TUTORIAL 8

1 Matching Randomized Algorithm

The goal is to propose a randomized algorithm to test if an input graph $G = (V, E)$ (not necessarily bipartite) has a perfect matching. Let $V = \{1, \dots, n\}$. Introduce a variable x_{ij} for every edge ij in the graph, with $i < j$. Form an $n \times n$ matrix M where $m_{ij} = 0$ if ij is not an edge, where $m_{ij} = x_{ij}$ if ij is an edge and $i < j$, and where $m_{ij} = -x_{ji}$ if ij is an edge and $i > j$.

1. Show that if G has a perfect matching, then $Det(M)$ is a non-zero polynomial.
2. Show that every term in $Det(M)$ corresponds to a subgraph of G which is a disjoint union of cycles and edges covering all the vertices (call this a *cycle factor*).
3. Show that if a cycle factor has only even components, then G has a perfect matching.
4. Show that if a cycle factor F has some odd components, then the term which corresponds to F cancel out.
5. Deduce that G has a perfect matching if and only if $Det(M) \neq 0$, and that testing the existence of a perfect matching can be done (probabilistically) in $O(n^\omega)$.

2 Counting compositions

Let $P \subseteq S_n$ be a set of n permutations. We denote by $P \circ P$ the set of permutations $\sigma \circ \pi$ where σ and π belong to P. The goal of this exercise is to provide an algorithm which returns the size of $P \circ P$ faster than the obvious $O(n^3)$ algorithm which actually computes $P \circ P$. We introduce for this variables $x_1, \dots, x_n$ and $y_1, \dots, y_n$

1. Given σ and π two permutations, we form the polynomial $f_{\sigma\pi} = \sum_{i=1}^{n} x_{\sigma(i)} y_{\pi^{-1}(i)}$. Show that $|P \circ P|$ is exactly the number of distinct polynomials $f_{\sigma\pi}$ where σ and π are in P.
2. Let $P = \{\sigma_1, \dots, \sigma_n\}$. Form the two $n \times n$ matrices A and B where $a_{ij} = x_{\sigma_j(i)}$ and $b_{ij} = y_{\sigma_j^{-1}(i)}$. What are the entries (c_{ij}) of $C = A^T B$?
3. Find a value N such that if all x_i and y_j are independently and uniformly chosen at random in $\{1, \dots, N\}$, then with probability 1/2 every distinct entry of C gives a distinct value.
4. Conclude.