Recurrences and Induction

1. Given the recurrence relations below. Find a closed form solution for each one.
   (a) $a_n = 2a_{n-1} + 3, \quad a_0 = 1$
   (b) $a_n = 3a_{n-1}, \quad a_0 = 2$
   (c) $a_n = a_{n-1} + 2, \quad a_0 = 3$
   (d) $a_n = a_{n-1} + n, \quad a_0 = 1$
   (e) $a_n = a_{n-1} + 2n + 3, \quad a_0 = 4$
   (f) $a_n = na_{n-1}, \quad a_0 = 5$

2. (a) Show that the sequence $a_n = 2^{n+2} - 3$ is a solution of the recurrence relation $a_n = 2a_{n-1} + 3$
   (b) Show that the sequence $a_n = 2 \cdot 3^n$ is a solution of the recurrence relation $a_n = 3a_{n-1}$
   (c) Show that the sequence $a_n = 3 + 2n$ is a solution of the recurrence relation $a_n = a_{n-1} + 2$
   (d) Show that the sequence $a_n = 1 + \frac{n(n+1)}{2}$ is a solution of the recurrence relation $a_n = a_{n-1} + n$
   (e) Show that the sequence $a_n = n^2 + 4n + 4$ is a solution of the recurrence relation $a_n = a_{n-1} + 2n + 3$
   (f) Show that the sequence $a_n = 5 \cdot n!$ is a solution of the recurrence relation $a_n = na_{n-1}$

3. (a) Given the recursive sequence $a_n = 2a_{n-1} + 3$, with initial condition $a_0 = 1$, use induction to prove that $a_n = 2^{n+2} - 3$.
   (b) Given the recursive sequence $a_n = 3a_{n-1}$, with initial condition $a_0 = 2$, use induction to prove that $a_n = 2 \cdot 3^n$.
   (c) Given the recursive sequence $a_n = a_{n-1} + 2$, with initial condition $a_0 = 3$, use induction to prove that $a_n = 3 + 2n$.
   (d) Given the recursive sequence $a_n = a_{n-1} + n$, with initial condition $a_0 = 1$, use induction to prove that $a_n = 1 + \frac{n(n+1)}{2}$.
   (e) Given the recursive sequence $a_n = a_{n-1} + 2n + 3$, with initial condition $a_0 = 4$, use induction to prove that $a_n = n^2 + 4n + 4$.
   (f) Given the recursive sequence $a_n = na_{n-1}$, with initial condition $a_0 = 5$, use induction to prove that $a_n = 5 \cdot n!$.

4. Prove by induction the following: If n is a non-negative integer, then 5 divides $n^5 - n$.

5. Prove by induction the following: If $n \in \mathbb{Z}$ and $n \geq 0$, then $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

6. Prove by induction the following: The inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for each $n \in \mathbb{N}$.

7. Use induction to prove that you can achieve any postage of 8 cents or more, exactly, using only 3-cent and 5-cent stamps. For example, for a postage of 47 cents, you could use nine 3-cent stamps and four 5-cent stamps.

8. Use induction to prove the following: If $n \in \mathbb{N}$, then 12 divides $(n^4 - n^2)$.