

Write **clearly** and **in the box**:

CSCI 2824
Midterm Exam 2 ‘
Spring 2019

Name:

Student ID:

Section number:

Read the following:

- **RIGHT NOW!** Write your name, student ID and section number on the top of your exam.
- You are allowed one 3×5 -in index card of notes (both sides). No magnifying glasses!
- You may use a calculator provided that it cannot access the internet or store large amounts of data.
- You may **NOT** use a smartphone as a calculator.
- Clearly mark answers to multiple choice questions in the provided answer box.
- Mark only one answer for multiple choice questions. If you think two answers are correct, mark the answer that **best** answers the question. No justification is required for multiple choice questions.
- If you do not know the answer to a question, skip it and come back to it later.
- For free response questions you must clearly justify all conclusions to receive full credit. A correct answer with no supporting work will receive no credit.
- If you need more space for free-response questions, there are blank pages at the end of the exam. If you choose to use the extra pages, make sure to **clearly** indicate which problem you are continuing.
- You have **90 minutes** for this exam.

Page	Points	Score
3	9	
4	12	
5	9	
6	10	
7	20	
9	20	
11	20	
Total	100	



Multiple choice problems: Write your answers in the boxes, or they will not be graded!

1. (3 points) Exams were given in two subject areas: Discrete Structures and Data Science. These exams were taken by 20 students. Suppose 9 people failed Discrete, 5 people failed both Discrete and Data Science, and 8 people failed Data Science. How many people **passed** both exams?

- A. 3
- B. 8
- C. 12
- D. 13
- E. unable to be determined

B

2. (3 points) Suppose A , B , and C are finite sets. Suppose that A has m distinct elements in it, B has n distinct elements in it, and C has p distinct elements in it. Suppose further that $C \subset A$. What is the cardinality of $\mathcal{P}(B \times C)$?

- A. $n \cdot p$
- B. $2^{n \cdot p}$
- C. $m \cdot p$
- D. $2^{m \cdot p}$
- E. $n \cdot (m - p)$
- F. $2^{n \cdot (m - p)}$

B

3. (3 points) Let $A = (-\infty, 3]$, $B = [-3, \infty)$, and $C = [-3, 3]$. Decide which **one** of the following is true:

- A. $A \cap B$ is the empty set
- B. $A \cap B$ is finite
- C. $A \cap B$ is countably infinite
- D. $A \cap B$ is uncountably infinite

D

Multiple choice problems: Write your answers in the boxes, or they will not be graded!

4. (3 points) What is the **smallest** integer p such that $f(n) = 3n^3 + n^3 \log(n^4) + n \log(n^5)$ is $\mathcal{O}(n^p)$.

A. $p = 2$

B. $p = 3$

C. $p = 4$

D. $p = 5$

E. $p = 6$

C

5. (3 points) Select the answer that is the closed form solution to this recurrence relation:

$$a_n = 8a_{n-1} - 16a_{n-2}$$

A. $a_n = 2^n$

B. $a_n = (-4)^n$

C. $a_n = n4^n$

D. $a_n = 1$

C

6. (3 points) Given the function $g : (\mathbb{Z} \times \mathbb{Z}) \rightarrow \mathbb{Z}$, $g(m, n) = 2m - 4n$. Which of the following is true?

A. g is one-to-one, but not onto

B. g is not one-to-one, but it is onto

C. g is both one-to-one and onto

D. g is neither one-to-one nor onto

D

Multiple choice problems: Write your answers in the boxes, or they will not be graded!

7. (3 points) Given the function $f : \mathbb{Z} \rightarrow \mathbb{Z}$, $f(n) = 4n - 1$. Which of the following is true?

- A. g is one-to-one, but not onto
- B. g is not one-to-one, but it is onto
- C. g is both one-to-one and onto
- D. g is neither one-to-one nor onto

A

8. (3 points) Consider the pseudocode for procedure DivineDivision(A, B, n), given below. The input to the procedure is two matrices A and B , and scalar n . $A[i][j]$ and $B[i][j]$ (the element in row i , column j of matrices A and B , respectively) are all integers. n gives the number of rows and columns of A and B (they are both $n \times n$ square matrices). Give an estimate of the complexity of this procedure, where complexity is measured by the number of **additions and subtractions** needed.

```

procedure DivineDivision( $A, B, n$ ):
    ctr = 0
    for  $i$  from 1 to  $n$ :
        for  $j$  from 1 to  $n$ :
            if ( $A[i][j] + B[i][j]$ ) % 3 == 0,
                then ctr = ctr+1
            else
                then ctr = ctr-1
    return ctr

```

- A. DivineDivision is order n
- B. DivineDivision is order n^2
- C. DivineDivision is order n^3
- D. DivineDivision is order n^4

