NONCOMMUTATIVE (CREPANT) DESINGULARIZATIONS AND THE GLOBAL SPECTRUM OF COMMUTATIVE RINGS

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Abstract. In this paper we study endomorphism rings of finite global dimension over not necessarily normal commutative rings. These objects have recently attracted attention as noncommutative (crepant) resolutions (NC(C)Rs) of singularities. Our results yield various necessary and sufficient conditions for their existence. We also introduce and study the global spectrum of a ring R, that is, the set of all possible finite global dimension of endomorphism rings of MCM R-modules.

Contents

1. Introduction

Let R be a commutative ring and M a finitely generated module over R. Let $A = \mathrm{End}_R(M)$. Recall that if R is normal Gorenstein, M is reflexive and A is maximal Cohen-Macaulay with finite global dimension, it is called a non-commutative crepant resolution (NCCR) of $\operatorname{Spec} R$. If one assumes only that M is faithful and

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A has finite global dimension, it is called a noncommutative resolution (NCR) of $\operatorname{Spec} R$.

Starting with the spectacular proof of the three dimensional case of the Bondal–Orlov conjecture [10], suitably interpreted by Van den Bergh [51], there has been strong evidence that these objects could be viewed as rather nice noncommutative analogues of resolution of singularities, as their names suggest. As such, their study is an intriguing blend of commutative algebra, noncommutative algebra and algebraic geometry and has recently attracted a lot of interest by many researchers. Questions of existence and construction of NC(C)Rs were considered e.g. in [51, 21, 23], see also [39] for an overview about non-commutative resolutions. In general, it is subtle to construct NC(C)Rs and explicit examples are e.g. in [16] (NCCR for the generic determinant), [37] (NCCRs via cluster tilting modules), [51] and [52], [38] (NCRs for curves), [36] (reconstruction algebras) or [20] (cluster tilting modules of curves).

In this note we study these objects while relaxing many of the assumptions usually assumed by previous work. First, our commutative rings R might not be normal or even a domain. Second, the module M might not be maximal Cohen-Macaulay or even reflexive. Let us now discuss why such directions are worthwhile.

Our investigation started with the following observation of Buchweitz: In complex analytic geometry, so-called free divisors in complex manifolds are studied. One says that a hypersurface D in $\mathbb{C}^n$ is free at a point $p \in \mathbb{C}^n$ if and only if its module of logarithmic derivations $\operatorname{Der}_{\mathbb{C}^n}(-\log D)$ is a free module over the local ring $\mathcal{O}_{\mathbb{C}^n,p}$, cf. [45]. This is equivalent to the condition of the Jacobian ideal of D being a maximal Cohen–Macaulay module over $\mathcal{O}_{D,p}$, see [1]. Free divisors occur frequently e.g. as discriminants of certain spaces [41, 30], as free hyperplane arrangements [50] or in quiver representations [17]. In [15] Buchweitz, Ebeling and von Bothmer study cases when the discriminant of a versal morphisms between analytic spaces yields a free divisor. They observe that in many of these cases, the normalization of this discriminant (and the critical locus) is given as the endomorphism ring of its Jacobian ideal, see [15, Thm. 2.5 and Rmk. 2.6]. Moreover, it is known that the critical locus of a versal morphism between smooth spaces is a determinantal variety, see [41]. In [16] it is shown that a generic determinantal variety has a non-commutative crepant desingularization. Write R for the coordinate ring of the generic determinantal singularity described by maximal minors of the generic $n \times m$ matrix. Let $A = \operatorname{End}_R(M)$ be the NCCR as constructed in [16]. So for any determinantal singularity X we have a map $X \to \operatorname{Spec} R$ obtained by specializing the generic matrices. So we can pull back the NCCR A via this map. Alternatively, we can pull back the module M. One may hope to that either of these will determine a NCCR of X, which may in particular be a free divisor, which is the discriminant of such a versal morphism. This insight raises an obvious question:

Question 1.1. *(Buchweitz [14]) When does a free divisor admit NCCRs?*

Very little is known about this question in general, even over the simplest class of normal crossing divisors, if one excludes the obvious and commutative(!) answer, namely the normalizations of such divisors. We are able to demonstrate certain criteria which provide concrete cases with positive and negative answers. One interesting problem which arises is to characterize when the normalization of a free divisor has *rational* singularities. In particular, we can give an example of a free divisor, which does not allow a NCCR.

The second main motivation for our work comes from the inherent nature of NC(C)Rs. If one wishes to view them as analogues of commutative desingularizations, a study of such notions over non-normal singularities is crucial. Similarly, there is no reason why one should study only the cases of maximal Cohen-Macaulay modules. Even over affine spaces, it is useful to know what commutative blow-ups are smooth, yet if one takes noncommutative blow-ups as endomorphism rings of MCM modules, the problem becomes boring: all those modules are free! Thus one could ask:

Question 1.2. *If R is a regular local ring or a polynomial ring over a field, which reflexive, non-free modules have endomorphism rings of finite global dimension?*

This question is surprisingly subtle, and we can only provide an example using recent (unpublished) work by Buchweitz–Pham [18].
Last but not least, from the viewpoint of noncommutative algebra, a fundamental question is:

Question 1.3. *Let A be a noncommutative ring of finite global dimension that is finitely generated as a module over its centre. What is the structure of the centre of A?*

Again, it is a subtle question even whether $Z(A)$ is normal, and our work provides a host of natural examples that will hopefully shed more light on this question.
As clear from the discussion above, a lot remains to be done, and we believe that our paper is merely a starting point.

The structure of this paper is as follows: section 2 deals with noncommutative (crepant) resolutions of not necessarily normal rings. Since the center of a NCCR is normal (Prop. 2.3), one can deduce several conditions for the existence, in particular of commutative and generator NC(C)Rs. Then we show that generator NCCRs (or NCRs of global dimension 1) for singular curves do not exist (Prop. 2.8) and further study Leuschke's NCRs for ADE-curves in 2.2.1. Moreover, the relationship between NCRs and rational singularities is examined: for two-dimensional rings it is shown that there exists a NCR if and only if the normalization of the ring has only rational singularities (Thm. 2.11), which yields interesting restrictions on the existence of NCRs. This is followed by an intermezzo about detecting rational singularities in the case of graded rings, which yields an example of a free divisor that does not allow a NCR (Example 2.19). Using the a-invariant, we find a criterion for the normalizations of certain graded Gorenstein rings to have rational singularities (Cor. 2.25).
In section 3 we define the global spectrum of a commutative ring. Adapting a lemma of [36] it is shown that the global dimension of a cluster tilting object of a curve is equal to 3 (Cor. 3.6). Then the global spectra of the Artinian local rings $S/(x^n)$, where $(S,(x))$ is a regular local ring of dimension 1, the node and the cusp are computed (Thm. 3.8, Prop. 3.9 and 3.10). Moreover it is shown that the global spectrum of a simple singularity of dimension 2 is $\{2\}$, see Thm. 3.11. Then the relation of the global spectrum of a ring and an extension is considered.
In section 4, several endomorphism rings of finite global dimension are studied: first it is shown how to transform an endomorphism ring of finite global dimension of a non-reflexive module over a regular ring into a finite dimensional endomorphism

ring of a reflexive but not free module, namely the direct sum over the syzygy modules of the residue field, see Thm. 4.1. Then, in section 4.2, NCRs for one of the simplest non-normal hypersurface singularities are constructed, the normal crossing divisor. Our construction yields NCRs which are never NCCRs. Finally, in section 4.3, we describe how to compute the global dimension of an endomorphism ring by constructing projective resolutions of its simple modules. The method is illustrated by the example of the E_6-curve singularity in example 4.12.

1.1. **Conventions and notation.** Usually, R will denote a commutative noetherian local Henselian ring with residue field k, whereas S is used for a commutative regular local ring. The R-algebras considered here will be central R-algebras. Modules and ideals will be right modules. Exceptions will be explicitly mentioned.

2. Non commutative (crepant) resolutions

Here we use van den Bergh's concept of *non-commutative crepant resolution (NCCR)* [51] in a more general setting: let R be a commutative noetherian ring. A reflexive R-module M gives a NCCR of R (or, geometrically interpreted, of $\mathrm{Spec}(R)$) if $A = \mathrm{End}_R M$ is *homologically homogeneous* (i.e., for all $\mathfrak{p} \in \mathrm{Spec}\, R$ we have $\mathrm{gldim}\, A_{\mathfrak{p}} = \dim R_{\mathfrak{p}}$). It is shown in [12] that this implies that $A_{\mathfrak{p}}$ is MCM.) Note that a module finite R-algebra Λ, which is a homologically homogeneous R-order, is also called a *non-singular order* cf. [36]. If R is equidimensional of dimension n, and all simple A-modules have the same projective dimension n then A is homologically homogeneous. It was shown in [51] that over a Gorenstein domain R, an endomorphism ring $A = \mathrm{End}_R M$ is a NCCR if M is reflexive, $\mathrm{gldim}\, A < \infty$ and A is MCM over R. Recently, the weaker concept of *non-commutative resolution (NCR)* was introduced in [23]: Let R be a commutative noetherian ring and M a finitely generated R-module. Then M is said to give a NCR of R (or $\mathrm{Spec}\, R$) if M is faithful and $A = \mathrm{End}_R M$ has finite global dimension. Then we also say that A is a NCR of R.

A module M is a generator of the category of R modules if given any R module N, there is a surjective map $\bigoplus_I M \to N$ where the index set I is possibly infinite. If R is commutative and Henselian local this is equivalent to R being a direct summand of M. Moreover, we say that $A = \mathrm{End}_R(M)$ is a *generator NC(C)R* if M is a generator of the category $\mathrm{mod}\, R$. For $M \in \mathrm{mod}\, R$ we denote by $\mathrm{add}\, M$ the subcategory of $\mathrm{mod}\, R$ that consists of direct summands and finite direct sums of copies of M. We denote by $\mathrm{MCM}(R)$ the full subcategory of $\mathrm{mod}\, R$ consisting of all maximal Cohen–Macaulay modules.

Before we start with existence questions of NC(C)Rs of not necessarily normal rings, let us therefore give some examples, in which non-normal rings play a prominent part.

Example 2.1. Let R be a one-dimensional Henselian local reduced noetherian ring. Then the normalization $\widetilde{R}$ is a NCCR of R, since it is equal to $\mathrm{End}_R(\widetilde{R}) = \mathrm{End}_{\widetilde{R}}(\widetilde{R})$. It is worth mentioning that in the curve case one also has $\widetilde{R} = \mathrm{End}_R(\mathcal{C})$, the endomorphism ring of the conductor ideal, see [24]. By [38] one can construct a generator NCR of R, which is (for R non-normal) by Prop. 2.8 never of global dimension 1, that is, it is never a NCCR.

