharge the integrating capacitor prior to each measurement. Also, the drift adjust setting should be checked occasionally. If the gain setting on the Magnetic Field Module is changed, the drift adjust control must be reset.

Magnetic field measurements are made by inserting or removing the search coil from the region containing the field to a field-free region or by leaving the search coil stationary and turning the field on or off.

Since the magnetic field in the large iron-core electromagnet is much greater than the field in an air-core solenoid, the Magnetic Field Module was designed with two gain settings. In the gain=100 position, the module is 100 times as sensitive as in the gain=1 position. For measuring fields generated by the air-core solenoid, set the gain at 100. For fields generated by the large electromagnet, set the gain at 1.

Connect the solenoid to the + and − terminals of the HP Power Supply. A three position (on-off-on) reversing switch is part of the solenoid circuit. Flip the switch to one of the on positions and adjust the power supply current such that two amperes is flowing through the solenoid.

Slide the search coil over the solenoid and, while holding the search coil at the center of the solenoid, discharge the Magnetic Field Module and then turn the current through the solenoid off using the three position switch (or turn the current from off to on). Take several readings and record the voltage on the integrator and the current through the solenoid. Then the result is

$$V_{\mathrm{sol}} = 100 \cdot K \cdot \frac{N}{R} \cdot \Delta\Phi_{\mathrm{sol}} = 100 \cdot K \cdot \frac{N}{R}(\mu_0 n I_{\mathrm{sol}}) A_{\mathrm{sol}} \tag{5}$$

Now set the gain to 1 on the Magnetic Field Module, and readjust the drift controls. Set the current for the large electromagnet to 5 amps, and use the search coil to measure the resulting B. Move the search coil *gently and smoothly* into the region of the magnetic field (do not move the coil hastily as you may damage it by striking against the magnet itself). Record V_{mag} . The corresponding equation is:

$$V_{\mathrm{mag}} = K \cdot \frac{N}{R} A_{\mathrm{coil}} B_{\mathrm{mag}} \tag{6}$$

Combine the results of Eq(5) and Eq(6) to determine the value of B for the large electromagnet when the current is 5 amps, and then do the same for the other magnet currents of 4, 3, and 2 amps. Compare these values for B with those obtained in Part I.

Appendices

Appendix A
Review of Error Analysis

Types of Uncertainties

Uncertainty in a measurement can arise from three possible origins: the measuring device, the procedure of how you measure, and the observed quantity itself. Usually the largest of these will determine the uncertainty in your data.

Uncertainties can be divided into two different types: systematic uncertainties and random (statistical) uncertainties[1].

Systematic Uncertainties

Systematic uncertainties or systematic errors always bias results in one specific direction. They will cause your measurement to consistently be higher or lower than the accepted value.

An *example* of a systematic error follows. Assume you want to measure the length of a table in cm using a meter stick. However, the stick is made of metal that has contracted due to the temperature in the room, so that it is less than one meter long. Therefore, all the intervals on the stick are smaller than they should be. Your numerical value for the length of the table will then always be larger than its actual length no matter how often or how carefully you measure. Another example might be measuring temperature using a mercury thermometer in which a bubble is present in the mercury column.

Systematic errors are usually due to imperfections in the equipment, improper or biased observation, or the presence of additional physical effects not taken into account. (An example might be an experiment on forces and acceleration in which there is friction in the setup and it is not taken into account!)

In performing experiments, try to estimate the effects of as many systematic errors as you can, and then remove or correct for the most important. By being aware of the sources of systematic error beforehand, it is often possible to perform experiments with sufficient care to compensate for weaknesses in the equipment.

[1] If you were to engage in further research, random uncertainty is typically referred to as statistical uncertainty.

Random Uncertainties

In contrast to systematic uncertainties, random uncertainties are an unavoidable result of measurement, no matter how well designed and calibrated the tools you are using. Whenever more than one measurement is taken, the values obtained will not be equal but will exhibit a spread around a mean value, which is considered the most reliable measurement. That spread is known as the random uncertainty. Random uncertainties are unbiased – meaning it is equally likely that an individual measurement is too high or too low.

From your everyday experience you might be thinking, "Stop! Whenever I measure the length of a table with a meter stick I get exactly the same value no matter how often I measure it!" This may happen if your meter stick is insensitive to random measurements, because you use a coarse scale (like mm) and you always read the length to the nearest mm. But if you would use a meter stick with a finer scale, or if you interpolate to fractions of a millimeter, you would definitely see the spread. As a general rule, if you do not get a spread in values, you can improve your measurements by using a finer scale or by interpolating between the finest scale marks on the ruler.

How can one reduce the effect of random uncertainties? Consider the following *example*. Ten people measure the time of a sprinter using stopwatches. It is very unlikely that each of the ten stopwatches will show exactly the same result. Even if all of the people started their watches at exactly the same time (unlikely) some of the people will have stopped the watch early, and others may have done so late. You will observe a spread in the results. If you *average* the times obtained by all ten stop watches, the *mean* value will be a better estimate of the true value than any individual measurement, since the uncertainty we are describing is random, the effects of the people who stop early will compensate for those who stop late. In general, making multiple measurements and averaging can reduce the effect of random uncertainty.

