CSCI 2824 – Spring 2019 **Homework 6**

This assignment is due on Friday March 1 to Gradescope by Noon. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number at the top of your assignment. You may **neatly** type your solutions for +1 extra credit on the assignment.

1. (a) Suppose P, Q, and R are non-empty sets. Prove that $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$ by showing that each side of this equation must be a **subset** of the other side, and concluding that the two sides must therefore be equal.

   (b) Suppose that P, Q, and R are non-empty sets. Prove that $P \times (Q \cap R) = (P \times Q) \cap (P \times R)$ by using **set builder notation** and set identities and definitions.

   (c) Let U be the set of all integers. Let E be the set of all even integers, D the set of all odd integers, P the set of positive integers, and N the set of all negative integers. Find the following sets.

      i. $E - P$
      ii. $P \cup N$
      iii. $D - E$
      iv. $\bar{U}$

Solution:

(a) ($\Longrightarrow$) Suppose $(x, y) \in P \times (Q \cap R)$
$\Longrightarrow$ $x \in P \wedge ((y \in Q \cap R))$ by definition of cartesian product
$\Longrightarrow$ $x \in P \wedge ((y \in Q) \wedge (y \in R))$ by definition of intersection
$\Longrightarrow$ $(x \in P \wedge y \in Q) \wedge (x \in P \wedge y \in R)$
$\Longrightarrow$ $((x, y) \in P \times Q) \wedge ((x, y) \in P \times R)$ by definition of cartesian product
$\Longrightarrow$ $(x, y) \in (P \times Q) \cap (P \times R)$ by definition of intersection
$\Longrightarrow$ $P \times (Q \cap R) \subseteq (P \times Q) \cap (P \times R)$ by definition of a subset

($\Longleftarrow$) Suppose $(x, y) \in (P \times Q) \cap (P \times R)$
$\Longrightarrow$ $(x, y) \in (P \times Q) \wedge (x, y) \in (P \times R)$ by definition of intersection
$\Longrightarrow$ $(x \in P \wedge y \in Q) \wedge (x \in P \wedge y \in R)$ by definition of a cartesian product
$\Longrightarrow$ $x \in P \wedge (y \in Q \wedge y \in R)$
$\Longrightarrow$ $x \in P \wedge (y \in Q \cap R)$ by definition of intersection
$\Longrightarrow$ $(x, y) \in P \times (Q \cap R)$ by definition of cartesian product
$\Longrightarrow$ $(P \times Q) \cap (P \times R) \subseteq P \times (Q \cap R)$ by definition of a subset

Since $LHS \subseteq RHS$ and $RHS \subseteq LHS$, the two sides must be equal.

(b)

$$\begin{aligned}
P \times (Q \cap R) &= \{(x, y) | x \in P \wedge y \in (Q \cap R)\} \\
&= \{(x, y) | x \in P \wedge ((y \in Q) \wedge (y \in R))\} \\
&= \{(x, y) | (x \in P \wedge y \in Q) \wedge (x \in P \wedge y \in R)\} \\
&= (P \times Q) \cap (P \times R)
\end{aligned}$$

(c)

(i) $E - P = \{0, -2, -4, -6, -8, ...\}$
(ii) $P \cup N = U - \{0\}$
(iii) $D - E = D$
(iv) $\bar{U} = \emptyset$

2. (a) Give an example of two uncountable sets A and B with a nonempty intersection, such that $A - B$ is

   i. finite
   ii. countably infinite
   iii. uncountably infinite

   (b) Use the Cantor diagonalization argument to prove that the number of real numbers in the interval $[3, 4]$ is uncountable.

   (c) Use a proof by contradiction to show that the set of irrational numbers that lie in the interval $[3, 4]$ is uncountable. (You can use the fact that the set of rational numbers ($\mathbb{Q}$) is countable and the set of reals ($\mathbb{R}$) is uncountable). Show all work.

Solution:

(a)
(i) $A = (-\infty, 1], B = (-\infty, 1)$ or some variation where only one number remains in the set.

