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CSCI 2824: Discrete Structures

Lecture 24: The Pigeon Hole Principle, Permutations, and Combinations

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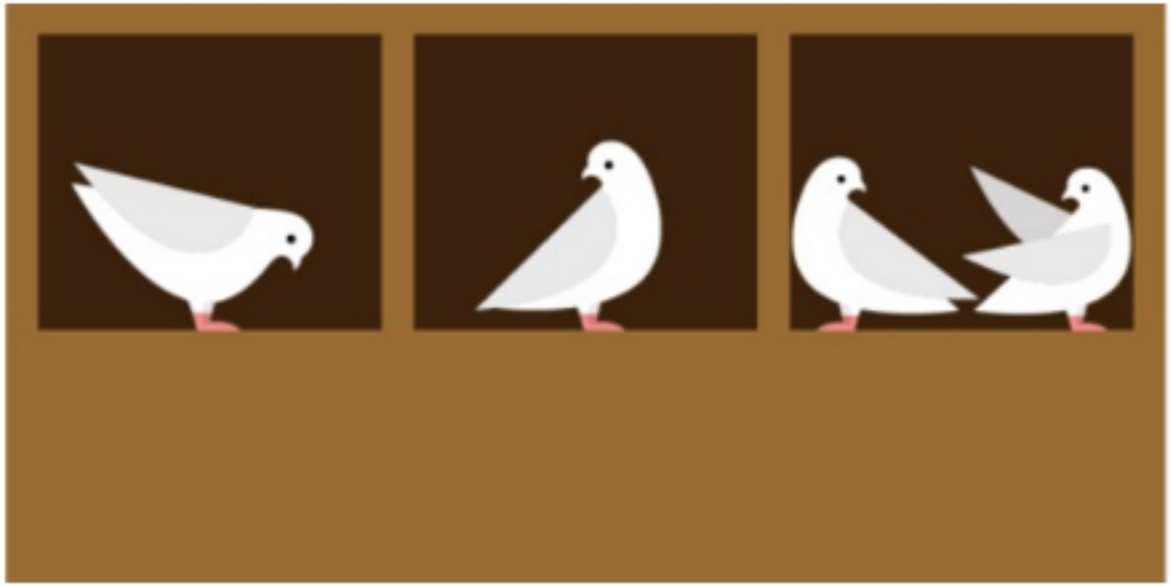
Pigeonhole Principle

Suppose that a flock of 4 pigeons flies into a set of 3 pigeonholes to roost. Because there are 4 pigeons and only 3 pigeonholes for them to go into, at least one of the pigeonholes must have at least two pigeons in it.



The Pigeonhole Principle: If k is a positive integer and k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Pigeonhole Principle

Example: Show that if there are 30 students in a class, then at least two of them have last names that start with the same letter.

Pigeonhole Principle

Example: A drawer contains a dozen brown socks and a dozen black socks, all unmatched. If you take out socks in the dark, how many must you grab to ensure that you have two socks that match?

How many socks must we remove to guarantee that we get 2 black socks?

Pigeonhole Principle

<u>Example:</u> How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

Pigeonhole Principle

<u>The Generalized Pigeonhole Principle</u>: If N objects are places into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

<u>Example:</u> (re-do) How many cards must be drawn from a standard deck of 52 cards to ensure that at least 3 of the cards are of the same suit?

If we have $N = 9$ cards, then there is at least one suit with at least

$$\left\lceil \frac{9}{4} \right\rceil = \lceil 2.25 \rceil = 3 \text{ cards of the same suit}$$

Pigeonhole Principle

Example: Show that there are at least seven people in California (pop. 38.8 million) with the same three initials that were born on the same day of the year. (Assume that everyone has three initials.)

Permutations and Combinations

<u>Example</u>: How many 4-digit PIN combinations are there?