Remark: We usually specify any measurement by including an estimate of the random uncertainty. (Since the random uncertainty is unbiased we note it with a $\pm$ sign). So if we measure a time of 7.6 seconds, but we expect a spread of about 0.2 seconds, we write as a result:

$$t = (7.6 \pm 0.2)\,\mathrm{s} \tag{1}$$

indicating that the uncertainty of this measurement is $0.2\,\mathrm{s}$ or about 3%.

Accuracy and Precision

An important distinction in physics is the difference between the *accuracy* and the *precision* of a measurement. Accuracy refers to the closeness of a measured value to a standard or known value. For example, if in lab you obtain a weight measurement of 3.2 kg for a given substance, but the actual or known weight is 10 kg, then your measurement is not accurate. In this case, your measurement is not close to the known value.

Precision refers to the closeness of two or more measurements to each other. Using the example above, if you weigh a given substance five times, and get 3.2 kg each time, then your measurement is very precise. Precision is independent of accuracy. You can be very precise but inaccurate, as described above. You can also be accurate but imprecise.

For example, if on average, your measurements for a given substance are close to the known value, but the measurements are far from each other, then you have accuracy without precision.

A good analogy for understanding accuracy and precision is to imagine a basketball player shooting baskets. If the player shoots with accuracy, his aim will always take the ball close to or into the basket. If the player shoots with precision, his aim will always take the ball to the same location which may or may not be close to the basket. A good player will be both accurate and precise by shooting the ball the same way each time and each time making it in the basket.

Numerical Estimates of Uncertainties

For this laboratory, we will estimate uncertainties with three approximation techniques, which we describe below. You should note which technique you are using in a particular experiment.

Upper Bound

Most of our measuring devices in this lab have scales that are coarser than the ability of our eyes to measure.

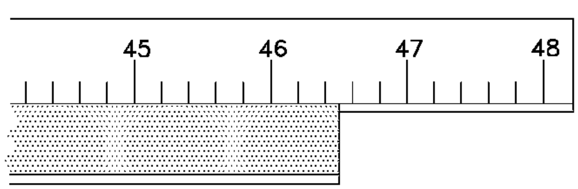

Figure 1: Measuring Length

For example in the figure above, where we are measuring the length of an object against a meter stick marked in cm, we can definitely say that our result is somewhere between 46.4 cm and 46.6 cm. We assume as an *upper* bound of our uncertainty, an amount equal to *half* this width (in this case 0.1 cm). The final result can be written as:

$$\ell = (46.5 \pm 0.1)\,\mathrm{cm} \tag{2}$$

There will be many circumstances when the error is more complicated than simply the coarseness of the measuring tool. For example, if you find yourself measuring something that is very long or hard to line up properly with a meter stick. In this case, you may need to use some judgement of the best possible measurement to make and the uncertainty will be greater than the millimeter precision of your meter stick. **It is always best to slightly overestimate error and allow yourself some wiggle room if you feel that better represents your measurement!**

Estimation from the Spread (2/3 method)

For data in which there is random uncertainty, we usually observe individual measurements to cluster around the mean and drop in frequency as the values get further from the mean (in both directions).[2] Find the interval around the mean that contains about 2/3 of the measured points: *half* the size of this interval is a good estimate of the uncertainty in each measurement.

The reasons for choosing a range that includes 2/3 of the values come from the underlying statistics of the normal (or Gaussian) distribution (see figure 4). This choice allows us to accurately add and multiply values with errors and has the advantage that the range is not affected much by outliers and occasional mistakes. A range that always includes all of the values is generally less meaningful.

Example: You measure the following values of a specific quantity:

9.7, 9.8, 10, 10.1, 10.1, 10.3

The mean of these six values is 10.0. The interval from 9.8 to 10.1 includes 4 of the 6 values; we therefore estimate the uncertainty to be 0.15. The result is that the best estimate of the quantity is 10.0 and the uncertainty of a single measurement is 0.2.[3][4]

[2] There is a precise mathematical procedure to obtain uncertainties (standard deviations) from a number of measured values. Here we will apply a simple "rule of thumb" that avoids the more complicated mathematics of that technique. The uncertainty using the standard deviation for the group of values in our example below is 0.2.

[3] Note that about 5% of the measured values will lie *outside* $\pm$ twice the uncertainty

[4] While the above method for calculating uncertainty is good enough for our purposes, it oversimplifies a bit the task of calculating the uncertainty of the *mean* of a quantity. For those who are interested, please see the appendix for elaboration and clarification.

