TOWARDS TRANSVERSALITY OF SINGULAR VARIETIES: SPLAYED DIVISORS

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ABSTRACT. We study a natural generalization of transversally intersecting smooth hypersurfaces in a complex manifold: hypersurfaces, whose components intersect in a transversal way but may be themselves singular. Such hypersurfaces will be called *splayed divisors*. A splayed divisor is characterized by a property of its Jacobian ideal. This yields an effective test for splayedness. Two further characterizations of a splayed divisor are shown, one reflecting the geometry of the intersection of its components, the other one using K. Saito's logarithmic derivations. As an application we prove that a union of smooth hypersurfaces has normal crossings if and only if it is a free divisor and has a radical Jacobian ideal. Further it is shown that the Hilbert–Samuel polynomials of splayed divisors satisfy the natural additivity property.

1. INTRODUCTION

Let M_1, M_2 be two submanifolds of a complex manifold S. Then M_1 and M_2 intersect *transversally* at a point $p \in S$ if their respective tangent spaces add up to the tangent space of S at p, that is,

$$(1) \qquad T_p M_1 + T_p M_2 = T_p S.$$

Transversality is a fundamental concept in algebraic geometry as well as in differential geometry. However, in many applications (e.g., embedded resolution of singularities, Hodge structures) one needs a notion of transversality for more than two subspaces. This leads to the notion of *normal crossings*, which means that several smooth components cross in a transversal way. Another way to phrase this is that the union of several smooth subspaces is isomorphic to a union of coordinate subspaces.

In this article we study a natural generalization of the concept of transversal intersection of two hypersurfaces in complex manifolds allowing singular components. The geometric idea is that two singular hypersurfaces D_1 and D_2 in a complex manifold S intersect transversally at a point p if their "tangent spaces" fill out the whole space and the ideal of their intersection is reduced. The notion of tangent space for singular hypersurfaces can be made precise by means of logarithmic derivations. In algebraic terms this means: one can find local coordinates at p such that the defining equations of the D_i can be chosen in separated variables. We call a union of such transversally intersecting hypersurfaces a *splayed divisor*. For an example in a threedimensional S, see fig. 1.
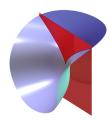
FIGURE 1. $D = \{x(y^2 - z^3) = 0\}$ (left) is splayed and $D' = \{x(x + y^2 - z^3) = 0\}$ (right) is not splayed.

In the first part of this work three different characterizations of splayed divisors are shown. Suppose that a divisor $D \subseteq S$ is given by $(D, p) = (D_1, p) \cup (D_2, p) = \{gh = 0\}$. First we consider the Jacobian ideal J_{gh} (the ideal generated by the partial derivatives of gh) of D at p. It is clear that for a splayed D, the Jacobian ideal satisfies

$$J_{gh} = gJ_h + hJ_g.$$

We show (Prop. 9) that this property already characterizes splayed divisors.
Further, for a splayed divisor D, the ideal $((gh) + J_{gh})$ defining its singular locus can be written as the intersection of the ideals defining the singular loci of the D_i and of the ideal defining the intersection $D_1 \cap D_2$:

$$((gh) + J_{gh}) = (g, h) \cap ((g) + J_g) \cap ((h) + J_h).$$

In Prop. 11 the sufficiency of this condition is proved.
Finally, equation (1) can be directly translated to singular subspaces via K. Saito's logarithmic derivations [Sai80], namely, D is splayed at p if and only if

$$\mathrm{Der}_{S,p}(\log D_1) + \mathrm{Der}_{S,p}(\log D_2) = \mathrm{Der}_{S,p}.$$

This is shown in Prop. 16.

In the second part, two applications are considered. First the relationship between splayed divisors and free divisors is studied. Free divisors are a generalization of normal crossing divisors and appear frequently in different areas of mathematics, e.g., in deformation theory as discriminants or in combinatorics as free hyperplane arrangements, see e.g. [Ale86, OT92, BEvB09, MS10, BM06, Sai81] for more examples. Then we give a partial answer to a question of H. Hauser about the characterization of normal crossing divisors by their Jacobian ideals: it is shown that if $D = \bigcup_{i=1}^n D_i$ is locally the union of smooth irreducible components then D has normal crossings if and only if D is locally free and its Jacobian ideal is radical. The computation of singularity invariants of splayed divisors is very interesting, in particular one should be able to deduce them from singularity invariants of their splayed components. We start with computing the Hilbert–Samuel polynomial $\chi_{D,p}$ of a splayed divisor $(D, p) = (D_1, p) \cup (D_2, p)$ and find that it satisfies an additivity relation: from the exact sequence

$$0 \to \mathcal{O}_{D,p} \to \mathcal{O}_{D_1,p} \oplus \mathcal{O}_{D_2,p} \to \mathcal{O}_{D_1 \cap D_2,p} \to 0$$

we deduce

$$\chi_{D,p}(t) + \chi_{D_1 \cap D_2,p}(t) = \chi_{D_1,p}(t) + \chi_{D_2,p}(t).$$

This additivity relation does not characterize splayed divisors, an example therefore is given at the end.

2. Setting and triviality lemma

We work in the complex analytic category. The main objects of our study are divisors (=hypersurfaces) in complex manifolds. We write (S, D) for a fixed divisor D in an n-dimensional complex manifold S. Denote by p a point in S and by $x = (x_1, \ldots, x_n)$ the complex coordinates of S around p. The divisor (D, p) will then be defined locally by an equation $\{h_p(x_1, \ldots, x_n) = 0\}$ where $h_p \in \mathcal{O}_{S,p} \cong \mathbb{C}\{x_1, \ldots, x_n\}$ (if the context is clear we leave the subscript $_p$). Note that we will always assume that h is reduced! To simplify the notation regarding splayed divisors, sometimes the coordinates are denoted by $(x_1, \ldots, x_k, y_{k+1}, \ldots, y_n)$ and $h(x, y) \in \mathbb{C}\{x, y\}$. The divisor D has *normal crossings* at a point p if one can find complex coordinates $(x_1, \ldots, x_n)$ at p such that the defining equation h of D is $h = x_1 \cdots x_k$ for $k \le n$. We also say that (D, p) is a *normal crossing divisor*. The *Jacobian ideal* of h is denoted by $J_h = (\partial_{x_1} h, \ldots, \partial_{x_n} h) \subseteq \mathcal{O}_{S,p}$. The image of J_h under the canonical epimorphism that sends $\mathcal{O}_{S,p}$ to $\mathcal{O}_{D,p} = \mathcal{O}_{S,p}/(h)$ is denoted by $\widetilde{J_h}$. The associated analytic coherent ideal sheaves are denoted by $J \subseteq \mathcal{O}_S$ and $\widetilde{J}$ in $\mathcal{O}_D$. The singular locus $\operatorname{Sing} D$ of D is defined by the ideal sheaf $\widetilde{J} \subseteq \mathcal{O}_D$. The singular locus $(\operatorname{Sing} D, p)$ is defined by the local ring $\mathcal{O}_{\operatorname{Sing} D, p} = \mathcal{O}_{S,p}/((h) + J_h)$.

The next lemma yields an ideal-theoretic characterization of Cartesian product structure. It is used frequently and can be found in various different formulations in the literature (e.g. in [dJP00, Sai80, GH85, CNM96]).

