

This assignment is due on Friday March 15 to Gradescope by Noon. You are expected to write or type up your solutions neatly. Remember that you are encouraged to discuss problems with your classmates, but you must work and write your solutions on your own.

**Important:** Make sure to clearly write your full name, the lecture section you belong to (001 or 002), and your student ID number at the top of your assignment. You may **neatly** type your solutions in LaTex for +1 extra credit on the assignment.

1. Use induction to prove that the following identities hold for all  $n \geq 1$ .

$$(a) \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$(b) \sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$$

$$(c) \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \dots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n} \quad (\text{for } n \geq 2)$$

Be sure to clearly state your induction hypothesis, and state whether you're using weak induction or strong induction for each part.

**Solution:**

(a) **Base Case:**  $n = 1$ .

$$\begin{aligned} \sum_{i=1}^1 i^2 &= 1^1 = 1 \\ \frac{1(1+1)(2 \cdot 1 + 1)}{6} &= \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1 \end{aligned}$$

$$1 = 1 \quad \checkmark$$

**Inductive Step:** Assume for  $n = k \geq 1$  that  $\sum_{i=1}^k i^2 = \frac{k(k+1)(2k+1)}{6}$ .

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \sum_{i=1}^k i^2 + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \quad \text{by the inductive hypothesis} \\ &= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\ &= \frac{(k+1)[k(2k+1) + 6(k+1)]}{6} \\ &= \frac{(k+1)[2k^2 + k + 6k + 6]}{6} \\ &= \frac{(k+1)(k+2)(2k+3)}{6} \end{aligned}$$

Thus, if the formula is true for  $n = k$ , then it must be true for  $n = k + 1$ , and by weak induction, it must be true for any integer  $n \geq 1$

(b) **Base Case:**  $n = 1$

$$\sum_{i=1}^1 i^3 = 1^3 = 1$$

$$\frac{1^1(1+1)^2}{4} = \frac{1 \cdot 2^2}{4} = \frac{4}{4} = 1$$

$$1 = 1 \quad \checkmark$$

**Inductive Step:** Assume for  $n = k \geq 1$  that  $\sum_{i=1}^k i^3 = \frac{k^2(k+1)^2}{4}$ .

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^k i^3 + (k+1)^3 \\ &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \quad \text{by the inductive hypothesis} \\ &= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\ &= \frac{(k+1)^2[k^2 + 4(k+1)]}{4} \\ &= \frac{(k+1)^2[k^2 + 4k + 4]}{4} \\ &= \frac{(k+1)^2(k+2)^2}{4} \\ &= \frac{(k+1)^2((k+1)+1)^2}{4} \end{aligned}$$

Thus, if the formula is true for  $n = k$ , then it must be true for  $n = k + 1$ , and by weak induction, it must be true for any integer  $n \geq 1$

(c) **Base Case:**  $n = 2 \quad \left(1 - \frac{1}{4}\right) = \frac{3}{4}$ . And,  $\frac{2+1}{2 \cdot 2} = \frac{3}{4}$

Since  $\frac{3}{4} = \frac{3}{4}$ , we have proven the base case.

**Inductive Hypothesis:** Assume that  $\left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \dots \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$  for  $n = k \geq 2$ .

Then,

$$\begin{aligned} \left(1 - \frac{1}{4}\right) \left(1 - \frac{1}{9}\right) \dots \left(1 - \frac{1}{k^2}\right) \cdot \left(1 - \frac{1}{(k+1)^2}\right) &= \frac{k+1}{2k} \cdot \left(1 - \frac{1}{(k+1)^2}\right) \\ &= \frac{k+1}{2k} \left(\frac{(k+1)^2 - 1}{(k+1)^2}\right) \\ &= \frac{1}{2k} \left(\frac{(k+1)^2 - 1}{k+1}\right) \\ &= \frac{k^2 + 2k + 1 - 1}{(2k)(k+1)} \\ &= \frac{k(k+2)}{(2k)(k+1)} \\ &= \frac{k+2}{2(k+1)} \end{aligned}$$