Square-Root Estimation in Counting

For inherently random phenomena that involve counting individual events or occurrences, we measure only a single number N. This kind of measurement is relevant to counting the number of radioactive decays in a specific time interval from a sample of material, for example. It is also relevant to counting the number of left-handed people in a random sample of the population. The (absolute) uncertainty of such a single measurement, N, is estimated as the square root of N (a counting measurement is expressed as $N \pm \sqrt{N}$). As an example, if we measure 50 radioactive decays in 1 second we should present the result as 50 ± 7 decays per second. (The quoted uncertainty indicates that a subsequent measurement performed identically could easily result in numbers differing by 7 from 50.)

Relative and Absolute Uncertainty

There are two ways to record uncertainties: the absolute value of the uncertainty or the uncertainty relative to the mean value. So in the example above, you can write $c = (5.1 \pm 0.3)\,\text{cm}$ or equally well $c = 5.1\,\text{cm}\ (1.00 \pm 0.06)$. You can see that if you multiply out the second form you will obtain the first, since $5.1 \times 0.06 = 0.3$. The second form may look a bit odd, but it tells you immediately that the uncertainty is 6% of the measured value. The number 0.3 cm is the absolute uncertainty and has the same units as the mean value (cm). The 0.06 (or 6%) is the relative uncertainty and has no units since it is the ratio of two lengths. It's important to use proper notation when describing uncertainty to remove any unwanted ambiguity, so make sure it's clear when you are using relative or absolute errors.

Propagation of Uncertainties

Often, we are not directly interested in a measured value, but we want to use it in a formula to calculate another quantity. In many cases, we measure many of the quantities in the formula and each has an associated uncertainty. We deal here with how to propagate uncertainties to obtain a well-defined uncertainty on a computed quantity.

Adding/Subtracting Quantities

When we **add or subtract** quantities, the combined uncertainty is the **sum of the absolute uncertainties** of the constituent parts[5].

[5] The propagation of random uncertainties is actually slightly more complicated, but the procedure outlined here usually represents a good approximation, and it never underestimates the uncertainty.

Take as an example measuring the length of a dog. We measure the distance between the left wall and the tail of the dog and subtract the distance from the wall to the dog's nose. So the total length of the dog is:

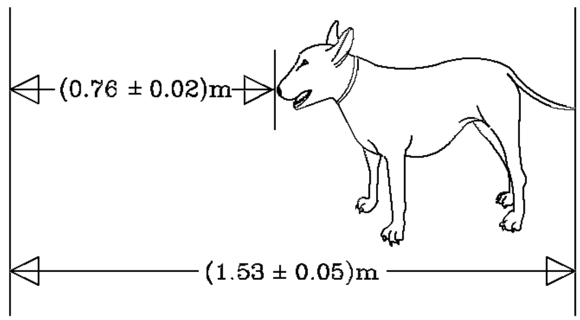

Figure 2: Measuring a Dog

$$
\begin{aligned}
\text{Length} &= (1.53 \pm 0.05)\,\text{m} - (0.76 \pm 0.02)\,\text{m} \\
&= (1.53 - 0.76) \pm (0.05 + 0.02)\ \text{m} \\
&= (0.77 \pm 0.07)\ \text{m}
\end{aligned} \tag{3}
$$

Multiplying/Dividing Quantities

When we **multiply or divide** quantities, the combined **relative** uncertainty is the **sum of the relative uncertainties** of the constituent parts.[6]

Take as an example the area of a rectangle, whose individual sides are measured to be:

$$
\begin{aligned}
a &= 25.0 \pm 0.5\,\text{cm} = 25.0\,\text{cm}\ (1.00 \pm 0.02) \\
b &= 10.0 \pm 0.3\,\text{cm} = 10.0\,\text{cm}\ (1.00 \pm 0.03)
\end{aligned} \tag{4}
$$

The area is obtained as follows:

$$
\begin{aligned}
\text{Area} &= (25.0 \pm 0.5\,\text{cm}) \cdot (10.0 \pm 0.3\,\text{cm}) \\
&= 25.0\,\text{cm}\ \ (1.00 \pm 0.02) \cdot 10.0\,\text{cm}\ \ (1.00 \pm 0.03) \\
&= (25.0\,\text{cm} \cdot 10.0\,\text{cm})\,(1.00 \pm (0.02 + 0.03)) \\
&= 250.0\,\text{cm}^2\ (1.00 \pm 0.05) \\
&= 250.0 \pm 12.5\,\text{cm}^2 \\
&= 250 \pm 10\,\text{cm}^2
\end{aligned} \tag{5}
$$

See the appendix for more information.

[6] Our calculation of the uncertainty actually overestimates it. The correct method does not add the absolute/relative uncertainty, but rather involves evaluating the square root of the sum of the squares. For more information please refer to the appendix of this lab manual.

Note that the final step has rounded both the result and the uncertainty to an appropriate number of significant digits, given the uncertainty on the lengths of the sides.

Remarks: Note that uncertainties on quantities used in a mathematical relationship always increase the uncertainty on the result. The quantity with the biggest uncertainty usually dominates the final result. Often one quantity will have a much bigger uncertainty than all the others. In such cases, we can simply use this main contribution.