Permutations and Combinations

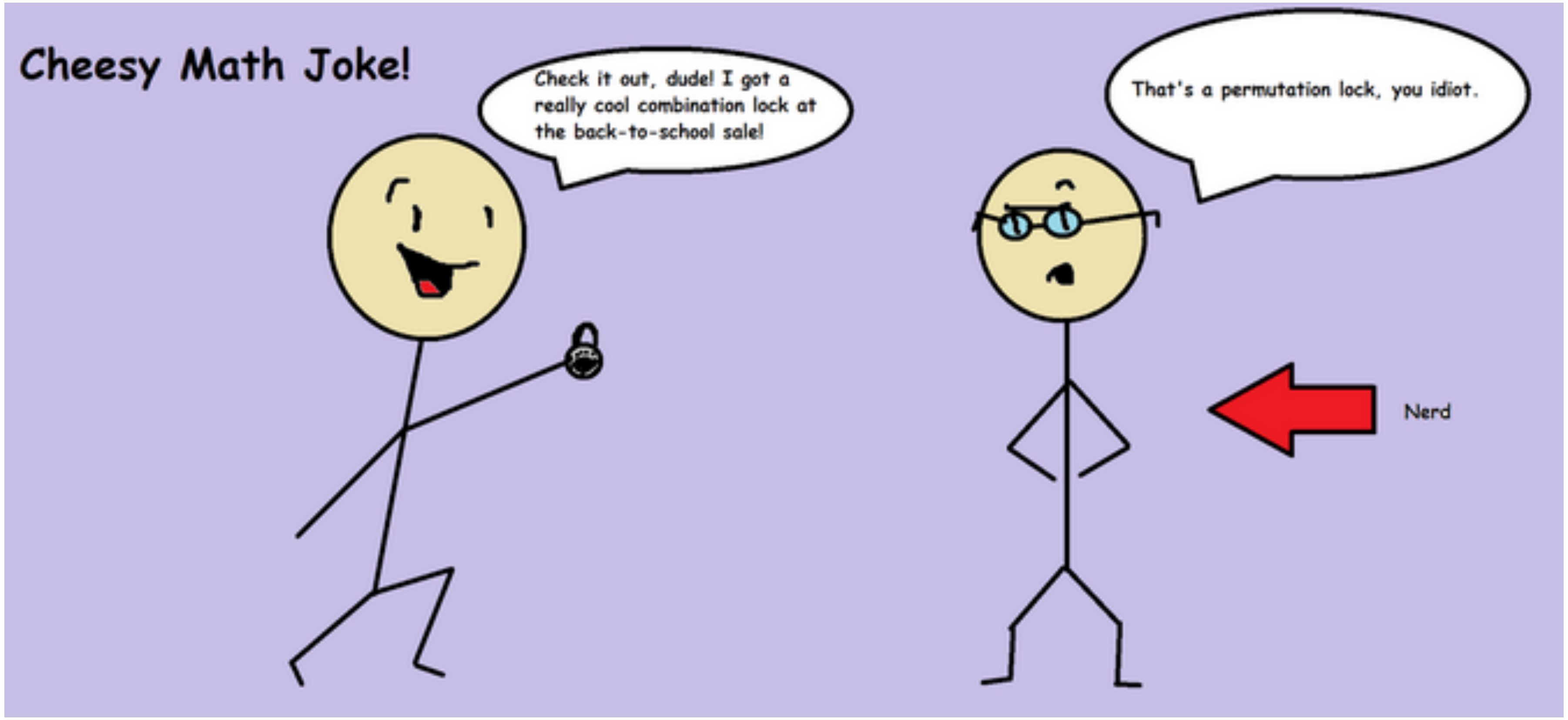
❖ Think of the digits as distinct items in a bag. We reach into the bag 4 times and select an item. After each item is selected we put it back in the bag and select again.

- What IF we select items from the bag without replacing them and take note of the order they come out?
- What IF we select items from the bag without replacing them and don't care about the order they come out?

Permutations and Combinations

The first case, where order matters, is called a **permutation**.

The second case, where order doesn't matter, is called a **combination**.



Permutations and Combinations

Example: How many three-character strings can we make if each character is a distinct letter?

Permutations and Combinations

When selecting r distinct items without replacing them, we call the possible selection an **r-permutation**.

Example: Find all 2-permutations of the set $S = \{a, b, c\}$.

Permutations and Combinations

Theorem: If n is a positive integer and r is an integer such that $1 \leq r \leq n$ then the number r-permutations from a set of size n is $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$

- Note that we define $P(n,0) = 1$ because there is exactly one way you can select NO items from a set of size n.

