THEOREM 13-18. *If x_1, y_1 is the fundamental solution of $x^2-dy^2=1$, then every positive solution of the equation is given by x_n, y_n, where x_n and y_n are the integers determined from*

$$x_n+y_n\sqrt{d}=(x_1+y_1\sqrt{d})^n \qquad (n=1, 2, 3, \ldots)$$

Proof: In anticipation of a contradiction, let us suppose that there exists a positive solution u, v which is not obtainable by the formula $(x_1+y_1\sqrt{d})^n$. Since $x_1+y_1\sqrt{d}>1$, the powers of $x_1+y_1\sqrt{d}$ become arbitrarily large; this means that $u+v\sqrt{d}$ must lie between two consecutive powers of $x_1+y_1\sqrt{d}$, say,

$$(x_1+y_1\sqrt{d})^n<u+v\sqrt{d}<(x_1+y_1\sqrt{d})^{n+1}$$

or, to phrase it in different terms,

$$x_n+y_n\sqrt{d}<u+v\sqrt{d}<(x_n+y_n\sqrt{d})(x_1+y_1\sqrt{d}).$$

On multiplying this inequality by the positive number $x_n-y_n\sqrt{d}$ and noting that $x_n^2-dy_n^2=1$, we are led to

$$1<(x_n-y_n\sqrt{d})(u+v\sqrt{d})<x_1+y_1\sqrt{d}.$$

Next define the integers r and s by $r+s\sqrt{d}=(x_n-y_n\sqrt{d})(u+v\sqrt{d})$; that is, let

$$r=x_nu-y_nvd, \qquad s=x_nv-y_nu.$$

An easy calculation reveals that

$$r^2-ds^2=(x_n^2-dy_n^2)(u^2-dv^2)=1$$

and so r, s is a solution of $x^2-dy^2=1$ satisfying $1<r+s\sqrt{d}<x_1+y_1\sqrt{d}$.

To complete the proof, it remains to show that r, s is a positive solution. Because $1<r+s\sqrt{d}$, we find that $0<r-s\sqrt{d}<1$. In consequence,

$$2r=(r+s\sqrt{d})+(r-s\sqrt{d})>1+0>0$$
$$s\sqrt{d}=(r+s\sqrt{d})-(r-s\sqrt{d})>1-1=0$$

which makes both r and s positive. The upshot is that since x_1, y_1 is the fundamental solution of $x^2-dy^2=1$, we must have $x_1<r$ and $y_1<s$; but then $x_1+y_1\sqrt{d}<r+s\sqrt{d}$, violating an earlier inequality. This contradiction ends our argument.

Pell's equation has attracted mathematicians throughout the ages. There is historical evidence that methods for solving the equation were known to the Greeks some 400 years before the beginning of the Christian era. A famous problem of indeterminate analysis known as the "cattle problem" is contained in an epigram sent by Archimedes to Eratosthenes as a challenge to Alexandrian scholars. In it, one is required to find the number of bulls and cows of each of four colors, the eight unknown quantities being connected by nine conditions. These conditions ultimately involve the solution of the Pell equation

$$x^2 - 4729494y^2 = 1,$$

which leads to enormous numbers; one of the eight unknown quantities is a figure having 206545 digits (assuming that 15 printed digits take up one inch of space, the number would be over 1/5 of a mile long). While it is generally agreed that the problem originated with the celebrated mathematician of Syracuse, no one contends that Archimedes actually carried through all the necessary computations.

Such equations and dogmatic rules, without any proof, for calculating their solutions spread to India more than a thousand years before they appeared in Europe. In the 7th century, Brahmagupta said that a person who can within a year solve the equation $x^2 - 92y^2 = 1$ is a mathematician; for those days, he would at least have to be a good arithmetician, since $x = 1151$, $y = 120$ is the smallest positive solution. A computationally more difficult task would be to find integers satisfying $x^2 - 94y^2 = 1$, for here the fundamental solution is given by $x = 2143295$, $y = 221064$.