Lemma 1 (Triviality lemma)**.** *Let (S, p) locally be isomorphic to $(\mathbb{C}^{n+m}, 0)$ and denote $\mathcal{O}_{S,p} = \mathbb{C}\{x_1, \ldots, x_n, y_1, \ldots, y_m\}$ and let $h(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be an element of $\mathcal{O}_{S,p}$. Then the following are equivalent:*
(a) The ideal $(\partial_{y_1} h, \ldots, \partial_{y_m} h)$ is contained in the ideal $(h, \partial_{x_1} h, \ldots, \partial_{x_n} h)$.
(b) There exists a local biholomorphic map $\varphi : (\mathbb{C}^{n+m}, 0) \to (\mathbb{C}^{n+m}, 0)$ and a holomorphic $v(x, y) \in \mathcal{O}_{S,p}^*$ such that

$$\varphi(x, y) = (\varphi_1(x, y), \ldots, \varphi_n(x, y), y_1, \ldots, y_m),$$

$\varphi(x, 0) = (x, 0)$, $v(x, 0) \equiv 1$ and $h \circ \varphi(x, y) = v(x, y) h(x, 0)$.
This means that $D = \{h(x, y) = 0\}$ is locally at p isomorphic to some $(D' \times \mathbb{C}^m, (0, 0))$ where D' is locally contained in $\mathbb{C}^n$.

Remark 2. This lemma is called "triviality lemma" because a stronger form of it characterizes local analytic triviality: if in (a) all partial derivatives $\partial_{y_i} h$ are even contained in $(x_1, \ldots, x_n, y_1, \ldots, y_m)(\partial_{x_1} h, \ldots, \partial_{x_n} h)$, then one can show that $(D, p) \cong (D_0 \times \mathbb{C}^m, (p', 0))$ where $D_0 = \{h(x, 0) = 0\}$ is the "fibre" at the origin.

Proof. See e.g. [Sai80]. □

The objects of our studies are divisors that are unions of Cartesian products. We call them splayed because they fill out the whole space.

Definition 3. Let D be a divisor in a complex manifold S, $\dim S = n$. The divisor D is called *splayed* at a point $p \in S$ (or (D, p) is *splayed*) if one can find coordinates $(x_1, \ldots, x_n)$ at p such that $(D, p) = (D_1, p) \cup (D_2, p)$ is defined by

$$h(x) = h_1(x_1, \ldots, x_k) h_2(x_{k+1}, \ldots, x_n),$$

$1 \le k \le n - 1$, where h_i is the defining reduced equation of D_i. Note that the h_i are not necessarily irreducible. The *splayed components* D_1 and D_2 are not unique. Splayed means that D is the union of two products: since h_1 is independent of $x_{k+1}, \ldots, x_n$, the divisor D_1 is locally at p isomorphic to a product $(D_1', 0) \times (\mathbb{C}^{n-k}, 0)$, where $(D_1', 0) \subseteq (\mathbb{C}^k, 0)$ (and similar for D_2).

Example 4. (1) Let $(D, 0)$ be the divisor in $(\mathbb{C}^2, 0)$ defined by $h_1 h_2 = x(y - x^2)$. Since D has normal crossings at the origin, D is splayed.
(2) The divisor $D = \{(x - y^2)zw = 0\}$ is splayed in $(\mathbb{C}^4, 0)$ but its splayed components are not unique, e.g. $h_1 = x - y^2$ and $h_2 = zw$ or $h_1 = (x - y^2)w$ and $h_2 = z$.
(3) The divisor $D = \{(xz(x + z - y^2) = 0\}$ is not splayed in $(\mathbb{C}^3, 0)$.

Remark 5. The concept of splayed divisors was also introduced by J. Damon in the context of free divisors under the name product union, see [Dam96]. In the theory of hyperplane arrangements one studies the analogue concept, under various names: e.g., in [OT92] splayed arrangements are called reducible and in [Bud10] they are referred to as decomposable.

3. Criteria for splayed divisors

Let (D, p) be a divisor in (S, p). If the decomposition into irreducible components of D is known, then there are effective methods to test whether D has normal crossings at p, see [Bod04]. The idea here is to find linearly independent tangent vectors, that is, to linearize the problem and thus reduce it to linear algebra. The algorithm of Bodnár does not work in order to test whether (D, p) is splayed, since the irreducible components may be singular themselves at p.
In Prop. 9 it is shown that $D = D_1 \cup D_2 = \{gh = 0\}$ is splayed at p if and only if its Jacobian ideal J_{gh} of (D, p) is equal to $gJ_h + hJ_g$, where g, h are the defining equations of D_1, D_2, respectively. This property can be tested in computer algebra systems like SINGULAR. The second characterization is more geometrical. In Prop. 11 we show that for a splayed divisor $\operatorname{Sing} D$ is the scheme-theoretic union of the intersection of D_1 and D_2, and of the singular loci $\operatorname{Sing} D_1$ and $\operatorname{Sing} D_2$. The third criterion (Prop. 16) is nearer to Bodnár's normal crossings test. Therefore we use the notion of logarithmic derivations and differential forms (in the sense of K. Saito [Sai80]), which will be a main ingredient in the section about freeness of splayed divisors.

3.1. **Jacobian ideal characterization 1 - algebra.** Here a characterization of splayedness by Jacobian Ideals is shown, which can be easily checked in concrete examples.

Definition 6. Let $D_1 = \{g = 0\}$, $D_2 = \{h = 0\}$ and $D = \{gh = 0\}$ at p be defined as above. We say that J_{gh} satisfies the *Leibniz property* if

$$J_{gh} = gJ_h + hJ_g.$$

If D is splayed then it is clear that its Jacobian ideal satisfies the Leibniz property. To establish the other implication, first an intermediate ideal-theoretic characterization of splayedness is proven (Lemma 7).

Lemma 7. *Let* $\dim S = n$ *and at a point* $p = (x_1, \ldots, x_n)$ *denote by* $\mathcal{O} = \mathcal{O}_{S,p} = \mathbb{C}\{x_1, \ldots, x_n\}$ *the local ring at* p. *Let* $D_1 = \{g(x) = 0\}$, $D_2 = \{h(x) = 0\}$ *and*

$D = \{gh(x) = 0\}$ *be divisors, where we assume that* $g, h \in \mathcal{O}$ *are reduced and have no common factors. Then* $(D, p) = (D_1, p) \cup D_2, p)$ *is splayed if and only if*

$$(2) \qquad (g) \cap ((gh) + J_{gh}) = g((h) + J_h).$$

Proof. First note that the inclusion $(g) \cap ((gh) + J_{gh}) \subseteq g((h) + J_h)$ always holds since g and h have no common factors.