Example 2.2. It is well-known that 2-dimensional simple singularities (rational double points) have a NCCR, coming from the McKay correspondence, see [11, 39]. From this one can build a NCCR of certain free divisors: let R be the 2-dimensional hypersurface singularity $\mathbb{C}\{x,y,z\}/(h)$, with $h = xy^4 + y^3z + z^3$. Then $D = \{h = 0\}$ is a free divisor in $(\mathbb{C}^3, 0)$ (Sekiguchi's B_5 [47]), whose normalization $\widetilde{D}$ is again a hypersurface given by the equation $\widetilde{h} = xy + uy + u^3$, where $u = \frac{z}{y}$. Note that $\widetilde{D} = \operatorname{Spec} \widetilde{R}$, where $\widetilde{R} = \mathbb{C}\{x,y,u\}/(\widetilde{h})$ has an A_2-singularity. Thus it has a NCCR $A = \operatorname{End}_{\widetilde{R}} M$, with $M = \bigoplus_{i=1}^{3} M_i$, where M_i are the indecomposable MCM-modules of $\widetilde{R}$. Since $\operatorname{End}_R M = \operatorname{End}_{\widetilde{R}} M$, the endomorphism ring A also yields a NCCR of R.

2.1. **Centres of NC(C)Rs.** In the introduction the question about the centre of a non-commutative ring was raised, see Question 1.3. In this section, we show that in the case of a NCCR of a commutative noetherian ring R, the centre is always the normalization of R. However, in the case of a generator NCR, the centre is R itself. This yields restrictions on the existence of NC(C)Rs, see Cor. 2.4, Propositions 2.6 and 2.8. It is shown in [48] and [34] that the centres of NCCRs are log terminal and therefore rational.
The following proposition uses the result of [12] that the centre of a homologically homogenous ring is a Krull domain.

Proposition 2.3. *Let R be a commutative noetherian ring and let M be a finitely generated, faithful, torsion-free R module. Suppose that $A = \operatorname{End}_R(M)$ is homologically homogeneous. Then $Z(A)$ is the normalization of R.*

Proof. The normalization of R is isomorphic to a finite product of normal domains: $\widetilde{R} \cong \prod_{i=1}^{k} \widetilde{R}_i$. Write $Q(R) = Q(\widetilde{R}) = \prod_i Q_i$ for the finite product of the fraction fields of the $\widetilde{R}_i$. We have that $Q \otimes M \simeq \bigoplus Q_i^{n_i}$ where n_i is the rank of M at Q_i. It follows that

$$Q \otimes_R A \simeq \operatorname{End}_Q(M \otimes Q) \simeq \bigoplus_{i,j} \operatorname{Hom}_Q(Q_i^{n_i}, Q_j^{n_j}) \simeq \prod_i \operatorname{End}_{Q_i}(Q_i^{n_i}) \simeq \prod Q_i^{n_i \times n_i}.$$

Since M is faithful, each $n_i > 0$ and we have that $Z(Q \otimes A) = Q$. Since M is torsion-free, we see that A is torsion-free. Hence $Z(A) \subseteq Q \otimes Z(A) = Z(A \otimes Q) = Q$, the first equality holds because we are inverting central elements. Since M is finitely generated, we have that A is finitely generated over R and so $Z(A)$ is integral over R. Furthermore, $Z(A)$ is noetherian since it is a submodule of the finitely generated (and thus noetherian) R-module A. By the above results we see that

$$R \subseteq Z(A) \subseteq Q,$$

where $Z(A)$ is an integral extension of R and thus $Z(A) \subseteq \widetilde{R}$. Since A is homologically homogeneous, by Theorem 5.3 of [12] A is a direct sum of prime homologically homogeneous rings and by Theorem 6.1 of loc. cit., $Z(A)$ is a Krull domain. Since $Z(A)$ is noetherian, this means nothing else but that $Z(A)$ is a direct sum of integrally closed integral domains (see e.g. [9, VII 1.3, Corollaire]). Thus we have $\widetilde{R} \subseteq Z(A)$ and the assertion follows. □

Corollary 2.4. *Let R be a commutative noetherian ring. Then R has a commutative NCR $\operatorname{End}_R M$ if and only if $\widetilde{R}$ is regular.*

Proof. If $A = \operatorname{End}_R M$ is commutative, then $Z(A) = A$ is a commutative ring of finite global dimension. By Serre's theorem A is a regular ring. If A is a NCR then the proof of Prop. 2.3 can be followed line by line, the only differences (if A is not a NCCR) are that A is torsion free because it is regular and $Z(A) = A$ is integrally closed for the same reason. Thus we can also conclude that $A = \widetilde{R}$. The assertion follows from regularity of A. The other implication is clear. □

Lemma 2.5. *Let R be a commutative noetherian ring and let $M \in \operatorname{mod}(R)$ and set $A = \operatorname{End}_R(R \oplus M)$. Then the centre of A is R.*

Proof. We may write A in matrix form as

$$\begin{pmatrix} R & M^* \\ \operatorname{Hom}_R(R, M) & \operatorname{End}_R(M) \end{pmatrix},$$

where $M^* = \operatorname{Hom}_R(M, R)$. Computing commutators of e.g. the elements $\begin{pmatrix} 1_R & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1_R & 0 \\ m & 0 \end{pmatrix}$ yields the assertion. □

Proposition 2.6. *Let R be a commutative noetherian ring and suppose that M is a generator giving a NCCR. Then R is normal.*

Proof. Let $A = \operatorname{End}_R M$ the NCCR of R. Since A is by definition homologically homogeneous, by Prop. 2.3, $Z(A) = \widetilde{R}$, the normalization of R. But by lemma 2.5 the centre of A is R. Thus the claim follows. □

2.2. **NC(C)Rs for commutative rings and rational singularities.** For this section, we mostly assume that R is a Henselian ring over an algebraically closed field k of characteristic 0. There seems to be an intimate relationship between the existence of a NC(C)R for a ring R and the type of singularities of R. In particular, rational singularities (at least in characteristic 0) occur, see Thm. 2.11: a non-normal two-dimensional ring has a NCR if and only if its normalization only has rational singularities. Here we also discuss some low-dimensional examples of NCRs, such as Leuschke's NCRs over simple curve singularities and some free divisors. We are able to give a partial answer to Question 1.1, namely, we find a free divisor which does not allow a NCR. Moreover, for certain Gorenstein graded rings, we derive a criterion which tells whether their normalizations have rational singularities, using the a-invariant. We write $\operatorname{MCM}(R)$ for the full subcategory of the category of finitely generated R-modules whose objects are the maximal Cohen-Macaulay modules. We say that R has finite MCM-type if there is a finite set of R-modules $\{M_i\}$ such that any module in $\operatorname{MCM}(R)$ is of the form $\bigoplus M_i^{n_i}$.

Lemma 2.7. *Let R be reduced, Henselian local ring of dimension at most 2. If there is a finitely generated module M which is a generator and gives a NCCR $A = \operatorname{End}_R(M)$, then R is normal and of finite* MCM*-type.*

Proof. First note that by Prop. 2.6, R is normal. Take now any $N \in \operatorname{MCM}(R)$. Then $\operatorname{Hom}_R(M, N) \in \operatorname{MCM}(R)$ since $\dim R \leq 2$. By Lemma 2.15 of [37] $\operatorname{Hom}_R(M, N)$ is projective over A. Since M is a generator, the functor $\operatorname{Hom}_R(M, -) : \operatorname{add} M \to \operatorname{proj} \operatorname{End}_R(M)$ is an equivalence. So N is contained in $\operatorname{add} M$. This implies that R is of finite MCM-type. □

The following statement follows from [42, 7.8.9], but we give here a self-contained proof.

Proposition 2.8. *Let R be reduced of dimension 1 and Henselian local. Then R has a generator giving a NCR of global dimension 1 if and only if R is regular.*

Proof. If R is normal, then R is regular and thus is its own NCR, via $\mathrm{End}_R R = R$. For the other implication, let $R \oplus M$ give a generator NCR. As R is reduced, M is generically free. So A has global dimension zero at the minimal primes and the global dimension of A is one at the maximal ideal. Thus by definition $A := \mathrm{End}_R(R \oplus M)$ is a NCCR. By Prop. 2.6 the ring R has to be normal. □

2.2.1. *Leuschke's NCRs over curves.* For a one-dimensional reduced Henselian local ring $(R, \mathfrak{m})$ there is a construction in section 1 of [38] of a module M such that $A = \mathrm{End}_R(M)$ has finite global dimension. The module M is built via the Grauert–Remmert normalization algorithm: in order to obtain the normalization $\widetilde{R}$ of R one builds a chain of rings

$$R = R_0 \subsetneq R_1^{(j_1)} \subsetneq \cdots \subsetneq R_n^{(j_n)} = \widetilde{R}, \tag{1}$$

where each $R_i^{(j_i)}$ is a direct factor of the endomorphism ring of the maximal ideal of $R_{i-1}^{(j_{i-1})}$. Note here that the endomorphism ring can only split into factors if R is reducible. The module $M = \bigoplus_i R_i^{(j_i)}$ is a finitely generated R-module and $A = \mathrm{End}_R(M)$ has global dimension at most $n+1$, where n is the length of the longest possible chain of algebras between R_0 and its normalization as in (1). Note that this length is bounded by the δ-invariant of R, which is the length of $\widetilde{R}/R$. We consider some examples of R and M, namely the ADE-curve singularities.

Remark 2.9. Note here that we have explicitly computed E_6, case (3) below, in example 4.12, and E_8, (5) below, with the same method. For E_7, case (4) below, and the D_n-curves we used add M-approximations, a method which will be described in subsequent work.

(1) The A_{n-1}-singularity, n odd: let $R = k[[x,y]]/(y^2+x^n)$, where k is algebraically closed. Then $R_1 = \mathrm{End}_R(\mathfrak{m}) \cong k[[x,y]]/(y^2+x^{n-2})$. By induction it follows that $R_i \cong k[[x,y]]/(y^2 - x^{n-2i})$, and consequently that $R_{(n-1)/2}$ is equal to $\widetilde{R}$. Note that R_i is (considered as R-module) isomorphic to the indecomposable MCM-module I_i (in the notation of [55], see Prop.(5.11) loc. cit.). Thus we see that
$$M = \bigoplus_{i=0}^{\frac{n-1}{2}} R_i$$
is the sum all indecomposable R-modules (see also [55, Prop. (5.11)]). Since R is a ring of finite MCM type and M is a representation generator (i.e., any indecomposable MCM module is a direct summand of M), Prop. 3.3 yields that the global dimension of $\mathrm{End}_R(M)$ is 2. Note that for n even, the singularity A_{n-1} can be analyzed similarly to yield $\mathrm{gldim}\, \mathrm{End}_R M = 2$.

