A NEW ALGORITHM FOR 3D LINE GENERATION

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Abstract. In this paper an efficient algorithm for straight line generation with 26-connectivity in Z^3 is given. The algorithm includes additions and tests of integer numbers only and is comparable with the known Kaufman's algorithm.

The algorithm may be applied in a 3D scene visualization, e.g. ray-casting technique.

Key words: 3-D line generation algorithm, volume rendreing, scene visualization.

1. Introduction

The presented algorithm generates a discrete representation of 3D continuous straight line (scan-converts the line). The discrete line is a set of nodes in a discrete voxel-image space. The algorithm may be applied in a 3D scene visualization, e.g. in a ray-casting technique.

There exists a variety of algorithms for the generation of straight line segments in a 2D space [1–7]. The best known is the Bresenham's algorithm, which is "optimal", and converts a straight line segment with 8-connectivity in Z^2.

In 3D space the problem of digital straight line generation is more difficult and the number of existing algorithms is less than in the 2D case. In the work [9] Kaufman described a 3D straight line generation algorithm, called "double Bresenham's algorithm", which generates a 26-connected digital straight line segment between P_o and P_t points. In the Kaufman's algorithm a next line point is chosen in two steps, and a Bresenham's algorithm principle is using twice: in both of two coordinate plane, which intersect along this coordinate axis (x, y or z), along which a difference between endpoint coordinates of the generated line is the largest. On the line point this grid node is chosen such, that its projections on the planes are of minimal distance from projections of the continuous line (on the planes) in sense of the Bresenham's algorithm principle.

In the generation loop of Kaufman's algorithm (limited to the case $\Delta x \geq \Delta y, \Delta z \geq 0$) are involved 2 additions and 3 tests for the one step along the generated line, and 15 operations over the generation loop: 6 tests for determination of the coordinate axis

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with the maximal difference between P_o and P_t points, 3 additions for absolute values calculations of these differences and 6 multiplications by 2.

In this paper we propose a new algorithm for 3D straight line generation with 26-connectivity in Z^3 (no limited in anywise), in which are chosen the grid nodes with a minimal sum of distances to two projecting planes, which a intersection border is the data (continuous) line and, as the distances, modules of function plane values are chosen, which describe the plane, [12].

The presented algorithm is a new version of the algorithm described in [13] however, it includes some additions and tests of only integer numbers. Moreover, it involves significantly less operations than previous one, and is comparable to the Kaufman's algorithm, but no limited in anywise.

2. Analysis of problem

Let the straight line segment l be defined with the aid of P_o and P_t points:

$$\begin{aligned} &(x - x_o)/\Delta x = (y - y_o)/\Delta y = (z - z_o)/\Delta z, \\ &\Delta x = x_t - x_o, \quad \Delta y = y_t - y_o, \quad \Delta z = z_t - z_o, \end{aligned} \tag{1a}$$

and the segment be the border of intersection of projecting planes: Q, R, S:

$$\begin{aligned} Q: \quad Q(x, y, z) &= 0, \\ R: \quad R(x, y, z) &= 0, \\ S: \quad S(x, y, z) &= 0; \end{aligned} \tag{1b}$$

the functions $Q(x, y, z)$, $R(x, y, z)$, and $S(x, y, z)$ are defined by formulas (1c):

$$\begin{aligned} Q(x, y, z) &\triangleq x\Delta z - z\Delta x - x_o\Delta z + z_o\Delta x, \\ R(x, y, z) &\triangleq y\Delta z - z\Delta y - y_o\Delta z + z_o\Delta y, \\ S(x, y, z) &\triangleq x\Delta y - y\Delta x - x_o\Delta y + y_o\Delta x; \end{aligned} \tag{1c}$$

x, y, z – coordinates of a current point, and in the origin point $P_o(x_o, y_o, z_o)$ the values of the Q, R, and S plane functions are equal 0, $Q_o = R_o = S_o \triangleq 0$.