Multiplication by a Constant

Multiplying a value by a constant leaves the relative error unchanged. This is equivalent to multiplying the absolute error by the same constant. For example, suppose we are trying to find the circumference of a circle knowing it's radius as $r = 1.0 \pm 0.1$ cm with error; we would calculate the circumference with error as follows.

$$\begin{aligned} C &= 2\pi r \\ C &= 2\pi(1.0 \pm 0.1) \\ C &= 6.3 \pm 0.6 \text{ cm} \end{aligned} \tag{6}$$

Powers and Roots

When raising a value to a certain power, its **relative uncertainty is multiplied by the exponent**. This applies to roots as well, since taking the root of a number is equivalent to raising that number to a fractional power.

Squaring a quantity involves multiplying its relative uncertainty by 2, while cubing a quantity causes its relative uncertainty to be multiplied by 3.

Taking the square root of a quantity (which is equivalent to raising the quantity to the 1/2 power) causes its relative uncertainty to be multiplied by 1/2. For example, if you know the area of a square to be:

$$\text{Area} = 100 \pm 8\,\text{m}^2 = 100\,\text{m}^2\ (1.00 \pm 0.08) \tag{7}$$

then it follows that the side of the square is:

$$\text{Side} = 10\,\text{m}\ \ (1.00 \pm 0.04) = 10.0 \pm 0.4\,\text{m} \tag{8}$$

The most general rule for finding the error in powers and roots is mathematically represented as follows.

$$f(x) = x^n \tag{9}$$

$$\frac{\sigma_{f(x)}}{f(x)} = |n|\frac{\sigma_x}{x} \tag{10}$$

Where σ is the *absolute* uncertainty and $f(x)$ is some power or root of x.

Other Functions

If you need to calculate the error of a calculation that does not fit into one of these rules (such as trigonometric functions or logarithmic ones), here is a manual method that you can use.

Based upon the error of the quantity that you determined, you can find the maximum and minimum values of the quantity that you are calculating. The value that you found should be roughly midway between these two quantities. Then if you split the difference between the maximum and minimum you should obtain a reasonable estimate of the error. Mathematically, you would do so as follows.

$$\sigma_{f(x)} = \frac{f(x+\sigma_x) - f(x-\sigma_x)}{2} \tag{11}$$

Here is an example: Suppose you measure an angle to be $(47.3 \pm 0.5)^\circ$ and you want to determine the error of $\sin(47.3 \pm 0.5)^\circ$. You find that $\sin(47.3) = 0.735$. Based upon your reported uncertainty, you know that your angle could be as large as 47.8° and as small as 46.8°, and therefore you should calculate $\sin(47.8) = 0.741$ and $\sin(46.8) = 0.729$. So your calculated value is 0.735 but it can be as low as 0.729 and as high as 0.741 and therefore, if you halve the difference between 0.729 and 0.741 you get a reasonable error estimate of 0.006. So you should report your value as 0.735 ± 0.006.

Best-Fit Line

In most research laboratories, plotting measurements is found to be the preferred method of reviewing the data and quantitatively measuring the relationship between the experimental variables. This is effective because we often have some idea of the expected relationship between the variables *a priori*. In these labs, this expected relationship is almost always arranged to be a straight line. But even if we know that the ideal points fit on a precise straight line, experimentally measured data points will not always lie on a single line – because the measurements always have intrinsic uncertainty. Therefore when the points are plotted, we should include error bars on both axes to indicate the uncertainties in the data. Because real measurements do not all lie on a single straight line, there are a variety of possible lines you might choose to fit the data.

How do we know which line represents the best fit? There is an exact mathematical procedure to obtain the best-fit line, but this is usually a very tedious calculation which is outside the scope of this lab. For experiments in this course, you will be using Excel's