(2) The D_n, $n \geq 4$, singularities $R = k[[x,y]]/(x^2y + y^{n-1})$: for odd n a SINGULAR computation shows that $R_1 \cong (x, y^{n-2}) \cong X_1$, , where we use Yoshino's [55, p.77ff] notation for the MCM-modules. X_1 is (as a ring) isomorphic to the transversal union of a line and a curve singularity of type A_{n-3}. The further

R_i's can be explicitly computed, see [8], Prop. 4.18: the ring R_i is the disjoint union of a line and an A_{n-2i-1}-singularity. As an R-module R_i is isomorphic to $A \oplus M_{i-1}$ for $2 \leq i \leq \frac{n-1}{2}$. This can be seen by looking at the rank of the matrix factorizations of the MCM-modules: denote by $R_y = R/y$ the smooth component of R and by $R_{x^2+y^{n-2}}$ the singular A_{n-3}-component. Then we denote the rank of an MCM-module N on R by (a,b), where a is the rank of $N \otimes_R R_y$ as R_y-module and b is the rank of $N \otimes_R R_{x^2+y^{n-2}}$ as $R_{x^2+y^{n-2}}$-module. The line clearly corresponds to the module A, whose rank is $(1,0)$ and this is the only indecomposable MCM(R)-module supported only on R_y. On the other hand, the only indecomposable MCM(R)-modules supported only on $R_{x^2+y^{n-2}}$ are the M_i, which are of rank $(0,1)$. By localizing at x^2+y^{n-2} it follows that on this component M_i is isomorphic to the ideal (x, y^i) and as in case (1) one sees that this module is isomorphic to the A_{n-2i-3}-singularity $k[[x,y]]/(y^2 - x^{n-2i-2})$.

According to Leuschke's formula we have to take M to be $R \oplus X_1 \oplus \bigoplus_{i=1}^{\frac{n-3}{2}} M_i$. The endomorphism ring $\operatorname{End}_R M$ has global dimension 3.

For even n a SINGULAR computation shows that $R_1 \cong X_1$, and similar to the odd case one deduces $R_i \cong A \oplus M_{i-1}$ for $2 \leq i \leq \frac{n-2}{2}$ and $R_{\frac{n}{2}} \cong A \oplus D_- \oplus D_+$ is the normalization. Then $M = \bigoplus_{i=0}^{\frac{n}{2}} R_i$ has an endomorphism ring of global dimension 3. Again, this example was computed using add M-approximations.

(3) $R = k[[x,y]]/(x^3+y^4)$, the E_6-singularity: in example 4.12 it is computed that $M = \bigoplus_{i=0}^{2} R_i$ and that the global dimension of $A = \operatorname{End}_R M$ equals 3.
(4) $R = k[[x,y]]/(x^3+xy^3)$, the E_7-singularity: Using SINGULAR, one computes $R_1 \cong M_1$, $R_2 \cong Y_1$ and R_3 is the normalization, which is isomorphic to $A \oplus D$. Again with add M-approximation (or the same method as in example 4.12) one can show that $\operatorname{gldim} \operatorname{End}_R(\bigoplus_{i=0}^{3} R_i)$ is 3.
(5) $R = k[[x,y]]/(x^3+y^5)$, the E_8-singularity: one can compute that $M = \bigoplus_{i=0}^{3} R_i$, where a SINGULAR computation shows that $R_1 \cong M_1$, $R_2 \cong A_1$ and $R_3 \cong A_2$ is the normalization. Similar to example 4.12 (or with add(M)-approximation) one sees that the global dimension of $A = \operatorname{End}_R M$ equals 3.

Let us now turn our attention towards the relationship of NCRs and rational singularities:

Lemma 2.10. *Let R be a commutative ring and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal primes of R. Suppose for each i, M_i is a faithful $R/\mathfrak{p}_i$-module such that $\operatorname{End}_{R/\mathfrak{p}_i}(M_i)$ has finite global dimension. Let $M = \oplus M_i$. Then $A = \operatorname{End}_R(M)$ has finite global dimension.*

Proof. First we observe that $\operatorname{Hom}_R(M_i, M_j) = 0$ for $i \neq j$ (pick an element $x \in \mathfrak{p}_i$ but not in $\mathfrak{p}_j$, any map from M_i to M_j must be killed by x which is a non-zerodivisor on M_j, showing that the map is zero). Thus $\operatorname{End}_R(M) = \prod \operatorname{End}_R(M_i) = \prod \operatorname{End}_{R/\mathfrak{p}_i}(M_i)$ so it has finite global dimension. $\square$

Theorem 2.11. *Let $(R, \mathfrak{m})$ be a reduced 2-dimensional, Henselian local ring. The following are equivalent:*

(1) R has a NCR.
(2) $\widetilde{R}$ has only rational singularities.

Proof. Assume that R has a NCR given by $A = \mathrm{End}_R(M)$. Since M is faithful and M is locally free on the minimal primes of $\mathrm{Spec}(R)$, we have that M is a locally a generator outside a closed subscheme of dimension at most one. By Cor. 2.2 of [23] the Grothendieck group $G(R)$ is finitely generated. Thus by [5] the group $G(\widetilde{R})_{\mathbb{Q}}$ is finitely generated. The normalization $\widetilde{R}$ is the direct product of the normalizations of the irreducible components of R. Thus it suffices to assume that R is a domain. Then by Cor 3.3 of [23] the normalization $\widetilde{R}$ has rational singularities.
For the other implication, suppose that $\widetilde{R}$ has only rational singularities. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ be the minimal primes of R. Let $\widetilde{R}_i$ be the normalization of $R/\mathfrak{p}_i$. Then by assumption, each $\widetilde{R}_i$ has rational singularities. By 3.3 of [23] we can find $M_1, \dots, M_r$ such that M_i is an NCR of R_i. Take $M = \bigoplus_{i=1}^r M_i$, then M gives a NCR of R by lemma 2.10. □

Corollary 2.12. *Let R be a reduced, Henselian local ring of dimension 2. Then there is a generator M giving a NCCR if and only if R is a quotient singularity.*

Proof. Assume that there is a generator M giving a NCCR. By Lemma 2.7 R has finite MCM-type. Therefore R is a quotient singularity, see [40, Thm. 7.19]. For the converse, if R is a quotient singularity, then by Herzog's theorem, cf. [40, Thm. 6.3], R is of finite MCM-type. Taking as M the sum of all indecomposables in $\mathrm{MCM}(R)$ gives $\mathrm{End}_R M$ homologically homogeneous of global dimension 2 (cf. Thm. P.2 of [39]; note here that since R is a quotient singularity, it is equidimensional and so $\mathrm{End}_R M$ is homologically homogeneous if and only if all simples have the same projective dimension). □

The above corollary yields necessary strong conditions for the existence of a NCCR of a normal crossing divisor. So it is reasonable to expect there does not exist a NCCR of a normal crossing divisor apart from its normalization:

Corollary 2.13. *Denote by $R = k[[x_1, \dots, x_n]]/(x_1 \cdots x_m)$ with $2 \le m \le n$ a normal crossing ring. Suppose $M = \bigoplus_{i=1}^k M_i$ gives a NCCR $A := \mathrm{End}_R(M)$. Then none of the "partial normalizations" $R/(x_{i_1} \cdots x_{i_l})$ for $2 \le l \le m$ and $1 \le i_1 < \cdots < i_l \le m$ is a direct summand of one of the M_i.*

Proof. To see this, suppose that $M_1 = (R/(x_{i_1} \cdots x_{i_l}))^{\oplus_s}$. Take the height 2 prime ideal $\mathfrak{p} = (x_{i_1}, x_{i_2}) \in R$ and localize R at $\mathfrak{p}$. Then $(M_1)_{\mathfrak{p}} = R_{\mathfrak{p}}^{\oplus_s}$ implies that $R_{\mathfrak{p}}$ is contained in $\mathrm{add}(M_{\mathfrak{p}})$. But since the definition of NCCR localizes and $R_{\mathfrak{p}}$ is a non-normal ring of dimension 2, this yields a contradiction to corollary 2.12. □

Corollary 2.14. *Suppose that R is a reduced d-dimensional, Henselian local ring. If R has a NCR, the normalization of R has only rational singularities in codimension 2. If R has a generator giving a NCCR, then R is already normal.*

Proof. NCR and NCCR localize, see for example [23, Lemma 3.5]. □

Question 2.15. *Assume that R has a NCR of global dimension 2. Does that imply that the normalization of R has only quotient singularities? (The converse is true by the previous corollary).*

Question 2.16. *Assume that R has dimension one and an NCR A of global dimension two. Classify all possible R and A.*

Remark 2.17. In [28] it is shown that $\dim A \le \mathrm{gldim}\, A$.

Question 2.18. *Do the results of [23] extend to algebras A that have finite global dimension? In other words does the centre of any algebra with finite global dimension have rational singularities (in characteristic 0)?*

If R is standard graded with an isolated singularity a result of Watanabe [54, Theorem 2.2] says that R has a rational singularity if and only if

$$a(R) := -\min\{n : [\omega_R]_n \neq 0\} < 0,$$

where ω_R is the graded canonical module. The easiest way to compute this invariant is to use the fact that $a(R)$ is equal to the degree of the rational function $H_R(t)$, the Hilbert series of R, see e.g. [13], Theorems 4.4.3 and 3.6.19.
One may explicitly calculate the a-invariant: Let R be a positively graded k-algebra, k a field. Then by e.g. [13] the Hilbert series of R is

$$H_R(t) = \frac{Q(t)}{\prod_{i=1}^{d}(1-t^{a_i})} \text{ with } Q(1) > 0,$$

where d is the dimension of R, the a_i are positive integers and $Q(t) \in \mathbb{Z}[t, t^{-1}]$. Then $a(R) = \deg H_R(t)$ is the degree of the rational function $H_R(t)$.
Note that in the case of a quasi-homogeneous isolated complete intersection singularity, rationality can be easily determined with *Flenner's rationality criterion*, see [27, Korollar 3.10], which can also be deduced from Watanabe's result: let k be a field of characteristic zero, $R = k[x_1, \ldots, x_{n+r}]/(f_1, \ldots, f_r)$ be a quasi-homogeneous complete intersection with weight$(x_i) = w_i > 0$ and $\deg f_i = d_i > 0$. Then R has a (isolated) rational singularity at the origin if and only if $w_1 + \cdots + w_{n+r} > d_1 + \cdots + d_r$.

