

Write **clearly** and **in the box**:

CSCI 2824  
Midterm Exam 2  
Spring 2019

Name:

Student ID:

Section number:

Read the following:

- **RIGHT NOW!** Write your name, student ID and section number on the top of your exam.
- You are allowed one  $3 \times 5$ -in index card of notes (both sides). No magnifying glasses!
- You may use a calculator provided that it cannot access the internet or store large amounts of data.
- You may **NOT** use a smartphone as a calculator.
- Clearly mark answers to multiple choice questions in the provided answer box.
- Mark only one answer for multiple choice questions. If you think two answers are correct, mark the answer that **best** answers the question. No justification is required for multiple choice questions.
- If you do not know the answer to a question, skip it and come back to it later.
- For free response questions you must clearly justify all conclusions to receive full credit. A correct answer with no supporting work will receive no credit.
- If you need more space for free-response questions, there are blank pages at the end of the exam. If you choose to use the extra pages, make sure to **clearly** indicate which problem you are continuing.
- You have **90 minutes** for this exam.

Page	Points	Score
3	9	
4	12	
5	9	
6	10	
7	20	
9	20	
11	20	
Total	100	





**Multiple choice problems: Write your answers in the boxes, or they will not be graded!**

1. (3 points) Exams were given in two subject areas: Discrete Structures and Data Science. These exams were taken by 20 students. Suppose 9 people failed Discrete, 5 people failed both Discrete and Data Science, and 8 people failed Data Science. How many people **passed** both exams?

- A. 3
- B. 8
- C. 12
- D. 13
- E. unable to be determined

B

2. (3 points) Suppose  $A$ ,  $B$ , and  $C$  are finite sets. Suppose that  $A$  has  $m$  distinct elements in it,  $B$  has  $n$  distinct elements in it, and  $C$  has  $p$  distinct elements in it. Suppose further that  $C \subset A$ . What is the cardinality of  $\mathcal{P}(B \times C)$ ?

- A.  $n \cdot p$
- B.  $2^{n \cdot p}$
- C.  $m \cdot p$
- D.  $2^{m \cdot p}$
- E.  $n \cdot (m - p)$
- F.  $2^{n \cdot (m-p)}$

B

3. (3 points) Let  $A = (-\infty, 3]$ ,  $B = [-3, \infty)$ , and  $C = [-3, 3]$ . Decide which **one** of the following is true:

- A.  $A \cap B$  is the empty set
- B.  $A \cap B$  is finite
- C.  $A \cap B$  is countably infinite
- D.  $A \cap B$  is uncountably infinite

D

**Multiple choice problems: Write your answers in the boxes, or they will not be graded!**

4. (3 points) What is the **smallest** integer  $p$  such that  $f(n) = 3n^3 + n^3 \log(n^4) + n \log(n^5)$  is  $\mathcal{O}(n^p)$ .

- A.  $p = 2$
- B.  $p = 3$
- C.  $p = 4$
- D.  $p = 5$
- E.  $p = 6$

C

5. (3 points) Select the answer that is the closed form solution to this recurrence relation:

$$a_n = 8a_{n-1} - 16a_{n-2}$$

- A.  $a_n = 2^n$
- B.  $a_n = (-4)^n$
- C.  $a_n = n4^n$
- D.  $a_n = 1$

C

6. (3 points) Given the function  $g : (\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathbb{Z}$ ,  $g(m, n) = 2m - 4n$ . Which of the following is true?

- A.  $g$  is one-to-one, but not onto
- B.  $g$  is not one-to-one, but it is onto
- C.  $g$  is both one-to-one and onto
- D.  $g$  is neither one-to-one nor onto

D

**Multiple choice problems: Write your answers in the boxes, or they will not be graded!**

7. (3 points) Given the function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$ ,  $f(n) = 4n - 1$ . Which of the following is true?

- A.  $g$  is one-to-one, but not onto
- B.  $g$  is not one-to-one, but it is onto
- C.  $g$  is both one-to-one and onto
- D.  $g$  is neither one-to-one nor onto

A

8. (3 points) Consider the pseudocode for procedure DivineDivision( $A, B, n$ ), given below. The input to the procedure is two matrices  $A$  and  $B$ , and scalar  $n$ .  $A[i][j]$  and  $B[i][j]$  (the element in row  $i$ , column  $j$  of matrices  $A$  and  $B$ , respectively) are all integers.  $n$  gives the number of rows and columns of  $A$  and  $B$  (they are both  $n \times n$  square matrices). Give an estimate of the complexity of this procedure, where complexity is measured by the number of **additions and subtractions** needed.

```
procedure DivineDivision(A, B, n):
    ctr = 0
    for i from 1 to n:
        for j from 1 to n:
            if (A[i][j] + B[i][j]) % 3 == 0,
                then ctr = ctr+1
            else
                then ctr = ctr-1
    return ctr
```

- A. DivineDivision is order  $n$
- B. DivineDivision is order  $n^2$
- C. DivineDivision is order  $n^3$
- D. DivineDivision is order  $n^4$

