Recurrences and Induction

1. Given the recurrence relations below. Find a closed form solution for each one.

    (a) $a_n = 2a_{n-1} + 3, \quad a_0 = 1$
    (b) $a_n = 3a_{n-1}, \qquad a_0 = 2$
    (c) $a_n = a_{n-1} + 2, \qquad a_0 = 3$
    (d) $a_n = a_{n-1} + n, \qquad a_0 = 1$
    (e) $a_n = a_{n-1} + 2n + 3, \quad a_0 = 4$
    (f) $a_n = na_{n-1}, \qquad a_0 = 5$

Solution:

(a)

$$
\begin{aligned}
a_n &= 2a_{n-1} + 3 \\
&= 2(2a_{n-2} + 3) + 3 = 2^2 a_{n-2} + 2 \cdot 3 + 3 \\
&= 2^2(2a_{n-3} + 3) + 2 \cdot 3 + 3 = 2^3 a_{n-3} + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= 2^3(2a_{n-4} + 3) + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 = 2^4 a_{n-4} + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= \dots \\
&= 2^n a_{n-n} + 2^{n-1} \cdot 3 + 2^{n-2} \cdot 3 + \dots + 2^3 \cdot 3 + 2^2 \cdot 3 + 2 \cdot 3 + 3 \\
&= 2^n a_0 + 3 \cdot (2^{n-1} + 2^{n-2} + \dots + 2^3 + 2^2 + 2 + 1) \\
&= 2^n(1) + 3\sum_{i=1}^{n-1} 2^i \\
&= 2^n + 3(2^n - 1) \quad \text{finite geometric sum formula} \\
&= 4 \cdot 2^n - 3 \\
&= 2^{n+2} - 3
\end{aligned}
$$

(b)

$$
\begin{aligned}
a_n &= 3a_{n-1} \\
&= 3(3a_{n-2}) \\
&= 3^2(3a_{n-3}) \\
&= 3^3(3a_{n-4}) \\
&= \dots \\
&= 3^n a_{n-n} \\
&= 3^n a_0 \\
&= 2 \cdot 3^n
\end{aligned}
$$

(c)

$$
\begin{aligned}
a_n &= a_{n-1} + 2 \\
&= (a_{n-2} + 2) + 2 \\
&= (a_{n-3} + 2) + 2 + 2 \\
&= (a_{n-4} + 2) + 2 + 2 + 2 \\
&= a_{n-5} + 5 \cdot 2 \\
&= \dots \\
&= a_{n-n} + n \cdot 2 \\
&= a_0 + 2n \\
&= 3 + 2n
\end{aligned}
$$

(d)Let's start by listing a few terms of this sequence.

$$a_n = a_{n-1} + n$$
$$a_{n-1} = a_{n-2} + (n-1)$$
$$a_{n-2} = a_{n-3} + (n-2)$$
$$a_{n-3} = a_{n-4} + (n-3)$$
$$a_{n-4} = a_{n-5} + (n-4)$$
$$= \ldots$$
$$a_{n-(n-1)} = a_1 = a_0 + 1$$

Now, we will use a backwards iterative approach to find the solution.

$$
\begin{aligned}
a_n &= a_{n-1} + n \\
&= a_{n-2} + (n-1) + n \\
&= a_{n-3} + (n-2) + (n-1) + n \\
&= a_{n-4} + (n-3) + (n-2) + (n-1) + n \\
&= a_{n-5} + (n-4) + (n-3) + (n-2) + (n-1) + n \\
&= \ldots \\
&= a_1 + 2 + 3 + \ldots + (n-4) + (n-3) + (n-2) + (n-1) + n \\
&= a_0 + 1 + 2 + 3 + \ldots + (n-4) + (n-3) + (n-2) + (n-1) + n \\
&= 1 + \sum_{i=1}^{n} i \\
&= 1 + \frac{n(n+1)}{2}
\end{aligned}
$$

(e) Again, let's start off by listing a few terms to get our bearings.

$$a_n = a_{n-1} + 2n + 3$$
$$a_{n-1} = a_{n-2} + 2(n-1) + 3$$
$$a_{n-2} = a_{n-3} + 2(n-2) + 3$$
$$a_{n-3} = a_{n-4} + 2(n-3) + 3$$
$$= \ldots$$
$$a_1 = a_0 + 2 + 3$$