(ii) $A = \{x | x \in \mathbb{R}\}, B = \{x | x \in \mathbb{R} \wedge x \notin \mathbb{Q}\}$ or some variation where a set containing all irrational numbers is subtracted from a larger set.

(iii) $A = [0, 2], B = [0, 1]$ or any variations where irrational numbers remain in the set $A - B$

(b)

Proof. We proceed by contradiction and assume that there does exist a function $f : \mathbb{N} \to [3, 4]$ that is 1-1 and onto. For each $m \in \mathbb{N}$, $f(m)$ is a real number between 3 and 4, and we represent it using the decimal notation

$$f(m) = 3.a_{m1}a_{m2}a_{m3}a_{m4}a_{m5}....$$

What is meant here is that for each $m, n \in \mathbb{N}$, a_{mn} is the digit from the set $\{0, 1, 2, 3, 4, ..., 9\}$ that represents the nth digit in the decimal expansion of $f(m)$. The 1-1 correspondence between $\mathbb{N}$ and $[3, 4]$ can be summarized in the doubly indexed array

$$\begin{array}{cccccccccccc}
N & & [3,4] & & & & & & & & \\
\hline
1 & \leftrightarrow & f(1) & = & .a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\
2 & \leftrightarrow & f(2) & = & .a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\
3 & \leftrightarrow & f(3) & = & .a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \cdots \\
4 & \leftrightarrow & f(4) & = & .a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & \cdots \\
5 & \leftrightarrow & f(5) & = & .a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & \cdots \\
6 & \leftrightarrow & f(6) & = & .a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & \cdots \\
\vdots & & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

The key assumption about this correspondence is that *every* real number in $[3, 4]$ is assumed to appear somewhere on this list. Now, define a real number $x \in [3, 4]$ with the decimal expansion $x = 3.b_1b_2b_3b_4...$ using the rule

$$b_n = \begin{cases} 2 & if\ a_{nn} \neq 2 \\ 3 & if\ a_{nn} = 2 \end{cases}$$

To compute b_1, we look at a_{11}. If this digit is a 2, we set $b_1 = 3$, if $a_11 \neq 2$ then $b_1 = 2$. Therefore, $x \neq f(1)$ (the first number listed).

To compute b_2, we look at a_{22}. If this digit is a 2, we set $b_2 = 3$, if $a_22 \neq 2$ then $b_2 = 2$. Therefore, $x \neq f(2)$ (the second number listed).

To compute b_3, we look at a_{33}. If this digit is a 2, we set $b_3 = 3$, if $a_33 \neq 2$ then $b_3 = 2$. Therefore, $x \neq f(3)$ (the third number listed).

We can continue on through the list in this fashion. We will find that $x \neq f(n)$ for any $n \in \mathbb{N}$. Thus it must be that x was not in our original list. This contradicts our assumption that we listed *every* real number in $[3, 4]$. Therefore, it must be that $[3, 4]$ is uncountably infinite.

□

(c)

Proof. Let $A =$ irrationals in $[3,4]$, let $B =$ rationals in $[3,4]$. Then $A \cup B = [3,4]$. Assume for the sake of contradiction that A is countable. We know that B is countable because we know that the rationals are countable. The union of 2 countable sets is countable. $\implies$ $[3,4]$ is countable. However, we know that any interval in $\mathbb{R}$ is uncountable (by Cantor's diagonal argument). $\rightarrow\leftarrow$ This is a contradiction to our previous assumption. Therefore, the irrationals in $[3,4]$ are uncountable. $\square$

3. (a) Find a closed form for the recurrence relation: $a_n = 2a_{n-1} - 2, a_0 = -1$
   (b) Find a closed form for the recurrence relation: $a_n = (n+2)a_{n-1}, a_0 = 3$
   (c) Show that $a_n = 5(-1)^n - n + 2$ is a solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2} + 2n - 9$.

