Homework: Dynamic Programming

This document defines the **homework assignments** for the "Algortihms" course @ Software University. Please submit a single **`zip`** / **`rar`** / **`7z`** archive holding the solutions (source code) of all below described problems.

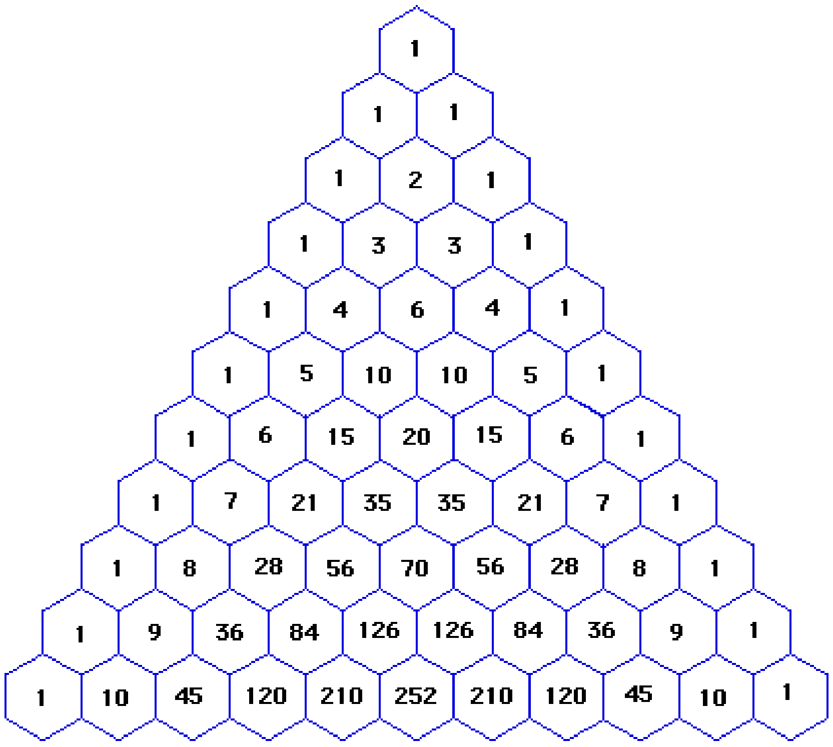
Problem 1. Binomial Coefficients

Write a program that finds the binomial coefficient $\binom{n}{k}$ for given non-negative integers **n** and **k**. The coefficient can be found recursively by adding the two numbers above using the formula:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{for all integers } n, k : 1 \le k \le n-1,$$

However, this leads to calculating the same coefficient multiple times (a problem which also occurs when solving the Fibonacci problem recursively). Use memoization to improve performance.

You can check your answers using the picture below (row and column indices start from 0):



`Examples:`

Input	Output

`n = 3`
`k = 2`
	`3`

`n = 4`
`k = 0`
	`1`

`n = 6`
`k = 2`
	`15`

`n = 10`
`k = 5`
	`252`

Problem 2. Longest Zigzag Subsequence

A zigzag sequence is one that alternately increases and decreases. More formally, such a sequence has to comply with one of the two rules below:

1) Every even element is smaller than its neighbors and every odd element is larger than its neighbors, or
2) Every odd element is smaller than its neighbors and every even element is larger than its neighbors

1 3 2 is a zigzag sequence, but 1 2 3 is not. Any sequence of one or two elements is zig zag.

Find the longest zigzag subsequence in a given sequence.

Examples:

Input	Output
`8,3,5,7,0,8,9,10,20,20,20,12,19,11`	`8 3 5 0 20 12 19 11`
`1,2,3`	`1 2`
`1,3,2`	`1 3 2`
`24,5,31,3,3,342,51,114,52,55,56,58`	`24 5 31 3 342 51 114 52 55`

Problem 3. Dividing Presents

Alan and Bob are twins. For their birthday they received some presents and now they need to split them amongst themselves. The goal is to **minimize the difference between the values of the presents received by the two brothers**, i.e. to divide the presents as equally as possible.

Assume the presents have values represented by positive integer numbers and that presents cannot be split in half (a present can only go to one brother or the other).

Find the minimal difference that can be obtained and print which presents each brother has received (you may only print the presents for one of them, it is obvious that the rest will go to the other brother). In the examples below Alan always takes a value less than or equal to Bob, but you may do it the other way around.

