

This assignment is due on Friday February, 15 to Gradescope by Noon. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number at the top of your assignment. You may **neatly** type your solutions for +1 extra credit on the assignment.

1. Use rules of inference to show that if $\forall x(P(x) \vee Q(x))$, $\forall x(\neg Q(x) \vee S(x))$, $\forall x(R(x) \rightarrow \neg S(x))$, and $\exists x\neg P(x)$ are true, then $\exists x\neg R(x)$ is true.

Solution:

1	$\forall x(P(x) \vee Q(x))$	Premise
2	$\forall x(\neg Q(x) \vee S(x))$	Premise
3	$\forall x(R(x) \rightarrow \neg S(x))$	Premise
4	$\exists x\neg P(x)$	Premise
5	$\neg P(c)$	Existential Instantiation (4)
6	$P(c) \vee Q(c)$	Universal Instantiation (1)
7	$\neg Q(c) \vee S(c)$	Universal Instantiation (2)
8	$R(c) \rightarrow \neg S(c)$	Universal Instantiation (3)
9	$\neg P(c) \rightarrow Q(c)$	RBI (5)
10	$Q(c)$	Modus Ponens (5,9)
11	$Q(c) \rightarrow S(c)$	RBI(6)
12	$S(c)$	Modus Ponens(10,11)
13	$S(c) \rightarrow \neg R(c)$	Contrapositive (7)
14	$\neg R(c)$	Modus Ponens
15	$\exists x\neg R(x)$	Existential Generalisation

2. Prove or disprove the following claims. Be sure to indicate whether you are using a Direct Proof, a Contrapositive Proof, a Proof by Cases, or a Proof by Contradiction, or a Counterexample.

- (a) If the average of a_1, a_2, \dots, a_n is some number \bar{a} , then at least one of the real numbers a_1, a_2, \dots, a_n must be greater than or equal to \bar{a} .
- (b) If n is an integer and $3n + 2$ is even, then n is even. [Note: Do NOT use a contrapositive proof.]
- (c) If the lengths of two sides of a triangle are irrational, then the third side must be irrational also.

Solution: 2 (a):

Proof. Using Proof by contradiction, assume that there exists a set of numbers a_1, a_2, \dots, a_n and their average \bar{a} , such that all the numbers a_1, a_2, \dots, a_n are less than \bar{a} . Thus, we have:

$$a_1 < \bar{a},$$

$$a_2 < \bar{a},$$

...

$$a_n < \bar{a}$$

$$a_1 + a_2 + \dots + a_n < \bar{a} + \bar{a} + \dots + \bar{a}$$

$$a_1 + a_2 + \dots + a_n < n \times \bar{a}$$

$$\frac{a_1 + a_2 + \dots + a_n}{n} < \bar{a}$$

Since we can't have $\bar{a} < \bar{a}$, we have found a contradiction. Hence, at least one number must be greater than or equal to \bar{a}

□

Alternate Solution using a Direct Proof:

Proof. Suppose that we have a set of numbers $\{a_1, a_2, a_3, \dots, a_n\}$ that has n elements. Suppose that the largest element is M . This means that for all a_i , $a_i \leq M$. The average of this set of numbers is \bar{a} . Therefore,

$$\frac{\sum_{i=1}^n a_i}{n} = \bar{a}$$

Since $a_i \leq M$, then $n\bar{a} \leq \sum_{i=1}^n M = nM$.

Thus we have that $n\bar{a} \leq nM \implies \bar{a} \leq M$. Therefore, there is at least one element of our set that is greater than or equal to \bar{a} as desired.

□

2 (b):

Proof. Again using a proof by contradiction, assume that $3n + 2$ is even and n is odd. Since n is odd, let $n = 2k + 1$ for some integer k . Then,

$$3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1$$

Thus we've found that $3n + 2$ is odd, a contradiction to our assumption that $3n + 2$ is even. Thus we've shown that it must be the case that if $3n + 2$ is even, then n is even, as desired.

□

Alternate intuition: $3n + 2$ is even.

$3n + 2 - 2 = 3n$ is even.

$3n$ can be even only if n is even as two odd numbers never result in an even number.

