Homework 20241204

Due Date: 20241216, 5 P.M.

- 考虑地震救灾场景，n个伤员需要被尽快送往医院. 在这个地区有k所医院，这n个人中每个人需要被送到距他们目前的地点半小时车程以内的医院（因此不同的人将对医院有不同的选择，依赖于他们当前所在的地方）.同时，人们不想由于太多的病人送来而使得任何一个医院超负荷. 医护人员通过移动电话联系，他们想共同解决是否可以为每个受伤的人选择一所医院，这种选择方式要求医院负荷是均衡的，即每个医院至多接受$\lceil n/k \rceil$的人. 给出一个多项式时间的算法，它以关于这些人所在位置的给定信息作为输入并且确定这是不是可能的.

考虑郊野环境中的移动通信场景，给定n个基站的位置，它们由平面上的点$b_1,b_2,\ldots,b_n$来指定，以及n个手机用户位置，它们也指定为平面上的点$p_1,p_2,\ldots,p_n$，最后，给定一个域参数$\Delta>0$. 如果能以下述这样的方式把每个电话分配给一个基站，我们就说这组便携式电话是**完全连通**的.

- 每个手机被分到不同的基站，且如果位于p_i的电话被分配到位于b_j的基站，那么在点p_i和b_j之间的直线距离至多是Δ.

假设在点p_i的用户决定开车向东经过z距离，需要修改手机对基站的分配（可能要几次）以便保持完全连通，给出一个多项式时间的算法.（假定在此期间其他手机固定.）如果可能，报告一个电话到基站的分配序列；如果不可能，报告使得完全连通性不可能再维持的一个点. 算法运行在$O(n^3)$时间.

4. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

> *Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e. If f is a maximum s-t flow in G, then f saturates every edge out of s with flow (i.e., for all edges e out of s, we have $f(e) = c_e$).*

5. Decide whether you think the following statement is true or false. If it is true, give a short explanation. If it is false, give a counterexample.

> *Let G be an arbitrary flow network, with a source s, a sink t, and a positive integer capacity c_e on every edge e; and let (A, B) be a mimimum s-t cut with respect to these capacities $\{c_e : e \in E\}$. Now suppose we add 1 to every capacity; then (A, B) is still a minimum s-t cut with respect to these new capacities $\{1 + c_e : e \in E\}$.*

24. Let $G = (V, E)$ be a directed graph, with source $s \in V$, sink $t \in V$, and nonnegative edge capacities $\{c_e\}$. Give a polynomial-time algorithm to decide whether G has a *unique* minimum s-t cut (i.e., an s-t of capacity strictly less than that of all other s-t cuts).

32. Given a graph $G = (V, E)$, and a natural number k, we can define a relation $\xrightarrow{G,k}$ on pairs of vertices of G as follows. If $x, y \in V$, we say that $x \xrightarrow{G,k} y$ if there exist k mutually edge-disjoint paths from x to y in G.

Is it true that for every G and every $k \geq 0$, the relation $\xrightarrow{G,k}$ is transitive? That is, is it always the case that if $x \xrightarrow{G,k} y$ and $y \xrightarrow{G,k} z$, then we have $x \xrightarrow{G,k} z$? Give a proof or a counterexample.

33. Let $G = (V, E)$ be a directed graph, and suppose that for each node v, the number of edges into v is equal to the number of edges out of v. That is, for all v,

$$|\{(u, v) : (u, v) \in E\}| = |\{(v, w) : (v, w) \in E\}|.$$

Let x, y be two nodes of G, and suppose that there exist k mutually edge-disjoint paths from x to y. Under these conditions, does it follow that there exist k mutually edge-disjoint paths from y to x? Give a proof or a counterexample with explanation.