The generation of the digital line depends on a choice of these coordinate grid nodes (integer points) $< x, y, z >$ of the Z^3 space, lying along the intersection border, for which the distance ρ from this border, defined as the sum of the plane of function values in these nodes in the following formula:

$$\rho = \left| \begin{array}{lcll} |S(x, y, z)| & + & |R(x, y, z)| & \text{or} \\ |S(x, y, z)| & + & |Q(x, y, z)| & \text{or} \\ |Q(x, y, z)| & + & |R(x, y, z)| & \end{array} \right. \tag{2a}$$

is minimal ($\rho --- >$ minimum).
(If the P_o point is not integer, we have to round it to the nearest integer: $P_{oi}(x_{oi}, y_{oi}, z_{oi}) = [P_o(x_o, y_o, z_o)]$, and compute the original values of the plane functions: $Q_{oi} = Q(P_{oi})$, $R_{oi} = R(P_{oi})$, $S_{oi} = S(P_{oi})$.)

The values of the S, Q and R functions of the consecutive point of the segment is calculated from formulas (2b), derived from the Taylor's expansions of these functions:

$$\begin{aligned}
Q(x+dx, y+dy, z+dz) &= Q(x,y,z) + L^x Q(x,y,z) + L^y Q(x,y,z) + L^z Q(x,y,z),\\
R(x+dx, y+dy, z+dz) &= R(x,y,z) + L^x R(x,y,z) + L^y R(x,y,z) + L^z R(x,y,z),\\
S(x+dx, y+dy, z+dz) &= S(x,y,z) + L^x S(x,y,z) + L^y S(x,y,z) + L^z S(x,y,z),
\end{aligned} \tag{2b}$$

where:

$$\begin{aligned}
&L^x Q = \Delta z dx, \quad L^y Q = 0, \quad L^z Q = -\Delta x dz,\\
&L^x R = 0, \quad L^y R = \Delta z dy, \quad L^z R = -\Delta y dz,\\
&L^x S = \Delta y dx, \quad L^y S = -\Delta x dy, \quad L^z S = 0,
\end{aligned} \tag{2c}$$

and a movement direction along the grid lines:

$$dx = SIGN(\Delta x), \qquad dy = SIGN(\Delta y), \qquad dz = SIGN(\Delta z). \tag{3}$$

The direction of the line is defined with the aid of a $\mathbf{V}^s$ vector, determined with the aid of the endpoints P_o and P_t: $\mathbf{V}^s \triangleq \mathbf{V}^s(\Delta x, \Delta y, \Delta z)$.

On the 26-connected grid, $G_{26c}(h^3)$, 26 elementary moves, pointing to 26 neighborhood nodes may be defined, [12]. One of the nodes is the consecutive point of the segment. The formulas (3), by choosing a 1 of 8 volume regions defined by the coordinate planes, eliminate the 19 of the 26 moves. In the chosen rectangle relations (4a):

$$\begin{aligned}
&(1)\ \mathbf{V}^s \subset O^1\ (\triangleq O^1(\mathbf{V}^1 \star \mathbf{V}^4 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta x) > ABS(\Delta y) > ABS(\Delta z))\\
&(2)\ \mathbf{V}^s \subset O^2\ (\triangleq O^2(\mathbf{V}^1 \star \mathbf{V}^5 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta x) \geq ABS(\Delta z) \geq ABS(\Delta y))\\
&(3)\ \mathbf{V}^s \subset O^3\ (\triangleq O^3(\mathbf{V}^2 \star \mathbf{V}^4 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta y) \geq ABS(\Delta x) \geq ABS(\Delta z))\\
&(4)\ \mathbf{V}^s \subset O^4\ (\triangleq O^4(\mathbf{V}^2 \star \mathbf{V}^6 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta y) > ABS(\Delta z) > ABS(\Delta x))\\
&(5)\ \mathbf{V}^s \subset O^5\ (\triangleq O^5(\mathbf{V}^3 \star \mathbf{V}^5 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta z) > ABS(\Delta x) > ABS(\Delta y))\\
&(6)\ \mathbf{V}^s \subset O^6\ (\triangleq O^6(\mathbf{V}^3 \star \mathbf{V}^6 \star \mathbf{V}^7)) \text{ if } (ABS(\Delta z) \geq ABS(\Delta y) \geq ABS(\Delta x))
\end{aligned} \tag{4a}$$

