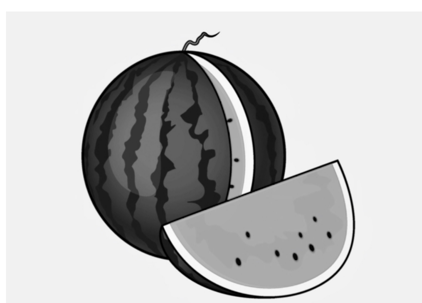
Write **clearly** and **in the box**:

CSCI 2824
Final Exam ‘
Spring 2019

Name:
Student ID:
Section number:

Read the following:

- **RIGHT NOW**! Write your name, student ID and section number on the top of your exam.
- You are allowed one 3×5-in index card of notes (both sides). No magnifying glasses!
- You may use a calculator provided that it cannot access the internet or store large amounts of data.
- You may **NOT** use a smartphone as a calculator.
- Clearly mark answers to multiple choice questions in the provided answer box.
- Mark only one answer for multiple choice questions. If you think two answers are correct, mark the answer that **best** answers the question. No justification is required for multiple choice questions.
- If you do not know the answer to a question, skip it and come back to it later.
- For free response questions you must clearly justify all conclusions to receive full credit. A correct answer with no supporting work will receive no credit.
- If you need more space for free-response questions, there are blank pages at the end of the exam. If you choose to use the extra pages, make sure to **clearly** indicate which problem you are continuing.
- You have **2 hours and 30 minutes** for this exam.



Multiple choice problems: **Write your answers in the boxes, or they will not be graded!**

1. (4 points) In Discrete Structures class, there are students who are 19, 20, 21, and 22 years old. Due to a newly instituted dress code, the students may only wear red, blue, green, or purple shirts to class. What is the minimum number of students that must attend class in order to guarantee that there are at least 3 students who are the same age wearing the same color to class.

   A. 17

   B. 18

   C. 32

   D. 33

   E. unable to be determined

   D

2. (4 points) Suppose you go to the zoo to see the elephants. There are 37 of them! What length bit-string is needed in order to assign each of them a unique binary number?

   A. 6

   B. 7

   C. 8

   D. 9

   E. 10

   F. 11

   A

3. (4 points) What rule of inference or logical fallacy is demonstrated by the following argument? *If it snows more than 6 inches, then the ground will be frozen. The ground is not frozen. Thus, it did not snow more than 6 inches.*

   A. disjunctive syllogism

   B. modus ponens

   C. modus tollens

   D. hypothetical syllogism

   E. affirming the conclusion

   F. denying the hypothesis

   C

Multiple choice problems: Write your answers in the boxes, or they will not be graded!

4. (4 points) What is the coefficient on $x^{50}y^{40}$ for $(-x-5y)^{90}$?

   A. 1

   B. $-1 \cdot -5 \cdot C(100, 40)$

   C. $-1 \cdot -5 \cdot P(100, 40)$

   D. $(-1)^{50} \cdot (-5)^{40} \cdot P(90, 40)$

   E. $(-1)^{40} \cdot (-5)^{50} \cdot P(90, 40)$

   F. $(-1)^{50} \cdot (-5)^{40} \cdot C(90, 40)$

   G. $(-1)^{40} \cdot (-5)^{50} \cdot C(90, 40)$

   H. $(-1)^{40} \cdot (-5)^{50} \cdot 90 \cdot 40$

   I. $(-1)^{50} \cdot (-5)^{40} \cdot 90 \cdot 40$

   J. 0 (because the given term does not exist in the expansion)

F

5. (4 points) Suppose that the Scooby Doo gang is going to see the Endgame movie. A row of seats at the movie theater is 11 seats wide. This row will be filled up completely by the 5 members of the Scooby Doo gang, and 6 strangers. How many arrangements of these 11 people are possible, such that the Scooby Doo gang can all sit adjacent to one another in this row?

   A. $8! \cdot 5!$

   B. $7! \cdot 5!$

   C. $11!$

   D. $7 \cdot 6!$

   E. $7 \cdot 5!$

B

6. (4 points) Let a set A be: $A = \{$the Easter Bunny, the Scooby Doo gang, 737, CU Boulder$\}$. $\varnothing$ denotes the empty set. Consider the following statements. Which of the following is/are necessarily true?