If D is splayed, we can suppose wlog. that $D_1 = \{g(x, 0) = 0\}$ and $D_2 = \{h(0, y) = 0\}$ where $(x, y) = (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n)$. Then it is easy to see, via the Leibniz property of J_{gh}, that $g((h) + J_h)$ is contained in $(g) \cap ((gh) + J_{gh})$. For the other direction, suppose that (2) holds. In this case it is easy to see that (2) is preserved under local isomorphisms of (S, p). One may assume wlog. that $h(x_1, \ldots, y_n) = h(0, \ldots, 0, y_{k+1}, \ldots, y_n)$ and that

$$(3) \qquad \partial_{y_i} h \notin (h, \partial_{y_{k+1}} h, \ldots, \widehat{\partial_{y_i} h}, \ldots, \partial_{y_n} h)$$

for all $i \in \{k + 1, \ldots, n\}$. (If not so, suppose that e.g. $\partial_{x_1} h$ is contained in $(h, \partial_{x_2} h, \ldots, \partial_{x_n} h)$. Then by the triviality lemma there exists a locally biholomorphic map $\varphi : (S, p) \to (S, p)$ such that $h \circ \varphi(x) = v(x)h(0, x_2, \ldots, x_n)$, with $v \in \mathcal{O}^*$. Then set $\widetilde{h} := h(0, x_2, \ldots, x_n)$ and $\widetilde{g} := g \circ \varphi$. The divisor defined by $\widetilde{g} \cdot \widetilde{h}$ is clearly isomorphic to D, and similarly D_1 and D_2 are defined by $\widetilde{g}$ and $\widetilde{h}$.)

Now let $h(y) = h(0, \ldots, 0, y_{k+1}, \ldots, y_n)$ such that (3) holds. From (2) it follows that $g(\partial_{y_i} h)$ is contained in $((gh) + J_{gh})$ and by definition $g\partial_{y_i} h + h\partial_{y_i} g$ is also contained in this ideal. Hence $h(\partial_{y_i} g) \in ((gh) + J_{gh})$ for all $k + 1 \le i \le n$. Thus we can write

$$h(\partial_{y_i} g) = a_i gh + \sum_j a_{ij}(g\partial_{y_j} h + h\partial_{y_j} g) + \sum_j b_{ij} h\partial_{x_j} g$$

where the summation indices run over the valid range. This equation can be rearranged as

$$(4) \qquad h(\partial_{y_i} g - a_i g - \sum_j a_{ij}\partial_{y_j} g - \sum_j b_{ij}\partial_{x_j} g) = g\sum_j a_{ij}(\partial_{y_j} h).$$

Since g and h have by assumption no common factors, we conclude from (4) that $\partial_{y_i} g - a_i g - \sum_j a_{ij}\partial_{y_j} g - \sum_j b_{ij}\partial_{x_j} g \in (g)$ and that $\sum_j a_{ij}(\partial_{y_j} h) = c_i h$ for any $k + 1 \le i \le n$ and some $c_i \in \mathcal{O}$. Then (3) implies that any $a_{ij} \in \mathfrak{m}$: if this were not the case, that is, if some $a_{im} \in \mathcal{O}^*$, then via the equation $\sum_j a_{ij}(\partial_{y_j} h) = c_i h$ one can express $\partial_{y_m} h$ in terms of h and the other $\partial_{y_j} h$, which is a contradiction to (3). Hence it follows that

$$\partial_{y_i} g \in (g, \partial_{x_1} g, \ldots, \partial_{x_k} g) + \mathfrak{m}(\partial_{y_{k+1}} g, \ldots, \partial_{y_n} g)$$

for any $k + 1 \le i \le n$. By the lemma below we obtain

$$\partial_{y_i} g \in (g, \partial_{x_1} g, \ldots, \partial_{x_k} g) \text{ for all } i = k + 1, \ldots, n.$$

By the triviality lemma there exists a locally biholomorphic $\psi : (S, p) \to (S, p)$ with $\psi(x, y) = (\psi_1(x, y), \ldots, \psi_k(x, y), y_{k+1}, \ldots, y_n)$ such that $g \circ \psi$ is equal to $vg(x_1, \ldots, x_k, 0)$ for a unit $v \in \mathcal{O}$. Set $\tilde{g} := v^{-1}(g \circ \psi)$ and $\tilde{h} := h \circ \psi = h = h(0, \ldots, 0, y_{k+1}, \ldots, y_n)$. By construction $\tilde{g}\tilde{h}$ defines a splayed divisor that is isomorphic to D.

$\square$

Lemma 8. *Let* $(R, \mathfrak{m})$ *be a local ring, let* $a_1, \ldots, a_k$ *be elements in* R *and let* I *be an ideal of* R. *Suppose for all* a_i *the following holds:*

$$(5) \qquad a_i \in I + \mathfrak{m}(a_1, \ldots, a_k),$$

all a_i *already lie in* I.

Proof. Since $I \subseteq \mathfrak{m}$ we can reduce modulo I and may thus assume that $I = 0$. Then (5) implies $(a_1, \ldots, a_k) \subseteq \mathfrak{m}(a_1, \ldots, a_k)$ and an application of Nakayama's lemma shows $(a_1, \ldots, a_k) = 0$. Thus, for nonzero I, all a_i lie in I. $\qquad\square$

Proposition 9. *Let (S, D) be a complex manifold S, $\dim S = n$, together with a divisor $D \subseteq S$ that is locally at a point defined by $\{gh = 0\}$, where g and h are reduced elements of $\mathcal{O}_{S,p}$ that are not necessarily irreducible but have no common factor. Then $D = \{g = 0\} \cup \{h = 0\}$ is splayed at p if and only if J_{gh} satisfies the Leibniz property*

$$J_{gh} = gJ_h + hJ_g.$$

Proof. As already remarked, if D is splayed then its clear that $J_{gh} = gJ_h + hJ_g$. Conversely, it is enough to show the equality $(g) \cap ((gh) + J_{gh}) = g((h) + J_h)$ from lemma 7. Since $(g) \cap ((gh) + J_{gh})$ is always contained in $g((h) + J_h)$ it remains to check the other inclusion: take a $\beta \in g((h) + J_h)$. Then a straightforward calculation shows that β is also contained in $(g) \cap ((gh) + J_{gh})$. $\qquad\square$

Example 10. Let D be the divisor in $\mathbb{C}^3$ given at a point p by $x(x + y^2 - z^3)$. Then D is the union of two smooth components $D_1 = \{h = x + y^2 - z^3 = 0\}$ and $H = \{g = x = 0\}$. The ideal $J_{gh} = (2x + y^2 - z^3, xy, xz^2)$ is strictly contained in $gJ_h + hJ_g = (x, y^2 - z^3)$. Thus D is not a splayed divisor (see fig. 1).

3.2. **Jacobian ideal characterization 2 - geometry.** Now we characterize a splayed divisor $D \subseteq S$, locally at a point p given by a $gh \in \mathcal{O}_{S,p}$, by $\mathcal{O}_{\operatorname{Sing} D,p} = \mathcal{O}_{S,p}/((gh) + J_{gh})$. This characterization reflects the geometry of (D, p), namely the two splayed components meeting transversally.