Now, we will use a backwards iterative approach to find the solution.

$$
\begin{aligned}
a_n &= a_{n-1} + 2n + 3 \\
&= (a_{n-2} + 2(n-1) + 3) + 2n + 3 && = a_{n-2} + 2(n-1) + 2n + 3 + 3 \\
&= (a_{n-3} + 2(n-2) + 3) + 2(n-1) + 2n + 3 + 3 && = a_{n-3} + 2(n-2) + 2(n-1) + 2n + 3 \cdot 3 \\
&= (a_{n-4} + 2(n-3) + 3) + 2(n-2) + 2(n-1) + 2n + 3 \cdot 3 && = a_{n-4} + 2((n-3) + (n-2) + (n-1) + n) + 4 \cdot 3 \\
&= \ldots \\
&= a_0 + 2(1 + 2 + \ldots + (n-2) + (n-1) + n) + 3n \\
&= 4 + 2\left(\frac{n(n+1)}{2}\right) + 3n \\
&= 4 + n(n+1) + 3n \\
&= n^2 + 4n + 4
\end{aligned}
$$

(f)

$$a_n = na_{n-1}$$
$$= n((n-1)a_{n-2})$$
$$= n(n-1)((n-2)a_{n-3})$$
$$= n(n-1)(n-2)((n-3)a_{n-4})$$
$$= ...$$
$$= n(n-1)(n-2)...3 \cdot 2 \cdot 1a_0$$
$$= 5 \cdot n!$$

2. (a) Show that the sequence $a_n = 2^{n+2} - 3$ is a solution of the recurrence relation $a_n = 2a_{n-1} + 3$

(b) Show that the sequence $a_n = 2 \cdot 3^n$ is a solution of the recurrence relation $a_n = 3a_{n-1}$

(c) Show that the sequence $a_n = 3 + 2n$ is a solution of the recurrence relation $a_n = a_{n-1} + 2$

(d) Show that the sequence $a_n = 1 + \frac{n(n+1)}{2}$ is a solution of the recurrence relation $a_n = a_{n-1} + n$

(e) Show that the sequence $a_n = n^2 + 4n + 4$ is a solution of the recurrence relation $a_n = a_{n-1} + 2n + 3$

(f) Show that the sequence $a_n = 5 \cdot n!$ is a solution of the recurrence relation $a_n = na_{n-1}$

Solution: (a)

$$2a_{n-1} + 3 = 2(2^{(n-1)+2} - 3) + 3$$
$$= 2(2^{n+1}) - 6 + 3$$
$$= 2^{n+2} - 3$$
$$= a_n$$

(b)

$$3a_{n-1} = 3(2 \cdot 3^{n-1})$$
$$= 2 \cdot 3 \cdot 3^{n-1}$$
$$= 2 \cdot 3^n$$
$$= a_n$$

(c)

$$a_{n-1} + 2 = 3 + 2(n-1) + 2$$
$$= 3 + 2n - 2 + 2$$
$$= 3 + 2n$$
$$= a_n$$

(d)

$$a_{n-1} + n = 1 + \frac{(n-1)n}{2} + n$$
$$= 1 + \frac{(n-1)n}{2} + \frac{2n}{2}$$
$$= 1 + \frac{(n-1)n + 2n}{2}$$
$$= 1 + \frac{n^2 - n + 2n}{2}$$
$$= 1 + \frac{n^2 + n}{2}$$
$$= 1 + \frac{n(n+1)}{2}$$
$$= a_n$$

(e)

$$
\begin{aligned}
a_{n-1} + 2n + 3 &= ((n-1)^2 + 4(n-1) + 4) + 2n + 3 \\
&= n^2 - 2n + 1 + 4n - 4 + 4 + 2n + 3 \\
&= n^2 + 4n + 4 \\
&= a_n
\end{aligned}
$$

(f)

$$
\begin{aligned}
na_{n-1} &= n(5(n-1)!) \\
&= 5 \cdot n \cdot (n-1)! \\
&= 5n! \\
&= a_n
\end{aligned}
$$

3. (a) Given the recursive sequence $a_n = 2a_{n-1} + 3$, with initial condition $a_0 = 1$, use induction to prove that $a_n = 2^{n+2} - 3$.

