Lecture 3:
A brief re-hash of the Finite Element Method

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Implementing the finite element method

Brief re-hash of the FEM, using the Poisson equation:

We start with the strong form:

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

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Brief re-hash of the FEM, using the Poisson equation:

We start with the strong form:

$$-\Delta u = f$$

...and transform this into the weak form by multiplying *from the left* with a test function:

$$(\nabla \varphi, \nabla u) = (\varphi, f) \quad \forall \varphi$$

The solution of this is a function *u(x)* from an infinite-dimensional function space.

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Since computers can't handle objects with infinitely many coefficients, we seek a finite dimensional function of the form

$$u_h = \sum_{j=1}^{N} U_j \varphi_j(x)$$

To determine the *N* coefficients, test with the *N* basis functions:

$$(\nabla \varphi_i, \nabla u_h) = (\varphi_i, f) \quad \forall i = 1 \ldots N$$

If basis functions are linearly independent, this yields *N* equations for *N* coefficients.
This is called the *Galerkin* method.

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Practical question 1: How to define the basis functions?

Answer: In the finite element method, this is done using the following concepts:

- Subdivision of the domain into a mesh
- Each cell of the mesh is a mapping of the reference cell
- Definition of basis functions on the reference cell
- Each shape function corresponds to a degree of freedom on the global mesh

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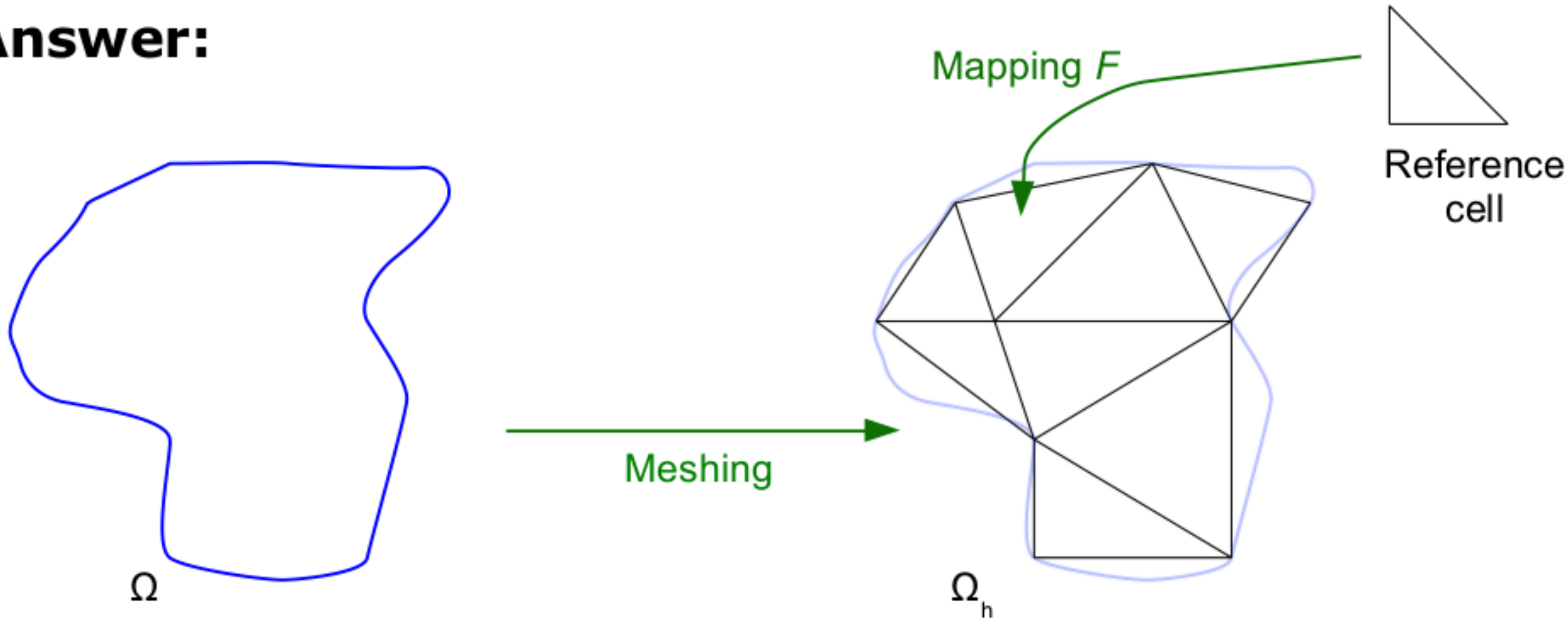
Practical question 1: How to define the basis functions?