Proposition 11. *The divisor $D = D_1 \cup D_2$, defined at p as above, is splayed if and only if*

$$(6) \qquad\qquad ((gh) + J_{gh}) = (g, h) \cap ((g) + J_g) \cap ((h) + J_h).$$

Proof. Recall here from lemma 7 that a divisor $\{gh = 0\}$ is splayed if and only if (2) holds. First suppose that (6) holds. A straightforward calculation shows (2).
For the other implication we use Grauert's division theorem (for the notation and statement see [dJP00, Theorem 7.1.9]): Suppose that $D = \{gh = 0\}$ is splayed. Then wlog. $g(x, y) = g(x)$ and $h(x, y) = h(y)$ in $\mathbb{C}\{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$. Clearly $((gh) + J_{gh}) \subseteq (g, h) \cap ((g) + J_g) \cap ((h) + J_h)$. So let α be an element of the right-hand side, that is, $\alpha = ag + bh = cg + \sum_{i=1}^{k} a_i \partial_{x_i} g$ for some $a, b, c, a_i \in \mathcal{O}$. Then also $\alpha - ag = bh = (c - a)g + \sum_{i=1}^{k} a_i \partial_{x_i} g$ is contained in $((g) + J_g)$. By Grauert's division theorem here exist some $\tilde{a}, \tilde{a}_i, r, \tilde{r}_i$ such that for all $i = 1, \ldots, k$ one has $c - a = \tilde{a}h + r$ and $a_i = \tilde{a}_i h + r_i$ and the leading monomial $L(h)$ does not divide any monomial of the unique remainders r, r_i. Then write

$$(b - \tilde{a}g - \sum_{i=1}^{k} \tilde{a}_i(\partial_{x_i}g))h = rg + \sum_{i=1}^{k} r_i(\partial_{x_i}g).$$

Since h only depends on y and g only on x, it follows that $L(h)$ does also not divide any of the monomials of the right-hand side of the equation. But this is only possible if $(b - \tilde{a}g - \sum_{i=1}^{k} \tilde{a}_i(\partial_{x_i}g))h = 0$. It follows that b is contained in $((g) + J_g)$. Interchanging the roles of g and h yields that $a \in ((h) + J_h)$ and thus $\alpha \in (gh, gJ_h + hJ_g)$. The Leibniz property of J_{gh} implies that $\alpha \in ((gh) + J_{gh})$. $\quad\square$

Remark 12. Here it can be seen that for splayed divisors the Jacobian ideal is the intersection of the two ideals defining the singular loci of the splayed components D_1 and D_2 plus the intersection of D_1 and D_2. For two smooth divisors D_1 and

D_2 this means that $D = D_1 \cup D_2$ is a splayed divisor if and only if the scheme-theoretical intersection $D_1 \cap D_2$ is smooth, which is in turn equivalent to saying that D_1 and D_2 intersect transversally, (cf. [Li09, FH10]). Note that for varieties of arbitrary codimension the condition that the scheme-theoretical intersection is smooth, leads to the formulation of *clean intersection*, see [Li09].

Example 13. Let $D \subseteq \mathbb{C}^3$ be given at the origin by $gh = x(x + y^2 + z^3) = 0$. Then D is not splayed at the origin. Here the intersection of the two components is given by the ideal $(g, h) = (x, y^2 + z^3)$. Also consider the divisor $D' \subseteq \mathbb{C}^3$ that is given by $g'h' = x(y^2 + z^3)$. Clearly D' is splayed at the origin and the intersection of the two components is given by the ideal $(g', h') = (x, y^2 + z^3)$, see fig. 1.

3.3. **Logarithmic derivation characterization - tangency.** As already mentioned in the introduction, two submanifolds of a manifold intersect transversally at a point if their respective tangent spaces at that point together generate the tangent space of the ambient manifold. In this section we will translate this definition for splayed divisors with the help of logarithmic derivations (Prop. 16). Logarithmic derivations were introduced by K. Saito in [Sai80] and are a useful tool in studying tangent behaviour for singular varieties. They lead to the definition of free divisors, which will be studied in more detail in section 4.1.

3.3.1. *A very brief recap of Saito's theory of free divisors (*[Sai80]*).* Let D be a divisor in S defined at p by $D = \{h = 0\}$. A *logarithmic vector field* (or *logarithmic derivation*) (along D) is a holomorphic vector field on S, that is, an element of Der_S, satisfying one of the two equivalent conditions:
(i) For any smooth point p of D, the vector $\delta(p)$ of p is tangent to D,
(ii) For any point p, where (D, p) is given by $h = 0$, the germ $\delta(h)$ is contained in the ideal (h) of $\mathcal{O}_{S,p}$. The module of germs of logarithmic derivations (along D) at p is denoted by

$$\mathrm{Der}_{S,p}(\log D) = \{\delta : \delta \in \mathrm{Der}_{S,p} \text{ such that } \delta h \in (h)\},$$

The $\mathrm{Der}_{S,p}(\log D)$ are the stalks at points p of the sheaf $\mathrm{Der}_S(\log D)$ of $\mathcal{O}_S$-modules. Similarly we define logarithmic differential forms: a meromorphic q-form ω is logarithmic (along D) at a point p if ωh and $hd\omega$ are holomorphic in an open neighbourhood around p. We denote

$$\Omega^q_{S,p}(\log D) = \{\omega : \omega \text{ germ of a logarithmic } q\text{-form at } p\},$$

One can show that $\mathrm{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$ are reflexive $\mathcal{O}_{S,p}$-modules dual to each other (see [Sai80]). One says that (D, p) is *free* or that D is free at p if $\mathrm{Der}_{S,p}(\log D)$ resp. $\Omega^1_{S,p}((\log D)$ is a free $\mathcal{O}_{S,p}$-module. In section 4.1 free divisors are discussed in more detail and the following theorem is used, which makes it possible to test whether D is free (cf. [Sai80, Thm.1.8]):

Theorem 14 (Saito's criterion)**.** *Let (S, D), p and h be as defined in the introduction. The $\mathcal{O}_{S,p}$-module $\mathrm{Der}_{S,p}(\log D)$ is free if and only if there exist n vector fields $\delta_i = \sum_{j=1}^n a_{ij}(x)\partial_{x_j}$ in $\mathrm{Der}_{S,p}(\log D)$, $i = 1, \ldots, n$, such that $\det(a_{ij}(x))$ is equal to h up to an invertible factor. Moreover, then the vector fields $\delta_1, \ldots, \delta_n$ form a basis for $\mathrm{Der}_{S,p}(\log D)$.*

Remark 15. The module $\mathrm{Der}_{S,p}(\log D)$ naturally carries the structure of a Lie algebra. This was considered under the name *tangent algebra* in [HM93], where also many properties of algebraic and analytic varieties were characterized in terms of tangent algebras. The Lie algebra structure of logarithmic derivations is also considered in [GS09, Sek08].

