

This assignment is due on Saturday April 13th to Gradescope by Noon. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

Important: Make sure to clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTeX for +1 extra credit on the assignment.

1. Easter is approaching, and the Easter Bunny has given out a massive amount of candy already. It's unreal. However, a little known fact about the Easter bunny is how much he loves candy himself. So, he has been eating a ton of candy while also giving candy out to youngsters, and in all the excitement he lost track of how much candy he gave to youngsters and how much he ate himself.

The Easter Bunny knows that 10 youngsters approached him looking for candy. The Easter Bunny also knows that he had 30 indistinguishable pieces of candy to start with, so he gave out at most 30 pieces of candy to these 10 youngsters. This can be represented as:

$$\sum_{k=1}^{10} x_k \leq 30$$

where x_k represents the number of pieces of candy that the k^{th} youngster receives.

- (a) How many ways could the Easter Bunny have distributed at most 30 pieces of candy among the 10 youngsters?

Solution: Let x_{11} represent how much candy the Easter Bunny ate for himself. Then we have $\sum_{k=1}^{11} x_k = 30$

The answer is the number of nonnegative integer solutions to that linear equation. This is stars and bars (combinations with repetition), with $r = 30$ stars to distribute among $n = 11$ bins.

$$\text{number of ways} = C(r + n - 1, r) = C(30 + 11 - 1, 30) = \boxed{C(40, 30) = 847,660,528}$$

- (b) Suddenly, the Easter Bunny remembers that he ate at least 11 pieces of candy. Now, how many ways are there for the Easter Bunny to have distributed the candy?

Solution: Same as part (a), but let's put aside 11 pieces of candy for the Easter Bunny to begin with. There are now 19 pieces of candy left to distribute among the 10 trick-or-treaters plus the Easter Bunny (so 11 bins still).

$$\text{number of ways} = C(r + n - 1, r) = C(19 + 11 - 1, 19) = \boxed{C(29, 19) = 20,030,010}$$

2. You are making a new account on your favorite social media website. This great new website offers the most fun way to steal all of your personal information! Long story short, you need a new password. Suppose passwords can include upper-case letters, lower-case letters, numbers and the symbols you can get by holding **Shift** + a number. Characters may be repeated. Please do **not** simplify your answers.

- (a) Suppose you want to make a 7, 8 or 9-character long password. How many such passwords are there?

Solutions:

We could make a 7, 8 **OR** 9 character password, so we can compute the ways to make each of these and add them together using the sum rule.

We have 26 uppercase letters, plus 26 lowercase letters, plus 10 numbers, plus 10 symbols for a total of $26 + 26 + 10 + 10 = 72$ possible characters.

$$\# \text{ 7 character passwords} = 72 \cdot 72 \cdots 72 = 72^7$$

$$\# \text{ 8 character passwords} = 72^8$$

$$\# \text{ 9 character passwords} = 72^9$$

$$\# \text{ 7, 8, or 9 character passwords} = \boxed{72^7 + 72^8 + 72^9}$$

- (b) Now suppose you want to generate a password of length n that includes exactly one symbol and one number. Find an expression in terms of n for the number of such passwords.

Solutions:

Let's generate a password of length n by breaking this down into steps. All of these steps will need to be done in order to construct a password of length n , so we combine them using the product rule.

First, out of the n possible characters in the password, we must choose 2 of them to be the symbol and the number.

The order matters because the number and symbol are distinguishable. So, this is **permutations**, and the number of ways to pick 2 out of the n characters total is $P(n, 2)$.

Next, we need to choose which number and symbol we have. There are 10 ways to pick the number, and 10 ways to pick the symbol, and we must choose one of each. This gives 10^2 ways to pick the number and symbol.

Finally, we must fill the $n - 2$ remaining slots in our password. There were 72 characters to choose from, but we must have *exactly* 1 number and 1 symbol, so there are only 52 choices for the remaining $n - 2$ slots. We can repeat characters, and the order matters, so this is permutations with repetition, and there are 52^{n-2} ways.

Putting it all together, we have $P(n, 2) \cdot 10^2 \cdot 52^{n-2} = \frac{n!}{(n-2)!} \cdot 100 \cdot 52^{n-2} = 100n(n-1) \cdot 52^{n-2}$
total possible passwords

- (c) Suppose you want to make a 9-character password, and passwords must contain at least one upper-case letter and at least one number. How many such passwords are there?

Solutions:

This will be the total number of possibly 9-character passwords without any restrictions, minus the number of passwords that violate these restriction. Let T = the set of all 9-character passwords without any restrictions.

So let U = the number of 9-character passwords that contain at least one upper-case letter, and let N = the number of 9-character passwords that contain at least one number.