localize the $\mathbf{V}^s$ vector in a 1 of 6 pyramids $O^1 \div O^6$ (operation $\star$ used to a triple of vectors, creates a pyramid with a top in a begin of the vectors and with a base – on a plane, defined by endpoints of the vector triple). Each of the pyramids is defined with the aid of 3 different length vectors, emanating from a node 0 (Fig. 1) :

(1) a vector $\mathbf{V}^7$ ($\triangleq \mathbf{V}^7(dx, dy, dz)$), (lying along main diagonal of the cube),

(2) a 1 of $\mathbf{V}^1$ ($\triangleq \mathbf{V}^1(dx, 0, 0)$), $\mathbf{V}^2$ ($\triangleq \mathbf{V}^2(0, dy, 0)$) and $\mathbf{V}^3$ ($\triangleq \mathbf{V}^3(0, 0, dz)$) vectors,(lying along the coordinate axes),

(3) a 1 of $\mathbf{V}^4$ ($\triangleq \mathbf{V}^4(dx, dy, 0)$), $\mathbf{V}^5$ ($\triangleq \mathbf{V}^5(dx, 0, dz)$), and $\mathbf{V}^6$ ($\triangleq \mathbf{V}^6(0, dy, dz)$) vectors, (along the plane diagonals) - the choice of the vector pyramid.

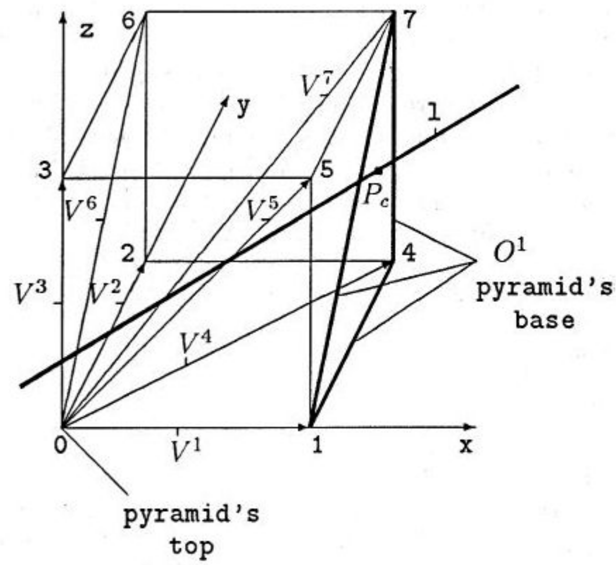

Fig. 1. The O^1 pyramid containing the l line

Relations (4a) (which determine the directions with minimal and maximal tangent of the segment), may be realised with the aid of a procedure PYRAMID, (4b), (which include 4 sums and max. 6 tests only).
The localization of the $\mathbf{V}^s$ vector in the pyramid O^i reduces the number of candidates for the consecutive node to the 3 nodes only. In order to choose the 1 of the 3 nodes we use the criterion (2a): $\rho --> min.$, i.e. the minimum distance of the $\mathbf{V}^s$ vector to the 3 nodes of the pyramid base.

3. The straight line generation algorithm

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For a determined straight line, the calculations of the movement direction along the grid lines (3) and the localization of the line direction $\mathbf{V}^s$ in the pyramid (4), are done once (over the generation loop). The remaining operations of the algorithm are done within the generation loop.