Fermat was not the first therefore to propose solving the equation $x^2 - dy^2 = 1$, or even to devise a general method of solution. He was perhaps the first to assert that the equation has an infinitude of solutions whatever the value of the nonsquare integer d. Moreover, his effort to elicit purely integral solutions to both this and other problems was a watershed in number theory, breaking away as it did from the classical tradition of Diophantus' *Arithmetica.*

PROBLEMS 13.5

1. If x_0, y_0 is a positive solution of the equation $x^2 - dy^2 = 1$, prove that $x_0 > y_0$.
2. By the technique of successively substituting $y = 1, 2, 3, \ldots$ into $dy^2 + 1$, determine the smallest positive solution of $x^2 - dy^2 = 1$ when d is
(a) 7; (b) 11; (c) 18; (d) 30; (e) 39.

3. Find all positive solutions of the following equations for which $y < 250$:
(a) $x^2 - 2y^2 = 1$; (b) $x^2 - 3y^2 = 1$; (c) $x^2 - 5y^2 = 1$.

4. Show that there is an infinitude of even integers n with the property that both $n + 1$ and $n/2 + 1$ are perfect squares. Exhibit two such integers.

5. Indicate two positive solutions of each of the equations below:
(a) $x^2 - 23y^2 = 1$; (b) $x^2 - 26y^2 = 1$; (c) $x^2 - 33y^2 = 1$.

6. Find the fundamental solutions of
(a) $x^2 - 29y^2 = 1$; (b) $x^2 - 41y^2 = 1$; (c) $x^2 - 74y^2 = 1$.
[*Hint:* $\sqrt{41} = [6; \overline{2, 2, 12}]$ and $\sqrt{74} = [8; \overline{1, 1, 1, 1, 16}]$.]

7. Exhibit a solution of each of the following equations:
(a) $x^2 - 13y^2 = -1$; (b) $x^2 - 29y^2 = -1$; (c) $x^2 - 41y^2 = -1$.

8. Establish that if x_0, y_0 is a solution of the equation $x^2 - dy^2 = -1$, then $x = 2x_0^2 + 1, y = 2x_0 y_0$ satisfies $x^2 - dy^2 = 1$. Brouncker used this fact in solving $x^2 - 313y^2 = 1$.

9. If d is divisible by a prime $p \equiv 3 \pmod 4$, show that the equation $x^2 - dy^2 = -1$ has no solution.

10. If x_1, y_1 is the fundamental solution of $x^2 - dy^2 = 1$ and

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n \qquad (n = 1, 2, 3, \ldots),$$

prove that the pair of integers x_n, y_n can be calculated from the formulas

$$x_n = \tfrac{1}{2}[(x_1 + y_1\sqrt{d})^n + (x_1 - y_1\sqrt{d})^n]$$
$$y_n = \frac{1}{2\sqrt{d}}[(x_1 + y_1\sqrt{d})^n - (x_1 - y_1\sqrt{d})^n].$$

11. Verify that the integers x_n, y_n in the previous problem can be defined inductively either by

$$x_{n+1} = x_1 x_n + d y_1 y_n$$
$$y_{n+1} = x_1 y_n + x_n y_1,$$

for $n = 1, 2, 3, \ldots$, or by

$$x_{n+1} = 2x_1 x_n - x_{n-1}$$
$$y_{n+1} = 2x_1 y_n - y_{n-1}$$

for $n = 2, 3, \ldots$.

12. Using the information that $x_1 = 15, y_1 = 2$ is the fundamental solution of $x^2 - 56y^2 = 1$, determine two more positive solutions.

13. (a) Prove that whenever the equation $x^2 - dy^2 = c$ is solvable, then it has infinitely many solutions. [*Hint:* If u, v satisfy $x^2 - dy^2 = c$ and r, s satisfy $x^2 - cy^2 = 1$, then $(ur \pm dvs)^2 - d(us \pm vr)^2 = (u^2 - dv^2)(r^2 - ds^2) = c$.]

(b) Given that $x=16, y=6$ is a solution of $x^2-7y^2=4$, obtain two other positive solutions.

(c) Given that $x=18, y=3$ is a solution of $x^2-35y^2=9$, obtain two other positive solutions.

14. Apply the theory of this section to confirm that there exist infinitely many primitive Pythagorean triples x, y, z in which x and y are consecutive integers.