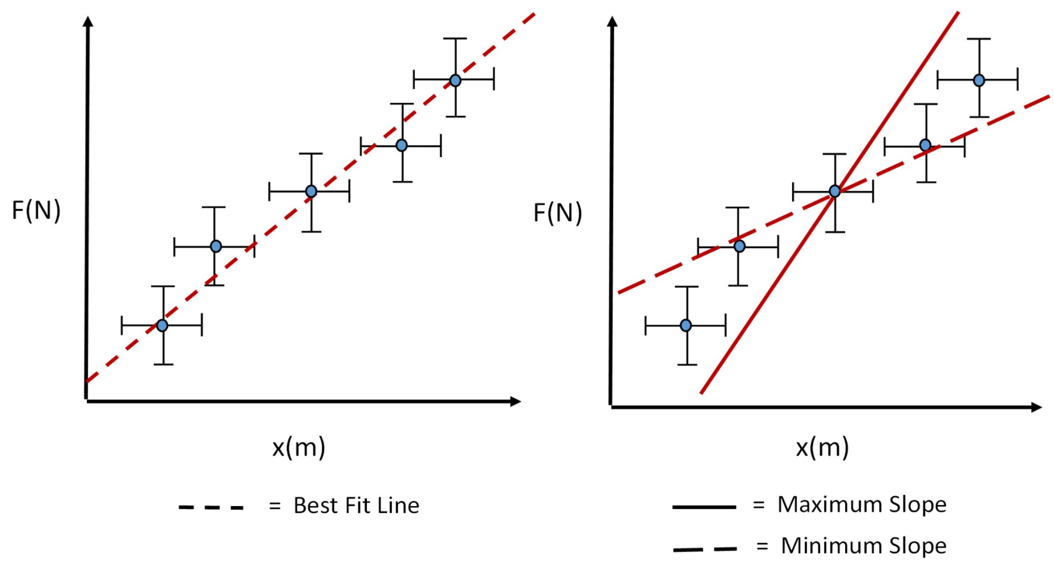

Figure 3: Left: an example best-fit line. Right: the maximum and minimum possible slope from our data used to calculate uncertainty in the best-fit line. Notice how we have drawn the lines on the outer bounds of the error bars to achieve the maximum and minimum possible slope within the error bars.

built-in fitting function for data. The process for doing which will be explained in your next lab. However, if you are interested in learning how to approximate the technique without a computer, please see the appendix.

Remark: Often in our experiments the data points will not look as nice as in the above examples. One or several points may not be close to any best-fit line you try. Such anomalous points may occur, for example, because of a mistake in measuring. In such cases, it is acceptable to ignore these anomalies when estimating the best-fit line (and of course you must note this fact down in your lab report). Dropping anomalous points must be done with extreme care and only rarely (if you know the point is not physically meaningful).[7] It is better to choose a line with as many points above the line as below. If you are not sure of your measurements, it is better to re-measure or to take more data points.

[7]More than once, data points that did not behave as theory predicted turned out to be new effects and led to Nobel prizes!

Numerical Statistics

The previous discussions of uncertainty and error tell us how we can quantitatively describe our inability to make perfect single measurements. However, in real physics experiments, very rarely do we draw conclusions from a single data points. As such, it is essential that we know how to quantify error in sets of data. The 2/3 methods as discussed in Section 3.2 provides a good estimation of data statistics, but we can more rigorously calculate data set statistics. In statistics, a data set can be well described by the following four fundamental quantities: mean, median, mode, and standard deviation. The mean of a data set is the sum of all numbers in the data set divided by the number of points in the set. It is defined in the following manner.

$$\text{Average} \equiv \bar{x} = \sum_i \frac{x_i}{N} \tag{12}$$

The median of a data set is the middle value in a set of numbers listed in increasing order. The mode is the number that occurs the most number of times in the data set. The standard deviation describes how the numbers in the data set are distributed around the mean. It is defined as follows.

$$\text{standard deviation} \equiv \sigma = \sqrt{\sum_i \frac{(x_i - \bar{x})^2}{N}} \tag{13}$$

These four statistical quantities give us enough information to characterize the distribution of our data set. For example, let's consider the two following Data Sets.

										Mean	Median	Mode	σ
Set 1	9	8	11	13	10	10	12	6	9	9.8	10	10	2.2
Set 2	11	0	10	40	2	3	10	10	4	9.8	10	10	11.4

Notice how both Data Sets have the same mean, median, and mode, which tells us the data points in each set are centered on the mean value of 9.8. However, the standard deviations are quite different. The large standard deviation in Data Set 2 tells us there must be outliers in the data set which increase the distribution. Whereas, the relatively small standard deviation in the first data set tells us the numbers in set 1 are clustered closely together. Standard deviation is especially important because it tells us exactly how distributed the number are around the mean value, which gives an indication of how error affects the spread of data points (for example, see figure 4 for the canonical "bell curve" distribution, also known as the gaussian distribution). The 2/3 method as discussed in Section 3.2 is an approximation to the standard deviation since 2/3 $\sim$ 66%, which roughly corresponds to the first standard deviation (see figure 4).

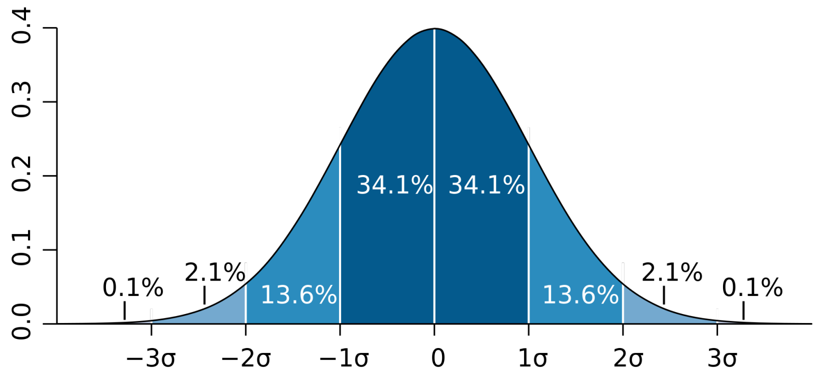

Figure 4: An example of a Gaussian distribution, also known as a bell curve. $\sim 68\%$ of the data points are within 1 standard deviation, $\sim 95\%$ of that data points are contained within 2 standard deviations, $\sim 99.5\%$ of the data points are contained within 3 standard deviations, etc...