To minimize a computational complexity of the algorithm, a sequence of the operations of the algorithm and the using formulas for different line parameters (i.e. the pyramids) are various. To calculate the ρ distance in the candidated nodes we need only the 2 of the 3 projecting plane values in these nodes. These plane values change among the pyramid base nodes in different way for different pyramids. To calculate ρ we will use these plane pair which the plane values change alternately in the base pyramid nodes. Alternately change guarantees the minimum calculations to the ρ evaluation. For the O^1 and O^2 pyramids this is (Q,S) pair. The value of the Q function does not change between the 1 and the 4 nodes, whereas the S function values - between the 4 and the 7 nodes. The value of the R function changes among all nodes of the base pyramid. For the O^3 and O^4 pyramids this is (R,S) pair, and for O^5 and O^6 pyramids this is (Q,R) pair. The choice of the plane pair is possible, because the direction of the generating line, including the pyramid, does not change along the line.

$PROCEDURE\_PYRAMID(x_o, y_o, z_o, x_t, y_t, z_t, Q, R, S, \mathbf{V}^s, O^i)$ (4b)
$BEGIN$
$COMPUTE: \Delta x, \Delta y, \Delta z,$ (formula (1a))
$COMPUTE\ ABS(\Delta x),\ ABS(\Delta y)\ ABS(\Delta z),$
$COMPUTE: \mathrm{d}x, \mathrm{d}y, \mathrm{d}z,$ (formula (3))
$EVALUATE:\ x := x_{oi},\ y := y_{oi},\ z := z_{oi},$
$REMEMBER: < x, y, z >,$ (the first point of the line)
$IF\ ABS(\Delta x) > ABS(\Delta y)\ AND\ ABS(\Delta x) \geq ABS(\Delta z)\ THEN\ COMPUTE:$
$CQ(1\text{-}7) = Q_{oi} + 2L^xQ + L^zQ,\ \Delta CQ(1) = L^xQ,\ \Delta CQ(7) = L^xQ + L^zQ,\ dxy = dxdy,\ \ CS(1\text{-}4) = S_{oi} + 2L^xS + L^yS,\ \Delta CS(1) = L^xS,\ \Delta CS(4) = L^XS + L^yS,\ dxz = dxdz,$
(CQ(1-7) = Q_1+Q_7, CQ(1) = Q_1, CQ(7) = Q_7)
(CS(1-4) = S_1+S_4, CS(1) = S_1, CS(4) = S_4)
$IF\ ABS(\Delta y) > ABS(\Delta z)\ THEN\ GOTO\ PROCEDURE\_LO1,$ $(\mathbf{V}^s \subset O^1)$
$ELSE$ $GOTO\ PROCEDURE\_LO2,$ $(\mathbf{V}^s \subset O^2)$
$ELSE\ IF\ ABS(\Delta y) > ABS(\Delta z)\ THEN\ COMPUTE:$
$CR(2\text{-}7) = R_{oi} + 2L^yR + L^zR,\ \Delta CR(2) = L^yR,\ \Delta CR(7) = L^yR + L^zR,\ dxy = dxdy,$
$CS(2\text{-}4) = S_{oi} + 2L^yS + L^xS,\ \Delta CS(2) = L^yS,\ \Delta CS(4) = L^xS + L^yS,\ dyz = dydz,$
(CR(2-7) = R_2+R_7, CR(2) = R_2, CR(7) = R_7)
(CS(2-4) = S_2+S_4, CS(2) = S_2, CS(4) = S_4)
$IF\ ABS(\Delta x) \geq ABS(\Delta z)\ THEN\ GOTO\ PROCEDURE\_LO3,$ $(\mathbf{V}^s \subset O^3)$
$ELSE$ $GOTO\ PROCEDURE\_LO4,$ $(\mathbf{V}^s \subset O^4)$
$ELSE\ COMPUTE:$
$CQ(3\text{-}5) = Q_{oi} + 2L^zQ + L^xQ,\ \Delta CQ(3) = L^zQ,\ \Delta CQ(5) = L^xQ + L^zQ,\ dxz = dxdz,\ \ CR(3\text{-}7) = R_{oi} + 2L^zR + L^yR,\ \Delta CR(3) = L^zR,\ \Delta CR(7) = L^yR + L^zR,\ dyz = dydz,$
(CQ(3-5) = Q_3+ Q_5, CQ(3) = Q_3, CQ(5) = Q_5)
(CR(3-7) = R_3+ R_7, CR(3) = R_3, CR(7) = R_7)

