TUTORIAL 7

1 Just a warm-up

Let $P \in \mathbb{F}[X, Y]$.

1. Case $\mathbb{F} = \mathbb{R}$. Show that if P vanishes on $\mathbb{R}^2$, then $P = 0$.
2. Case $\mathbb{F}$ finite. Show that P can vanish on $\mathbb{F}^2$ with $P \neq 0$.
3. Show that the Schwartz-Zippel lemma is sharp by providing some examples achieving the $d|S|^{n-1}$ bound.

2 Alon's combinatorial nullstellensatz

Let $P \in \mathbb{F}[X_1, \dots, X_n]$, $P \neq 0$, of degree d, and $S_1, \dots, S_n$ be finite subsets of $\mathbb{F}$.

1. (Alon-Tarsi) Assume that $|S_i| > deg_i(P)$ for all $i = 1, \dots, n$, where $deg_i(P)$ is the maximum degree of x_i in P. Show that P cannot vanish on $S_1 \times S_2 \times \ldots \times S_n$.
2. Assume P vanishes on $S_1 \times \cdots \times S_n$ and let $g_i = \prod_{s \in S_i}(X_i - s)$. Show that there exist polynomials $h_1, \dots, h_n \in \mathbb{F}[X_1, \dots, X_n]$ such that $deg(h_i) \leq d - |S_i|$ and
$$P = \sum_{i=1}^{n} h_i g_i \ .$$
3. Assume that P contains a term $c \cdot X_1^{\alpha_1} \cdot X_2^{\alpha_2} \cdots X_n^{\alpha_n}$ with $c \neq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = d$. Show that if $|S_i| > \alpha_i$ for all $i = 1, \dots, n$, then P does not vanish on $S_1 \times S_2 \times \cdots \times S_n$.

3 Alon-Füredi

Consider $H_n = \{0, 1\}^n$, the set of vertices of the n-cube of $\mathbb{R}^n$. Note that two hyperplanes can cover H_n, namely $x_1 = 0$ and $x_1 = 1$. A surprising result asserts that the minimum number of hyperplanes needed to cover $H_n \setminus \{0^n\}$, but *not* 0, actually jumps to n.

1. Convince yourself of this fact in low dimension.
2. Assume for contradiction that there exist $m < n$ hyperplanes $a^{(i)} \cdot x = b^{(i)}$ for $i = 1, \dots, m$ covering $H_n \setminus \{0^n\}$ but not 0^n (i.e. all $b^{(i)}$ are non zero). Form a polynomial P with degree n which vanishes on H_n and such that $X_1 \cdot X_2 \cdots X_n$ appears as a term with non-zero coefficient.
3. Conclude.