Corollary: If n and r are integers with $0 \leq r \leq n$, then $P(n,r) = \frac{n!}{(n-r)!}$

- Special Case: $P(n,n) = n!$

Permutations and Combinations

<u>Example</u>: How many different 1^{st}-2^{nd}-3^{rd} place permutations can occur in a race with 5 runners?

Permutations and Combinations

Example: How many permutations of the letters ABCDEFGH contain the string ABC?

Example: How many ways are there to arrange n men and n women in a row if the men and the women alternate?

Permutations and Combinations

When we make an ***unordered*** selection of distinct items it's called a combination.

Example: Find all 2-letter combinations from the set $S = \{a, b, c\}$

In the previous example, we listed all of the 2-permutations of S.

$$ab, ac, ba, bc, ca, cb$$

But now we want to throw out items that are unordered repeats:

ab, ac, bc repeats: bc, ca, cb

Permutations and Combinations

Let's derive a formula for the number of r-combinations by starting with the number of r-permutations and then throwing out duplicates.

$$P(n,r) = \frac{n!}{(n-r)!}$$

The number of duplicates depends on the number of selected items.

E.g. Suppose we selected three letters from the alphabet. The permutations involving abc are:

$$abc, bca, cba, acb, cab, bca$$

Number of r −combinations $= \frac{P(n,r)}{r!} = \frac{n!}{(n-r)!r!}$

Note that we are overcounting the combinations of abc by a factor of 6 which exactly equals the number of permutations of 3 items.

Permutations and Combinations

Thereom: If n is a nonnegative integer and r is an integer with $0 \leq r \leq n$ then the number of r −combinations is

$$C(n,r) = \frac{n!}{(n-r)!\,r!}$$

Alternate notation: $C(n,r) = \binom{n}{r}$

Permutations and Combinations

Example: How many poker hands of five cards can be drawn from a 52-card deck?

Note: We don’t care about order here, so are interested in combinations instead of permutations.

Permutations and Combinations

Example: A club has 25 members. How many ways are there to choose 4 members to serve on the club's executive committee?

Example: How many bit strings of length 10:

a) Contain exactly four 1’s

b) Contain at most four 1’s

Permutations and Combinations

The hardest part in working with permutations and combinations is knowing if you should use permutations, combinations, or both.

<u>Example</u>: The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels?

Extra Practice

EX. 1 A club has 25 members. How many ways are there to choose a president, vice president, secretary, and treasurer?

EX. 2 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels.

EX. 3 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* one vowel?

EX. 4 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* two vowels?

EX. 5 Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men as women?

EX. 1 A club has 25 members. How many ways are there to choose a president, vice president, secretary, and treasurer?

This is similar to the example in the lecture, but this time we actually care about order since we're now differentiating between positions on the executive committee.

Instead of computing the combinations we now compute the permutations. So we have

$$P(25, 4) = \frac{25!}{21!} = 25 \times 24 \times 23 \times 22 = 303,600$$

EX. 2 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain exactly 2 vowels.

First pick the position of the vowels, $C(6,2) = 15$

For each vowel-positions there are 5 possible vowels, so for both vowel positions there are 5^2 possible combinations

There are 21^4 ways the consonants can be arranged

By the product rule we have

$$15 \times 5^2 \times 21^4 = 72,930,375 \text{ strings}$$

EX. 3 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* one vowel?

The easiest way to do this one is to compute the total possible number of strings and then subtract off the ones that don't have any vowels.

$$26^6 - 21^6 = 223,149,655 \text{ strings}$$

EX. 4 The English alphabet contains 21 consonants and 5 vowels. How many strings of six letters contain *at least* two vowels?

Similar to the previous problem, we could compute the total number of strings and then subtract off the number of strings with no vowels and the number of strings with exactly 1 vowel (which we've already computed)

$$26^6 - 21^6 - 6 \times 5 \times 21^5 = 100,626,625$$

EX. 5 Suppose that a department contains 10 men and 15 women. How many ways are there to form a committee with six members if it must have the same number of men as women?

Note that order does not matter here, so we'll be using combinations.

We need to choose the 3 male members of the committee and the 3 female members of the committee separately.

There are $C(10, 3) = 120$ ways to choose the men

There are $C(15, 3) = 455$ ways to choose the women

Then by the product rule, there are $C(10, 3) \cdot C(15, 3) = 54,600$ ways to choose the committee