2 (c): This statement is false. We use a counterexample to show this.

Consider a right triangle with two sides of length $\sqrt{2}$. Thus, the hypotenuse is $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$. Since 2 is a rational number, we have disproven this claim.

3. The divisibility rule by 3 is a rule that a number N is divisible by 3 if the sum of the digits is divisible by 3. For example, is 132 divisible by 3? $1 + 3 + 2 = 6$ and 6 is a number divisible by 3. So 132 must also be divisible by 3.

Please prove this "rule" for three digit numbers:

Let the positive integer N have the form $N = 100a + 10b + c$ where $a + b + c = 3n$ and a, b , and c are digits with $a \neq 0$. Prove that N is divisible by 3.

Solution:

Proof. Let three digit positive integer N be of the form $N = 100a + 10b + c$.

We have to prove that if the sum of digits of number is divisible by 3 then the number is divisible by 3.
Given : $a + b + c = 3n$, we can manipulate the expression for N .

$$\begin{aligned} N &= 100a + 10b + c \\ &= 99a + a + 9b + b + c \\ &= 99a + 9b + (a + b + c) \\ &= 99a + 9b + 3n \\ &= 3(33a + 3b + n) \end{aligned}$$

Thus $\frac{N}{3} = m$ where $m = 33a + 3b + n$ is some integer. Therefore, N is divisible by 3, as desired.

□

4. Prove the following:

- (a) Prove that n is even if and only if $n^2 - 6n + 5$ is odd.
(b) Prove that if $2n^2 + 3n + 1$ is even, then n is odd.

Solution:

- (a) *Proof.* (\implies) For the forward direction, we assume that n is even. Let $n = 2k$ for some integer k . Then,

$$\begin{aligned} n^2 - 6n + 5 &= (2k)^2 - 6(2k) + 5 \\ &= 4k^2 - 12k + 5 \\ &= 4k^2 - 12k + 4 + 1 \\ &= 2(2k^2 - 6k + 2) + 1 \\ &= 2m + 1 \end{aligned}$$

where $m = 2k^2 - 6k + 2$ is some integer. Therefore, $n^2 - 6n + 5$ is odd, as desired.

- (\impliedby) For the backward direction, we use contraposition. Assume that n is odd. Let $n = 2k + 1$. Then,

$$\begin{aligned} n^2 - 6n + 5 &= (2k + 1)^2 - 6(2k + 1) + 5 \\ &= 4k^2 + 4k + 1 - 12k - 6 + 5 \\ &= 4k^2 - 8k \\ &= 2(2k^2 - 4k) \end{aligned}$$

Thus $n^2 - 6n + 5$ is even and we are done.

□

- (b) *Proof.* Using a contrapositive proof, let n be an even number of the form $2k$. Let us substitute this into the expression $2n^2 + 3n + 1$.

The expression then becomes $2(2k)^2 + 3(2k) + 1$.

On expansion, we get $8k^2 + 6k + 1$ which is equal to $2(4k^2 + 3k) + 1$.

This expression is always odd. Hence we have proven the claim using contraposition. \square

5. Use proof by cases to prove that $x + |x - 8| \geq 8$ for all real numbers x . [Hint: $|x| = x$ when $x \geq 0$ and $|x| = -x$ when $x < 0$.]

Solution:

Proof. To prove $x + |x - 8| \geq 8$ we consider two cases:
 $x \geq 8$ and $x < 8$

Case 1: for $x \geq 8$, $|x - 8| = x - 8$

so,

$$\begin{aligned} x + |x - 8| &= x + x - 8 \\ &= 2x - 8 \\ &\geq 2 \cdot 8 - 8 \\ &= 8 \end{aligned}$$

since $x \geq 8$.

Thus, $x + |x - 8| \geq 8$, as desired.

Case 2: for $x < 8$, $|x - 8| = 8 - x$

Therefore,

$$\begin{aligned} x + |x - 8| &= x + 8 - x \\ &= 8 \\ &\geq 8 \end{aligned}$$

Thus, $x + |x - 8| \geq 8$, as desired.

From Case 1 and Case 2, we have found that $x + |x - 8| \geq 8$ for all real numbers x . \square