Input	Output

`3,2,3,2,2,77,89,23,90,11`	`Difference: 30`
	`Alan:136 Bob:166`
	`Alan takes: 11 90 23 2 2 3 2 3`
	`Bob takes the rest.`

`2,2,4,4,1,1`	`Difference: 0`
	`Alan:7 Bob:7`
	`Alan takes: 1 4 2`
	`Bob takes the rest.`

`7,17,45,91,11,32,102,33,6,3`	`Difference: 1`
	`Alan:173 Bob:174`
	`Alan takes: 33 32 91 17`
	`Bob takes the rest.`

`1,1,1,1,1,1,1,1,1,22`	`Difference: 13`
	`Alan:9 Bob:22`
	`Alan takes: 1 1 1 1 1 1 1 1 1`
	`Bob takes the rest.`

Problem 4. Representing a Sum with Unlimited Amount of Coins

We have a set of coins with predetermined values, e.g. 1, 2, 5, 10, 20, 50. Given a sum **S**, the task is to find how many combinations of coins will sum up to **S**. For each value, we can use an **unlimited number of coins**, e.g. we can use S coins of value 1 or S/2 coins of value 2 (if S is even), etc.

Examples:

Input	Output	Comments

S = 6
Coins = {1,2,3,4,6}
	10	The 10 combinations are:
		6 = 6
		6 = 4 + 2
		6 = 4 + 1 + 1
		6 = 3 + 3
		6 = 3 + 2 + 1
		6 = 3 + 1 + 1 + 1
		6 = 2 + 2 + 2
		6 = 2 + 2 + 1 + 1
		6 = 2 + 1 + 1 + 1 + 1
		6 = 1 + 1 + 1 + 1 + 1 + 1

S = 5
Coins = {1,2,5}
	4	The 4 combinations are:
		5 = 5
		5 = 2 + 2 + 1
		5 = 2 + 1 + 1 + 1
		5 = 1 + 1 + 1 + 1 + 1

S = 13
Coins = {1,2,5,10}
	16	

S = 100
Coins = {1,2,5,10,20,50,100}
	4563	

Problem 5. Representing a Sum with Limited Amount of Coins

In the previous problem, the coins represented values, not actual coins (we could take as many coins of a certain value as we wanted). In this problem, we'll regard the coins as actual coins, e.g. 1, 2, 5 are three coins and we can use each of them only once. We can, of course, have more coins of a given value, e.g. – 1, 1, 2, 2, 10.

The task is the same - find the number of ways we can combine the coins to obtain a certain sum **S**.

Examples:

Input	Output	Comments

S = 6
Coins = {1,2,2,3,3,4,6}
	4	The 4 combinations are:
		6 = 6
		6 = 4 + 2
		6 = 3 + 3
		6 = 3 + 2 + 1

S = 5
Coins = {1,2,2,5}
	2	The 2 combinations are:
		5 = 5
		5 = 2 + 2 + 1

S = 13
Coins = {1,2,2,5,5,10}
	2	The 2 combinations are:
		13 = 10 + 2 + 1
		13 = 5 + 5 + 2 + 1

`S = 100`
`Coins = {50,20,20,20,20,20,10}`
	`2`	`The 2 combinations are:`
		`100 = 50 + 20 + 20 + 10`
		`100 = 20 + 20 + 20 + 20 + 20`

Problem 6. * Minimum Edit Distance

We have two strings, **s1** and **s2**. The goal is to obtain **s2** from **s1** by applying the following operations:

- **replace(i, x)** – in **s1**, replaces the symbol at index **i** with the character **x**
- **insert(i, x)** – in **s1**, inserts the character **x** at index **i**
- **delete(i)** – from **s1**, removes the character at index **i**

We are only allowed to modify **s1**, **s2** stays unchanged at all times. Each of the three operations has a certain **cost** associated with it (positive integer number). **Note**: the cost of the **replace(i, x)** operation is 0 if it doesn't actually change the character.

The goal is to find the sequence of operations which will produce **s2** from **s1** with **minimal cost**.