Number of Significant Digits

The number of significant digits in a result refers to the number of digits that are relevant. The digits may occur after a string of zeroes. For example, the measurement of 2.3 mm has two significant digits. This does not change if you express the result in meters as 0.0023 m. The number 100.10, by contrast, has 5 significant digits[8].

When you record a result, you should use the calculated error to determine how many significant digits to keep. Let's illustrate the procedure with the following example. Assume you measure the diameter of a circle to be $d = 1.6232\,\text{cm}$, with an uncertainty of 0.102 cm. You now round your uncertainty to one or two significant digits (up to you). So (using one significant digit) we initially quote $d = (1.6232 \pm 0.1)\,\text{cm}$. Now we compare the mean value with the uncertainty, and keep only those digits that the uncertainty indicates are relevant. Finally, we quote the result as $d = (1.6 \pm 0.1)\,\text{cm}$ for our measurement.

Suppose further that we wish to use this measurement to calculate the circumference c of the circle with the relation $c = \pi \cdot d$. If we use a standard calculator, we might get a 10 digit display indicating:

$$c = 5.099433195 \pm 0.3204424507\,\text{cm} \tag{14}$$

This is not a reasonable way to write the result! The uncertainty in the diameter had only one significant digit, so the uncertainty of the circumference calculated from the

[8] Another way to find the number of significant digits is to convert to scientific notation, and count the number of digits in the mantissa (also significand or coefficient). For example: for 1.2×10^2, there are two significant digits in 1.2.

diameter cannot be substantially better. Therefore we should record the final result as:

$$c = 5.1 \pm 0.3\,\mathrm{cm} \tag{15}$$

(If you do intermediate calculations, it is a good idea to keep as many figures as your calculator can store. The above argument applies when you <u>record</u> your results!)

Appendix B
Error Analysis With Excel

Plotting with Excel

An important set of data analysis tools in Excel are plotting and linear fit functions. You will need to plot and fit data many times throughout this lab course, so make sure you are familiar with this section. Below is a walkthrough of plotting and fitting a set of data with error in excel.

1. Before plotting, you need to have 4 columns with data: x data, y data, x error data, and y error data. Make sure you have entered the information into excel.

2. First select your x data and y data (you can select multiple boxes in excel by holding down the ctrl button while selecting). Make sure to select your x data first or your x and y axes will be switched.

3. Choose the subheading "insert", then "Scatter", then "Scatter with straight lines and markers". Now your x and y data should be plotted without error bars (see figure 1).

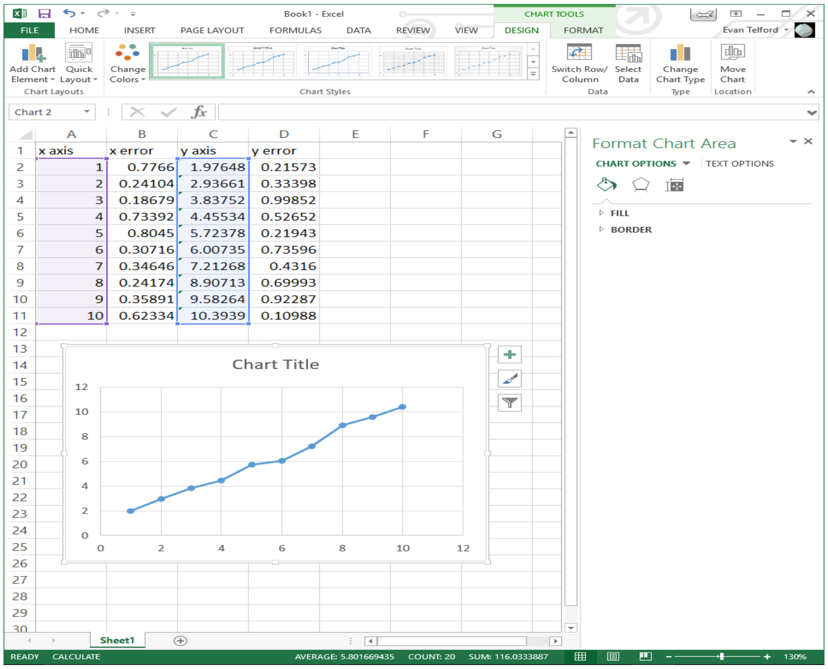

Figure 1: Selecting x and y data and creating a lined scatter plot in excel.

4. To include error bars select your chart, then click the "plus" marker on the top right of the chart. Check the box titled "error bars". Now some basic error bars should appear on the plot. These are not based on the error bar data in your excel document, they are standard error bars.

5. To change them so they match your error bar data, select the x axis error bars on your chart, and format the error bars by clicking the "Custom" selection, then "specify value".

6. It should now prompt you for positive error values and negative error values. Delete "{1}" from the two boxes, and select your error bars using the cursor. Your chart will now have the correct error bars (see figure 2).

