TUTORIAL 9

1 Joint problems

Let L be a finite set of lines in $\mathbb{R}^3$. A *joint* of L is a point j of $\mathbb{R}^3$ which is contained in three lines of L whose directions are linearly independent (i.e. the set of lines of L containing j is not contained in a plane). Let J be the set of joints of L. The goal of this exercise is to show that $|J|$ is at most $O(|L|^{3/2})$.

1. Show that there are arbitrarily large sets L of lines in $\mathbb{R}^3$ with J achieving the order of magnitude of this upper bound.
2. Consider now a set of lines L and its set of joints J. Show that there exists a non-zero polynomial $P \in \mathbb{R}[x, y, z]$ with degree at most $(6|J|)^{1/3}$ which vanishes on J.
3. Choose P with minimum degree. Show that if a line contains strictly more than $(6|J|)^{1/3}$ points of J, then P vanishes on it.
4. Show that if P vanishes on all lines of L containing a point $j \in J$, then each of the three derivatives of P (with respect to x, y and z) vanishes on j.
5. Deduce that there exists a line of L which contains at most $(6|J|)^{1/3}$ elements of J.
6. Conclude.

2 Rabinowitsch trick

Let $p_1, \dots, p_t$ be polynomials in $\mathbb{C}[x_1, \dots, x_n]$. Assume that some polynomial p vanishes on all points x of $\mathbb{C}^n$ such that $p_1(x) = \cdots = p_t(x) = 0$.

1. We add a new variable y and consider our polynomials in $\mathbb{C}[x_1, \dots, x_n, y]$. Show that there exists $q_1, \dots, q_t, q$ in $\mathbb{C}[x_1, \dots, x_n, y]$ such that $q_1p_1 + q_2p_2 + \cdots + q_tp_t + q(1 - yp) = 1$.
2. Substitute y by a rational fraction in order to show that there exists some integer k such that $p^k = h_1p_1 + h_2p_2 + \cdots + h_tp_t$ for some $h_1, \dots, h_t$ in $\mathbb{C}[x_1, \dots, x_n]$.
3. Conclude that for any ideal I of $\mathbb{C}[x_1, \dots, x_n]$, we have $I(V(I)) = \sqrt{I}$.