   (b) Given the recursive sequence $a_n = 3a_{n-1}$, with initial condition $a_0 = 2$, use induction to prove that $a_n = 2 \cdot 3^n$.

   (c) Given the recursive sequence $a_n = a_{n-1} + 2$, with initial condition $a_0 = 3$, use induction to prove that $a_n = 3 + 2n$.

   (d) Given the recursive sequence $a_n = a_{n-1} + n$, with initial condition $a_0 = 1$, use induction to prove that $a_n = 1 + \frac{n(n+1)}{2}$.

   (e) Given the recursive sequence $a_n = a_{n-1} + 2n + 3$, with initial condition $a_0 = 4$, use induction to prove that $a_n = n^2 + 4n + 4$.

   (f) Given the recursive sequence $a_n = na_{n-1}$, with initial condition $a_0 = 5$, use induction to prove that $a_n = 5 \cdot n!$.

Solution: (a) **Base Case**: $a_0 = 1$, so our base case is $n = 0$.

$$
a_0 = 2^{0+2} - 3 = 4 - 3 = 1
$$

$$
1 = 1 \checkmark
$$

Induction Step: Assume that for some $k \geq 1$, $a_k = 2^{k+2} - 3$.

Using our recurrence relation,

$$
\begin{aligned}
a_{k+1} &= 2a_k + 3 \\
&= 2(2^{k+2} - 3) + 3 \quad \text{by the induction hypothesis} \\
&= 2^{k+2+1} - 6 + 3 \\
&= 2^{(k+1)+2} - 3
\end{aligned}
$$

Thus, by weak induction, we have proven that $a_n = 2^{n+2} - 3$ for all $n \geq 0$.

(b) **Base Case:** $a_0 = 2$, so our base case is $n = 0$.

$$
a_0 = 2 \cdot 3^0 = 2 \cdot 1 = 2
$$

$$
2 = 2 \checkmark
$$

Induction Step: Assume for $k \geq 0$, that $a_k = 2 \cdot 3^k$.

Using our recurrence relation,

$$
\begin{aligned}
a_{k+1} &= 3a_k \\
&= 3(2 \cdot 3^k) \quad \text{by the induction hypothesis} \\
&= 2 \cdot 3^{k+1}
\end{aligned}
$$

Thus by weak induction, we have proven that $a_n = 2 \cdot 3^n$ for all $n \geq 0$.

(c) **Base Case:** $a_0 = 3$, so our base case is $n = 0$.

$$a_0 = 3 + 2 \cdot 0 = 3$$

$$3 = 3 \checkmark$$

Induction Step: Assume for $k \geq 0$, that $a_k = 3 + 2 \cdot k$.

Using our recurrence relation,

$$
\begin{aligned}
a_{k+1} &= a_k + 2 \\
&= 3 + 2 \cdot k + 2 \quad \text{by the induction hypothesis} \\
&= 3 + 2(k+1)
\end{aligned}
$$

Thus by weak induction, we have proven that $a_n = 3 + 2n$ for all $n \geq 0$.

(d) **Base Case:** $a_0 = 1$, so the base case is $n = 0$.

$$a_0 = 1 + \frac{0(0+1)}{2} = 1 + 0 = 1$$

$$1 = 1 \checkmark$$

Induction Step: Assume for $k \geq 0$, that $a_k = 1 + \frac{k(k+1)}{2}$.

Using our recurrence relation,

$$
\begin{aligned}
a_{k+1} &= a_k + k + 1 \\
&= 1 + \frac{k(k+1)}{2} + k + 1 \quad \text{by the induction hypothesis} \\
&= 1 + \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= 1 + \frac{k(k+1) + 2(k+1)}{2} \\
&= 1 + \frac{(k+1)(k+2)}{2} \\
&= 1 + \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

Thus by weak induction, we have proven that $a_n = 1 + \frac{n(n+1)}{2}$ for all $n \geq 0$.

(e) **Base Case:** $a_0 = 4$, the base case is $n = 0$.

$$a_0 = 0^2 + 4 \cdot 0 + 4 = 4$$

$$4 = 4 \checkmark$$

Induction Step: Assume for $k \geq 0$, that $a_k = k^2 + 4k + 4$.