7. Repeat steps 5-6 for your y data.

8. To linear fit your data, right click on your data in the plot and select "Add Trendline". Check the "Linear", "Display Equation on chart", and "Display R-square value on chart" boxes.

9. Now the slope, intercept, and R squared values will be displayed on your chart. R squared is a measure of how well the line fits your data. It should be close to 1 and at the very least greater than 0.9.

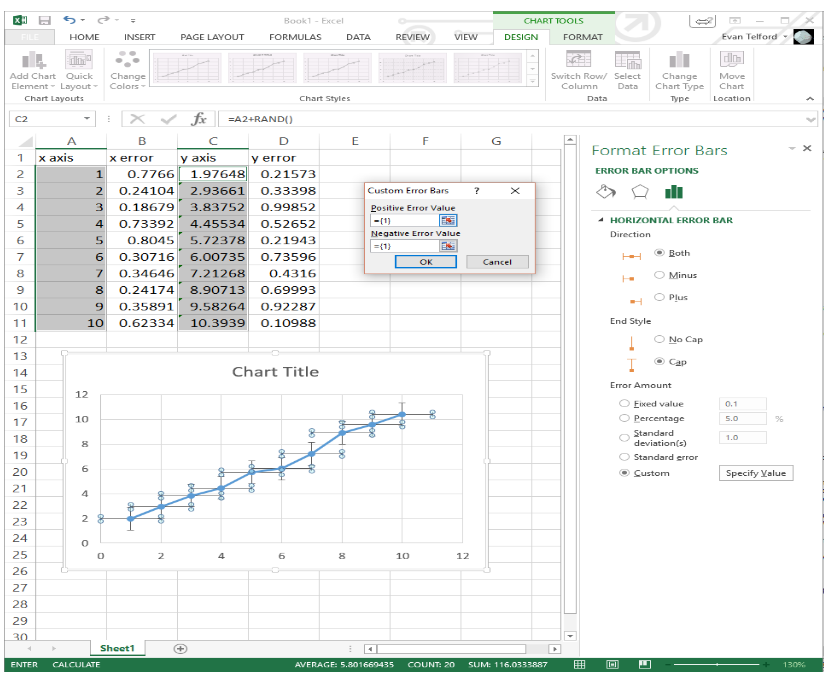

Figure 2: Using your own data set to create x and y error bars in excel.

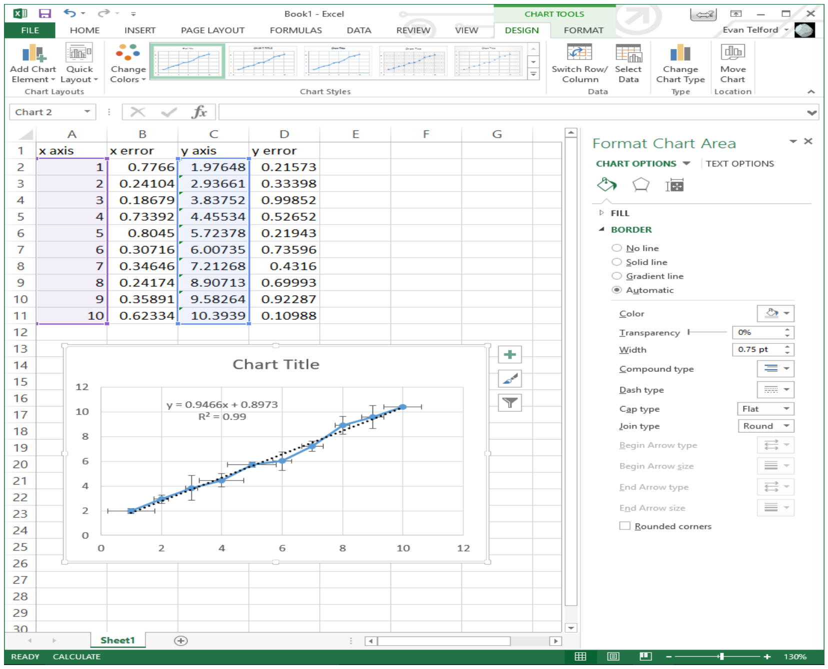

Figure 3: Using excel to perform a linear fit and return the intercept and slope.

Finding Error in Slope

The previous steps will help plot your data, but in order to draw conclusions, you must add uncertainty. Excel has a built in function called LINEST which finds the standard error for a linear fit. Using the same columns of data from before, the walkthrough below will help you calculate the error so you can propagate it further in the experiment.

1. The LINEST function is an array-type function, meaning it will output more than one number. Start, by highlighting a 2-by-3 section of empty cells (two columns, three rows).

2. With these six cells highlighted, in the input box at the top of the screen type "= LINEST(" and add the proper arguments. The arguments should be the list of y-values, the list of x-values, TRUE, and TRUE. For example: = `LINEST(C2:C11, A2:A11, TRUE, TRUE)`.