B

9. (3 points) Consider the pseudocode below for the function mysteryFunction(input). What should the function output be if the input is  $input = [9, 3, 13, 3, 4, 7, 5, 10]$ ? You may assume that array indexing begins at 0, which is the left-most element in the given input object (i.e.  $input[0]=9$ ).

```
def mysteryFunction(input)
    k=0
    x=input[k]
    for i = 1, 2, ..., length(input)
        if (input[i] < x), then (k = i, and x = input[i])
    return (k)
```

- A. 1
- B. 2
- C. 3
- D. 13

A

**Short-answer problems:** If your answers do not fit in the given box, MAKE A NOTE of where the work is continued or it will NOT be graded!

10. (6 points) Consider the sequence defined as  $T_{n+3} = T_{n+2} + T_{n+1} + T_n$  with initial conditions:  
 $T_1 = 1, \quad T_2 = 1, \quad T_3 = 1$

Suppose we want to prove that  $T_n < 2^n$  for all  $n \geq 4$ . Answer the following two questions. Note: You do **not** need to complete the proof. You only need to answer the specified questions.

- Specify any and all base cases. Show the base case(s) in their entirety.
- What is your induction hypothesis?

**Solution:**

- (a) **Base Cases:** Since our recursion depends on the term  $n - 3$ , we need 3 base cases.

$$n = 4 \quad T_4 = 1 + 1 + 1 = 3 \quad \text{and} \quad 2^4 = 16, \quad \text{since } 3 < 16, \text{ we have verified that } T_4 < 2^4 \quad \checkmark$$

$$n = 5 \quad T_5 = 3 + 1 + 1 = 5 \quad \text{and} \quad 2^5 = 32, \quad \text{since } 5 < 32, \text{ we have verified that } T_5 < 2^5 \quad \checkmark$$

$$n = 6 \quad T_6 = 5 + 3 + 1 = 9 \quad \text{and} \quad 2^6 = 64, \quad \text{since } 9 < 64, \text{ we have verified that } T_6 < 2^6 \quad \checkmark$$

- (b) **Induction Hypothesis:** Assume for  $4 \leq k \leq m$  that  $T_k < 2^k$ .

11. (7 points) Suppose you have two sets:  $A = \{b, c, d\}$  and  $B = \{a, b\}$ . Find the set:  $\mathcal{P}(A) - \mathcal{P}(B)$ .

**Solution:**

$$\mathcal{P}(A) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

$$\mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Therefore,

$$\mathcal{P}(A) - \mathcal{P}(B) = \{\{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

12. (20 points) (a) Given the recursive sequence  $a_n = 2a_{n-1} + 3$  with  $a_0 = 1$ . Find the closed form solution.

(b) Given the recursive sequence  $a_n = a_{n-1} + 2n + 3$  with initial condition  $a_0 = 4$ , use induction to prove that  $a_n = n^2 + 4n + 4$  for  $n \geq 0$ .

**Solution:**

(a)

$$\begin{aligned}
a_n &= 2a_{n-1} + 3 \\
&= 2(2a_{n-2} + 3) + 3 = 2^2a_{n-2} + 2 \cdot 3 + 3 \\
&= 2^2(2a_{n-3} + 3) + 2 \cdot 3 + 3 = 2^3a_{n-3} + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= 2^3(2a_{n-4} + 3) + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 = 2^4a_{n-4} + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= \dots \\
&= 2^n a_{n-n} + 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3 + \dots + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= 2^n a_0 + 3 \cdot (2^{n-1} + 2^{n-2} + \dots + 2^3 + 2^2 + 2 + 1) \\
&= 2^n(1) + 3 \sum_{i=0}^{n-1} 2^i \\
&= 2^n + 3(2^n - 1) \quad \text{finite geometric sum formula} \\
&= 4 \cdot 2^n - 3 \\
&= 2^{n+2} - 3
\end{aligned}$$

Therefore, the closed form solution of the given recursion is  $\boxed{a_n = 2^{n+2} - 3}$ .

(b) **Base Case:**  $a_0 = 4$ , the base case is  $n = 0$ .

$$\begin{aligned}
a_0 &= 0^2 + 4 \cdot 0 + 4 = 4 \\
4 &= 4 \checkmark
\end{aligned}$$

**Induction Step:** Assume for  $k \geq 0$ , that  $a_k = k^2 + 4k + 4$ .

Using our recurrence relation,

$$\begin{aligned}
a_{k+1} &= a_k + 2(k+1) + 3 \\
&= k^2 + 4k + 4 + 2k + 2 + 3 \quad \text{by the inductive hypothesis} \\
&= k^2 + 4(k+1) + 2k + 1 + 4 \\
&= k^2 + 2k + 1 + 4(k+1) + 4 \\
&= (k+1)^2 + 4(k+1) + 4
\end{aligned}$$

Thus by weak induction, we have proven that  $a_n = n^2 + 4n + 4$  for all  $n \geq 0$ .