The following example sheds more light on Buchweitz's question (question 1.1 about NC(C)Rs of free divisors). However, we have not been able to produce an example of an *irreducible* free divisor that does not have a NCR.

Example 2.19. (free divisor with non-rational normalization) Let $(D, 0) \subseteq (\mathbb{C}^n, 0)$ be a divisor given by a reduced equation $h = 0$. Let $R := \mathbb{C}\{x_1, \ldots, x_n\}/(h)$. Suppose that $h = h_2 \cdots h_n$, where $h_i = \sum_{j=1}^{i} x_j^k$ for some $k > n$. By Prop. 5.1 (or Example 5.3) of [19], h defines a free divisor D. If $k > n$, then the normalization $\widetilde{D}$ of D does not have rational singularities. The normalization $\widetilde{R}$ of $R = \mathbb{C}\{x_1, \ldots, x_n\}/(h_2 \cdots h_n)$ is $\bigoplus_{i=2}^{n} \widetilde{R}_i$, where $R_i = \mathbb{C}\{x_1, \ldots, x_n\}/(h_i)$. Each h_i is homogeneous and by Flenner's rationality criterion each $\widetilde{R}_i$ has a rational singularity if and only if $k \leq n$. By Thm. 2.11 R does not have a NCR.

The results above suggest:

Question 2.20. *Let $(D, 0) \subseteq (\mathbb{C}^n, 0)$ be an irreducible free divisor given by a reduced equation $h = 0$. Let $R := \mathbb{C}\{x_1, \ldots, x_n\}/(h)$. Does the normalization $\widetilde{R}$ always have rational singularities?*

Example 2.21. (Simis' quintic) The homogeneous polynomial $h = -x^5 + 2x^2y^3 + xy^4 + 3y^5 + y^4z$ gives rise to an irreducible free divisor D in $\mathbb{C}^3$. It's normalization is a homogeneous Cohen–Macaulay ring, but not Gorenstein. The a-invariant of the normalization is -1, which shows that it has a rational singularity. By theorem 2.11, D has a NCR. The a-invariant of the original ring $\mathbb{C}[x, y, z]/(h)$ is 2.

Example 2.22. (linear free divisor) The linear free divisor *discriminant in the space of cubics* (cf. [29]) in $\mathbb{C}^4$ has equation $h = y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27x^2w^2$. Its a-invariant is 0. By Lemma 2.24 the normalization of $\mathbb{C}[x, y, z, w]/(h)$ has a rational singularity.

Remark 2.23. The question about possible singularities of normalizations of free divisors is quite subtle. For example, in the case of discriminants of versal deformations of isolated hypersurface singularities (which are always free divisors, by [46]), one knows that the normalization is always smooth, see [49]. In [26, Conjecture 26] it was asked whether the normalization of a free divisor with radical Jacobian ideal is always smooth.

Lemma 2.24. *Let R be a Gorenstein graded ring, and $R \longrightarrow S$ be a birational graded integral extension which is not an isomorphism in codimension* 1*. Then*

$$a(S) < a(R).$$

Proof. Consider the exact sequence

$$0 \longrightarrow R \longrightarrow S \longrightarrow C \longrightarrow 0, \tag{2}$$

where C is the cokernel of $R \longrightarrow S$. Then since S is birational over R, C is torsion. Apply $\operatorname{Hom}_R(-, \omega_R)$. Thus $\operatorname{Hom}_R(C, \omega_R) = 0$. It follows that $\omega_S \cong \operatorname{Hom}_R(S, \omega_R)$ embeds into $\operatorname{Hom}_R(R, \omega_R) = \omega_R$. Since the a-invariant is equal to minus the smallest degree of the graded canonical module, it follows that $a(S) \leq a(R)$. But ω_R is 1-generated, so if equality holds, then $\omega_S = \omega_R$. We show that this is not possible: Let $\mathfrak{p}$ be a prime of height 1 in R. Now apply $\operatorname{Hom}_{R_\mathfrak{p}}(-, \omega_{R_\mathfrak{p}})$ to sequence (2). Then we get a short exact sequence

$$0 \longrightarrow \omega_{S_\mathfrak{p}} \longrightarrow \omega_{R_\mathfrak{p}} \longrightarrow \operatorname{Ext}^1(C_\mathfrak{p}, \omega_{R_\mathfrak{p}}) \longrightarrow 0$$

since $\operatorname{Ext}^2_R(S, \omega_R)_p = 0$. But the first map is an isomorphism, so $\operatorname{Ext}^1(C_\mathfrak{p}, \omega_{R_\mathfrak{p}})$ is 0. As $C_\mathfrak{p}$ is torsion and $R_\mathfrak{p}$ is one dimensional, $C_\mathfrak{p}$ is a module of finite length over $R_\mathfrak{p}$. As such $\operatorname{Ext}^1(C_\mathfrak{p}, \omega_{R_\mathfrak{p}})$ is Matlis dual to $C_\mathfrak{p}$ itself by local duality, so it is zero if and only if $C_\mathfrak{p}$ is zero. Therefore $R \longrightarrow S$ is an isomorphism in codimension 1. □

Corollary 2.25. *Let R be a graded non-normal Gorenstein ring with $a(R) = 0$ and let $\widetilde{R}$ be its normalization. If $\widetilde{R}$ has an isolated singularity (e.g., if $\dim R = 2$), then the singularity is rational.*

Corollary 2.26. *Let $R = \mathbb{C}[[x, y, z]]/(f)$, where f is irreducible homogeneous of degree* 3*. Then R has an NCR if and only if it is non-normal.*

Proof. If R is not normal, then by the previous corollary, $\widetilde{R}$ is rational and therefore has an NCR, which becomes an NCR over R, see [38, Lemma 1]. If R is normal, then it is the cone over an elliptic curve, therefore it does not have a NCR, see example 3.4 of [23]. □

3. The global spectrum

So far we have considered NCRs and NCCRs of the form $\operatorname{End}_R M$, for some reflexive R-module M. One definition of NCCRs involve the endomorphism ring being homologically homogenous, meaning $\operatorname{gldim} \operatorname{End}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Thus, an understanding of possible global dimensions is important. In fact, understanding dimension one is *critical* since NCCRs localize, so obstructions in dimension one can be used to determine obstructions to the existence of NCCR in general.

Hence, taking a more general point of view, we introduce the *global spectrum of a singularity.*

Definition 3.1. Let R be a commutative Cohen-Macaulay ring. We define the *global spectrum* of R, $\mathrm{gs}(R)$, to be the range of all *finite* $\operatorname{gldim}\operatorname{End}_R(M)$ where M is a MCM-module over R.

This concept is somewhat related to the concept of *representation dimension* by Auslander, which is the infimum of $\operatorname{gldim}\operatorname{End}_R(M)$ where R is an Artinian algebra and M is a generator-cogenerator (note however that being a cogenerator does not make sense for higher-dimensional R). So the main question to consider is the following:

Question 3.2. *What is the global spectrum of a ring R?*

Computation of the global spectrum appears to be subtle even in the case R when is Artinian or has finite MCM-type, see Theorem 3.8 and Prop. 3.9. We start here with a study of possible global dimension for endomorphism rings for curves and obtain numbers contained in the global spectrum: for 1-dimensional reduced rings R the normalization is an endomorphism ring of finite global dimension, and thus 1 is always contained in $\mathrm{gs}(R)$. Other particular cases are rings of finite MCM-type (Prop. 3.3, Thm. 3.11) and one-dimensional rings with cluster-tilting objects, see Prop. 3.6.

Proposition 3.3. *Let $(R, \mathfrak{m})$ be a one-dimensional reduced Henselian local ring of finite* MCM*-type, which is not regular. Let M be the direct sum of all indecomposable* MCM*-modules of R. Then the ring $\operatorname{End}_R(M)$ has global dimension 2.*

Proof. The result follows from [35, Prop. 4.3.1] (take the Auslander triple (R, M, T), where T is some cotilting module. In our case we have $(d, m, n) = (1, 1, 1)$ and the triple is not trivial, so $\operatorname{gldim}(\operatorname{End}_R(M)) \geq n+1 = 2$). A direct proof is also written out in [39, Theorem P.2]. $\square$

The following result appeared in [36], however the version stated there did not specify the assumptions on R.

Proposition 3.4. *Let R be a Cohen Macaulay Henselian local ring which is Gorenstein in codimension one and assume $\dim R \leq 2$. Let $M \in \mathrm{MCM}(R)$ be a generator with $A = \operatorname{End}_R(M)$. For $n \geq 0$, the following are equivalent:*

(1) $\operatorname{gldim} A \leq n+2$.

(2) For any $X \in \mathrm{MCM}(R)$, there exists an exact sequence

$$0 \to M_n \to \cdots \to M_0 \to X \to 0$$

such that $M_i \in \mathrm{add}(M)$ and the induced sequence:

$$0 \to \mathrm{Hom}(M, M_n) \to \cdots \to \mathrm{Hom}(M, M_0) \to \mathrm{Hom}(M, X) \to 0$$

is exact.

Proof. The proof is verbatim to that of [36, Prop 2.11]. The place where our assumption on R is used is the fact that $\mathrm{MCM}(R)$ coincides with the category of second syzygies in $\operatorname{mod} R$, see Thm. 3.6 of [25] and an A-module is a second syzygy if and only if it has the form $\mathrm{Hom}_R(M, X)$, where X is a second syzygy in $\operatorname{mod} R$. $\square$

Recall that an MCM module M is called cluster-tilting if

$$\begin{aligned}\operatorname{add}(M) &= \{X \in \operatorname{MCM}(R) \mid \operatorname{Ext}^1_R(X, M) = 0\} \\ &= \{X \in \operatorname{MCM}(R) \mid \operatorname{Ext}^1_R(M, X) = 0\}.\end{aligned}$$

Here we give the precise value for the global dimension of endomorphism rings of cluster-tilting modules.