3.3.2. *Logarithmic derivations and splayed divisors.*

Proposition 16. *Let $(D, p) = (D_1, p) \cup (D_2, p) \subseteq (S, p) \cong (\mathbb{C}^n, 0)$ be a divisor such that $(D_1, p) = \{g = 0\}$ and $(D_2, p) = \{h = 0\}$ for some g, h not necessarily irreducible but with no common factor. Then D is splayed at p if and only if*

$$(7) \qquad \qquad \mathrm{Der}_{S,p}(\log D_1) + \mathrm{Der}_{S,p}(\log D_2) = \mathrm{Der}_{S,p}.$$

Proof. As above, we denote by $(x, y) := (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n)$ the coordinates at p and by $\mathcal{O} = \mathbb{C}\{x, y\}$ the corresponding local ring. If D is splayed then wlog. one has $g = g(x, 0)$ and $\mathrm{Der}_{S,p}(\log D_1)$ is generated by some $\delta_1, \ldots, \delta_m$, $m \geq k$, $\partial_{y_{k+1}}, \ldots, \partial_{y_n}$. Similarly $h = h(0, y)$ and $\mathrm{Der}_{S,p}(\log D_2)$ is generated by some $\partial_{x_1}, \ldots, \partial_{x_k}, \varepsilon_1, \ldots, \varepsilon_l$ for some $l \geq n - k$. Then clearly we have

$$\mathrm{Der}_{S,p}(\log D_1) + \mathrm{Der}_{S,p}(\log D_2) = \mathcal{O}\langle \partial_{x_1}, \ldots, \partial_{x_k}, \partial_{y_{k+1}}, \ldots, \partial_{y_n} \rangle = \mathrm{Der}_{S,p}.$$

For the other implication we may assume that $g(x, y) = g(x, 0)$ and

$$\partial_{x_i} g \notin (\partial_{x_1} g, \ldots, \widehat{\partial_{x_i} g}, \ldots, \partial_{x_k} g)$$

for all $i \leq k$ by a similar argument as in lemma 7. Then it is clear that $\mathrm{Der}_{S,p}(\log D_1)$ can be generated by some $\delta_1, \ldots, \delta_m$ and $\partial_{y_{k+1}}, \ldots, \partial_{y_n}$, where $m \geq k$ and the coefficients of the δ_i only depend on $x_1, \ldots, x_k$ and lie in the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}$. Using $\partial_{x_i} \in {}_{\mathcal{O}}\langle \varepsilon_1, \ldots, \varepsilon_l, \delta_1, \ldots, \delta_m, \partial_{y_{k+1}}, \ldots, \partial_{y_n} \rangle$ and computing $\partial_{x_i} h$ we obtain (similar to the proof of lemma 7) that

$$(\partial_{x_1} h, \ldots, \partial_{x_k} h) \subseteq (h, \partial_{y_{k+1}} h, \ldots, \partial_{y_n} h).$$

An application of the triviality lemma shows that D_2 can be chosen independently of the variables $x_1, \ldots, x_k$, and thus that D is splayed. $\qquad \square$

Question 17. *In view of Proposition 16 and the duality of $\mathrm{Der}_{S,p}(\log D)$ and $\Omega^1_{S,p}(\log D)$ we ask: can one express the splayedness of a divisor $(D_1 \cup D_2, p)$ in terms of $\Omega^1_{S,p}(\log D_1)$, $\Omega^1_{S,p}(\log D_2)$ and $\Omega^1_{S,p}$?*

4. Applications

4.1. **Free divisors and normal crossings.** In some sense free divisors are a generalization of normal crossing divisors: their modules of tangent vector fields are free. However, irreducible (non-smooth) free divisors are highly singular, that is, they are non-normal and their singular loci are Cohen–Macaulay, see [Ale90, Sim06]. Here we consider the relationship between free and splayed divisors and in particular between splayed and normal crossing divisors. First we show that a splayed divisor is free if and only if its splayed components are free (Prop. 18). Then we turn to normal crossing divisors, in particular to the problem of characterizing normal crossing divisors by their Jacobian ideal. This problem was stated by H. Hauser and considered in [Fab11], where a general answer was found. Here we use the ideal theoretic characterizations of splayed divisors in order to show that a divisor consisting of smooth irreducible components has normal crossings at a point p if and only if it is free at p and its Jacobian ideal is radical (Cor. 24).

First let us describe the structure of $\mathrm{Der}_{S,p}(\log D)$ for splayed divisors in a different way: let S, T be complex manifolds of dimensions n, m and suppose that $(S \times T, 0) \cong (\mathbb{C}^{n+m}, 0)$, with complex coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$ at the origin. Let $(D_1^x, 0)$ be a divisor in $(S, 0)$, which is defined by a reduced $g'(x) \in \mathcal{O}_{S,0} \cong \mathbb{C}\{x_1, \ldots, x_n\}$ and which has a logarithmic derivation module over $\mathbb{C}\{x\}$ denoted by $\mathrm{Der}_{S,0}(\log D_1^x)$. Then we may consider the cylinder over D_1^x in the T-direction

in $(S \times T, 0)$, namely the hypersurface D_1 defined by $g(x, y) = g(x, 0) := g'(x) \in \mathbb{C}\{x, y\}$. It is easy to see that

$$\mathrm{Der}_{S \times T, 0}(\log D_1) = (\mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x, y\}) \oplus (\mathrm{Der}_{T,0} \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x, y\}).$$

Similarly define D_2^y and D_2 with equations $h'(y) = h(x, y)$ and also $\mathrm{Der}_{S \times T, 0}(\log D_2)$. Thus we define the (splayed) divisor $D = D_1 \cup D_2$ in $S \times T$ that is given at 0 by the equation $gh = 0$. Since g and h have separated variables, there is a natural splitting of $\mathrm{Der}_{S \times T, 0}(\log D)$:

$$(8) \quad \mathrm{Der}_{S \times T, 0}(\log D) = (\mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x, y\}) \oplus (\mathrm{Der}_{T,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x, y\}).$$

Proposition 18. *Let $(D, p) = (D_1, p) \cup (D_2, p)$ be a splayed divisor in $S \cong \mathbb{C}^{n+m}$ defined as above. The divisor $D = \{g(x)h(y) = 0\}$ is free if and only if $D_1 = \{g(x) = 0\}$ and $D_2 = \{h(y) = 0\}$ are both free.*

Proof. If both D_1 and D_2 are free then there exist bases of $\mathrm{Der}_{S \times T, p}(\log D_1)$ and $\mathrm{Der}_{S \times T, p}(\log D_2)$ of the form

$$\delta_1 = \sum_{i=1}^{n} a_{1i}\partial_{x_i}, \ldots, \delta_n = \sum_{i=1}^{n} a_{ni}\partial_{x_i}, \delta_{n+1} = \partial_{y_1}, \ldots, \delta_{n+m} = \partial_{y_m}$$

and

$$\varepsilon_1 = \partial_{x_1}, \ldots, \varepsilon_n = \partial_{x_n}, \varepsilon_{n+1} = \sum_{i=1}^{m} b_{n+1,i}\partial_{y_i}, \ldots, \varepsilon_{n+m} = \sum_{i=1}^{m} b_{n+m,i}\partial_{y_i}.$$

It is easy to see that any δ_i for $1 \leq i \leq n$ and any ε_j for $n+1 \leq j \leq n+m$ is also an element of $\mathrm{Der}_{S \times T, p}(\log D)$. By Saito's criterion (Thm. 14) it follows that $\delta_1, \ldots, \delta_n, \varepsilon_{n+1}, \ldots, \varepsilon_{n+m}$ form a basis of $\mathrm{Der}_{S \times T, p}(\log D)$. Conversely, suppose that $\mathrm{Der}_{S \times T, p}(\log D)$ is free. Since $\mathrm{Der}_{S \times T, p}(\log D)$ is free, it follows by (8) that $\mathrm{Der}_{S,0}(\log D_1^x) \otimes_{\mathbb{C}\{x\}} \mathbb{C}\{x, y\}$ and $\mathrm{Der}_{T,0}(\log D_2^y) \otimes_{\mathbb{C}\{y\}} \mathbb{C}\{x, y\}$ are projective $\mathcal{O}_{S \times T, p}$-modules. Since the notion of projective and free module over regular local rings coincide, these two modules are even free. $\square$

4.1.1. *Radical Jacobian ideal - normal crossings.* In [Fab11] the following problem (stated by H. Hauser) was considered: suppose that $D \subseteq S$, $\dim S = n$, is a divisor that is locally at a point $p \in S$ given by $\{h = 0\}$. Can we determine if D has normal crossings at p by the knowledge of its Jacobian ideal J_h?
It was shown that D has normal crossings in case $\mathcal{O}_{\mathrm{Sing}\,D}$ is Cohen–Macaulay of dimension $n - 2$, the Jacobian ideal is radical and the normalization of D is smooth (see [Fab11, Thm. 2.1]).

