6.3 THE GREATEST INTEGER FUNCTION

The greatest integer or "bracket" function [ ] is especially suitable for treating divisibility problems. While not strictly a number-theoretic function, its study has a natural place in this chapter.

DEFINITION 6-4. For an arbitrary real number x, we denote by $[x]$ the largest integer less than or equal to x; that is, $[x]$ is the unique integer satisfying $x-1<[x]\leq x$.

By way of illustration, [ ] assumes the particular values

$$[-3/2]=-2,\ [\sqrt{2}]=1,\ [1/3]=0,\ [\pi]=3,\ [-\pi]=-4.$$

The important observation to be made here is that the equality $[x]=x$ holds if and only if x is an integer. Definition 6-4 also makes plain that any real number x can be written as

$$x=[x]+\theta$$

for a suitable choice of θ, with $0\leq\theta<1$.

We now plan to investigate the question of how many times a particular prime p appears in $n!$. For instance, if $p=3$ and $n=9$, then

$$\begin{aligned}9! &= 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\\ &= 2^7\cdot 3^4\cdot 5\cdot 7,\end{aligned}$$

so that the exact power of 3 which divides $9!$ is 4. It is desirable to have a formula that will give this count, without the necessity of always writing $n!$ in canonical form. This is accomplished by

THEOREM 6-9. *If n is a positive integer and p a prime, then the exponent of the highest power of p that divides $n!$ is*

$$\sum_{k=1}^{\infty}[n/p^k].$$

(This is not an infinite series, since $[n/p^k]=0$ for $p^k>n$.)

Proof: Among the first n positive integers, those which are divisible by p are $p, 2p, \ldots, tp$, where t is the largest integer such that $tp\leq n$; in other words, t is the largest integer less than or equal to n/p

(which is to say $t = [n/p]$). Thus, there are exactly $[n/p]$ multiples of p occurring in the product that defines $n!$, namely,

$$(1) \qquad p, 2p, \ldots, [n/p]p.$$

The exponent of p in the prime factorization of $n!$ is obtained by adding to the number of integers in (1), the number of integers among $1, 2, \ldots, n$ which are divisible by p^2, and then the number divisible by p^3, and so on. Reasoning as in the first paragraph, the integers between 1 and n that are divisible by p^2 are

$$(2) \qquad p^2, 2p^2, \ldots, [n/p^2]p^2,$$

which are $[n/p^2]$ in number. Of these, $[n/p^3]$ are again divisible by p:

$$(3) \qquad p^3, 2p^3, \ldots, [n/p^3]p^3.$$

After a finite number of repetitions of this process, we are led to conclude that the total number of times p divides $n!$ is $\sum_{k=1}^{\infty} [n/p^k]$.

This result can be cast as the following equation, which usually appears under the name of Legendre's formula:

$$n! = \prod_{p \leq n} p^{\sum_{k=1}^{\infty} [n/p^k]}.$$

Example 6-2

We would like to find the number of zeroes with which the decimal representation of 50! terminates. In determining the number of times 10 enters into the product 50!, it is enough to find the exponents of 2 and 5 in the prime factorization of 50!, and then to select the smaller figure.

By direct calculation we see that

$$\begin{aligned}[50/2] + [50/2^2] + [50/2^3] + [50/2^4] + [50/2^5] \\ = 25 + 12 + 6 + 3 + 1 = 47.\end{aligned}$$

Theorem 6-9 tells us that 2^{47} divides 50!, but 2^{48} does not. Similarly,

$$[50/5] + [50/5^2] = 10 + 2 = 12$$

and so the highest power of 5 dividing 50! is 12. This means that 50! ends with 12 zeroes.

We cannot resist using Theorem 6-9 to prove the following fact.

THEOREM 6-10. *If n and r are positive integers with $1 \leq r < n$, then the binomial coefficient*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is also an integer.

Proof: The argument rests on the observation that if a and b are arbitrary real numbers, then $[a+b] \geq [a] + [b]$. In particular, for each prime factor of p of $r!\,(n-r)!$,

$$[n/p^k] \geq [r/p^k] + [(n-r)/p^k], \qquad k = 1, 2, \ldots.$$

Adding these inequalities together, we obtain

$$(1) \qquad \sum_{k \geq 1} [n/p^k] \geq \sum_{k \geq 1} [r/p^k] + \sum_{k \geq 1} [(n-r)/p^k].$$

The left-hand side of (1) gives the exponent of the highest power of the prime p that divides $n!$, whereas the right-hand side equals the highest power of this prime contained in $r!(n-r)!$. Hence, p appears in the numerator of $n!/r!(n-r)!$ at least as many times as it occurs in the denominator. Since this holds true for every prime divisor of the denominator, $r!(n-r)!$ must divide $n!$, making $n!/r!(n-r)!$ an integer.

COROLLARY. *For a positive integer r, the product of any r consecutive positive integers is divisible by $r!$.*

Proof: The product of r consecutive positive integers, the largest of which is n, is

$$n(n-1)(n-2)\cdots(n-r+1).$$

Now we have

$$n(n-1)\cdots(n-r+1) = \left(\frac{n!}{r!(n-r)!}\right) r!.$$

Since $n!/r!(n-r)!$ is an integer, it follows that $r!$ must divide the product $n(n-1)\cdots(n-r+1)$, as asserted.

We pick up a few loose threads. Having introduced the greatest integer function, let us see what it has to do with the study of number-theoretic functions. Their relationship is brought out by

THEOREM 6-11. *Let f and F be number-theoretic functions such that*

$$F(n) = \sum_{d \mid n} f(d).$$

Then, for any positive integer N,

$$\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k)[N/k].$$

Proof: We begin by noting that

$$(1) \qquad \sum_{n=1}^{N} F(n) = \sum_{n=1}^{N} \sum_{d \mid n} f(d).$$

The strategy is to collect terms with equal values of $f(d)$ in this double sum. For a fixed positive integer $k \leq N$, the term $f(k)$ appears in $\sum_{d \mid n} f(d)$ if and only if k is a divisor of n. (Since each integer has itself as a divisor, the right-hand side of (1) includes $f(k)$, at least once.) Now, in order to calculate the number of sums $\sum_{d \mid n} f(d)$ in which $f(k)$ occurs as a term, it is sufficient to find the number of integers among $1, 2, \ldots, N$ which are divisible by k. There are exactly $[N/k]$ of them:

$$k, 2k, 3k, \ldots, [N/k]k.$$

Thus, for each k such that $1 \leq k \leq N$, $f(k)$ is a term of the sum $\sum_{d \mid n} f(d)$ for $[N/k]$ different positive integers less than or equal to N. Knowing this, we may rewrite the double sum in (1) as

$$\sum_{n=1}^{N} \sum_{d \mid n} f(d) = \sum_{k=1}^{N} f(k)[N/k]$$

and our task is complete.

As an immediate application of Theorem 6-11, we deduce

COROLLARY 1. *If N is a positive integer, then*

$$\sum_{n=1}^{N} \tau(n) = \sum_{n=1}^{N} [N/n].$$