Lemma 3.5. *Let R be a non-regular Cohen Macaulay local ring which is Gorenstein in codimension one and assume $\dim R \leq 2$. Let $M \in \operatorname{MCM}(R)$ be a cluster-tilting object and $A = \operatorname{End}_R(M)$. Then $\operatorname{gldim} A = 3$.*

Proof. By the definition of cluster-tilting objects, M is a generator. Thus, one can apply Proposition 3.4. To show that $\operatorname{gldim} A \leq 3$, we check condition (2). Take any $X \in \operatorname{MCM}(R)$, we take the left approximation of X by $\operatorname{add} M$. By Construction 3.5 in [22], we get a sequence $0 \to M_1 \to M_0 \to X \to 0$ such that $M_0 \in \operatorname{add}(M)$ and the induced sequence $0 \to \operatorname{Hom}(M, M_1) \to \operatorname{Hom}(M, M_0) \to \operatorname{Hom}(M, X) \to 0$ is exact. This means $\operatorname{Ext}^1(M, M_1) = 0$, so $M_1 \in \operatorname{add}(M)$.
Now, we need to rule out the case $\operatorname{gldim} A \leq 2$. Assume so, then Proposition 3.4 applies again to show that $\operatorname{MCM}(R) = \operatorname{add}(M)$. Since R is not regular, we can pick $X \in \operatorname{MCM}(R)$ which is not projective. Let ΩX be the first syzygy of X, obviously X and ΩX are in $\operatorname{MCM}(R)$, which is $\operatorname{add}(M)$. As M is cluster tilting, we must have $\operatorname{Ext}^1_R(X, \Omega X) = 0$. But the sequence $0 \to \Omega X \to F \to X \to 0$ is not split, so $\operatorname{Ext}^1_R(X, \Omega X)$ is not zero, contradiction. $\square$

Corollary 3.6. *Let $(S, \mathfrak{m})$ be a Henselian regular local ring of dimension 2. Let $R = S/(f)$ be a reduced hypersurface, and assume that $f = f_1 \cdots f_n$ is a factorization of f into prime elements with $f_i \notin \mathfrak{m}^2$ for each i. Let $S_i = S/(\prod_{j=1}^{i} f_j)$ be the partial normalizations of R and set $T = \bigoplus_{i=1}^{n} S_i$. Then $\operatorname{gldim} \operatorname{End}_R(T) = 3$.*

Proof. By [20, Theorem 4.1] (the case $S = k[[x, y]]$ and k infinite) or [22, Theorem 4.7] we know that S is cluster-tilting, so the previous Lemma applies. $\square$

Question 3.7. *Let R be as in 3.6. Is it true that $\operatorname{gs}(R) = \{1, 2, 3\}$?*

3.1. **Some computations of global spectra.** Here we give a few examples of global spectra: we compute the global spectrum of the zero dimensional ring $S/(x^n)$, where $(S, (x))$ is a regular local ring of dimension 1, of the node and the cusp and of a simple 2-dimensional singularity. We also consider the behavior of the global spectrum under separable field extensions.

Theorem 3.8. *Let $(S, (x))$ be a regular local ring of dimension one. Let $R = S/(x^n)$ and $M_i = R/(x^i)$ for $1 \leq i \leq n$. Let M be an R-module. Then $\operatorname{gldim} \operatorname{End}_R(M)$ is finite if and only if $\operatorname{add}(M) = \{M_1\}$, in which case the global dimension is 0, or $\operatorname{add}(M) = \{M_1, \ldots, M_l\}$, for some $1 \leq l \leq n$, in which case the global dimension is two. In particular the global spectrum of R is $\{0, 2\}$.*

Proof. The modules $M_1, \ldots, M_n = R$ are all the indecomposable modules over R. First if $l = 1$, then $\operatorname{End}_R(M_i)$ has finite global dimension if and only if $i = 1$. For the rest of the proof we assume that $l \geq 2$. For the purpose of this theorem we may suppose that $M = \bigoplus_{i=1}^{l} M_{a_i}$ with $a_1 < \ldots < a_l$. Observe that $A = \operatorname{End}_R(M) \cong \operatorname{End}_{S/(x^{a_l})}(M)$. Thus without loss of generality, we can assume that $a_l = n$ and all we need to show is that $\{a_1, \ldots, a_l\} = \{1, \ldots, n\}$. Suppose that this is not the case.

Pick any $c \in \{1, \dots, n\} \setminus \{a_1, \dots, a_l\}$. We will prove that $\operatorname{proj.dim}_A \operatorname{Hom}(M, M_c)$ is infinite by showing that a syzygy of $\operatorname{Hom}(M, M_c)$ contains as a direct summand a module $\operatorname{Hom}(M, M_{c'})$ with $c' \in \{1, \dots, n\} \setminus \{a_1, \dots, a_l\}$. That would demonstrate that an arbitrarily high syzygy of $\operatorname{Hom}(M, M_c)$ is not projective.
It remains to prove the claim. We will build a syzygy of $\operatorname{Hom}(M, M_c)$ using the construction 3.5 of [22]. Since $\operatorname{Hom}(M, M_c)$ has exactly l-minimal generators and $a_l = n$, which means M is a generator, we have a short exact sequence

$$0 \longrightarrow X \longrightarrow M^l \longrightarrow M_c \longrightarrow 0.$$

By the construction, for any $1 \leq j \leq n$ the sequence remains exact when we apply $\operatorname{Hom}_R(M_j, -)$. In particular, $\operatorname{Hom}_R(M, X)$ is an A-syzygy for $\operatorname{Hom}_R(M, M_c)$. Suppose the claim is not true. Then X must be in $\operatorname{add}(M)$. Let $X = \bigoplus_{i=1}^l M_{a_i}^{x_i}$. Now for each j we are going to count lengths of the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M_j, X) \longrightarrow \operatorname{Hom}_R(M_j, M^l) \longrightarrow \operatorname{Hom}_R(M_j, M_c) \longrightarrow 0.$$

It is easy to see that $\operatorname{length}(\operatorname{Hom}_R(M_a, M_b)) = \min\{a, b\}$. This gives us the following system of l linear equations (for $j = 1, \dots, l$):

$$G_j : \sum_{i=1}^{l} x_i \min(a_i, a_j) = l \left(\sum_i \min(a_i, a_j) \right) - \min(c, a_j).$$

Subtracting G_{j-1} from G_j we get that

$$\sum_{i=j}^{l} x_i (a_j - a_{j-1}) = l(l - j + 1)(a_j - a_{j-1}) + \min(c, a_{j-1}) - \min(c, a_j). \tag{3}$$

Consider 2 cases (recall that $a_l = n$):
(i) $a_1 < c < a_n$, let t be such that $a_{t-1} < c < a_t$. Then the above equation becomes

$$\sum_{i=t}^{l} x_i (a_t - a_{t-1}) = l(l - t + 1)(a_t - a_{t-1}) + (a_t - c).$$

Equivalently, $\sum_{i=t}^{l} x_t - l(l - t + 1) = \frac{a_{t-1} - c}{a_t - a_{t-1}}$. However, the right hand side is not an integer.
(ii) $0 < c < a_1$, so (3) gives us $\sum_{i=j}^{l} x_j = l(l - j + 1)$ for $2 \leq j \leq l$. Substituting this into the original equation, this gives us

$$\sum_{i=1}^{l} x_i a_1 = l^2 a_1 - c.$$

It follows that $(\sum_{i=1}^{l} x_i - l^2) = -\frac{c}{a_1}$, but again the right hand side is not an integer. Observe now that since we have proved that M must be a representation generator, $\operatorname{gldim} \operatorname{End}_R(M) = 2$. □

Proposition 3.9. *Let $R = k[[x, y]]/(xy)$. Then $\operatorname{gs} R = \{1, 2, 3\}$.*

Proof. The ring R has three indecomposable MCM-modules, in Yoshino's [55, p.75ff] notation: $N_+ = R/x$, $N_- = R/y$ and R itself. Thus any MCM-module is of the form $M = N_+^a \oplus N_-^b \oplus R^c$. There can occur essentially three different situations, which yield $\operatorname{gldim} \operatorname{End}_R M \leq \infty$:
(i) $M = N_+$ or $M = N_-$: then $\operatorname{End}_R M \cong M$ and is a regular commutative ring of global dimension 1. Note that $\operatorname{End}_R M$ is not an NCR since M is not faithful.

(ii) $M = N_+ \oplus N_-$: then $\operatorname{End}_R M \cong \widetilde{R}$ and $\operatorname{gldim} \operatorname{End}_R M = 1$. Note that this is the only NCCR of R.
(iii) $M = R \oplus N_+$ or $M = R \oplus N_-$: Then M is cluster-tilting and by Prop. 3.6 $\operatorname{gldim} \operatorname{End}_R M = 3$. Here $\operatorname{End}_R M$ is an NCR of R.
The quiver has the following form:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

with the relation $ca = bc = 0$.
(iv) $M = R \oplus N_+ \oplus N_-$: then M is a representation generator and by Prop. 3.3 the global dimension of $\operatorname{End}_R M$ is equal to 2. We get path algebra of the following quiver

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 12 \arrow[l, bend left, "b"] \arrow[r, bend left, "c"] & 2 \arrow[l, bend left, "d"] \end{tikzcd}$$

subject to the relations

$$ca = bd = 0.$$

□

Proposition 3.10. *Let $R = k[[x, y]]/(x^3 + y^2)$ be the cusp. Then $\operatorname{gs} R = \{1, 2\}$.*

Proof. Here $\operatorname{MCM}(R)$ has two indecomposables: R and $\mathfrak{m}$. Similar to the proof of Prop. 3.9 there are two cases which yield an endomorphism ring of finite global dimension:
(i) $M = R \oplus \mathfrak{m}$: then $\operatorname{gldim} \operatorname{End}_R M = 2$ by Prop. 3.3. The quiver has the following form:

$$\begin{tikzcd} 1 \arrow[r, bend left, "a"] & 2 \arrow[l, bend left, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

with the relation $ab = c^2$.

(ii) $M = \mathfrak{m}$: Then $\operatorname{End}_R M \cong \widetilde{R}$ and its global dimension is 1. □

In dimension 2 we can determine the global spectrum of simple singularities:

Theorem 3.11. *Assume that R is a Henselian local 2 dimensional simple singularity. Let $M \in \operatorname{MCM}(R)$, then $\operatorname{gldim}(\operatorname{End}_R(M))$ is finite if and only if $\operatorname{add}(M) = \operatorname{MCM}(R)$. The global spectrum of R is $\{2\}$.*

Proof. Let $A = \operatorname{End}_R(M)$ and assume $\operatorname{gldim}(A) < \infty$. By lemma 5.4 of [3] we know that $\omega_A = \operatorname{Hom}_R(A, R) \cong A$, thus A is a Gorenstein order. Hence by lemma 2.15 of [37] the Auslander–Buchsbaum formula holds for mod A. Take any $N \in \operatorname{MCM}(R)$. Then $\operatorname{Hom}_R(M, N)$ is an A-module of depth 2, so must be projective. Therefore $N \in \operatorname{add}(M)$ and we are done. □

Question 3.12. *Does the other direction of the theorem also hold: let R be a Henselian local normal 2-dimensional ring. If $\operatorname{gs} R = \{2\}$, is then R a simple singularity? More generally, does the equivalence $\operatorname{gs} R = \{d\}$ if and only if R is a simple d-dimensional singularity (for $d > 1$), hold?*

It is interesting to relate the global spectrum of a ring and an extension. For separable field extensions we have an inclusion of global spectra.