Thus, if the formula is true for  $n = k$ , then it must be true for  $n = k + 1$ , and by weak induction, it must be true for any integer  $n \geq 2$

2. Let the sequence  $T_n$  be defined by  $T_1 = T_2 = T_3 = 1$  and  $T_n = T_{n-1} + T_{n-2} + T_{n-3}$  for  $n \geq 4$ . Use induction to prove that

$$T_n < 2^n \text{ for } n \geq 4$$

Be sure to clearly state your induction hypothesis, and state whether you're using weak induction or strong induction for each part.

**Solution: Base Case:** Since our recursion depends on the term  $n - 3$ , we need 3 base cases.

$$n = 4 \quad T_4 = 1 + 1 + 1 = 3 \quad \text{and} \quad 2^4 = 16, \quad \text{since } 3 < 16, \text{ we have verified that } T_4 < 2^4 \quad \checkmark$$

$$n = 5 \quad T_5 = 3 + 1 + 1 = 5 \quad \text{and} \quad 2^5 = 32, \quad \text{since } 5 < 32, \text{ we have verified that } T_5 < 2^5 \quad \checkmark$$

$$n = 5 \quad T_5 = 5 + 3 + 1 = 9 \quad \text{and} \quad 2^5 = 32, \quad \text{since } 9 < 64, \text{ we have verified that } T_6 < 2^6 \quad \checkmark$$

**Inductive Step:** Assume for  $4 \leq k \leq m$  that  $T_k = T_{k-1} + T_{k-2} + T_{k-3}$  and  $T_k < 2^k$ .

Then,

$$\begin{aligned} T_{k+1} &= T_k + T_{k-1} + T_{k-2} \\ &< 2^k + 2^{k-1} + 2^{k-2} \quad \text{by the strong induction hypothesis} \\ &= 2^{k-2}(2^2 + 2 + 1) \\ &= 2^{k-2} \cdot 7 \\ &< 2^{k-2} \cdot 8 \\ &= 2^{k-2} \cdot 2^3 \\ &= 2^{k+1} \end{aligned}$$

By strong induction,  $T_n < 2^n$  for  $n \geq 4$ .

3. Consider the function  $f(n) = 50n^3 + 6n^3 \log(n^3) - n \log(n^2)$  which represents the complexity of some algorithm.

- (a) Find a tight big-**O** bound of the form  $g(n) = n^p$  for the given function  $f$  with some natural number  $p$ . What are the constants  $C$  and  $k$  from the big-**O** definition?
- (b) Find a tight big-**Ω** bound of the form  $g(n) = n^p$  for the given function  $f$  with some natural number  $p$ . What are the constants  $C$  and  $k$  from the big-**Ω** definition?
- (c) Can we conclude that  $f$  is big-**Θ**( $n^p$ ) for any natural number  $p$ ?

**Solution:** (a) First, we simplify using our rules for logarithms:  $f(n) = 50n^3 + 18n^3 \log(n) - 2n \log(n)$ .

Next, the natural first guess should be the **leading order term**, which is  $n^3$ . But two terms have  $n^3$  in them. Since the  $\log n$  is only going to make the second term larger than  $n^3$  for  $\log n > 1$  ( $n > e$ ), this means the second term is the dominant one (upper bound).

The lowest power of  $n$  that can be an upper bound for that second term is  $n^4$ . We get upper bounds for each term in terms of  $n^4$ :

$$\begin{aligned} 50n^3 &\leq 50n^4, \quad \text{for } n \geq 1 \\ 18n^3 \log(n) &\leq 18n^3 \cdot n = 18n^4, \quad \text{for } n \geq 1 \\ -2n \log(n) &\leq 0, \quad \text{for } n \geq 1 \end{aligned}$$

So we have:

$$f(n) \leq 50n^4 + 18n^4 + 0 = 68n^4, \quad \text{for } n \geq 1$$

Thus with  $C = 68$  and  $k = 1$ ,  $f$  is **O**( $n^4$ ).

