MATH3195/5195M EXERCISE SHEET 3
SOLUTIONS

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Radical, Modules, Nakayama and exact sequences

Problem 1. Let R be a ring and consider $R[[x]]$, the ring of formal power series with coefficients in R (an element $f \in R[[x]]$ is of the form $f = \sum_{n=0}^{\infty} a_n x^n$ with $a_n \in R$). Show the following:

(a) f is a unit in $R[[x]]$ if and only if a_0 is a unit in R.
(b) $f \in J(R[[x]])$ if and only if $a_0 \in J(R)$.
(c) Let K be a field. Then $K[[x]]$ is a local ring with maximal ideal $\langle x \rangle$. (One can also show that $K[[x_1, \ldots, x_n]]$ is a local ring with maximal ideal $\langle x_1, \ldots, x_n \rangle$).

Solution. (a) First suppose that a_0 is invertible in R. Let $b_0 = a_0^{-1}$ and inductively define an inverse $g = \sum_{n \geq 0} b_n x^n$ to f: from the condition $fg = 1$ we get that $b_n = -a_0^{-1}(a_1 b_{n-1} + \cdots + a_n a_0^{-1})$.
On the other hand, if f is a unit in $R[[x]]$, then its inverse is of the form $g = \sum_{n \geq 0} b_n x^n$. From $fg = 1$, we see that $a_0 b_0 = b_0 a_0 = 1$, thus a_0 is invertible in R.
(b) Let $f \in J(R[[x]])$, then by Lemma 7.9 for all $g \in R[[x]]$ we have that $1 + fg$ is a unit in $R[[x]]$. In particular we can set $g = r \in R$ to be an element in R. By part (a) this means that $1 + a_0 r$ is a unit in R for any $r \in R$. But then again by Lemma 7.9, this means that $a_0 \in J(R)$.
Conversely, if $a_0 \in J(R)$, then $1 + a_0 r$ is a unit in R for any $r \in R$. But then for any $h = \sum_{n \geq 0} h_n x^i$ the element $1 + fh$ has $1 + a_0 h_0$ with $h_0 \in R$ as constant term, which is a unit. Again by Lemma 7.9, this means that f is in $J(R[[x]])$.
(c) We may use the fact from Exercise sheet 2, Problem 5: we have to show that the non-units of $K[[x]]$ form a maximal ideal. By part (a), the non-units of $K[[x]]$ are power series of the form $f = \sum_{n \geq 1} a_n x^n$ since any $a_0 \neq 0$ is invertible in K. Thus the non-units form the ideal $\langle x \rangle$. This ideal is maximal, since $K[[x]]/\langle x \rangle = K$ is a field.

Problem 2. (a) Let $R = \mathbb{Q}[[x, y]]$ and let $J = \langle xy + y^3, x + x^2 y, xy + 3y, x^4 - 5y^2 + x^2 y \rangle$ be an ideal in R. Show that J is minimally generated by two elements in R.
(b) Let $R = K[t]$ and consider $M = K[t, t^{-1}]$ as R-module and let $I = tR$ be an ideal in R. Show that $M = IM$ but $M \neq 0$. Why does this example not contradict Nakayama's lemma?

Solution. (a) Use the Nakayama lemma to show that $J = \mathfrak{m} = \langle x, y\rangle$: the lemma says that if for a finitely generated R-module M and a submodule $N \subseteq M$ and an ideal $I \subseteq J(R)$ one has the equality $M = N + IM$, then $N = M$. In this example R is a local ring with maximal ideal $\mathfrak{m}$, thus we have $\mathfrak{m} = J(R)$. Clearly $J \subseteq \mathfrak{m}$. We show that $\mathfrak{m} = J + \mathfrak{m}\mathfrak{m}$, then by Nakayama's lemma it follows that $\mathfrak{m} = J$: first calculate $\mathfrak{m}^2 = \langle x^2, xy, y^2\rangle$. Then

$$J + \mathfrak{m}^2 = xy + y^3, x + x^2y, xy + 3y, x^4 - 5y^2 + x^2, x^2, xy, y^2\rangle = \langle x, 3y\rangle\ ,$$