The problem wants $|U \cap N|$

We can get this from: $|T| = |U \cap N| + |\overline{U \cap N}|$

$|\overline{U \cap N}|$ is the number of 9-character passwords that have no uppercase letters or no numbers, or both.

$$|\overline{U \cap N}| = |\overline{U \cup N}| = |\overline{U}| + |\overline{N}| - |\overline{U \cap N}|$$

$|\overline{U}|$ = the number of passwords without any uppercase letters. This is $(72 - 26)^9 = 46^9$

$|\overline{N}|$ = the number of passwords without any numbers. This is $(72 - 10)^9 = 62^9$

$|\overline{U \cap N}|$ = the number of passwords without any numbers nor any uppercase letters. This is $(72 - 26 - 10)^9 = 36^9$

So, the number of passwords we can have is:

$$|T| - (|\overline{U}| + |\overline{N}| - |\overline{U \cap N}|) = \boxed{72^9 - (46^9 + 62^9 - 36^9)}$$

- (d) A password's entropy can be calculated as the \log_2 of the number of characters in the character set used, multiplied by the number of characters in the password itself. How many bits of entropy does a 9-character password have, if it may be chosen from the set of upper-case letters, lower-case letters, numbers and symbols (**Shift** + number)?

Solutions:

We have a character set of 72 possible characters total, and a 9-character long password. So entropy is given by:

$$\text{entropy} = 9 \cdot \log_2 72 \approx 55.52 \text{ bits}$$

3. Yahtzee is a game in which a player rolls five six-sided dice simultaneously. The actual rules are a bit more involved, but for the sake of simplicity, let us only consider the case of rolling all five dice at once.

- (a) What is the probability of obtaining all unique outcomes on each of the five dice rolled? For example, $\{1, 2, 4, 5, 6\}$ is all unique outcomes, but $\{1, 2, 4, 5, 2\}$ repeats a 2, so those outcomes are not all unique.

Solutions: This probability is the number of ways to construct a 5-dice roll with all unique outcomes, divided by the total number of ways 5 dice can be rolled.

The number of ways to roll all unique outcomes is the number of 5-permutations of the 6 possible outcomes for each die, *without repetition*: $P(6, 5) = \frac{6!}{(6-5)!} = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$

The total number of ways to roll 5 dice is the number of 5-permutations of 6 possible outcomes for each die, with repetition allowed: 6^5

So the probability of getting all unique outcomes is

$$\begin{aligned} \frac{6!/(6-5)!}{6^5} &= \frac{6}{6} \cdot \frac{5}{6} \cdot \frac{4}{6} \cdot \frac{3}{6} \cdot \frac{2}{6} \\ &= \frac{5}{6} \cdot \frac{2}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{5}{3} \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} \\ &= \frac{5}{54} \approx 0.093 \end{aligned}$$

- (b) A “small straight” consists of four numbers all in a sequence, plus one other die that can be anything. For example, one such outcome could be $\{2, 3, 4, 5, 3\}$, and another could be $\{2, 3, 4, 5, 6\}$ (we are ignoring the distinction between a small and a large straight). What is the probability of rolling a small straight when you roll all five dice?

Caution: beware of counting specific outcomes multiple times!

Solution: This is the total number of small straights divided by the total number of outcomes (which is the same as in Part (a)).

The total number of small straights can be divided into the union of two disjoint sets: S = the set of all small straights that are *not* large straights, and L = the set of all large straights. So let's find $|S|$ and $|L|$, shall we?

Small straights that are not large straights: There are 3 different ways we could roll a small straight that isn't a large straight. They are:

- 1234 and another outcome that isn't 5
- 2345 and another outcome that isn't a 1 or 6
- 3456 and another outcome that isn't a 2

In the first case, there are two possibilities: (1) our 5th die is a repeat of 1-4, or (2) our 5th die is a 6.

- If the 5th die is a repeat, then there are 4 ways to pick which value to repeat. Then, there are $C(5, 2)$ ways to choose which 2 dice are the repeated value, and $3!$ ways to choose the remaining 3 values among the remaining 3 dice. This is a total of $4 \cdot C(5, 2) \cdot 3! = 2 \cdot 5!$
- If the 5th die is a 6, then there are $5!$ ways to pick the 5 values among the 5 dice.
- This gives a total of $3 \cdot 5!$ ways to get the first case.
- This is the same as the third case, for another $3 \cdot 5!$
- The second case is just like the repeats situation from the first case, so another $2 \cdot 5!$

This gives a total of $8 \cdot 5!$ small straights that are not large straights.

Large straights: There are 2 different ways we could roll a large straight:

- 12345
- 23456

For each of those, we have 5 options for which die is the lowest number, 4 options for which die is the second-lowest, and so on, until only 1 choice for which die is the highest number. This is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5!$ ways to get each of those large straights, for a total of $2 \cdot 5!$ large straights.

Adding everything up, we have a total of $8 \cdot 5! + 2 \cdot 5! = 10 \cdot 5!$ ways to roll a small straight (that includes large straights)

This gives a probability of:

$$p(\text{small straight}) = \frac{10 \cdot 5!}{6^5} \approx 0.1543$$

- (c) You roll all five dice but then your phone alerts you that you've caught a new Pokémon with your Fortnite. Nice! You become so engrossed in the video games that you don't see what the outcome of the dice rolls is. Your kind friend tells you that the dice are all unique. Given this information, what then is the probability of that you have rolled a small straight?