Using our recurrence relation,

$$
\begin{aligned}
a_{k+1} &= a_k + 2(k+1) + 3 \\
&= k^2 + 4k + 4 + 2k + 2 + 3 \\
&= k^2 + 4(k+1) + 2k + 1 + 4 \\
&= k^2 + 2k + 1 + 4(k+1) + 4 \\
&= (k+1)^2 + 4(k+1) + 4
\end{aligned}
$$

Thus by weak induction, we have proven that $a_n = n^2 + 4n + 4$ for all $n \geq 0$.

(f) **Base Case:** $a_0 = 5$, the base case is $n = 0$.

$$
a_0 = 5 \cdot 0! = 5 \cdot 1 = 5
$$
$$
5 = 5\checkmark
$$

4. Prove by induction the following: If n is a non-negative integer, then 5 divides $n^5 - n$.

   Solution:

   Proof: Base Case: The first non-negative integer is 0, so the base case involves $n = 0$. If $n = 0$, then $0^5 - 0 = 0$ and 5 divides 0, since $\frac{0}{5} = 0$.

   For the induction step, let $k \geq 0$. We need to prove that **if** 5 divides $(k^5 - k)$, **then** 5 divides $((k+1)^5 - (k+1))$. We will use direct proof. Suppose 5 divides $(k^5 - k)$. [Note: This is our induction hypothesis.]

   If 5 divides $(k^5 - k)$, then $k^5 - k = 5m$ for some $m \in \mathbb{Z}$.

$$
\begin{aligned}
(k+1)^5 - (k+1) &= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1 \\
&= (k^5 - k) + 5k^4 + 10k^3 + 10k^2 + 5k \\
&= 5m + 5k^4 + 10k^3 + 10k^2 + 5k \quad \text{by the induction hypothesis} \\
&= 5(m + k^4 + 2k^3 + 2k^2 + k)
\end{aligned}
$$

   This shows that $(k+1)^5 - (k+1)$ is an integer multiple of 5, so 5 divides $((k+1)^5 - (k+1))$.

   We have now shown that 5 divides $(k^5 - k)$ implies that 5 divides $((k+1)^5 - (k+1))$.

   It follows by weak induction that 5 divides $n^5 - n$ for all non-negative integers n. $\quad Q.E.D.$

5. Prove by induction the following: If $n \in \mathbb{Z}$ and $n \geq 0$, then $\displaystyle\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$.

   Solution:

   Proof: Base Case: If $n = 0$, this statement is $\displaystyle\sum_{i=0}^{0} i \cdot i! = (0+1)! - 1$. The left hand side is $0 \cdot 0! = 0$, and the right-hand side is $1! - 1 = 0$. Thus the equation holds, as both sides are zero.

Consider any integer $k \geq 0$. We must show that S_k implies S_{k+1}. That is, we must show that

$$\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1 \quad \text{implies} \quad \sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1$$

For our induction hypothesis, suppose that $\sum_{i=0}^{k} i \cdot i! = (k+1)! - 1$. Observe that

$$
\begin{aligned}
\sum_{i=0}^{k+1} i \cdot i! &= \left(\sum_{i=0}^{k} i \cdot i! \right) + (k+1)(k+1)! \\
&= ((k+1)! - 1) + (k+1)(k+1)! \\
&= (k+1)! + (k+1)(k+1)! - 1 \\
&= (1 + (k+1))(k+1)! - 1 \\
&= (k+2)(k+1)! - 1 \\
&= (k+2)! - 1 \\
&= ((k+1)+1)! - 1
\end{aligned}
$$

Therefore, $\sum_{i=0}^{k+1} i \cdot i! = ((k+1)+1)! - 1$.

We have now proved by weak induction that $\sum_{i=0}^{n} i \cdot i! = (n+1)! - 1$ for every integer $n \geq 0$. $\quad Q.E.D$

6. Prove by induction the following: The inequality $2^n \leq 2^{n+1} - 2^{n-1} - 1$ holds for each $n \in \mathbb{N}$.

Solution:

Proof: Base Case: If $n = 1$, this statement is $2^1 \leq 2^{1+1} - 2^{1-1} - 1$, and this simplifies to $2 \leq 4 - 1 - 1$.