which is equal to $\mathfrak{m}$, since $3 \in \mathbb{Q}^*$. For the minimal number of generators we can use the third version of Nakayama's lemma: $(R, \mathfrak{m})$ is a local ring and $\mathfrak{m}/\mathfrak{m}^2 \cong \mathbb{Q}x \oplus \mathbb{Q}y$ as a $R/\mathfrak{m} \cong \mathbb{Q}$-vector space. So x and y form a basis of this vector space and Nakayama's lemma allows to conclude that hey are a minimal set of generators of $\mathfrak{m}$.
(b) First calculate IM: these are all elements of the form $f(t)tg(t, t^{-1})$, where $f(t) \in R$, $g(t, t^{-1}) \in M$. Clearly this element is again a polynomial in t and t^{-1}, so is contained in M. On the other hand, it is also clear that $M \subseteq IM$, since any element $g(t, t^{-1})$ of M can be written as $t(t^{-1}g(t, t^{-1}))$, with $(t^{-1}g(t, t^{-1})) \in M$. Thus we have $IM = M$.
There are various conditions of Nakayama's lemma that are not satisfied: I is not a subset of $J(R) = \langle 0\rangle$. Also, M is not finitely generated as an R-module (as R-module, $M = R + Rt^{-1} + Rt^{-2} + \cdots$. Note that this is not a *direct* sum!).

Problem 3. Prove the isomorphism theorems for modules (without using the snake lemma).

Solution. (1) Use the notation from the lecture: let $\phi : M \to N$ be an R-module homomorphism. Define a map $\tilde{\phi} : M/\ker\phi \to \mathrm{im}\phi$ by $\tilde{\phi}(m + \ker\phi) = \phi(m)$.
- **(Well defined)** If $m + \ker\phi = m' + \ker\phi$, then $m - m' \in \ker\phi$. So $\tilde{\phi}(m + \ker\phi) = \phi(m) - \phi(m - m') = \phi(m') = \tilde{\phi}(m' + \ker\phi)$.
- **(R-homomorphism)** $\tilde{\phi}(r(m + \ker\phi) + s(n + \ker\phi)) = \tilde{\phi}((rm + sn) + \ker\phi) = \phi(rm + sn) = r\phi(m) + s\phi(n) = r\tilde{\phi}(m + \ker\phi) + s\phi(n + \ker\phi)$.
- **(Injective)** If $\phi(m + \ker\phi) = 0$ then $\phi(m) = 0$, so $m \in \ker\phi$ and $m + \ker\phi = 0$.
- **(Surjective)** Clear.

So $\tilde{\phi}$ is an R-isomorphism.

(2) Here assume that $M \supseteq N \supseteq L$ are R-modules. Define a map $\phi : M/L \to M/N$ by $\phi(m + L) = m + N$.
- **(Well defined)** If $m + L = m' + L$, then $m - m' \in L \subset N$, so $m + N = m' + N$.
- **(R-homomorphism)** $\phi(r(m + L) + r'(m' + L)) = \phi((rm + r'm') + L) = (rm + r'm') + N = r(m + N) + r'(m' + N) = r\phi(m + L) + r'\phi(m' + L)$.
- **(Kernel)** $m + L \in \ker\phi \iff \phi(m + L) = 0 \iff m + N = 0 \iff m \in N \iff m + L \in N/L$.
- **(Image)** Clearly ϕ is surjective.

So by (i), $\tilde{\phi}$ defines an isomorphism $(M/L)/(N/L) \cong (M/N)$.

(3) Define $\phi : M \to (M + L)/L$ by $\phi(m) \to m + L$. (Note that $m \in M \subset M + L$).
- **(R-homomorphism)** $\phi(rm + sn) = (rm + sn) + L = r(m + L) + s(n + L) = r\phi(m) + s\phi(n)$.
- **(Kernel)** $m \in \ker\phi \iff \phi(m) = 0 \iff m + L = 0 \iff m \in L \iff m \in M \cap L$.
- **(Image)** Let $(m + \ell) + L$ in $(M + L)/L$. Then $(m + \ell) - m \in L$, so $m + L = (m + \ell) + L$. Now $\phi(m) = m + L = (m + \ell) + L$, so ϕ is surjective.

By (i), $\tilde{\phi}$ defines an isomorphism $M/(M \cap L) \cong (M + L)/L$.

Problem 4. (a) Prove the 3×3-lemma: Let R be a ring. Assume that

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_1 \arrow[r, "\alpha"] \arrow[d, "f_1"'] & A_2 \arrow[r, "\alpha'"] \arrow[d, "f_2"'] & A_3 \arrow[r] \arrow[d, "f_3"'] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\beta"] \arrow[d, "g_1"'] & B_2 \arrow[r, "\beta'"] \arrow[d, "g_2"'] & B_3 \arrow[r] \arrow[d, "g_3"'] & 0 \\
0 \arrow[r] & C_1 \arrow[r, "\gamma"] \arrow[d] & C_2 \arrow[r, "\gamma'"] \arrow[d] & C_3 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is a commutative diagram of R-modules and all columns and the middle row is exact. Show that the top row is exact if and only if the bottom row is exact.