3. Press CONTROL+SHIFT+ENTER. *Note that on a Mac, this is CMD+SHIFT+ENTER.*

Your results should have filled in that 2-by-3 section in the following way:

Slope	X Intercept
Error of Slope	Error of Intercept
R^2	Error in Y

Table B.1: LINEST function output in Excel.

Helpful Commands

Your TA will guide you through the relevant excel commands necessary for data analysis, however a list of some relevant excel commands are listed below. A list of all excel commands can be found on the Microsoft Office website[1]

`ABS`	Returns the absolute value of a number
`AVERAGE`	Computes the average of the selected data set
`COS`	Calculates cosine of a number
`DEGREES`	Converts radians to degrees
`EXP`	Returns e raised to the power of a given number
`LN`	Returns the natural logarithm of a number
`MEDIAN`	Finds the median of a data set
`MODE.SNGL`	Finds the most commonly occurring number in a data set
`PI`	Returns the value of pi
`POWER`	Returns the result of a number raised to a power
`SIN`	Calculates the sine of a number
`SQRT`	Calculates the square root of a number
`STDEV.P`	Calculates the standard deviation based on the entire population
`STDEV.S`	Estimates the standard deviation based on a sample
`SUM`	Calculates the sum of a data set
`TAN`	Calculates the tangent of a number

[1] https://support.office.com/en-us/article/Excel-functions-alphabetical-b3944572-255d-4efb-bb96-c6d90033e188

Appendix C
Advanced Error Analysis

Clarification of 2/3 Rule

To find the true uncertainty, we are really interested in the *standard error of the mean*, i.e., how likely it would be for a newly measured average value to be close to our original value were we to perform the experiment again. The proper way to figure this out would be to get say a thousand friends to perform this experiment in the same way, each using the same number of data points, and then compare the results of everyone. Each student would calculate his or her own mean, and they would likely all be clustered around some central average. We could then examine the spread of this cluster of means using the 2/3 rule, and we'd have a quantitative measure of the uncertainty surrounding any single student's measurement.

While it's usually impractical to get 1000 friends together to repeat an experiment a thousand times, it turns out that the uncertainty (or "standard error") of the mean can be estimated with the following formula:

$$\text{Standard Error Of The Mean} = \frac{\text{Uncertainty Of Single Measurement}}{\sqrt{N}} \tag{1}$$

where "N" is the number of data points in your sample, and "Uncertainty Of Single Measurement" is the uncertainty calculated via the 2/3 method. The "$\sqrt{N}$" term should make sense qualitatively – as we take more and more data points, our measured average becomes less and less uncertain as we approach what should be the "global" mean.

The "Correct" Way to Add Uncertainties

The rules we've given for propagating uncertainties through a calculation are essentially correct, and intuitively make sense. When adding two quantities together, if one has an uncertainty of Δx and another has an uncertainty of Δy, the sum could indeed range from $(x + y) - (\Delta x + \Delta y)$ to $(x + y) + (\Delta x + \Delta y)$. This implies that the proper way to find the uncertainty of $(x + y)$ is to add their respective absolute uncertainties.

There is, however, a small problem – this overestimates the uncertainty! Since x and y are equally likely to be wrong by either a *positive* amount or a *negative* amount, there's a good chance that the respective errors of each variable will partly cancel one another out. To account for this, a more accurate way to estimate uncertainty turns out to be to add uncertainties *in quadrature*. This means:

Adding/Subtracting Quantities

$$(A \pm \Delta A) + (B \pm \Delta B) = (A + B) \pm \sqrt{(\Delta A)^2 + (\Delta B)^2} \tag{2}$$

Multiplying/Dividing Quantities

$$A\left(1 \pm \frac{\Delta A}{A}\right) \times B\left(1 \pm \frac{\Delta B}{B}\right) = (A \times B)\left(1 \pm \sqrt{\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\Delta B}{B}\right)^2}\right) \tag{3}$$

While this method gives a closer approximation to what the true propagated uncertainty should be, it is clearly a more complex calculation. In the limited time available to complete your experiment and lab report, you may use the simpler, earlier uncertainty calculation method provided, and avoid this complicated calculation. But do remember that the simpler method *overestimates* the total uncertainty.

Max-Min Method for Best-fit Line

This alternate technique will show you how to draw an approximate best-fit line for a set of data without a computer and it is sufficiently precise for most purposes.

First, try to draw a line with as many points (with uncertainties included) lying above the line as below it. The gauge of how close the line is to a point is given by the uncertainty associated with that measured point. However, all the points at the left end should not lie on one side of the line with all the points at the right end lying on the other side. As a rule of thumb, roughly 2/3 of the points should have the line passing through the uncertainties (just as with the 2/3 rule).

Clearly, this "eyeball" method has inherent uncertainty, so how do we estimate the uncertainty on the slope of the best-fit line? To do this we should estimate the spread of the slope, or maximum and minimum possible slopes that one can conceivably interpret from the graph. Half the difference between the minimum and maximum slopes is a good estimate of the slope uncertainty ($\sigma = \frac{m_{max} - m_{min}}{2}$).