Solution: We are looking for $p(\text{small straight} \mid \text{all unique outcomes})$, which for brevity's sake we will call $p(S \mid U)$.

The definition of conditional probability tells us:

$$p(S \mid U) = \frac{p(S \cap U)}{p(U)}$$

We found $p(U) = 5/54$ in part (a). So we just need to find $p(S \cap U)$. Basic probability tells us:

$$p(S \cap U) = \frac{\text{number of outcomes that have a small straight and all unique dice}}{\text{total number of possible outcomes}}$$

The denominator is 6^5 , same as in part (a). As for the all unique outcomes that have small straights, we could have:

- 12345 or 12346, and there are $5!$ ways to do either of those (5 choice of die for the 1, 4 choices of die for the 2, 3 choices of die for the 3, 2 choices of die for the 4 and 1 choice of die for the 5 (or one choice of die for the 6 in the second case). So there are $2 \cdot 5!$ ways here.
- 23456 or 13456, and same as the last case, there are $2 \cdot 5!$ ways here.
- This gives a total of $4 \cdot 5!$ ways to roll a small straight with all unique dice outcomes.

The probability of rolling a small straight, given we have all unique outcomes is then:

$$p(S \mid U) = \frac{\frac{4 \cdot 5!}{5}}{\frac{6^5}{54}} = \frac{4 \cdot 5! \cdot 54}{5 \cdot 6^5} = \frac{4 \cdot 4! \cdot 9}{6^4} = \frac{4 \cdot 4 \cdot 3 \cdot 2 \cdot 9}{6^4} = \boxed{\frac{2}{3}}$$

- (d) Are the events [roll a small straight] and [roll all unique outcomes] independent? Fully justify your answer **with math**.

Solution:

No they are not independent (they are dependent). We know this because

$$p(S | U) = \frac{2}{3} \neq p(S) \approx 0.1543$$

We could have also checked either of $p(S \cap U) = p(S)p(U)$ or $p(U | S) = p(U)$

4. Solve the following problems and be sure to show all your work.

- (a) What is the coefficient on $x^{199}y^{301}$ in the expansion of $(x + y)^{500}$? Do not simplify your answer.

Solution:

The Binomial Theorem gives $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$. So we have $n = 500$ and want to find the coefficient on the $k = 301$ term (found by matching the exponent on y). We get:

$$\binom{500}{301} x^{199} y^{301} \rightarrow \text{coefficient} = C(500, 199) = C(500, 301) = \boxed{\frac{500!}{199! 301!}}$$

- (b) What is the coefficient on $x^{199}y^{301}$ in the expansion of $(2x - y)^{500}$? Do not simplify your answer.

Solution:

Now we need to replace $x \rightarrow 2x$ and $y \rightarrow -y$. We find:

$$\binom{500}{301} (2x)^{199} (-y)^{301} = \frac{500!}{199! 301!} 2^{199} (-1)^{301} x^{199} y^{301} \rightarrow \text{coefficient} = \boxed{-\frac{500!}{199! 301!} 2^{199}}$$

- (c) In the expansion of $(x + y)^{500}$, is there a term whose only x and y components are $x^{198}y^{301}$? Why or why not?

Solution:

This term does **not** exist in the given expansion. From the binomial theorem, all terms must have powers on x and y that sum to 500. (That is, $(n - k) + (k)$ must equal n .) In $x^{198}y^{301}$, the sum of the exponents is only 499.

- (d) How many possible distinct rearrangements of the letters STRUCTURE are there? Do not simplify your answer.

Hint: Any two of the same letter are indistinguishable from one another. For example, the two Ts are indistinguishable.

Solution:

Consider the letters as objects that we must place in some arrangement, where there are 9 open slots initially. The task of creating an arrangement of these letters amounts to performing *all* of 4 sub-tasks:

- Out of 9 spots, choose 2 for the 2 R's $\rightarrow C(9, 2)$ ways
- Out of 7 spots remaining, choose 2 for the 2 T's $\rightarrow C(7, 2)$ ways
- Out of 5 spots remaining, choose 2 for the 2 U's $\rightarrow C(5, 2)$ ways
- There are 3 spots left and 3 distinguishable letters left; just need to arrange what order they are in $\rightarrow P(3, 3)$ ways

Since the task of creating an arrangement of the letters involved performing all of these tasks, we use the product rule to combine:

$$\# \text{ ways} = C(9, 2) \cdot C(7, 2) \cdot C(5, 2) \cdot P(3, 3) = \frac{9!}{7! 2!} \frac{7!}{5! 2!} \frac{5!}{3! 2!} 3! = \boxed{\frac{9!}{(2!)^3}}$$