Answer:



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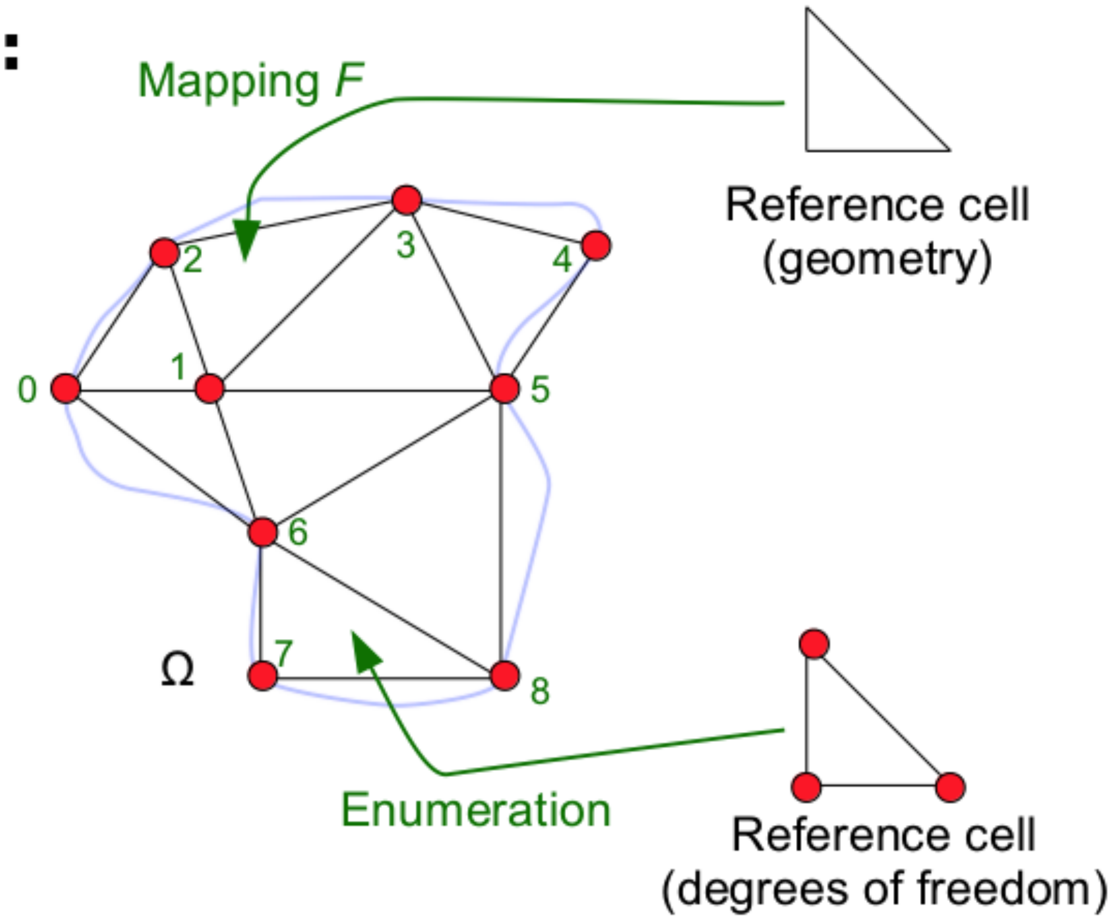

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Concepts in red will correspond to things we need to implement in software, explicitly or implicitly.

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Given the definition $u_h = \sum_{j=1}^{N} U_j \varphi_j(x)$, we can expand the bilinear form

$$(\nabla \varphi_i, \nabla u_h) = (\varphi_i, f) \quad \forall i = 1 \ldots N$$

to obtain:

$$\sum_{j=1}^{N} (\nabla \varphi_i, \nabla \varphi_j) U_j = (\varphi_i, f) \quad \forall i = 1 \ldots N$$

This is a linear system

$$AU = F$$

with

$$A_{ij} = (\nabla \varphi_i, \nabla \varphi_j) \qquad F_i = (\varphi_i, f)$$

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Practical question 2: How to compute

$$A_{ij}=(\nabla\varphi_i,\nabla\varphi_j) \qquad F_i=(\varphi_i,f)$$

Answer: By mapping back to the reference cell...

$$\begin{aligned}A_{ij} &= (\nabla\varphi_i,\nabla\varphi_j)\\ &= \sum_K \int_K \nabla\varphi_i(x)\cdot\nabla\varphi_j(x)\\ &= \sum_K \int_{\hat K} J_K^{-1}(\hat x)\hat\nabla\hat\varphi_i(\hat x)\;\cdot\; J_K^{-1}(\hat x)\hat\nabla\hat\varphi_j(\hat x)\;|\det J_K(\hat x)|\end{aligned}$$

...and quadrature:

$$A_{ij} \approx \sum_K \sum_{q=1}^{Q} J_K^{-1}(\hat x_q)\hat\nabla\hat\varphi_i(\hat x_q)\;\cdot\; J_K^{-1}(\hat x_q)\hat\nabla\hat\varphi_j(\hat x_q)\;\underbrace{|\det J_K(\hat x_q)|\;w_q}_{=:\mathrm{JxW}}$$

Similarly for the right hand side *F*.