Here we use splayed divisors to show this theorem for the case where $(D, p) = \bigcup_{i=1}^{m}(D_i, p)$ is a union of *smooth* irreducible components D_i. Note that in this special case the hypothesis on the normalization is not needed. Moreover, the problem can be reduced to the case where (D, p) is irreducible (Prop. 23).

Remark 19. One can show, using the theorem of Briançon–Skoda that J_h radical implies that h is already contained in J_h, see [LT81].

Lemma 20. *Let $D = \{gh = 0\}$ be splayed at a point $p = (x_1, \ldots, x_k, y_{k+1}, \ldots, y_n)$ in $S \cong \mathbb{C}^n$ and denote by $D_1 = \{g = g(x, 0) = 0\}$ and $D_2 = \{h = h(0, y) = 0\}$ its splayed components. The Jacobian ideal of D, denoted by $J_{gh} = (\partial_{x_1}gh + g\partial_{x_1}h, \ldots, \partial_{y_n}gh + g\partial_{y_n}h)$ is radical if and only if both J_h and J_g are also radical.*

Proof. Suppose that J_g and J_h are radical. Similarly to the proof of Prop. 11 one concludes that the ideals J_{gh} and $(g,h) \cap J_g \cap J_h$ are equal. We compute the radical

$$\sqrt{J_{gh}} = \sqrt{(g,h) \cap J_g \cap J_h} = \sqrt{(g,h)} \cap \sqrt{J_g} \cap \sqrt{J_h} = (g,h) \cap J_g \cap J_h = J_{gh},$$

where the third equality holds because of our assumptions. Conversely, suppose that $J_{gh} = \sqrt{J_{gh}}$. Then gh is an element of J_{gh} (cf. Remark 19). Localization of $\mathbb{C}\{x,y\}$ in g yields $(J_{gh})_g = ((h) + J_h)_g$, which is radical, since J_{gh} is radical. Note that for an ideal $I \subseteq \mathbb{C}\{x,y\}$, we denote by I_g the localization of I in g (cf. [Mat86]). Using that $((h) + J_h)_g$ is an intersection of prime ideals in the localization, a direct computation shows that any element of $\sqrt{((h) + J_h)}$ is already contained in $((h) + J_h)$. Similarly one proves $((g) + J_g) = \sqrt{((g) + J_g)}$. □

Lemma 21. *Let $D \subseteq S$ be a divisor given at $p \in S$ by $\{gh = 0\}$ with $gh \in \mathcal{O}_{S,p}$ reduced and suppose that J_{gh} is radical. Then*

$$J_{gh} = gJ_h + hJ_g.$$

In particular, $D = D_1 \cup D_2$ defined by $D_1 = \{g = 0\}$ and $D_2 = \{h = 0\}$ is splayed at p.

Proof. Consider the element $g\partial_{x_i}(gh)$, which is contained in J_{gh} for any $i = 1, \ldots, n$. Using that J_{gh} is an intersection of prime ideals, one obtains that any $g(\partial_{x_i} h)$ is contained in J_{gh}. This yields

$$J_{gh} = (g\partial_{x_1}(h), \ldots, g\partial_{x_n}(h), h\partial_{x_1}(g), \ldots, h\partial_{x_n}(g)) = gJ_h + hJ_g.$$

By Prop. 9 (D, p) is splayed. □

Example 22. Splayed divisors need not have radical Jacobian ideals, as the following example shows. Let D be the divisor in $(\mathbb{C}^3, 0)$ with coordinates (x, y, z) at 0, that is defined by $gh = x(y^2 + z^3)$. Then clearly $(D, 0)$ is splayed. The Jacobian ideal is $J_{gh} = (y^2 + z^3, xy, xz^2) = (y, z^2) \cap (x, y^2 + z^3)$, which is not radical. Note that D is a free divisor.

Proposition 23. *Let $D = D_1 \cup D_2$ be a divisor in an n dimensional complex manifold S and let D, D_1 and D_2 at a point $p \in S$ be defined by the equations gh, g and h, respectively. Suppose that J_{gh} is radical. Then D is splayed and J_h and J_g are also radical. If moreover D is free at p then also D_1 and D_2 are free at p.*

Proof. This follows from proposition 9, proposition 18 and lemma 21. □

Corollary 24. *Let (S, D) be a complex manifold, $\dim S = n$, together with a divisor $D \subseteq S$ and suppose that locally at a point $p \in S$ the divisor (D, p) has a decomposition into irreducible components $\bigcup_{i=1}^{m}(D_i, p)$ such that each (D_i, p) is smooth. Let the corresponding equation of D at p be $h = h_1 \cdots h_m$. If D is free at p and $J_h = \sqrt{J_h}$ then D has normal crossings at p.*

Proof. We use induction on n. If $n = 2$, then an explicit computation or an application of the theorem of Mather–Yau ([MY82]) shows the assertion. Now suppose the assertion is true for divisors in manifolds of dimension $n - 1$. For a smooth component D_1 of D, one can find local coordinates $(x_1, \ldots, x_n)$ such that $D_1 = \{x_1 = 0\}$. Prop. 23 shows that the divisor $(D \backslash D_1) := h_2 \cdots h_m$ is also free and has a radical Jacobian ideal. Moreover, D is locally splayed, that is, $D \backslash D_1$ is locally isomorphic to some divisor depending only on the last $n - 1$ coordinates. Thus by induction hypothesis $D \backslash D_1$ is isomorphic to a normal crossings divisor $y_2 \cdots y_m = 0$, where the y_i are the result of a coordinate transformation of $(x_1, \ldots, x_n)$ such that $x_1 = y_1$. This implies that $m \leq n - 1$. Hence D is isomorphic to the normal crossings divisor $x_1 y_2 \cdots y_m$. □

4.2. **Hilbert–Samuel polynomials.** Splayed divisors are particularly interesting for computational reasons. We start here a study of properties of splayed divisors by considering their Hilbert–Samuel polynomials. We find that multiplicities behave the same for splayed as for non–splayed divisors but that the Hilbert–Samuel polynomials for splayed divisors are additive, which means the following: let $(D, p) = (D_1, p) \cup (D_2, p)$ be a splayed divisor at a point p in a complex manifold. Then from the exact sequence

$$0 \to \mathcal{O}_{D,p} \to \mathcal{O}_{D_1,p} \oplus \mathcal{O}_{D_2,p} \to \mathcal{O}_{D_1 \cap D_2,p} \to 0$$

follows

$$\chi_{D,p} + \chi_{D_1 \cap D_2,p} = \chi_{D_1,p} + \chi_{D_2,p},$$

where $\chi_{D,p}$ denotes the Hilbert–Samuel polynomial of D at p. Since one can compute Hilbert–Samuel polynomials of divisors in an easy way ([dJP00, Lemma 4.2.20]), this yields a method to compute $\chi_{D_1 \cap D_2,p}$.