Lemma 3.13. *Let $k \to K$ be a separable field extension and R a k-algebra. Then for any $M \operatorname{mod} R$, $\operatorname{gldim} \operatorname{End}_R(M) = \operatorname{gldim} \operatorname{End}_{R\otimes_k K}(M \otimes_k K)$. In particular, $\operatorname{gs}(R) \subseteq \operatorname{gs}(R \otimes_k K)$ and they are equal if any MCM module over $R \otimes_k K$ is extended from R.*

Proof. Denote $A = \operatorname{End}_R(M)$ and $-_K = - \otimes_k K$. Given an A-module X, we have $\operatorname{proj.dim}_A X = \operatorname{proj.dim}_{A_K} X_K$ as A_K is A=projective and contains A as a A-direct summand. Thus $\operatorname{gldim} A_K \geq \operatorname{gldim} A$. On the other hand, any simple A_K-module is a direct summand of some X_K, as in proof of [23, Lemma 3.6], so we also have $\operatorname{gldim} A_K \leq \operatorname{gldim} A$. □

It is not always true that $\operatorname{gs}(R) = \operatorname{gs}(R_K)$ even in the separable case.

Example 3.14. Let $R = \mathbb{R}[[u, v]]/(u^2 + v^2)$. Then $\operatorname{gs}(R) = \{1, 2\}$ but $\operatorname{gs}(R_{\mathbb{C}}) = \{1, 2, 3\}$.

Proof. The global spectrum of $R_{\mathbb{C}}$ has been computed in 3.9 where $x = u + iv, y = u - iv$. The irreducible maximal Cohen-Macaulay modules over R are only R and the maximal ideal $\mathfrak{m}$ by Example 14.12 [55]. Note that $\mathfrak{m} \otimes \mathbb{C} \simeq N_+ \oplus N_-$ as in 3.9. So up to Morita equivalence we only get cases (ii) and (iv). □

4. Some endomorphism rings of finite global dimension

This section deals with question 1.2 and, more generally, with explicit computation of the global dimension of an endomorphism ring. In theorem 4.1, we prove the finiteness of the global dimensions of endomorphism rings of certain reflexive R-modules, where R is a quasi-normal ring. In conjunction with a result of Buchweitz–Pham, an endomorphism ring of a reflexive module over a regular ring with finite global dimension can be constructed. This is only a first step in a more general study of endomorphism rings of modules over regular rings of finite global dimension. However, in 4.2 a NCR for a normal crossing divisor is derived from this result.
In Thm. 4.5 we construct a different NCR for the normal crossing divisor, obtained from a Koszul algebra, and compute its global dimension. Finally, in 4.3, we discuss a method for explicit computation of the global dimension for endomorphism rings over Henselian local rings.

4.1. **Global dimension of syzygies of k.** A commutative noetherian ring R is called *quasi-normal* if it is Gorenstein in codimension one (i.e., $R_{\mathfrak{p}}$ is Gorenstein for any prime ideal $\mathfrak{p}$ of height at most one) and if it satisfies condition (S_2), i.e., for any prime ideal $\mathfrak{p}$ of R one has $\operatorname{depth}(R_{\mathfrak{p}}) \geq \min\{2, \dim R_{\mathfrak{p}}\}$. We aim to prove the following

Theorem 4.1. *Let R be a quasi-normal ring and $\mathfrak{m}$ be an ideal of grade at least 2 on R. Let M be a reflexive module with R as a summand and assume that $\operatorname{gldim} \operatorname{End}_R(M \oplus \mathfrak{m})$ is finite. Then $\operatorname{gldim} \operatorname{End}_R(M)$ is also finite.*

As a corollary, using results by Buchweitz and Pham [18], we obtain:

Corollary 4.2. *Let $n \geq 2$, k a field and $R = k[x_1, \dots, x_n]$ or its localization at $\mathfrak{m} = (x_1, \dots, x_n)$ or $k[[x_1, \dots, x_n]]$ and $M = \sum_{i=2}^{n} \Omega^i k$. Then $\operatorname{End}_R M$ has finite global dimension.*

The reason is that Buchweitz and Pham already showed that $\operatorname{End}_R(\mathfrak{m} \oplus M) = \operatorname{End}_R(\sum_{i=1}^{n} \Omega^i k)$ has finite global dimension.
We need a couple of lemmas:

Lemma 4.3. *Let A be a ring and e an idempotent such that $\operatorname{proj.dim}_B eA(1-e) < \infty$ where $B = (1-e)A(1-e)$. Then if $\operatorname{gldim} A < \infty$, so is $\operatorname{gldim} B$.*

Proof. Let X be any (left) module over B. We need to show that $\operatorname{proj.dim}_B X$ is finite. Let $Y = X \otimes_B (1-e)A \in \operatorname{mod}(A)$, then we have $Y(1-e) = X$. As $\operatorname{gldim} A < \infty$, one has a projective resolution

$$0 \longrightarrow A_n \longrightarrow \cdots \longrightarrow A_0 \longrightarrow Y \longrightarrow 0.$$

Multiply with $(1-e)$ one gets a long exact sequence:

$$0 \longrightarrow A_n(1-e) \longrightarrow \cdots \longrightarrow A_0(1-e) \longrightarrow X \longrightarrow 0$$

in $\operatorname{mod}(B)$. However, note that $A(1-e) = eA(1-e) \oplus B$, so each $A_i(1-e)$ has finite projective dimension over B, and thus so does X. $\square$

Lemma 4.4. *Let R be a commutative noetherian ring. Then for any ideal $\mathfrak{m}$ and module M such that $\operatorname{grade}(\mathfrak{m}, M) \geq 2$ we have $\operatorname{Hom}_R(\mathfrak{m}, M) \cong \operatorname{Hom}_R(R, M) \cong M$.*

Proof. Start with the short exact sequence $0 \longrightarrow \mathfrak{m} \longrightarrow R \longrightarrow R/\mathfrak{m} \longrightarrow 0$ and take $\operatorname{Hom}_R(-, M)$. The desired isomorphism follows since $\operatorname{Ext}^i_R(R/\mathfrak{m}, M) = 0$ for $i < 2$ (see e.g. Theorems 1.6.16 and 1.6.17 of [13]). $\square$

Now we prove Theorem 4.1.

Proof. Let $A = \operatorname{End}_R(M \oplus \mathfrak{m})$ and e, f be the idempotents corresponding to the summands $\mathfrak{m}$ and R respectively. Then $\operatorname{End}_R(M) = (1-e)A(1-e)$, so to apply Lemma 4.3 we only need to check that $\operatorname{proj.dim}_B eA(1-e) < \infty$. However $eA(1-e) = fA(1-e)$ since the former is $\operatorname{Hom}_R(\mathfrak{m}, M)$ and the later is $\operatorname{Hom}_R(R, M)$. Note that here the condition of R to be quasi-normal is used: one needs that an R-sequence of two or less elements is also an M-sequence, see [53, Thm 1.4]. But R is a summand of M, so $fA(1-e)$ is a summand of B, and we are done. $\square$

4.2. **NCRs for the normal crossing divisor.** A hypersurface in a smooth ambient space has (simple) normal crossings at a point if it is locally isomorphic to the union of coordinate hyperplanes. Here we will consider the normal crossing divisor[1] over $k[x_1, \dots, x_n]$ (but everything works similar for $k[[x_1, \dots, x_n]]$ or $k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$). Thus, the normal crossing divisor is given by the ring $R = k[x_1, \dots, x_n]/(x_1 \cdots x_n)$, which is of Krull-dimension $n-1$.
In order to obtain a NCR of a normal crossing divisor, one can apply Lemma 2.10 together with Corollary 4.2. Namely, let $R = k[x_1, \dots, x_n]/(x_1 \cdots x_n)$. Take a NCR given by a module M_i over each $R_i = k[x_1, \dots, x_n]/(x_i)$. Then by lemma 2.10,

---

[1]We tacitly assume that we take all possible hyperplanes, i.e., in an n-dimensional ambient space, the coordinate ring of the normal crossing divisor is $k[x_1, \dots, x_n]/(x_1 \cdots x_n)$. So one may speak of *the* normal crossing divisor.

$M = \bigoplus_{i=1}^{n} M_i$ gives a NCR for R. So one can take e.g. the reflexive module from Cor. 4.2 as M_i, since each R_i is smooth, or $M_i = R_i$, which yields $M = \bigoplus_{i=1}^{n} M_i$ with NCR $\operatorname{End}_R M \cong \bigoplus_{i=1}^{n} R_i \cong \widetilde{R}$, the normalization.

However, below we construct a different NCR of the normal crossings singularity, and compute its global dimension.

Theorem 4.5. *Let $R = k[x_1, \ldots, x_n]/(x_1 \cdots x_n)$ and let $M = \bigoplus_{I \subseteq [n]} R/\left(\prod_{i \in I} x_i\right)$. Then $\operatorname{End}_R(M)$ is an NCR of R with global dimension n. Furthermore, let $K \subseteq [n]$, where $[n] = \{1, \ldots, n\}$, and let*

$$M = \bigoplus_{\substack{I \subseteq [n] \\ I \neq K}} \frac{R}{\left(\prod_{i \in I} x_i\right)}$$

then $\operatorname{End}_R M$ is an NCR of R of global dimension $\leq 2n - 1$.