(b) Give an example of two short exact sequences $0 \to A' \xrightarrow{\mu} A \xrightarrow{\varepsilon} A'' \to 0$ and $0 \to B' \xrightarrow{\mu'} B \xrightarrow{\varepsilon'} B'' \to 0$ with $A' \cong B'$ and $A'' \cong B''$ but where A is not isomorphic to B.

Solution. (a) Use the snake lemma: First assume that the top row is exact. Then since the diagram commutes and the second row is exact, the snake lemma yields the exact sequence:

$$0 \to \ker(f_1) \to \ker(f_2) \to \ker(f_3) \to \operatorname{coker}(f_1) \to \operatorname{coker}(f_2) \to \operatorname{coker}(f_3) \to 0\,.$$

Since the i-th column is exact, $\ker(f_i) = 0$ and $\operatorname{coker}(f_i) \cong C_i$ for $i = 1, 2, 3$. Thus the above exact sequence becomes the short exact sequence

$$0 \to C_1 \to C_2 \to C_3 \to 0\,,$$

which had to be shown.

Similarly, if the bottom row is exact, the snake lemma gives us the exact sequence

$$0 \to \ker(g_1) \to \ker(g_2) \to \ker(g_3) \to \operatorname{coker}(g_1) \to \operatorname{coker}(g_2) \to \operatorname{coker}(g_3) \to 0\,.$$

Since the i-th column is exact, $\ker(g_i) = A_i$ and $\operatorname{coker}(g_i) \cong 0$ for $i = 1, 2, 3$. This simplifies to the short exact sequence

$$0 \to A_1 \to A_2 \to A_3 \to 0\,,$$

which had to be shown.

(b) Consider e.g. the exact sequences of $\mathbb{Z}$-modules: The first one is

$$0 \to \mathbb{Z}_2 \xrightarrow{i} \mathbb{Z}_2 \times \mathbb{Z}_2 \xrightarrow{\pi} \mathbb{Z}_2 \to 0\,,$$

where i is the natural inclusion $x \mapsto (x, 0)$ and π is the natural projection $(x, y) \mapsto y$ (here x, y are elements of the form $n + 2\mathbb{Z}$, $n \in \mathbb{Z}$). Thus the above sequence is split exact.

On the other hand, consider the sequence

$$0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} \mathbb{Z}_2 \to 0\,,$$

where f sends $n + 2\mathbb{Z}$ to $2n + 4\mathbb{Z}$ (this is a $\mathbb{Z}$-homomorphism and well-defined, since if $n + 2\mathbb{Z} = n' + 2\mathbb{Z}$, then $n - n' \in 2\mathbb{Z}$ and $2n - 2n' \in 4\mathbb{Z}$, so $2n + 4\mathbb{Z} = 2n' + 4\mathbb{Z}$) and $g(n + 4\mathbb{Z}) = n + 2\mathbb{Z}$. The sequence is exact at both sides, since f is injective and g is clearly surjective. In the middle, one sees that the image of f is $\{0 + 4\mathbb{Z}, 2 + 4\mathbb{Z}\}$, but this

is also $\ker g$.
As is well-known from algebra, the two groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic (and they are also not isomorphic as $\mathbb{Z}$-modules!).

Problem 5. (Localisation of a module) Let R be a ring and $A \subset R$ be multiplicatively closed. Let M be an R-module.

(a) Show that $(m, a) \sim (n, b)$ if and only if $mbc = nac$ for some $c \in A$ defines an equivalence relation on $M \times A$.
(b) Writing $A^{-1}M$ for the set of equivalence classes of $\sim$, and $\frac{m}{a}$ for the class containing (m, a), show that the operation
$$\frac{m}{a} + \frac{n}{b} = \frac{bm + an}{ab}$$
is well defined and hence that $A^{-1}M$ is an abelian group.
(c) By defining an appropriate multiplication rule, show that $A^{-1}M$ has the structure of an $A^{-1}R$-module.