The Hilbert–Samuel function and the Hilbert–Samuel polynomial do not only depend on the module one is considering but also on a chosen filtration. However, the degree and leading coefficient of the Hilbert–Samuel polynomial are independent of the filtration. We use notation from [Mat86] and [dJP00].

Let R be a noetherian local ring with maximal ideal $\mathfrak{m}$. Let $I \subseteq R$ be an ideal and let M be a module over R. A set $\{M_n\}_{n \geq 0}$ of submodules of M is called an *I-filtration* of M if $M = M_0 \supset M_1 \supset M_2 \supset \ldots$ and $IM_n \subseteq M_{n+1}$ for all $n \geq 0$. In the following we will always suppose that M is a finitely generated R module. Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of R and let $\{M_i\}$ be a $\mathfrak{q}$-filtration. Then the *Hilbert–Samuel function of the filtration* $\{M_i\}$ is

$$HS_{\{M_i\}} : \mathbb{N} \to \mathbb{N}, d \mapsto \text{length}_{R/\mathfrak{m}} M/M_d.$$

If $M_i = \mathfrak{m}^i M$ then we simply write HS_M. One can show that there exists a polynomial $\chi_{\{M_i\}}$ with rational coefficients such that $HS_{\{M_i\}}(d) = \chi_{\{M_i\}}(d)$ for d sufficiently large. We call $\chi_M^{\mathfrak{q}} := \chi_{\{\mathfrak{q}^n M\}_{n \geq 0}}$ the *Hilbert–Samuel polynomial of M with respect to* $\mathfrak{q}$. The degree d of $\chi_M^{\mathfrak{q}}(k) = \sum_{i=0}^{d} a_i k^i$ only depends on M and not on $\mathfrak{q}$.

When considering Hilbert–Samuel polynomials of modules over a local ring $\mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$ w.r.t. the maximal ideal $\mathfrak{m}$, one can use standard bases to simplify computations. For definitions and notation about standard bases we refer to [dJP00]. Here we only need the following facts:

Fact (i) Take the degree lexicographical ordering $<$. Then for an ideal $I \subseteq \mathcal{O}$ one has

$$HS_{\mathcal{O}/I}(k) = HS_{\mathcal{O}/L(I)}(k),$$

where $L(I)$ denotes the leading ideal of I. In particular, $\mathcal{O}/I$ and $\mathcal{O}/L(I)$ have the same Hilbert–Samuel polynomial with respect to $\mathfrak{m}$.

Fact (ii) Let $f, g \in \mathcal{O}$ and assume that $L(f)$ and $L(g)$ are coprime. Then $L((f,g)) = (L(f), L(g))$, that is, f, g are a standard basis of the ideal (f,g).

In contrast to the graded case, the Hilbert–Samuel polynomial is not additive on exact sequences, one has a certain error polynomial, whose degree can be determined with a theorem of Flenner and Vogel, see [FV93]:

Theorem 25 (Flenner–Vogel)**.** *Let $(R, \mathfrak{m})$ be a noetherian local ring, $0 \to M' \to M \to M'' \to 0$ an exact sequence of finitely generated R-modules and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal. Denote further*

$$\mathrm{Gr}_{\mathfrak{q}}(N) = \bigoplus_{i=0}^{\infty} \mathfrak{q}^i N / \mathfrak{q}^{i+1} N$$

the associated graded ring of a finite R-module N. Then the following holds:
(a) $\mathrm{supp}\,\ker(\mathrm{Gr}_{\mathfrak{q}}(M') \to \mathrm{Gr}_{\mathfrak{q}}(M)) = \mathrm{supp}\,\ker(\mathrm{Gr}_{\mathfrak{q}}(M)/\mathrm{Gr}_{\mathfrak{q}}(M') \to \mathrm{Gr}_{\mathfrak{q}}(M''))$.
(b) Denote d the dimension of these supports. Then for all $n \geq 0$

$$S(n) := \chi_{M'}^{\mathfrak{q}}(n) + \chi_{M''}^{\mathfrak{q}}(n) - \chi_M^{\mathfrak{q}}(n),$$

where $S(n)$ is a polynomial of degree $d - 1$ for $n \gg 0$. In particular, if $d = 0$, then set $S = 0$.

Before turning to splayed divisors, consider the additivity problem of the Hilbert–Samuel polynomial for arbitrary finitely generated modules over a local ring $(R, \mathfrak{m})$. If

$$0 \to N \to M \to M/N \to 0$$

is an exact sequence of finitely generated R-modules and $\mathfrak{q}$ an $\mathfrak{m}$-primary ideal, then

$$(9) \qquad 0 \to N/(\mathfrak{q}^n M \cap N) \to M/\mathfrak{q}^n M \to (M/N)/\mathfrak{q}^n(M/N) \to 0$$

is an exact sequence, which implies

$$\chi_M^{\mathfrak{q}} = \chi_{M/N}^{\mathfrak{q}} + \chi_{\{\mathfrak{q}^n M \cap N\}}.$$

Here $\chi_{\{\mathfrak{q}^n M \cap N\}}$ denotes the Hilbert–Samuel polynomial w.r.t. the filtration $N/(\mathfrak{q}^n M \cap N)$ and in general one has $\chi_N^{\mathfrak{q}} \neq \chi_{\{\mathfrak{q}^n M \cap N\}}$. However, for split exact sequences the Hilbert–Samuel polynomial is always additive:

Lemma 26. *Let $(R, \mathfrak{m})$ be a local ring and let M, N be finitely generated R-modules. Consider the exact sequence*

$$0 \to N \to N \oplus M \to M \to 0.$$

Then $\chi_{N \oplus M}^{\mathfrak{q}} = \chi_M^{\mathfrak{q}} + \chi_N^{\mathfrak{q}}$ for any $\mathfrak{m}$-primary ideal $\mathfrak{q}$.

Proof. A direct computation shows that $N \cap \mathfrak{q}^n(N \oplus M) = \mathfrak{q}^n N$ for any n. Using the exact sequence (9) the assertion follows. $\square$

Let now $D = D_1 \cup D_2 \subseteq \mathbb{C}^n$ be a not necessarily splayed divisor that is locally at a point $p = (x)$ defined by $h_1(x)h_2(x) \in \mathcal{O} = \mathbb{C}\{x\}$, with components $(D_1, p) = \{h_1(x) = 0\}$ and $D_2 = \{h_2(x) = 0\}$. The multiplicities of D_i at p are denoted by $m_p(D_i) := \mathfrak{e}(\mathcal{O}_{S,p}/(h_i), \mathfrak{m})$. The Hilbert–Samuel polynomial of D_i at p is denoted by $\chi_{D_i,p} := \chi_{\mathcal{O}/(h_i)}^{\mathfrak{m}}$, and similarly the multiplicity and Hilbert–Samuel polynomial for D.