Solution: (a)

$$\begin{aligned}
a_n &= 2a_{n-1} - 2 \\
&= 2(2a_{n-2} - 2) - 2 = 2^2 a_{n-2} - 2^2 - 2^1 \\
&= 2(2(2a_{n-3} - 2) - 2) - 2 = 2^3 a_{n-3} - 2^3 - 2^2 - 2^1 \\
&. \\
&. \\
&. \\
&= 2^n a_{n-n} - 2^n - 2^{n-1} - \ldots - 2^3 - 2^2 - 2^1 \\
&= 2^n a_0 - 2^n - 2^{n-1} - \ldots - 2^3 - 2^2 - 2^1 \\
&= 2^n \cdot a_0 - (2^n + 2^{n-1} + \ldots + 2^3 + 2^2 + 2^1) \\
&= 2^n \cdot a_0 - \sum_{i=1}^{n} 2^i \\
&= 2^n \cdot a_0 - (2^{n+1} - 2) \\
&= 2^n - 2^{n+1} + 2 \\
&= 2^n - 2 \cdot 2^n + 2 \\
&= -3 \cdot 2^n + 2
\end{aligned}$$

So $\boxed{a_n = -3 \cdot 2^n + 2}$.

(b)

$$\begin{aligned}
a_n &= (n+2)a_{n-1} \\
&= (n+2)(n+1)a_{n-2} \\
&= (n+2)(n+1)(n)a_{n-3} \\
&= (n+2)(n+1)(n)(n-1)a_{n-4} \\
&. \\
&. \\
&. \\
&= (n+2)(n+1)(n)\ldots(4) \cdot (3) \cdot a_{n-n} \\
&= \frac{(n+2)!}{2} a_0 \\
&= \frac{3}{2}(n+2)!
\end{aligned}$$

So $\boxed{a_n = \frac{3}{2}(n+2)!}$

(c)

$$
\begin{aligned}
a_n &= a_{n-1} + 2a_{n-2} + 2n - 9 \\
&= 5(-1)^{n-1} - (n-1) + 2 + 2(5(-1)^{n-2} - (n-2) + 2) + 2n - 9 \\
&= -5(-1)^{n-2} - n + 1 + 2 + 10(-1)^{n-2} - 2n + 4 + 4 + 2n - 9 \\
&= 5(-1)^{n-2} - n + 2 \\
&= 5 \cdot (-1)^2 \cdot (-1)^{n-2} - n + 2 \\
&= 5(-1)^n - n + 2 \\
&= a_n
\end{aligned}
$$

4. (a) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m, n) = 2m - n$. Is this function onto?
   (b) Consider the function $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ where $f(m, n) = m^2 - n^2$. Is this function onto?
   (c) Define the set C = the set of all residents of Colorado. Define in words a function $f : C \to \mathbb{Z}$. Is your function one-to-one? Is it onto? Be sure that the f you defined is indeed a **function**. Be creative and have fun!
   (d) Again, define the set C = the set of all residents of Colorado. Define in words a function $f : C \to \mathbb{Z}$. However this time, make sure that your function is one-to-one. (Make sure to give a different example from part (c)).

Solution:

(a) This function is onto. The definition of onto tells us that for every $b \in \mathbb{Z}$ there is an $a = (m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(a) = b$. Let $a = (0, -n)$. Then $f(a) = f((0, -n)) = 2 \cdot 0 - (-n) = n$ for any $n \in \mathbb{Z}$

(b) This function is **not** onto. A counterexample would be noticing that $4 \in \mathbb{Z}$ however there do not exist $m, n \in \mathbb{Z}$ such that $m^2 - n^2 = 4$.

(c) Many possible functions. Here's one. Let f be the function that represents the number of pairs of socks that each person in Colorado owns. This function is not one-to-one because there are at least two people who own the same amount of socks. Additionally, the function is not onto, because we can find the person with the most number of pairs of socks in CO. Suppose this person owns M pairs of socks. Then the number $M + 1$ never gets mapped to, so the function is not onto.

(d) Many possible functions. If you consider a social security number without the dashes, then you can represent anyone's social security number as an integer. So one possible one-to-one function would be matching each person in CO to their unique social security number.