B

9. (3 points) Consider the pseudocode below for the function mysteryFunction(input). What should the function output be if the input is $input = [9, 3, 13, 3, 4, 7, 5, 10]$? You may assume that array indexing begins at 0, which is the left-most element in the given input object (i.e. $input[0]=9$).

```

def mysteryFunction(input)
    k=0
    x=input[k]
    for i = 1, 2, ..., length(input)
        if (input[i] < x), then ( $k = i$ , and  $x = input[i]$ )
    return (k)

```

- A. 1
- B. 2
- C. 3
- D. 13

A

Short-answer problems: If your answers do not fit in the given box, MAKE A NOTE of where the work is continued or it will NOT be graded!

10. (6 points) Consider the sequence defined as $T_{n+3} = T_{n+2} + T_{n+1} + T_n$ with initial conditions:
 $T_1 = 1, T_2 = 1, T_3 = 1$

Suppose we want to prove that $T_n < 2^n$ for all $n \geq 4$. Answer the following two questions. Note: You do **not** need to complete the proof. You only need to answer the specified questions.

- (a) Specify any and all base cases. Show the base case(s) in their entirety.
- (b) What is your induction hypothesis?

Solution:

- (a) **Base Cases:** Since our recursion depends on the term $n - 3$, we need 3 base cases.

$$n = 4 \quad T_4 = 1 + 1 + 1 = 3 \quad \text{and} \quad 2^4 = 16, \quad \text{since } 3 < 16, \text{ we have verified that } T_4 < 2^4 \quad \checkmark$$

$$n = 5 \quad T_5 = 3 + 1 + 1 = 5 \quad \text{and} \quad 2^5 = 32, \quad \text{since } 5 < 32, \text{ we have verified that } T_5 < 2^5 \quad \checkmark$$

$$n = 6 \quad T_6 = 5 + 3 + 1 = 9 \quad \text{and} \quad 2^6 = 64, \quad \text{since } 9 < 64, \text{ we have verified that } T_6 < 2^6 \quad \checkmark$$

- (b) **Induction Hypothesis:** Assume for $4 \leq k \leq m$ that $T_k < 2^k$.

11. (7 points) Suppose you have two sets: $A = \{b, c, d\}$ and $B = \{a, b\}$. Find the set: $\mathcal{P}(A) - \mathcal{P}(B)$.

Solution:

$$\mathcal{P}(A) = \{\emptyset, \{b\}, \{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

$$\mathcal{P}(B) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Therefore,

$$\mathcal{P}(A) - \mathcal{P}(B) = \{\{c\}, \{d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{b, c, d\}\}$$

12. (20 points) (a) Given the recursive sequence $a_n = 2a_{n-1} + 3$ with $a_0 = 1$. Find the closed form solution.
- (b) Given the recursive sequence $a_n = a_{n-1} + 2n + 3$ with initial condition $a_0 = 4$, use induction to prove that $a_n = n^2 + 4n + 4$ for $n \geq 0$.

Solution:

(a)

$$\begin{aligned}
 a_n &= 2a_{n-1} + 3 \\
 &= 2(2a_{n-2} + 3) + 3 = 2^2a_{n-2} + 2 \cdot 3 + 3 \\
 &= 2^2(2a_{n-3} + 3) + 2 \cdot 3 + 3 = 2^3a_{n-3} + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
 &= 2^3(2a_{n-4} + 3) + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 = 2^4a_{n-4} + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
 &= \dots \\
 &= 2^n a_{n-n} + 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3 + \dots + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
 &= 2^n a_0 + 3 \cdot (2^{n-1} + 2^{n-2} + \dots + 2^3 + 2^2 + 2 + 1) \\
 &= 2^n(1) + 3 \sum_{i=0}^{n-1} 2^i \\
 &= 2^n + 3(2^n - 1) \quad \text{finite geometric sum formula} \\
 &= 4 \cdot 2^n - 3 \\
 &= 2^{n+2} - 3
 \end{aligned}$$

Therefore, the closed form solution of the given recursion is $\boxed{a_n = 2^{n+2} - 3}$.

(b) **Base Case:** $a_0 = 4$, the base case is $n = 0$.

$$\begin{aligned}
 a_0 &= 0^2 + 4 \cdot 0 + 4 = 4 \\
 4 &= 4 \checkmark
 \end{aligned}$$

Induction Step: Assume for $k \geq 0$, that $a_k = k^2 + 4k + 4$.