(b) Again, the natural first guess should be the **leading order term**, which is  $n^3$ . But two terms have  $n^3$  in them. Since the  $\log n$  is only going to make the second term larger than  $n^3$  for  $\log n > 1$  ( $n > e$ ), this means the *first* term will be smaller, and so this is our first guess for the big- $\Omega$  bound.

The lowest power of  $n$  that can serve as a lower bound is  $n^3$ . We get the lower bounds for each term in terms of  $n^3$ :

$$\begin{aligned} 50n^3 &\geq 50n^3, \quad \text{for } n \geq 1 \\ 18n^3 \log(n) &\geq 18n^3, \quad \text{for } n \geq e \\ 2n \log n &\leq 2n^3 \rightarrow -2n \log(n) \geq -2n^3, \quad \text{for } n \geq 1 \end{aligned}$$

where the second line comes from the fact that  $\log n > 1$  exactly when  $n > e$  (by exponentiating both sides and using the fact that  $e^{\log n} = n$ ).

So we have:

$$f(n) \geq 50n^3 + 18n^3 - 2n^3 = 66n^3, \quad \text{for } n \geq e$$

Thus, with  $C = 66$  and  $k = e$ ,  $f$  is  $\Omega(n^3)$ .

(c) No, because it is not both big-O and big-*Omega*  $n^p$  for some function  $n^p$ .

4. Multiply the following matrices:

(a)

$$\begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 8 & 1 & 2 \\ 7 & 6 & 5 \end{bmatrix} =$$

(b) Leave your answer in terms of  $a_{ij} \cdot b_{ij}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} =$$

**Solution:** (a) The matrix multiplication of a 2x3 and a 3x2 matrix results in a 3x3 matrix

$$\begin{bmatrix} 3 & 4 \\ 1 & 0 \\ 2 & 7 \end{bmatrix} \cdot \begin{bmatrix} 8 & 1 & 2 \\ 7 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 3 \cdot 8 + 4 \cdot 7 & 3 \cdot 1 + 4 \cdot 6 & 3 \cdot 2 + 4 \cdot 5 \\ 1 \cdot 8 + 0 \cdot 7 & 1 \cdot 1 + 0 \cdot 6 & 1 \cdot 2 + 0 \cdot 5 \\ 2 \cdot 8 + 7 \cdot 7 & 2 \cdot 1 + 7 \cdot 6 & 2 \cdot 2 + 7 \cdot 5 \end{bmatrix} = \begin{bmatrix} 52 & 27 & 26 \\ 8 & 1 & 2 \\ 65 & 44 & 39 \end{bmatrix}$$

(b) The matrix multiplication of two 3x3 matrices will result in another 3x3 matrix

$$\begin{bmatrix} a_{11} \cdot b_{11} + a_{12} \cdot b_{21} + a_{13} \cdot b_{31} & a_{11} \cdot b_{12} + a_{12} \cdot b_{22} + a_{13} \cdot b_{32} & a_{11} \cdot b_{13} + a_{12} \cdot b_{23} + a_{13} \cdot b_{33} \\ a_{21} \cdot b_{11} + a_{22} \cdot b_{21} + a_{23} \cdot b_{31} & a_{21} \cdot b_{12} + a_{22} \cdot b_{22} + a_{23} \cdot b_{32} & a_{21} \cdot b_{13} + a_{22} \cdot b_{23} + a_{23} \cdot b_{33} \\ a_{31} \cdot b_{11} + a_{32} \cdot b_{21} + a_{33} \cdot b_{31} & a_{31} \cdot b_{12} + a_{32} \cdot b_{22} + a_{33} \cdot b_{32} & a_{31} \cdot b_{13} + a_{32} \cdot b_{23} + a_{33} \cdot b_{33} \end{bmatrix}$$