The proof will be established after studying the intermediate algebra Λ_n which we construct below. Consider the quiver $\square^1$ given by a two cycle with two arrows and two vertices:

$$\begin{tikzcd} \cdot \arrow[r, bend left] & \cdot \arrow[l, bend left] \end{tikzcd}$$

We write $\square^n$ for the quiver given by the 1-skeleton of the n dimensional cube with arrows in both directions. Alternatively $\square^n$ is the Hasse diagram of the lattice of subsets of $[n]$. We will index vertices of this quiver by these subsets. So the set of vertices is $2^{[n]} = \{I \subseteq [n]\}$. We define a metric $d(I, J)$ on $2^{[n]}$ to be the minimal number of insertions and deletions of single elements required to move from I to J. Let $\Lambda_n = \bigotimes_{i=1}^{n} k(\square^1)$ be the tensor product of the path algebra of $\square^1$. Several properties of Λ_n are described below:

Proposition 4.6. *The algebra $\Lambda_n = \bigotimes_{i=1}^{n} k(\square^1)$ is isomorphic to:*

1. *the path algebra $k(\square^n)/\mathcal{R}$ with relations $\mathcal{R}$ that every square commutes.*
2. *the order*

$$\left(\prod_{i \in J \setminus I} x_i k[x_1, \ldots, x_n] \right)_{I,J} \subset k[x_1, \ldots, x_n]^{2^{[n]} \times 2^{[n]}}.$$

Proof. The second description is immediate from the definition. An arrow can be interpreted as removing or adding a single element to a set. We map the arrow $I \to I \cup \{j\}$ to $x_j e_{I, I \cup \{j\}}$ and $I \to I \setminus \{j\}$ to $e_{I, I \setminus \{j\}}$. The inverse map can be described as mapping the monomial $x_1^{k_1} \cdots x_n^{k_n} e_{I,J}$ to a choice of shortest path from I to J composed with loops that add and remove i the appropriate number of times k_i. An alternate proof is to verify that

$$k(\square^1) \simeq \begin{pmatrix} k[x] & xk[x] \\ k[x] & k[x] \end{pmatrix} \subset k[x]^{2 \times 2}$$

via the isomorphism described above. Next one can show that the tensor product of this isomorphism yields the above description. $\square$

For example, the order Λ_2 is

$$\begin{aligned} k(\Box^2)/\mathcal{R} &\simeq \begin{pmatrix} k[x_1] & x_1k[x_1] \\ k[x_1] & k[x_1] \end{pmatrix} \otimes \begin{pmatrix} k[x_2] & x_2k[x_2] \\ k[x_2] & k[x_2] \end{pmatrix} \\ &\simeq \begin{pmatrix} k[x_1,x_2] & x_1k[x_1,x_2] & x_2k[x_1,x_2] & x_1x_2k[x_1,x_2] \\ k[x_1,x_2] & k[x_1,x_2] & x_2k[x_1,x_2] & x_2k[x_1,x_2] \\ k[x_1,x_2] & x_1k[x_1,x_2] & k[x_1,x_2] & x_1k[x_1,x_2] \\ k[x_1,x_2] & k[x_1,x_2] & k[x_1,x_2] & k[x_1,x_2] \end{pmatrix} \subset k[x_1,x_2]^{4\times 4} \end{aligned}$$

We next need to establish some good properties that Λ_n satisfies. In particular we will be using Koszul algebras as described in [7].

Proposition 4.7. *Λ_n satisfies:*

(1) Λ_n is Koszul.
(2) If we grade Λ_n by path length then $H_{\Lambda_n}(t) = \frac{1}{(1-t^2)^n}(t^{d(I,J)})_{I,J}$
(3) Let e be a primitive idempotent, then $\Lambda_n e \Lambda_n$ is projective as a left Λ_n module.

Proof. The Koszul property is immediate since Λ_1 is a path algebra with no extra relations, and Λ_n is a tensor product of Koszul algebras, Theorem 3.7 [31]. We associate the subset $I \subseteq [n]$ to the vector $\vec{I} = (i_1, \dots, i_n) \in \mathbb{F}_2^n$ where $i_k = 1$ if $k \in I$ and 0 otherwise. Observe the following identity:

$$\left(\bigotimes_{i=1}^{n} \begin{pmatrix} 1 & t_i \\ t_i & 1 \end{pmatrix} \right)_{I,J} = t_1^{i_1+j_1} \cdots t_n^{i_n+j_n}. \tag{4}$$

Write $|\vec{I}|$ for the number of non-zero entries of $\vec{I} \in \mathbb{F}_2^n$. The Hilbert Series can be computed as the tensor product of the Hilbert series of the path algebra $k(\Box^1)$ using the above identity evaluated at $t_i = t$ and using the facts:

$$\begin{aligned} d(I,J) &= |\vec{I} + \vec{J}|, \\ H_{\Lambda_1}(t) &= \frac{1}{1-t^2}\begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \\ H_{A\otimes B}(t) &= H_A(t) \otimes H_B(t). \end{aligned}$$

Lastly, to check that $\Lambda_n e \Lambda_n$ is projective, we may choose any primitive idempotent due to the symmetry. So let $e = e_\emptyset$. The module $\Lambda_n e \Lambda_n = (\Lambda_n e)(e\Lambda_n)$ is the outer product of the first row and first column. So we get that

$$\Lambda_n e \Lambda_n = \bigoplus_{J\subseteq[n]} \Lambda_n e \Lambda_n e_J = \bigoplus_{J\subseteq[n]} x^J \Lambda_n e \simeq \bigoplus_{J\subseteq[n]} \Lambda_n e \simeq (\Lambda_n e)^{\oplus 2^n}.$$

$\Box$

Corollary 4.8. *The algebra Λ_n enjoys the following properties:*

(1) Λ_n has global dimension n
(2) Λ_n is homologically homogeneous.
(3) The simple module at e has the following resolution.

$$0 \longleftarrow S_I \longleftarrow P_I \longleftarrow \bigoplus_{\substack{J\subseteq[n] \\ d(I,J)=1}} P_J \longleftarrow \bigoplus_{\substack{J\subseteq[n] \\ d(I,J)=2}} P_J \longleftarrow \cdots$$

Proof. The Hilbert Series of the Koszul dual $\Lambda_n^!$ is determined by $H_{\Lambda_n}(t)H_{\Lambda_n^!}(-t) = 1$. Furthermore, we see that

$$H_{\Lambda_1^!}(t) = \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}.$$

So

$$H_{\Lambda_n^!}(t) = H_{\Lambda_n}(-t)^{-1} = \left(\bigotimes_{i=1}^n H_{\Lambda_1}(-t) \right)^{-1} = \bigotimes_{i=1}^n H_{\Lambda_1}(-t)^{-1}.$$

which we compute to be $H_{\Lambda_n^!}(t) = (t^{d(I,J)})_{I,J}$ by the combinatorial identity (4) of the above proof. Any Koszul algebra A has a resolution of the form

$$0 \longleftarrow A_0 \longleftarrow A \otimes (A_0^!)^* \longleftarrow A \otimes (A_1^!)^* \longleftarrow A \otimes (A_2^!)^* \longleftarrow \cdots$$

This resolution is a sum of the resolutions of the simple modules at the vertices. We multiply the resolution on the left by the primitive idempotent e_I, to obtain a resolution of the simple right module $S_I = e_I(\Lambda_n)_0$. The Hilbert series of $\Lambda^!$ shows that each simple has projective dimension n and has a resolution of the above form. □

We now pass to the algebra of interest. Let $e = e_\emptyset$ be the idempotent at the empty set. Set $S = k[x_1, \ldots, x_n]$ and write $x^I = \prod_{i \in I} x_i$.
The following proposition establishes Theorem 4.5.

Proposition 4.9. *Let e be a primitive idempotent of Λ_n. Let $B_n = \Lambda_n / \Lambda_n e \Lambda_n$, then B_n is isomorphic to*

$$\mathrm{End}_{S/(x_1 \cdots x_n)} \left(\bigoplus_{I \subset [n]} S/(x^I) \right).$$

Also, B_n is Koszul and has global dimension n. Furthermore, if f is a primitive idempotent of B_n, then $(1-f)B_n(1-f)$ has finite global dimension.

Proof. We can set $e = e_\emptyset$ as in the proof of the above proposition. The above proof also shows

$$e_I \Lambda_n e \Lambda_n e_J = x^J k[x_1, \ldots, x_n].$$

So

$$\begin{aligned} e_I B_n e_J = \frac{x^{J/I} k[x_1, \ldots, x_n]}{x^J} &= \left(\frac{x^J}{x^{I \cap J}} \right) \frac{k[x_1, \ldots, x_n]}{(x^J)} \\ &= \frac{(x^J : x^I)}{(x^J)} = \mathrm{Hom}(R/(x^I), R/(x^J)). \end{aligned}$$

Now Theorem 1.6 and Example 1 of [4] shows that since $\Lambda_n e \Lambda_n$ is projective, we obtain the projective resolutions of the simple modules of B_n by simply deleting the projectives $P_\emptyset$ from the resolution. So we obtain a linear resolution of length at most n. Hence B_n is Koszul with global dimension n. Lastly, since each projective only appears once in the resolution of a given simple module, if we remove an idempotent $f = e_K$, we can replace $B_n f = P_K$ with its projective resolution over B_n, which will not involve P_K. This is explained in more detail in [33]. □

4.3. **Computation of global dimension.** In this section we consider the problem of computing the global dimension of an endomorphism ring $\operatorname{End}_R M$ of a finitely generated module M over a commutative noetherian ring R. First we state some well-known facts about the structure of finitely generated algebras over local noetherian Henselian rings R and about quivers related to these rings. We always assume that R local noetherian and Henselian, since one needs Krull-Schmidt (see e.g. [44] exercise 6.6). Using quivers of $\operatorname{End}_R M$ we give an algorithm for the explicit computation of the global dimensions of the algebras.

If Λ is a noetherian ring with $\operatorname{gldim} \Lambda < \infty$, then its global dimension is is given as

$$\operatorname{gldim}(\Lambda) = \sup\{\operatorname{proj.dim}_\Lambda S : S \text{ is a simple } \Lambda - \text{module}\}, \tag{5}$$

see e.g., [42] 7.1.14. If A is a finitely generated algebra over a noetherian ring then the result holds with no assumptions on the global dimension of A, cf. [6]: $rt.\operatorname{gldim} A = \sup\{\operatorname{proj.dim}_A S\colon S \text{ a simple right } A\text{-module}\}$. Recall that the right global dimension of a ring A is equal to its left global dimension if A is noetherian [2], so we can speak of its global dimension.