Examples:

Input	Output	Comments

`cost-replace = 3`
`cost-insert = 2`
`cost-delete = 1`
`s1 = abracadabra`
`s2 = mabragabra`
	`Minimum edit distance: 7`
	`INSERT(0, m)`
	`DELETE(3)`
	`DELETE(4)`
	`REPLACE(6, g)`
		`Indices refer to the original s1 string – DELETE(3) deletes the symbol at index 3 from abracadabra, not from the modified string mabracadabra after the INSERT(0, m) operation.`

`cost-replace = 5`
`cost-insert = 2`
`cost-delete = 1`
`s1 = nqma bira`
`s2 = ima bira`
	`Minimum edit distance: 4`
	`DELETE(0)`
	`DELETE(1)`
	`INSERT(1, i)`
		`We can obtain s2 with two operations – DELETE(0) + REPLACE(1, i), but the cost of the REPLACE operation is high, that's why the solution involves three operations, their total cost is smaller. The INSERT can be performed also at index 0 and index 2.`

`cost-replace = 3`
`cost-insert = 3`
`cost-delete = 3`
`s1 = equal`
`s2 = equal`
	`Minimum edit distance: 0`	

`cost-replace = 1`
`cost-insert = 1`
`cost-delete = 1`
`s1 = equal`
`s2 = different`
	`Minimum edit distance: 8`
	`INSERT(0,d)`
	`INSERT(1,i)`
	`INSERT(2,f)`
	`INSERT(3,f)`
	`REPLACE(1,r)`
	`REPLACE(2,e)`
	`REPLACE(3,n)`
	`REPLACE(4,t)`	

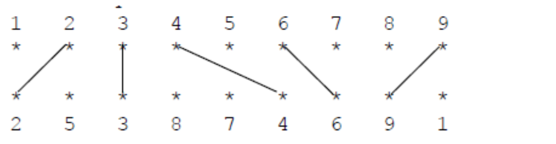
Problem 7. * Connecting Cables

We are in a rectangular room. On opposite sides of the room there are sets of **n** cables (n < 1000). The cables are indexed from 1 to n.

On each side of the room there is a permutation of the cables, e.g. on one side we have {1, 2, 3, 4, 5} and on the other side we have {5, 1, 3, 4, 2}. We are trying to connect each cable from one side with the corresponding cable on the other side – connect 1 with 1, 2 with 2, etc. **Cables are straight and should not overlap!**

The task is to find the maximum number of pairs we can connect given the restrictions above.

Examples:

Input	Output	Comments
`Side1 = {1,2,3,4,5,6,7,8,9}` `Side2 = {2,5,3,8,7,4,6,9,1}`	`Maximum pairs connected: 5` `Connected pairs: 2 3 4 6 9`	
`Side1 = {1,2,3,4}` `Side2 = {4,3,2,1}`	`Maximum pairs connected: 1` `Connected pairs: 1`	`Any other pair can be conntected as well.`
`Side1 = {1,2,3}` `Side2 = {1,2,3}`	`Maximum pairs connected: 3` `Connected pairs: 1 2 3`	

Problem 8. * Symbol Multiplication

We have an **alphabet** of k symbols (a finite number) and a **multiplication table** showing the result of multiplying each two symbols of the alphabet. E.g., the alphabet is {a, b, c} and the multiplication table is:

	a	***b***	***c***
a	*b*	*b*	*a*
b	*c*	*b*	*a*
c	*a*	*c*	*c*

This shows that a*a = b, a*b = b, b*a = c, etc. As shown in the example, multiplication is **not commutative or associative** – a*b != b*a, therefore, the order of multiplication is essential.

We have a string **S** comprised of characters from the alphabet. The task is to find whether we can obtain the symbol **'a'** by inserting brackets in the string – all symbols in brackets are multiplied. If so, print the string with the brackets inserted. Print "No solution" otherwise. Assume 'a' will always be in the alphabet.

Examples:

Input	Output	Comments
`Alphabet = {a,b,c}` `Table =` `bba` `cba` `aac` `S = abc`	`((a*b)*c)`	`((a*b)*c) = (b*c) = a`
`Alphabet = {a,b,c}` `Table =` `bba` `cba` `aac` `S = bacacbcabbbcacab`	`(((b*a)*(c*a))*(((c*(b*c))*a)*((b*((b*b)*(c*a)))*(c*(a*b)))))`	
`Alphabet = {a,b}` `Table =` `bb` `bb` `S = abbbaaba`	`No solution`	`No combination of two symbols produces 'a' after multiplication.`