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Practical question 3: How to store the matrix and vectors of the linear system

$$AU = F$$

Answers:

- *A* is sparse, so store it in compressed row format
- *U*,*F* are just vectors, store them as arrays
- Implement efficient algorithms on them, e.g. matrix-vector products, preconditioners, etc.
- For large-scale computations, data structures and algorithms must be parallel

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Practical question 4: How to solve the linear system

$$AU = F$$

Answers: In practical computations, we need a variety of

- Direct solvers
- Iterative solvers
- Parallel solvers

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Practical question 5: What to do with the solution of the linear system

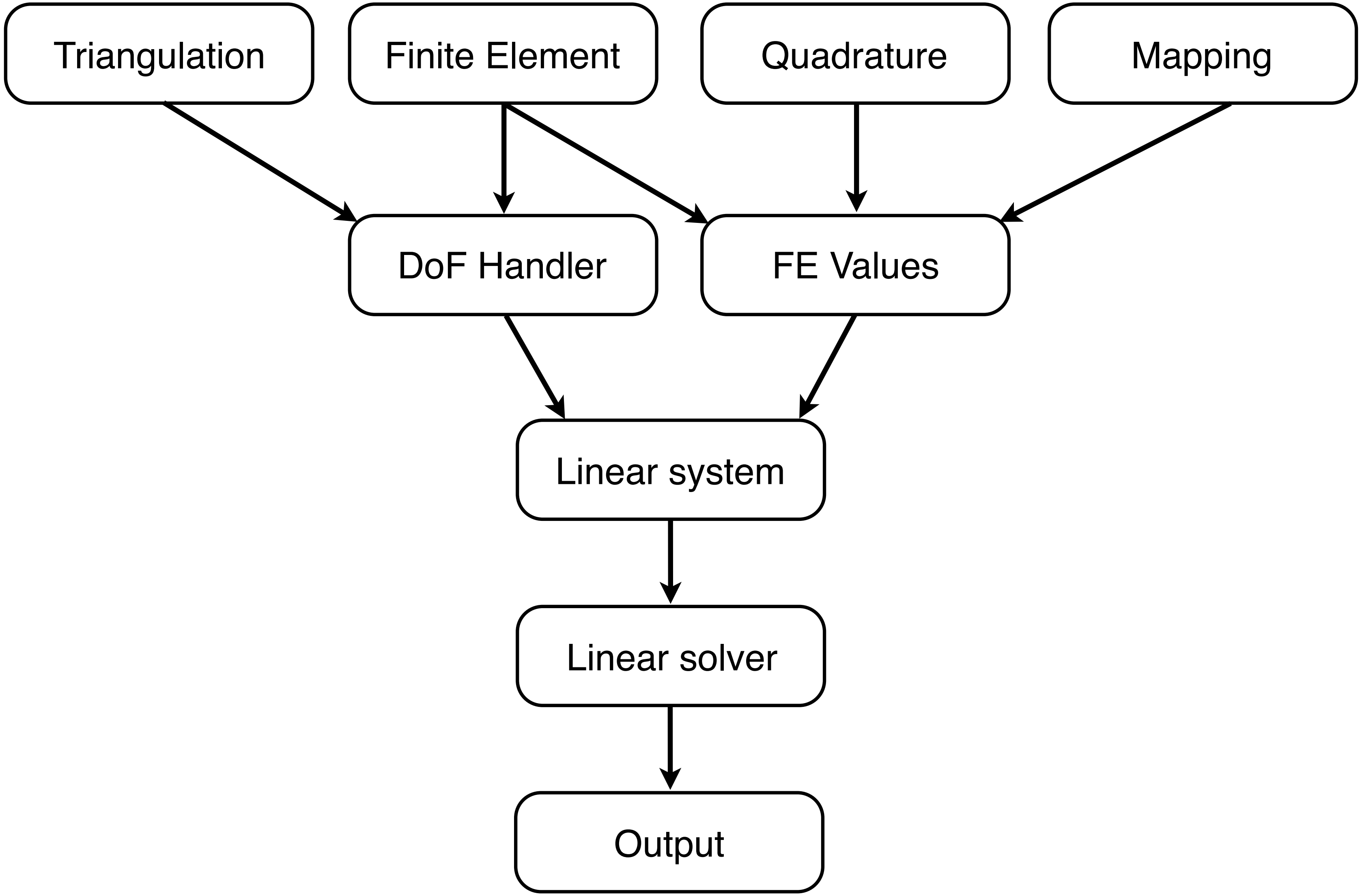
$$AU = F$$

Answers: The goal is not to solve the linear system, but to do something with its solution:

- Visualize
- Evaluate for quantities of interest
- Estimate the error

These steps are often called *postprocessing the solution*.

Structure of a prototypical FE problem



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Summary:

- By going through the mathematical description of the FEM, we have identified *concepts* that need to be represented by *software components*.

- Other components relate to what we *want to do* with numerical solutions of PDEs.

- The next few lectures will show the software realization of these concepts.