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$IF\ ABS(\Delta x) > ABS(\Delta y)\ THEN\ GOTO\ PROCEDURE\_LO5,$ $(\mathbf{V}^s \subset \mathrm{O}^5)$
$ELSE$ $GOTO\ PROCEDURE\_LO6,$ $(\mathbf{V}^s \subset \mathrm{O}^6)$
$END.$

For the $O^1(\mathbf{V}^1,\mathbf{V}^4,\mathbf{V}^7)$ pyramid following dependencies and formulas obligue:

$$\begin{aligned} &Q_1 = Q + L^x Q, && Q_4 = Q_1, && Q_7 = Q_4 + L^z Q, \\ &R_1 = R, && R_4 = R_1 + L^y R, && R_7 = R_4 + L^z R, \\ &S_1 = S + L^x S, && S_4 = S_1 + L^y S, && S_7 = S_4, \end{aligned} \tag{5a}$$

where Q, R and S are the values of the functions $Q(x,y,z)$, $R(x,y,z)$ and $S(x,y,z)$ in the previous chosen node, $< x, y, z >$, Q_1, R_1 and S_1 are the values of these functions in the 1 node, and Q_4, R_4 and S_4 - in the 4 node.

We use the (Q, S) plane pair, because for the O^1 pyramid the Q and S function values change alternately among the 1, 4, and 7 pyramid base nodes. The choice goes in the two steps: between the 1 and 4 nodes with the aid of the S function, and between the 4 and 7 nodes with the aid of the Q function, if in the first step the 4 node is chosen. Because $S_1 > 0$ and $S_4 < 0$, we use the condition $S_1 + S_4 = CS < 0$ instead of $ABS(S_1) < ABS(S_4)$, and because $Q_1 > 0$ and $Q_7 < 0$, we use the condition $Q_1 + Q_7 = CQ < 0$ instead of $ABS(Q_1) < ABS(Q_7)$. The operations in the loop, for the one generation step, are contained in PROCEDURE_LO1:

$PROCEDURE\_LO1(Q, S, L^x Q, L^z Q, L^x S, L^y S, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z)$ (5b)
$WHILE(x \neq x_t)$
$BEGIN$
$x : +dx,$ (step to node 1)
$IF\ CS(1\text{-}4)\, dxy < 0\ THEN:$ $(\equiv \mathrm{ABS}(\mathrm{S}_1) < \mathrm{ABS}\ (\mathrm{S}_4))$
$CQ(1\text{-}7){:}\ +\Delta CQ(1),\ CS(1\text{-}4){:}\ +\Delta CS(1),$ $(\equiv \mathrm{Q} := \mathrm{Q}_1,\ \mathrm{S} := \mathrm{S}_1)$
(if P_c coincident point of the l line and pyramid base plane 1–4–7 is closer to edge 1–5 than edge 4–7, then P_c is closer to the 1 than the 7 node too; Q, R, and S functions are linear)
$ELSE$: $y : +dy,$ (step to node 4)
$IF\ CQ(1\text{-}7)\, dxz < 0\ THEN:$ $(\equiv \mathrm{ABS}(\mathrm{Q}_1) < \mathrm{ABS}(\mathrm{Q}_7))$
$CQ(1\text{-}7){:}\ +\Delta CQ(1),\ CS(1\text{-}4){:}\ +\Delta CS(4),$ $(\equiv \mathrm{Q} = \mathrm{Q}_1, \mathrm{S} = \mathrm{S}_4)$
(else if P_c point is closer to 1-4 edge than to 5-7 edge, then it is closer to the 4 than 7 node too)
$ELSE : z : +dz$ (step to node 7)
$CQ(1\text{-}7){:}\ +\Delta CQ(7),\ CS(1\text{-}4){:}\ +\Delta CS(4),$ $(\equiv \mathrm{Q} = \mathrm{Q}_7, \mathrm{S} = \mathrm{S}_4)$
(else P_c point is closer to the 7 node)
$REMEMBER : \langle x, y, z \rangle,$ (the next point of the line)
$END.$

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Dependences, formulas and procedures refer to the rest $O^2 - O^6$ pyramids, are contained in the Appendix.