For Artinian algebras there is a well-known structure theorem of projective modules (see e.g., Theorem 6.3 and Corollary 6.3a of [43]). In particular the indecomposable projective modules of an Artinian algebra A are in one to one correspondence with the simple $A/\mathbf{J}(A)$-modules, where $\mathbf{J}(A) = \operatorname{rad}(A)$ is the Jacobson radical of A.
In the following let A be a finitely generated R-algebra. In this case a similar structure theorem holds (see [44] Theorem 6.18, 6.21 and Cor. 6.22):

Theorem 4.10. *Let A be a R-algebra, which is finitely generated as R-module, where R is local noetherian commutative Henselian. Denote by $\bar{A} = A/\mathbf{J}$, where $\mathbf{J} = \operatorname{rad} A$ is the Jacobson radical of A. Then $\bar{A}$ is a semi-simple Artinian ring.*
Suppose that $1 = e_1 + \cdots + e_n$ is a decomposition of $1 \in A$ into orthogonal primitive idempotents in A. Then

$$A = e_1 A \oplus \cdots \oplus e_n A$$

is a decomposition in indecomposable right ideals of A and

$$\bar{A} = \bar{e}_1 \bar{A} \oplus \cdots \oplus \bar{e}_n \bar{A}$$

is a decomposition of $\bar{A}$ into minimal right ideals. Moreover, $e_i A \cong e_j A$ if and only if $\bar{e}_i \bar{A} \cong \bar{e}_j \bar{A}$.

This theorem says that the indecomposables summands of A are of the form $P_i = e_i A$. By definition, the P_i are the indecomposable projective modules over A. The modules $S_i = P_i/\mathbf{J}$ are the simple modules over A (as well as over the semi-simple algebra $\bar{A}$) and $P_i \longrightarrow S_i \longrightarrow 0$ is a projective cover.

Remark 4.11. The most general setting for which Thm. 4.10 holds, are semi-perfect rings, i.e., rings over which any finitely generated (right) module has a projective cover, see [32], section 10.3f.

For endomorphism rings $A = \operatorname{End}_R(M)$, where $M = \bigoplus_{i=1}^n M_i$ and R is as before, the above discussion leads to a method for the computation of $\operatorname{gldim} A$. Write A

in matrix form

$$A = \begin{pmatrix} \mathrm{Hom}_R(M_1, M_1) & \cdots & \mathrm{Hom}_R(M_n, M_1) \\ \vdots & \ddots & \vdots \\ \mathrm{Hom}_R(M_1, M_n) & \cdots & \mathrm{Hom}_R(M_n, M_n) \end{pmatrix}. \tag{6}$$

Then $e_1, \ldots, e_n$, where e_i is the $n \times n$ matrix with 1 as its ii-entry and 0 else, form a complete set of orthogonal idempotents. By Thm. 4.10 the projectives of A are the rows $P_i = e_i A$ and the simples are $S_i = P_i/\mathbf{J}$, $i = 1, \ldots, n$. By (5), $\mathrm{gldim}(A) = \max_i\{\mathrm{proj.\,dim}_A(S_i)\}$.

4.3.1. *Projective resolutions of the simples.* In order to find the resolutions of the simples, we present the following method, which uses the quiver of $\mathrm{End}_R M$ (we mostly follow the exposition in [32], chapter 11). Let R be as above a local Henselian ring and suppose that $M = \oplus_{i=1}^n M_i$ is a finitely generated R-module such that any indecomposable summand M_i appears with multiplicity 1 (then $A = \mathrm{End}_R M$ is called *basic*). This is no serious restriction since $\mathrm{End}_R(\oplus_{i=1}^n M_i^{a_i})$ is Morita-equivalent to $\mathrm{End}_R M$. Moreover, assume that A is *split*, i.e., $A/\mathbf{J}$ is a product of matrix algebras over the residue field. This is the case when the residue field of R is separably closed (e.g., residue field of characteristic 0).

The quiver of A: Since A is split and basic we have that $A/\mathbf{J}$ is a product of copies of the residue field, one for each indecomposable direct summand M_i. By Thm. 4.10 we have a complete set of idempotents $e_i = id_{M_i} \in \mathrm{End}(M_i)$ which lifts the idempotents in $A/\mathbf{J}$. By Prop. 11.1.1 of [32] the Jacobson radical of A is of the form

$$\mathbf{J}_{ii} = \mathrm{rad}(e_i A e_i) = \mathrm{rad}(\mathrm{End}_R M_i) \ \text{ and } \ \mathbf{J}_{ij} = e_i A e_j = \mathrm{Hom}(M_j, M_i) \text{ for } i \neq j.$$

We define the quiver of A to be the quiver of the Artinian algebra $A/\mathbf{J}^2$. To obtain the square of $\mathbf{J}$ one computes $\sum_k e_i \mathbf{J} e_k \mathbf{J} e_j$. The quiver of A has vertices corresponding to the M_i which we will label simply as i. The number of arrows from $i \to j$ is given by the length of $e_i \mathbf{J}/\mathbf{J}^2 e_j$. We already know that each vertex of the quiver corresponds to a simple module and also to its projective cover.

Projective covers: Semiperfect rings A are characterized by the fact that all finitely generated A modules have projective covers, Theorem 10.4.8 [32]. To describe the projective cover we introduce the notion of the *top of a module*, denoted by $\top M = M/\mathbf{J}M$. The top is the largest semi-simple quotient of M. If the $\top M = \oplus S_i^{n_i}$, then the projective cover of M is the projective cover of $\top M$, namely $\bigoplus_i P_i^{n_i} = \bigoplus_i (e_i A)^{n_i}$, see the remark proceeding Theorem 10.4.10 [32]. Furthermore, note that $\mathrm{Hom}(P_i, P_j) = e_j A e_i$.

Projective resolution of the simple modules: Choose a vertex i. Let S_i be the simple module at i. Its projective cover is $P_i = e_i A$. Write K_i for the kernel of the natural map $P_i \to S_i$. Now $K_i = \mathbf{J P_i}$ and so $\top K_i = (\mathbf{J} P_i)/(\mathbf{J}^2 P_i)$. So the projective cover of P_i is given by the projectives in the quiver of A with arrows $i \leftarrow j$. The projective resolution of S_i begins as follows:

$$\bigoplus_{j \leftarrow i} P_j \to P_i \to S_i \to 0.$$

The maps from $P_j \to P_i$ are lifts of arrows in $e_i(\mathbf{J}/\mathbf{J}^2)e_j$. Now to continue, one needs to compute the top of the kernel of this map and repeat.

Example 4.12. (Global dimension of Leuschke's endomorphism rings for the E_6 curve) Let $R = k[[x, y]]/(x^3 + y^4)$, where k is a field of characteristic 0, be the coordinate ring of the E_6-curve. It is well-known that R is of finite MCM-type, see [55, p.79f] for a description of the indecomposables and notation. One can write R as $k[[t^4, t^3]]$, where $x = t^4, y = t^3$ is a parametrization of the E_6-curve. Computing the chain of rings of [38] one obtains that $R_0 = R$, $R_1 = t^{-3}M_1$, where $M_1 = (x^2, y)$ and $R_2 = \widetilde{R} = t^{-6}B$, where $B = (x^2, xy, y^2)$. In parametric description R_1 is the semi-group generated by $1, t^3, t^4, t^5$, that is, R_1 is isomorphic to the coordinate ring of the singular space curve $k[[t^3, t^4, t^5]] \cong k[[x, y, z]]/(x^3-yz, y^2-xz, z^2-x^2y)$. The next ring $R_2 \cong k[[t]]$ is the semigroup generated by $1, t$. Write now $M = \bigoplus_{i=0}^2 R_i$. By [38] the endomorphism ring $A = \mathrm{End}_R M$ has finite global dimension. We claim that $\mathrm{gldim}\, A = 3$. By the above, it is sufficient to compute the projective resolutions of the three simples S_1, S_2, S_3. The matrix description of A is

$$A = \begin{pmatrix} R & \mathfrak{m}_R & t^6R_2 \\ R_1 & R_1 & t^3R_2 \\ R_2 & R_2 & R_2 \end{pmatrix},$$

where $\mathfrak{m}_R$ is the maximal ideal of R, that is, the semigroup $t^3, t^4, t^6, \dots$. For the computation of the quiver, we change to additive notation for the semigroups: write $m + \langle n_1, \dots, n_k \rangle$ as an ideal in a subalgebra of $k[[t]]$. Using this notation we have

$$A = \begin{pmatrix} \langle 0,3,4 \rangle & \langle 3,4 \rangle & 6 + \langle 0,1 \rangle \\ \langle 0,3,4,5 \rangle & \langle 0,3,4,5 \rangle & 3 + \langle 0,1 \rangle \\ \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 0,1 \rangle \end{pmatrix}.$$

The radical of A are the off-diagonal elements and the respective maximal ideals on the diagonal, see the discussion above:

$$\mathbf{J} = \begin{pmatrix} \langle 3,4 \rangle & \langle 3,4 \rangle & 6 + \langle 0,1 \rangle \\ \langle 0,3,4,5 \rangle & \langle 3,4,5 \rangle & 3 + \langle 0,1 \rangle \\ \langle 0,1 \rangle & \langle 0,1 \rangle & \langle 1 \rangle \end{pmatrix}$$

Hence we compute $\mathbf{J}/\mathbf{J}^2$ to be

$$\mathbf{J}/\mathbf{J}^2 = \begin{pmatrix} \cdot & 3,4 & \cdot \\ 0 & \cdot & 3 \\ \cdot & 0 & 1 \end{pmatrix}.$$

So the quiver of A is:

$$\begin{tikzcd} 1 \arrow[r, bend left=40, "t^4"] \arrow[r, bend left=15, "t^3"] & 2 \arrow[l, bend left=20, "1"] \arrow[r, bend left=20, "t^3"] & 3 \arrow[l, bend left=20, "1"] \arrow[loop right, "t"] \end{tikzcd}$$

where the relations are clear from the labels. Thus the three simples are $S_1 = (k, 0, 0), S_2 = (0, k, 0), S_3 = (0, 0, k)$. The minimal projective resolutions are as follows (with $P_i = e_iA$):

$$0 \longleftarrow S_1 \longleftarrow P_1 \xleftarrow{(t^3,t^4)} P_2 \oplus P_2 \xleftarrow{\begin{pmatrix} t^4 \\ -t^3 \end{pmatrix}} P_3 \longleftarrow 0,$$

$$0 \longleftarrow S_2 \longleftarrow P_2 \xleftarrow{(1,t^3)} P_1 \oplus P_3 \xleftarrow{\begin{pmatrix} t^3 & t^4 \\ -1 & -t \end{pmatrix}} P_2 \oplus P_2 \xleftarrow{\begin{pmatrix} t^4 \\ -t^3 \end{pmatrix}} P_3 \longleftarrow 0$$

$$0 \longleftarrow S_3 \longleftarrow P_3 \xleftarrow{(1,t)} P_2 \oplus P_3 \xleftarrow{\begin{pmatrix} t^3 \\ -t^2 \end{pmatrix}} P_3 \longleftarrow 0.$$

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