Remark 27. In order to compute the Hilbert–Samuel polynomial of $\mathcal{O}/I$ for any ideal $I \subseteq \mathcal{O}$ we can consider $\mathcal{O}/I$ either as ring or as an $\mathcal{O}$-module. This does not make a difference for the Hilbert–Samuel functions, since they only depend on the graded structure of $\mathcal{O}/I$.

It is well-known that multiplicities of divisors are additive, more precisely one can show that for D and D_i defined as above, $m_p(D) = m_p(D_1) + m_p(D_2)$ holds (see e.g. [dJP00]). For splayed divisors the additivity also holds for Hilbert–Samuel polynomials:

Proposition 28. *Let $(D, p) = (D_1, p) \cup (D_2, p)$ be splayed at $p \in S$, where $(S, p) \cong (\mathbb{C}^n, 0)$. Then the Hilbert–Samuel polynomials of the components D_1 and D_2 are additive, that is,*

$$\chi_{D,p}(t) + \chi_{D_1 \cap D_2, p}(t) = \chi_{D_1, p}(t) + \chi_{D_2, p}(t).$$

Proof. Denote by $\mathcal{O} := \mathbb{C}\{x, y\} = \mathbb{C}\{x_1, \ldots, x_k, y_{k+1}, \ldots, y_n\}$ the coordinate ring of $(\mathbb{C}^n, 0)$. There is an exact sequence (see e.g. [GP08])

(10) $$0 \to \mathcal{O}/(h_1 h_2) \to \mathcal{O}/(h_1) \oplus \mathcal{O}/(h_2) \to \mathcal{O}/(h_1, h_2) \to 0.$$

With facts (i) and (ii) from above the question can be reduced to leading ideals because (10) remains exact if we just consider the leading ideals. The divisor D is splayed, so we can assume that it is defined by $g(x)h(y)$. Choosing any valid monomial ordering one finds $L(g) = x^\alpha$, $L(h) = y^\beta$, $L((gh)) = x^\alpha y^\beta$, and by fact (ii) from above follows $L((g, h)) = (x^\alpha, y^\beta)$. From Lemma 26 it follows that

$$\chi^{\mathfrak{m}}_{\mathcal{O}/(x^\alpha)} + \chi^{\mathfrak{m}}_{\mathcal{O}/(y^\beta)} = \chi^{\mathfrak{m}}_{\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta)}.$$

Thus it remains to prove that the Hilbert–Samuel polynomials w.r.t. $\mathfrak{m}$ of the exact sequence

$$0 \to \mathcal{O}/(x^\alpha \cdot y^\beta) \to \mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta) \to \mathcal{O}/(x^\alpha, y^\beta) \to 0$$

are additive. In order to apply the theorem of Flenner–Vogel we show that the map

$$\mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x^\alpha y^\beta)) \xrightarrow{\ \varphi\ } \mathrm{Gr}_{\mathfrak{m}}(\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta))$$

is injective. The map φ clearly preserves the degree, so it is enough to show the assertion for a homogeneous element of degree d. Therefore take some $\bar{a} \in \mathfrak{m}^d(\mathcal{O}/(x^\alpha y^\beta))/\mathfrak{m}^{d+1}(\mathcal{O}/(x^\alpha y^\beta))$. Consider $\bar{a}$ as an element in $\mathcal{O}$: from Grauert's division theorem follows that $\bar{a}$ can be written as $r + \alpha_1 x^\alpha + \alpha_2 y^\beta$, where r is the unique remainder from the division through x^α and y^β and $\alpha_1 \in \mathcal{O}$ is not divisible by y^β and α_2 is not divisible by x^α. Suppose that $\varphi(\bar{a}) = (0, 0)$. Then write $\varphi(\bar{a}) = (r + \alpha_2 y^\beta, r + \alpha_1 x^\alpha)$ in $\mathcal{O}/(x^\alpha) \oplus \mathcal{O}/(y^\beta)$. In $\mathcal{O}$ this reads as $r + \alpha_2 y^\beta = c x^\alpha$ and $r + \alpha_1 x^\alpha = c' y^\beta$ for some $c, c' \in \mathcal{O}$ with the right order. It follows that $r \in (x^\alpha, y^\beta)$. But r is the unique remainder from the division through the standard basis (x^α, y^β), so $r = 0$ in $\mathcal{O}$. Since x^α, y^β are clearly a regular sequence in $\mathcal{O}$, their syzygies are trivial and from the conditions on α_1, α_2 it follows that $\alpha_1 = \alpha_2 = 0$. This implies the injectivity of φ. Hence by the theorem of Flenner–Vogel, the remainder polynomial S is zero and the assertion of the proposition follows. $\square$

In general the Hilbert–Samuel polynomial of a divisor $(D, p) = (D_1, p) \cup (D_2, p)$ is not additive, as is seen in the following example.

Example 29. By Lemma 4.2.20 of [dJP00] one can explicitly compute the Hilbert–Samuel polynomial of $\mathcal{O}/(f)$, where $\mathcal{O} = \{x_1, \ldots, x_n\}$ and $\mathrm{ord}(f) = m$, namely

(11) $$\chi^{\mathfrak{m}}_{\mathcal{O}/(f)}(d) = \sum_{j=1}^{m} \binom{n + d - j - 1}{n - 1}.$$

Consider now $(\mathbb{C}^2, 0)$ with coordinate ring $\mathcal{O} = \mathbb{C}\{x, y\}$ and with $h_1 = x^2 - y$ and $h_2 = y$. Then the germ of the divisor $(D, 0) = (D_1, 0) \cup (D_2, 0)$ that is locally given by $\{y(x^2 - y) = 0\}$ with $(D_1, 0) = \{x^2 - y = 0\}$ and $(D_2, 0) = \{y = 0\}$ is not splayed. The intersection $(D_1 \cap D_2, 0)$ is locally given by the ideal (x^2, y) and coordinate ring $\mathcal{O}/(x^2, y) = \mathbb{C}\{x\}/(x^2)$. By formula (11) we can compute the Hilbert–Samuel polynomials of D, D_1, D_2 and $D_1 \cap D_2$ and obtain $\chi_{D,p}(t) = 2t - 1$, which is clearly not equal to $\chi_{D_1, p}(t) + \chi_{D_2, p}(t) - \chi_{D_1 \cap D_2, p}(t) = t + t - 2$.

One might ask if the additivity of the Hilbert–Samuel polynomials characterizes splayed divisors. However, here is a counterexample to this assertion:

Example 30. Consider $D \subseteq \mathbb{C}^3$ locally defined by $gh = (x^2 - y^3)(y^2 - x^2 z)$. Then (D, p) is the union of the cylinder over a cusp (D_1, p) and of the Whitney Umbrella (D_2, p). Clearly (D, p) is not splayed (use for example the Leibniz–property). However, $L(g) = x^2$ and $L(h) = y^2$, so the leading monomials of g and h are coprime and one can repeat the argument in the proof of the preceding proposition to find that

$$\chi_{D,p} + \chi_{D_1 \cap D_2, p} = \chi_{D_1, p} + \chi_{D_2, p}.$$

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