13. (20 points) Consider the function  $f(n) = 9n^3 + 2n^3 \log_e(n^5) - n \log_e(n)$

- (a) Find the smallest nonnegative integer  $p$  for which  $n^p$  is a tight big-**O** bound on  $f(n)$ . Be sure to justify any inequalities you use and provide the  $C$  and  $k$  from the big-**O** definition.
- (b) Find the largest nonnegative integer  $p$  for which  $n^p$  is a tight big-**Ω** bound on  $f(n)$ . Be sure to justify any inequalities you use and provide the  $C$  and  $k$  from the big-**Ω** definition.
- (c) Can we conclude that  $f(n)$  is  $\Theta(n^p)$  for some nonnegative integer  $p$ ? Explain why or why not.

**Solution:**

(a) First, we simplify using our rules for logarithms:  $f(n) = 9n^3 + 10n^3 \log_e(n) - n \log_e(n)$ .

Next, the natural first guess should be the **leading order term**, which is  $n^3$ . But two terms have  $n^3$  in them. Since the  $\log_e n$  is only going to make the second term larger than  $n^3$  for  $\log n > 1$  ( $n > e$ ), this means the second term is the dominant one (upper bound).

The lowest power of  $n$  that can be an upper bound for that second term is  $n^4$ . We get upper bounds for each term in terms of  $n^4$ :

$$\begin{aligned} 9n^3 &\leq 9n^4, \quad \text{for } n \geq 1 \\ 10n^3 \log(n) &\leq 10n^3 \cdot n = 10n^4, \quad \text{for } n \geq 1 \\ -n \log_e(n) &\leq 0, \quad \text{for } n \geq 1 \end{aligned}$$

So we have:

$$f(n) \leq 9n^4 + 10n^4 + 0 = 19n^4, \quad \text{for } n \geq 1$$

Thus with  $C = 19$  and  $k = 1$ ,  $f$  is **O**( $n^4$ ).

(b) Again, the natural first guess should be the **leading order term**, which is  $n^3$ . But two terms have  $n^3$  in them. Since the  $\log_e n$  is only going to make the second term larger than  $n^3$  for  $\log n > 1$  ( $n > e$ ), this means the *first* term will be smaller, and so this is our first guess for the big-**Ω** bound.

The lowest power of  $n$  that can serve as a lower bound is  $n^3$ . We get the lower bounds for each term in terms of  $n^3$ :

$$\begin{aligned} 9n^3 &\geq 9n^3, \quad \text{for } n \geq 1 \\ 10n^3 \log(n) &\geq 10n^3, \quad \text{for } n \geq e \\ n \log n &\leq n^3 \rightarrow -n \log(n) \geq -n^3, \quad \text{for } n \geq 1 \end{aligned}$$

where the second line comes from the fact that  $\log_e n > 1$  exactly when  $n > e$  (by exponentiating both sides and using the fact that  $e^{\log_e n} = n$ ).

So we have:

$$f(n) \geq 9n^3 + 10n^3 - n^3 = 18n^3, \quad \text{for } n \geq e$$

Thus, with  $C = 18$  and  $k = e$ ,  $f$  is **Ω**( $n^3$ ).

(c) No, because it is not both big-**O** and big-**Ω**  $n^p$  with the same  $p$  value.

14. (20 points) (a) Prove DeMorgan's Law for sets  $\overline{A \cup B} = \overline{A} \cap \overline{B}$  using **set builder notation**.

(b) Let  $A$ ,  $B$ , and  $C$  be sets. Show that  $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$ .

**Solution:**

(a)

$$\begin{aligned}\overline{A \cup B} &= \{x | x \in \overline{A \cup B}\} \quad \text{set builder notation for } \overline{A \cup B} \\ &= \{x | x \notin (A \cup B)\} \quad \text{definition of complement} \\ &= \{x | \neg(x \in (A \cup B))\} \quad \text{definition of "not in"} \\ &= \{x | \neg(x \in A \vee x \in B)\} \quad \text{definition of union} \\ &= \{x | x \notin A \wedge x \notin B\} \quad \text{DeMorgans Law} \\ &= \{x | x \in \overline{A} \wedge x \in \overline{B}\} \quad \text{definition of complement} \\ &= \{x | x \in \overline{A} \cap \overline{B}\} \quad \text{definition of intersection} \\ &= \overline{A} \cap \overline{B} \quad \text{set builder notation for } \overline{A} \cap \overline{B}\end{aligned}$$

(b)

$$\begin{aligned}\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} \quad \text{DeMorgan's Law for Sets} \\ &= \overline{A} \cap (\overline{B} \cup \overline{C}) \quad \text{DeMorgan's Law for Sets} \\ &= (\overline{B} \cup \overline{C}) \cap \overline{A} \quad \text{by Commutativity of an Intersection} \\ &= (\overline{C} \cup \overline{B}) \cap \overline{A} \quad \text{by Commutativity of a Union}\end{aligned}$$