Solution. (a) Reflexive and symmetric are clear. For transitive, suppose $(m, a) \sim (n, b)$ and $(n, b) \sim (\ell, c)$. Then we have $r, s \in A$ such that $mbr = nar$ and $ncs = \ell bs$, and since A is multiplicatively closed we have $brs \in A$, hence $mc(brs) = narcs = \ell a(brs)$.
(b) It suffices to prove the result for $\frac{m'}{a'} = \frac{m}{a}$. Then we have $c \in A$ such that $m'ac = ma'c$. Now $\frac{m'}{a'} + \frac{n}{b} = \frac{bm'+a'n}{a'b}$, but
$$(bm' + a'n)(ab)c = (m'ac)b^2 + aa'bcn = (ma'c)b^2 + aa'bcn = (bm + an)(a'b)c$$
so each sum is equal to the same class, so the addition is well defined. Associativity is clear by associativity of M, the inverse of $\frac{m}{a}$ is $\frac{-m}{a}$ and the identity is $\frac{0}{1}$.

(c) We define $\frac{r}{a}\frac{m}{b} = \frac{rm}{ab}$. It is easy to then check that this is a module.

Problem 6. Let R be a ring and $A \subset R$ be multiplicatively closed.

(a) Suppose that $\phi : M \to N$ is a homomorphism of R modules. Show ϕ induces an $A^{-1}R$-homomorphism $A^{-1}M \to A^{-1}N$.
(b) Suppose $0 \to L \to M \to N \to 0$ is an exact sequence of R-modules. Show that $0 \to A^{-1}L \to A^{-1}M \to A^{-1}N \to 0$, with the induced maps from (i), is an exact sequence of $A^{-1}R$-modules. (*Remark*: This means that localization is an exact functor from the category of R-modules to the category of $A^{-1}R$-modules)

Solution. (a) We define $\tilde{\phi} : A^{-1}M \to A^{-1}N$ by $\tilde{\phi}(\frac{m}{a}) = \frac{\phi(m)}{a}$.

- **(Well defined)** Suppose $\frac{m'}{a'} = \frac{m}{a}$, then there is some $c \in A$ such that $ma'c = m'ac$. Now $\phi(m)a'c = \phi(ma'c) = \phi(m'ac) = \phi(m')ac$, so $\frac{\phi(m)}{a} = \frac{\phi(m')}{a'}$.

- ($A^{-1}R$-**hom**) For $r \in R$, $m, n \in M$ and $a, b, c \in A$ we have

$$\begin{aligned}\tilde{\phi}\left(\frac{r}{a}\frac{m}{b} + \frac{n}{c}\right) &= \tilde{\phi}\left(\frac{rmc + nab}{abc}\right) \\ &= \frac{\phi(rmc + nab)}{abc} \\ &= \frac{r\phi(m)c + \phi(n)ab}{abc} \\ &= \frac{r}{a}\frac{\phi(m)}{b} + \frac{\phi(n)}{c} \\ &= \frac{r}{a}\tilde{\phi}\left(\frac{m}{b}\right) + \tilde{\phi}\left(\frac{n}{c}\right).\end{aligned}$$

(b) Let $\phi : L \to M$, $\psi : M \to N$ be the above maps.

- (**Exact at** $A^{-1}L$) We show that $\tilde{\phi}$ is injective, so suppose that $\tilde{\phi}(\frac{\ell}{a}) = \frac{\phi(\ell)}{a} = 0$. Then there is some $c \in A$ such that $\phi(\ell)c = \phi(\ell c) = 0$. Since ϕ is injective, we have $\ell c = 0$ and hence $\frac{\ell}{a} = 0$.
- (**Exact at** $A^{-1}M$) Firstly, $\tilde{\psi}(\tilde{\phi}(\frac{\ell}{a})) = \frac{\psi(\phi(\ell))}{a} = 0$ since the original sequence is exact. Thus $\mathrm{im}\tilde{\phi} \subset \ker\tilde{\psi}$. Now suppose $\frac{m}{a} \in \ker\tilde{\psi}$, so $\tilde{\psi}(\frac{m}{a}) = \frac{\psi(m)}{a} = 0$. Thus there is some $c \in A$ such that $\psi(m)c = \psi(mc) = 0$, so $mc \in \ker\psi = \mathrm{im}\phi$ and we can write $mc = \phi(\ell)$ for some $\ell \in L$. But now $\tilde{\phi}(\frac{\ell}{ac}) = \frac{\phi(\ell)}{ac} = \frac{mc}{ac} = \frac{m}{a}$, so $\ker\tilde{\psi} \subset \mathrm{im}\tilde{\phi}$.
- (**Exact at** $A^{-1}N$) We show $\tilde{\psi}$ is surjective, so suppose $\frac{n}{a} \in A^{-1}N$. Now since ψ is surjective we have $m \in M$ such that $\psi(m) = n$. Then $\tilde{\psi}(\frac{m}{a}) = \frac{\psi(m)}{a} = \frac{n}{a}$.