   (1) $\varnothing \in A$
   (2) $\varnothing \subset A$

   A. Statement (1) is true, but statement (2) is not necessarily true.

   B. Statement (1) is not necessarily true, but statement (2) is true.

   C. Both statements are true.

   D. Both statements are not necessarily true.

B

Multiple choice problems: Write your answers in the boxes, or they will not be graded!

7. (4 points) What is the **largest** integer p such that $f(n) = 3\log(n) + 9\log(n^3) + 5n^2(\log n)$ is $O(n^p)$?

A. $p = 1$

B. $p = 2$

C. $p = 3$

D. $p = 4$

E. $p = 5$

C

8. (4 points) What is the following matrix product? $\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} 6 & 1 \\ 5 & 1 \end{bmatrix}$

A. $\begin{bmatrix} 32 & 6 \\ 21 & 4 \end{bmatrix}$

B. $\begin{bmatrix} 17 & 15 \\ 2 & 4 \\ 1 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 17 \\ 10 \end{bmatrix}$

D. not possible to multiply

E. none of the above

A

9. (4 points) Consider the pseudocode for procedure *count_dog_hairs*(D, n), given below. The input to the procedure is a 3-dimensional array D and scalar n. Each element $D[i][j]$ (the element in row i and column j) is a binary variable representing whether or not there is a pollen grain at the coordinates given by that element of the matrix, n gives the number of rows and columns of D (it is an $n \times n$ square array, the same length in both dimensions). Give an estimate of the complexity of this procedure, where complexity is measured by the number of **additions** needed.

```
procedure count_dog_hairs(D, n):
    count = 0
    for i from 1 to n:
        for j from 1 to n:
            count = count+D[i][j][k]
    return count
```

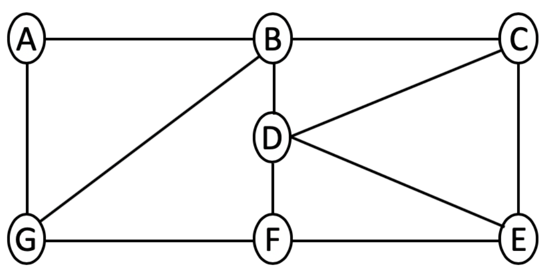
A. *count_dog_hairs* is order 1

B. *count_dog_hairs* is order n

C. *count_dog_hairs* is order n^2

D. *count_dog_hairs* is order n^3

C



10. (4 points) Suppose we use the Greedy Coloring Algorithm to color the vertices of graph G given above. The vertices are ordered A, B, C, D, E, F, G, using Red (color 1), Blue (color 2), Green (color 3), and Purple (color 4) (in that order of preference). What color will the vertex E have?

A. Red

B. Blue

C. Green

D. Purple

E. Another color not listed.

B

Short-answer problems: If your answers do not fit in the given box, <u>MAKE A NOTE</u> of where the work is continued or it will NOT be graded!

11. (5 points) Consider the claim: For all $n \geq 2$, $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2} < 2 - \frac{1}{n}$. Suppose you wanted to prove this claim using induction. Answer the following questions regarding key components of a proof by induction. Note: You do not need to do the whole proof here. Just answer the questions.

   (a) Write out the full base case(s) that you would need.

   (b) Write out your induction hypothesis.

   Solution:
   (a) Base Case: $n = 2$. We verify the base case by plugging $n = 2$ into both the left hand and right hand sides of the inequality above and showing that the inequality is true when $n = 2$.

$$\frac{1}{1} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$$

$$2 - \frac{1}{2} = \frac{3}{2}$$

$$\frac{5}{4} < \frac{3}{2} \quad \checkmark$$

   (b) Inductive hypothesis (weak induction): For some $k \geq 2$, $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} < 2 - \frac{1}{k}$.

12. (5 points) Determine whether the following relation $R \subseteq A \times A$, where A is the set of all CU students, is reflexive, symmetric, transitive, and/or an equivalence relation. Briefly justify each conclusion.

$$(a, b) \in R \text{ if and only if } a \text{ attends the 9am Discrete Structures lecture with } b$$

   Solution:
   Reflexive: yes. You can take the lecture with yourself.
   Symmetric: yes. If you take the lecture with another student, that other student necessarily also takes the lecture with you.
   Transitive: yes. If you take the lecture with Person A, and Person A takes the lecture with Person B, it is necessary that you are also in the lecture with Person B.