Say $k \geq 1$. Suppose, for the induction hypothesis, that $2^k \leq 2^{k+1} - 2^{k-1} - 1$. Then,

$$
\begin{aligned}
2^k &\leq 2^{k+1} - 2^{k-1} - 1 \\
2(2^k) &\leq 2(2^{k+1} - 2^{k-1} - 1) \quad \text{multiply both sides by 2)} \\
2^{k+1} &\leq 2^{k+2} - 2^k - 2 \\
2^{k+1} &\leq 2^{k+2} - 2^k - 2 + 1 \quad \text{(add 1 to the bigger side)} \\
2^{k+1} &\leq 2^{k+2} - 2^k - 1 \\
2^{k+1} &\leq 2^{(k+1)+1} - 2^{(k+1)-1} - 1
\end{aligned}
$$

It follows by weak induction that $2^n \leq 2^{n+1} - 2^{n-1} - 1$ for each $n \in \mathbb{N}$. $\quad Q.E.D$

7. Use induction to prove that you can achieve any postage of 8 cents or more, exactly, using only 3-cent and 5-cent stamps. For example, for a postage of 47 cents, you could use nine 3-cent stamps and four 5-cent stamps.

Solution: Let S_n be the statement S_n : You can get a postage of exactly $n-$cents using only 3-cent and 5-cent stamps. We will use strong induction.

For the base cases, we will show that the claim holds for postages of 8, 9, and 10 cents: For 8-cents, use one 3-cent stamp and one 5-cent stamp. For 9-cents, use three 3-cent stamps. For 10-cents, use two 5-cent stamps.

Now, for the induction step, let $k \geq 10$, and for each $8 \leq m \leq k$, assume a postage of m cents can be obtained exactly with 3-cent and 5-cent stamps. That is, assume that $S_8, S_9, ..., S_k$ are all true. This is our induction hypothesis. Now, we must show that S_{k+1} is true.

By our assumption, S_{k-2} is true. Thus we can get $(k-2)$-cents postage with 3-cent and 5-cent stamps. Now just add one more 3-cent stamp, and we have $(k-2) + 3 = k + 1$ cents postage with 3-cent and 5-cent stamps. This proves the claim with strong induction. $Q.E.D.$

8. Use induction to prove the following: If $n \in \mathbb{N}$, then 12 divides $(n^4 - n^2)$.

   Solution:

   Proof. We will prove this with strong induction.
   (1) First note that the statement is true for the first six positive integers:
      For $n = 1$, 12 divides $1^4 - 1^2 = 0$.   For $n = 4$, 12 divides $4^4 - 4^2 = 240$.
      For $n = 2$, 12 divides $2^4 - 2^2 = 12$.   For $n = 5$, 12 divides $5^4 - 5^2 = 600$.
      For $n = 3$, 12 divides $3^4 - 3^2 = 72$.   For $n = 6$, 12 divides $6^4 - 6^2 = 1260$.

   (2) For $k \geq 6$, assume $12 \mid (m^4 - m^2)$ for $1 \leq m \leq k$ (i.e., $S_1, S_2, ..., S_k$ are true). We must show S_{k+1} is true, that is, $12 \mid ((k+1)^4 - (k+1)^2)$. Now, S_{k-5} being true means $12 \mid ((k-5)^4 - (k-5)^2)$. To simplify, put $\boxed{k-5 = \ell}$ so $12 \mid (\ell^4 - \ell^2)$, meaning $\boxed{\ell^4 - \ell^2 = 12a}$ for $a \in \mathbb{Z}$, and $\boxed{k+1 = \ell + 6}$. Then:

   $$
   \begin{aligned}
   (k+1)^4 - (k+1)^2 &= (\ell+6)^4 - (\ell+6)^2 \\
   &= \ell^4 + 24\ell^3 + 216\ell^2 + 864\ell + 1296 - (\ell^2 + 12\ell + 36) \\
   &= (\ell^4 - \ell^2) + 24\ell^3 + 216\ell^2 + 852\ell + 1260 \\
   &= 12a + 24\ell^3 + 216\ell^2 + 852\ell + 1260 \\
   &= 12(a + 2\ell^3 + 18\ell^2 + 71\ell + 105).
   \end{aligned}
   $$

   Because $(a + 2\ell^3 + 18\ell^2 + 71\ell + 105) \in \mathbb{Z}$, we get $12 \mid ((k+1)^4 - (k+1)^2)$.   ■