Using our recurrence relation,

$$\begin{aligned}
 a_{k+1} &= a_k + 2(k+1) + 3 \\
 &= k^2 + 4k + 4 + 2k + 2 + 3 \quad \text{by the inductive hypothesis} \\
 &= k^2 + 4(k+1) + 2k + 1 + 4 \\
 &= k^2 + 2k + 1 + 4(k+1) + 4 \\
 &= (k+1)^2 + 4(k+1) + 4
 \end{aligned}$$

Thus by weak induction, we have proven that $a_n = n^2 + 4n + 4$ for all $n \geq 0$.

13. (20 points) Consider the function $f(n) = 9n^3 + 2n^3 \log_e(n^5) - n \log_e(n)$

- (a) Find the smallest nonnegative integer p for which n^p is a tight big-**O** bound on $f(n)$. Be sure to justify any inequalities you use and provide the C and k from the big-**O** definition.
- (b) Find the largest nonnegative integer p for which n^p is a tight big- **Ω** bound on $f(n)$. Be sure to justify any inequalities you use and provide the C and k from the big- **Ω** definition.
- (c) Can we conclude that $f(n)$ is **$\Theta(n^p)$** for some nonnegative integer p ? Explain why or why not.

Solution:

(a) First, we simplify using our rules for logarithms: $f(n) = 9n^3 + 10n^3 \log_e(n) - n \log_e(n)$.

Next, the natural first guess should be the **leading order term**, which is n^3 . But two terms have n^3 in them. Since the $\log_e n$ is only going to make the second term larger than n^3 for $\log n > 1$ ($n > e$), this means the second term is the dominant one (upper bound).

The lowest power of n that can be an upper bound for that second term is n^4 . We get upper bounds for each term in terms of n^4 :

$$\begin{aligned} 9n^3 &\leq 9n^4, \text{ for } n \geq 1 \\ 10n^3 \log(n) &\leq 10n^3 \cdot n = 10n^4, \text{ for } n \geq 1 \\ -n \log_e(n) &\leq 0, \text{ for } n \geq 1 \end{aligned}$$

So we have:

$$f(n) \leq 9n^4 + 10n^4 + 0 = 19n^4, \text{ for } n \geq 1$$

Thus with $C = 19$ and $k = 1$, f is **$O(n^4)$** .

(b) Again, the natural first guess should be the **leading order term**, which is n^3 . But two terms have n^3 in them. Since the $\log_e n$ is only going to make the second term larger than n^3 for $\log n > 1$ ($n > e$), this means the *first* term will be smaller, and so this is our first guess for the big- **Ω** bound.

The lowest power of n that can serve as a lower bound is n^3 . We get the lower bounds for each term in terms of n^3 :

$$\begin{aligned} 9n^3 &\geq 9n^3, \text{ for } n \geq 1 \\ 10n^3 \log(n) &\geq 10n^3, \text{ for } n \geq e \\ n \log n \leq n^3 &\longrightarrow -n \log(n) \geq -n^3, \text{ for } n \geq 1 \end{aligned}$$

where the second line comes from the fact that $\log_e n > 1$ exactly when $n > e$ (by exponentiating both sides and using the fact that $e^{\log_e n} = n$).

So we have:

$$f(n) \geq 9n^3 + 10n^3 - n^3 = 18n^3, \text{ for } n \geq e$$

Thus, with $C = 18$ and $k = e$, f is **$\Omega(n^3)$** .

(c) No, because it is not both big-**O** and big- **Ω** n^p with the same p value.

14. (20 points) (a) Prove DeMorgan's Law for sets $\overline{A \cup B} = \overline{A} \cap \overline{B}$ using **set builder notation**.
 (b) Let A , B , and C be sets. Show that $\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

Solution:

(a)

$$\begin{aligned}
 \overline{A \cup B} &= \{x | x \in \overline{A \cup B}\} && \text{set builder notation for } \overline{A \cup B} \\
 &= \{x | x \notin (A \cup B)\} && \text{definition of complement} \\
 &= \{x | \neg(x \in (A \cup B))\} && \text{definition of "not in"} \\
 &= \{x | \neg(x \in A \vee x \in B)\} && \text{definition of union} \\
 &= \{x | x \notin A \wedge x \notin B\} && \text{DeMorgans Law} \\
 &= \{x | x \in \overline{A} \wedge x \in \overline{B}\} && \text{definition of complement} \\
 &= \{x | x \in \overline{A} \cap \overline{B}\} && \text{definition of intersection} \\
 &= \overline{A} \cap \overline{B} && \text{set builder notation for } \overline{A} \cap \overline{B}
 \end{aligned}$$

(b)

$$\begin{aligned}
 \overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{DeMorgan's Law for Sets} \\
 &= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{DeMorgan's Law for Sets} \\
 &= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by Commutativity of an Intersection} \\
 &= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by Commutativity of a Union}
 \end{aligned}$$