   This **IS an equivalence relation**. It possesses the reflexive, symmetric, and transitive property.

Short-answer problems: If your answers do not fit in the given box, MAKE A NOTE of where the work is continued or it will NOT be graded!

13. (5 points) Let D be the set of all CSCI 2824 students currently taking this final exam. Let C be the set of all chairs in the exam room. Define $f : D \to C$ by $f(x) =$ the seat that student x is sitting in.

    (a) Is f one-to-one? (b) Is f onto? Briefly justify your answers.

    Solution:
    (a) yes. f is one-to-one because there is only one student in each chair.

    (b) no. f is not onto because there are empty chairs.

14. (13 points) Consider the following recurrence relation: $a_n = a_{n-1} + 6a_{n-2} + 6^n$. Answer the following questions. Make sure to show all of your work. Label important steps along the way (e.g. label the characteristic polynomial).

   (a) Solve for the general solution, $a_n^{(h)}$, to the associated homogenous recurrence relation.

   (b) Find the particular solution, $a_n^{(p)}$ (make the guess and solve for any constants).

   (c) What is the full general solution to the given recurrence relation?

   (d) Suppose **only for this part** that you are given the initial conditions $a_0 = 1$ and $a_1 = 8$. Set up **but do not solve** the system of equations that would allow you to calculate the values of any remaining unknown constants in your answer to part (c).

Solution:
(a) We plug in the usual guess of $a_n = r^n$ to get:

$$r^n = r^{n-1} + 6r^{n-2} \iff r^2 = r + 6 \iff r^2 - r - 6 = 0 \iff (r-3)(r+2) = 0$$

So our characteristic roots are $r = 3$ and $r = -2$.
Thus, our homogeneous solution is $\boxed{a_n^{(h)} = A \cdot 3^n + B \cdot (-2)^n}$

(b) The non-homogeneous part is 6^n, so we guess something that looks similar: $a_n^{(p)} = C \cdot 6^n$.

We plug this into the non-homogeneous recurrence relation and solve for C:

$$\begin{aligned} C \cdot 6^n &= C \cdot 6^{n-1} + 6C \cdot 6^{n-2} + 6^n \\ C \cdot 6^2 &= C \cdot 6 + 6C + 6^2 \\ 36C &= 12C + 36 \\ 24C &= 36 \\ C &= \frac{36}{24} = \frac{3}{2} \end{aligned}$$

So the particular solution is $\boxed{a_n^{(p)} = \frac{3}{2} \cdot 6^n}$

(c) Combine the answers to parts (a) and (b): $a_n = a_n^{(h)} + a_n^{(p)}$

$\boxed{a_n = A \cdot 3^n + B \cdot (-2)^n + \frac{3}{2} \cdot 6^n}$

(d)

$$\begin{aligned} a_0 &= A \cdot 3^0 + B \cdot (-2)^0 + \frac{3}{2} \cdot 6^0 \\ &\implies 1 = A + B + \frac{3}{2} \\ &\implies A + B = \frac{-1}{2} \end{aligned}$$

And

$$\begin{aligned} a_1 &= A \cdot 3^1 + B \cdot (-2)^1 + \frac{3}{2} \cdot 6^1 \\ &\implies 8 = 3A - 2B + 9 \\ &\implies 3A - 2B = -1 \end{aligned}$$

So the system to solve would be $\boxed{A + B = \frac{-1}{2}}$ and $\boxed{3A - 2B = -1}$.

15. (12 points) Prove or disprove the following claims:

(a) If $x^2(y+3)$ is even, then x is even or y is odd.

(b) If x and y are both real numbers, then $|x+y| = |x| + |y|$.

Solution:
(a) **Proof:** Using contraposition, we assume that x is odd and y is even. Let $x = 2k+1$ and $y = 2m$ for arbitrary integers k and m.

$$\begin{aligned}
x^2(y+3) &= (2k+1)^2(2m+3) \\
&= (4k^2+4k+1)(2m+3) \\
&= 8k^2m + 8km + 2m + 12k^2 + 12k + 3 \\
&= 8k^2m + 8km + 2m + 12k^2 + 12k + 2 + 1 \\
&= 2(4k^2m + 4km + m + 6k^2 + 6k + 1) + 1 \\
&= 2l + 1 \quad \text{for an integer } \; l = 4k^2m + 4km + m + 6k^2 + 6k + 1
\end{aligned}$$

So $x^2(y+3)$ is odd. Thus, using a proof by contrapositive we have proven our claim.