4. Conclusion

The algorithm is not limited to some case as the Kaufman's algorithm. It was implemented and tested for many thousand pairs of automatically generated line endpoints. It is less complex than the Kaufman's algorithm, but for the particular cases of the Δx, Δy and Δz values it performs as the Kaufman's algorithm. For the one generation step (in the generation loop), it includes: min.: 3 additions and 1 test, or max. (like the Kaufman's algorithm): 4 additions and 2 tests.

APPENDIX

For the $O^2(\mathbf{V}^1, \mathbf{V}^5, \mathbf{V}^7)$ pyramid the (Q, S) pair of function values are used too and the following formulas:
$Q_1 = Q + L^x Q$, $Q_5 = Q_1 + L^z Q$, $Q_7 = Q_5$, $S_1 = S + L^x S$, $S_5 = S_1$, $S_7 = S_5 + L^y S$.
The operations for the one generation step are contained in PROCEDURE_LO2:

$PROCEDURE\_LO2(Q, S, L^xQ, L^zQ, L^xS, L^yS, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z,)$
$WHILE\ (x \neq x_t)$
$BEGIN$
$x: +dx,$
$IF\ CQ(1\text{-}5)\, dxz < 0 \quad THEN:$
$\quad CQ(1\text{-}5):\ +\Delta CQ(1),\ CS(1\text{-}7):\ +\Delta CS(1),$
$ELSE: z:\ +dz,$
$\qquad IF\ CS(1\text{-}7)\, dxy\ <\ 0 \quad THEN:$
$\qquad\quad CQ(1\text{-}5):\ +\Delta CQ(5),\ CS(1\text{-}7):\ +\Delta CS(1),$
$\qquad ELSE: y:\ +dy,$
$\qquad\quad CQ(1\text{-}5):\ +\Delta CQ(5),\ CS(1\text{-}7):\ +\Delta CS(7),$
$REMEMBER: \langle x, y, z\rangle,$
$END.$

For the $O^3(\mathbf{V}^2, \mathbf{V}^4, \mathbf{V}^7)$ pyramid the (R,S) pair of function values are used and the following formulas:
$R_2 = R + L^y R$, $R_4 = R_2$, $R_7 = R_4 + L^z R$, $S_2 = S + L^y S$, $S_4 = S_2 + L^x S$, $S_7 = S_4$.
The operations for the one generation step are contained in PROCEDURE_LO3:

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$PROCEDURE\_LO3(R, S, L^y R, L^z R, L^x S, L^y S, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z)$
$WHILE\ (y \neq y_t)$
$BEGIN$
$y:\ + dy,$
$IF\ CS(2\text{-}4)\, dxy\ >\ 0\quad THEN:$
$\quad CR(2\text{-}7):\ + \Delta CR(2),\ CS(2\text{-}4):\ + \Delta CS(2),$
$ELSE:\quad x: + dx,$
$\qquad IF\ CR(2\text{-}7)\, dyz\ <\ 0\quad THEN:$
$\qquad\quad CR(2\text{-}7):\ + \Delta CR(2),\ CS(2\text{-}4):\ + \Delta CS(4),$
$\qquad ELSE: z:\ + dz,$
$\qquad\quad CR(2\text{-}7):\ + \Delta CR(7),\ CS(2\text{-}4):\ + \Delta CS(4),$
$REMEMBER: \langle x, y, z\rangle,$
$END.$

For the $O^4(\mathbf{V}^2, \mathbf{V}^6, \mathbf{V}^7)$ pyramid the (R, S) pair of function values are used too and the following formulas:
$R_2 = R + L^y R,\ R_6 = R_2 + L^z R,\ R_7 = R_6,\ S_2 = S + L^y S,\ S_6 = S_2,\ S_7 = S_6 + L^x S,$
and the operations for the one generation step are contained in PROCEDURE_LO4:

