Practice 6

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Question

A lottery ticket costs 2 dollars and the probability of winning is 0.1. If you win a lottery you receive 10 dollars, if you lose you receive 0 dollars.

1. Let X be a random variable that represents your money gain after playing one round of lottery, i.e. $X = 8$ if you win and $X = -2$ if you lose. Find a and b such that $X = a \cdot Y + b$ where $Y \sim Bernoulli(p)$, i.e. Y is a Bernoulli random variable. What is the value of p?

$X = 10 \cdot Y - 2$ and $p = 0.1$

2. Use the properties of expectation and variance to find $E(X)$ and $Var(X)$.

$E(X) = 10 \cdot E(Y) - 2 = 10 \cdot 0.1 - 2 = -1$

$Var(X) = 10^2 \cdot Var(Y) = 10^2 \cdot 0.1 \cdot 0.9 = 9$

3. If you play this lottery many-many times, do you think your average money gain will be positive or negative?

Negative. Although it is possible to win money in one single round, if you play this lottery many-many times the average money gain (i.e. the sample mean) will be close to the $E(Y) = -1$. This means that, on average, we will be loosing 1 dollar per play.

4. You decided to test your luck and bought 5 lottery tickets. Let Z denote the number of winning tickets. What is the expectation and variance of Z? Hint: use the link between Binomial and Bernoulli random variables.

First, we note that $Z \sim Binomial(5, 0.1)$. If $Y_1, \ldots, Y_5 \sim Bernoulli(0.1)$ then $Z = Y_1 + \ldots + Y_5$.

Thus, $E(Z) = 5 \cdot 0.1 = 0.5$ and $Var(Z) = 5 \cdot 0.1 \cdot 0.9 = 0.45$.

5. What is the probability that at least one of these five tickets will win?

Using Binomial table we find

$$P(Z \geq 1) = 1 - P(Z = 0) = 1 - 0.5905 = 0.4095$$

6. Let W be the *average* money gain for your five tickets. Find the expectation and variance of W. Hint: use $X_1, \ldots X_5$ to represent the money gain of each ticket and find the formula that expresses W in terms of $X_1, \ldots, X_5$

Following the hint we find $W = \frac{X_1+\ldots+X_5}{5}$. Since $E(X_i) = -1$ and $Var(X_i) = 9$ we conclude that $E(W) = -1$ and $Var(W) = \frac{9}{5} = 1.8$.

7. Find the chances that your average money gain is not negative, i.e. $P(W \geq 0)$? Is it higher than 50%? Hint: first find the formula that expresses W in terms of Z.

If $Y_1, \ldots, Y_5$ are Bernoulli random variables representing the outcomes of each lottery tickets then

$$W = \frac{X_1 + \ldots + X_5}{5} = \frac{10 \cdot Y_1 - 2 + \ldots + 10 \cdot Y_5 - 2}{5} = \frac{10 \cdot Z - 10}{5} = 2 \cdot Z - 2$$

Therefore

$$P(W \geq 0) = P(2 \cdot Z - 2 \geq 0) = P(Z \geq 1) = 0.4095$$

It is lower that 50%, so buying 5 tickets does not sound like a good idea.

8. Now you decided to buy 100 tickets. Let W be the *average* money gain for your 100 tickets. What is the expectation and variance of W?

As $W = \frac{X_1+\ldots+X_{100}}{100}$ we conclude that $E(W) = -1$ and $Var(W) = \frac{9}{100} = 0.09$.

9. What is the approximate distribution of W?

By Central Limit Theorem the distribution of W is approximately Normal, i.e. $W \sim Normal(-1, 0.09) = Normal(-1, 0.3^2)$.

10. Use the answer in 9 to find the chances that your new average money gain is not negative, i.e. $P(W \geq 0)$? Is it higher than 50%?

We need to apply standardization to W. If V is standard normal, then using the distribution table we find

$$P(W \geq 0) = P\left(\frac{W+1}{0.3} \geq \frac{0+1}{0.3}\right) = P\left(V \geq 3.33\right) = 1 - P\left(V < 3.33\right) = 1 - 0.9996 = 0.0004$$

No, it is very small (we have no chance!). Note that this result is consisted with our answer in 3.

11. Use the 68–95–99.7 rule to find the interval $[c, d]$ that contains 95% of W values, i.e. such that $P(c \leq W \leq d) = 0.95$.

We are interested in the bottom 95% interval of the distribution curve, thus

$$c = \mu - 2 \cdot \sigma = -1 - 2 \cdot 0.3 = -1.6$$

$$d = \mu + 2 \cdot \sigma = -1 + 2 \cdot 0.3 = -0.4$$

12. Use the 68–95–99.7 rule to find 2.5-th percentile for W. In other words we need to find the value t such that 2.5% of W values are less than t, i.e. $P(W \leq t) = 0.025$.

We are interested in the bottom 2.5% values of the distribution curve, thus

$$t = \mu - 2 \cdot \sigma = -1 - 2 \cdot 0.3 = -1.6$$

13. Use standardization and the distribution table to find the 2.5-th percentile for W.

$$P(W \leq t) = P\left(V \leq \frac{t+1}{0.3}\right) = 0.025$$

We search 0.025 value in the distribution table, it corresponds to $P(V \leq -1.96)$. Thus $\frac{t+1}{0.3} = -1.96$ and $t = -1.96 \cdot 0.3 - 1 = -1.588$. Note that we got slightly different answer from one we got using the empirical rule (this rule is approximate!).