(b) This can be disproven with a counter-example. Consider $x = -3$ and $y = 4$. Then, $|-3+4| = |1| = 1$, however $|-3| + |4| = 3 + 4 = 7$. Since, $1 \neq 7$, we have disproven the claim with a counterexample.

16. (10 points) Suppose you have a bag of 10 dice:

- Five are regular, fair 6-sided dice; let R represent the event that you draw one of these dice from the bag.
- Three are 6-sided weighted dice where the probability of rolling a 3 is three times as likely as rolling a 1. And the probability of rolling a 6 is also three times as likely as rolling a 1. You may assume that rolling a 1, 2, 4, or 5 are all equally probable. Let W represent the event that you draw one of these dice from the bag.
- Two are 6-sided dice, with an extra pip drawn on the three-pip side so that each die has two 4's but no 3's (so they are fair, but the sides are numbered $\{1, 2, 4, 4, 5, 6\}$); let M represent the event that you draw one of these marked-up dice from the bag.

Suppose you draw a die at random from the bag and then you roll it. Define r_3 as the event that you roll a 3.

(a) What is the probability that you roll a 3? You may leave your answer as a sum of fractions.

(b) Suppose you roll a 3. Given this information, what is the probability that you drew one of the Weighted dice (event W)? You may leave your answer as an unsimplified fractional expression.

(c) Are the events r_3 and W independent? Fully justify your answer using math.

Solution:

(a) In order to answer this, we need to calculate the probability of rolling a 3 on the weighted dice. Since there are 6 possible outcomes, we have $p(1) + p(2) + p(3) + p(4) + p(5) + p(6) = 1$. We are also told that on the weighted dice, $p(3) = p(6) = 3p(1)$. Combining these two equations, we see that $p(1) + p(1) + 3p(1) + p(1) + p(1) + 3p(1) = 1 \implies 8p(1) = 1 \implies p(1) = \frac{1}{10}$. Rolling a 3: $p(3) = 3 \cdot \frac{1}{10}$. Using the law of total probability, we have

$$p(3) = p(3|R) \cdot p(R) + p(3|W) \cdot p(W) + p(3|M) \cdot p(M)$$

Therefore,

$$\begin{aligned} p(4) &= \frac{1}{6} \cdot \frac{5}{10} + \frac{3}{10} \cdot \frac{3}{10} + 0 \cdot \frac{2}{10} \\ &= \frac{1}{12} + \frac{9}{100} \end{aligned}$$

(b) Using Bayes' Theorem,

$$\begin{aligned} p(W|3) &= \frac{p(3|W) \cdot p(W)}{p(3)} \\ &= \frac{\frac{3}{10} \cdot \frac{3}{10}}{\frac{1}{12} + \frac{9}{100}} \end{aligned}$$

(c) $p(3|W) = \frac{3}{10}$ and $p(3) = \frac{1}{12} + \frac{9}{100}$. Since $\frac{3}{10} \neq \frac{1}{12} + \frac{9}{100}$, these two events are **not independent**.

17. (10 points) Consider the linear equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 24$$

where all the x_i are non-negative integers, $x_i \geq 0$.

(a) How many solutions does the equation have? Fully justify your answer.

(b) How many solutions does the equation have, subject to the additional constraint that each $x_i \geq 2$. (Each variable is at least two.)

Solution:
(a) Consider each of the 5 variables as a different "bin", into which you can place the 24 ones ($24 \times 1 = 24$.

You need $5 - 1 = 4$ dividers between the 5 variables.

Then, there are a total of $24 + 4$ objects and you need to chose 4 of them to be dividers, giving $\boxed{C(28, 4) = C(28, 24) = \frac{28!}{24!4!} = \frac{28 \cdot 27 \cdot 26 \cdot 25}{4!} = 20,475}$ total ways to solve the equation.

(b) If each $x_i \geq 2$, then there are 14 ones left to distribute among the 5 variables. Therefore, there are $\boxed{C(14 + 5 - 1, 4) = C(18, 4) = C(18, 14) = \frac{18!}{14!4!} = 3060}$ ways to solve the equation given the additional constraint that each $x_i \geq 2$.