$PROCEDURE\_LO4(R, S, L^y R, L^z R, L^x S, L^z S, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z)$
$WHILE\ (y \neq y_t)$
$BEGIN$
$y:\ + dy,$
$IF\ CR(2\text{-}6)\, dyz\ <\ 0\quad THEN:$
$\quad CR(2\text{-}6):\ + \Delta CR(2),\ CS\,(2)7):\ + \Delta CS(0),$
$ELSE: z: + dz,$
$\qquad IF\ CS(2\text{-}7)\, dxy > 0\quad THEN:$
$\qquad\quad CR(2\text{-}6):\ + \Delta CR(6),\ CS(2\text{-}7):\ + \Delta CS\,(2),$
$\qquad ELSE: x:\ + dx,$
$\qquad\quad CR(2\text{-}6):\ + \Delta CR(6),\ CS(2\text{-}7):\ + \Delta CS\,(7),$
$REMEMBER: \langle x, y, z\rangle,$
$END.$

For the $O^5(\mathbf{V}^3, \mathbf{V}^5, \mathbf{V}^7)$ pyramid the (Q, R) pair of the function values are used and the following formulas:
$Q_3 = Q + L^z Q,\ Q_5 = Q_3 + L^x Q,\ Q_7 = Q_5,\ R_3 = R + L^z R,\ R_5 = R_3,\ R_7 = R_5 + L^y R$
and the operations for the one generation step are contained in PROCEDURE_LO5:

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$PROCEDURE\_LO5(Q, R, L^xQ, L^zQ, L^yR, L^zR, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z)$
$WHILE\ (z \neq z_t)$
$BEGIN$
$z:\ + dz,$
$IF\ CQ(3\text{-}5)\,dxz > 0\quad THEN:$
$\qquad CQ(3\text{-}5):\ + \Delta CQ(3),\ CR(3\text{-}7):\ + \Delta CR(3),$
$ELSE: x:\ + dx,$
$\qquad IF\ CR(3\text{-}7)\,dyz > 0\quad THEN:$
$\qquad\quad CQ(3\text{-}5):\ + \Delta CQ(5)),\ CR(3\text{-}7):\ + \Delta CR(3),$
$\qquad ELSE: y:\ + dy,$
$\qquad\quad CQ(3\text{-}5):\ + \Delta CQ(5),\ CR(3\text{-}7):\ + \Delta CR(7),$
$REMEMBER: \langle x, y, z\rangle,$
$END.$

For the $O^6(\mathbf{V}^3, \mathbf{V}^6, \mathbf{V}^7)$ pyramid the (Q, R) pair of the function values are used too and the following formulas:
$Q_3 = Q + L^zQ,\ Q_6 = Q_3,\ Q_7 = Q_6 + L^zQ,\ R_3 = R + L^zR,\ R_6 = R_3 + L^yR,\ R_7 = R_6$
and the operations for the one generation step are contained in PROCEDURE_LO6:

$PROCEDURE\_LO6(Q, R, L^xQ, L^zQ, L^yR, L^zR, \mathrm{d}x, \mathrm{d}y, \mathrm{d}z, x, y, z)$
$WHILE\ (z \neq z_t)$
$BEGIN$
$z:\ + dz,$
$IF\ CR(3\text{-}6)\,dyz > 0\quad THEN:$
$\quad CQ(3\text{-}7):\ + \Delta CQ(3),\ CR(3\text{-}6):\ + \Delta CR(3),$
$ELSE: y:\ + dy,$
$\qquad IF\ CQ(3\text{-}7)\,dxz > 0\quad THEN:$
$\qquad\quad CQ(3\text{-}7):\ + \Delta CQ(3),\ CR(3\text{-}6):\ + \Delta CR(6),$
$\qquad ELSE: x:\ + dx,$
$\qquad\qquad CQ(3\text{-}7):\ + \Delta CQ(7),\ CR(3\text{-}6):\ + \Delta R(6),$
$REMEMBER: \langle x, y, z\rangle,$
$END.$

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