

1 Greedy algorithms

A *greedy algorithm* breaks a problem down into a sequence of simple steps, each of which is chosen such that some measure of the “quality” of the intermediate solution increases by a maximal amount at each step from the base case to the full solution.

A classic example of a greedy approach is navigation in a k -dimensional Euclidian space. Let \mathbf{y} denote the location of our destination, and let \mathbf{x} denote our current position, with $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$. The distance remaining to our destination is then simply the Euclidian distance between \mathbf{x} and \mathbf{y} :

$$L_2(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^k (x_i - y_i)^2} , \quad (1)$$

and the greedy step to take at each moment in our journey is the one that minimizes $L_2(\mathbf{x}, \mathbf{y})$.¹

This greedy approach is only *optimal*—that is, guarantees that we will always reach the target—when the space is smooth, with no impassable regions. It is not hard to construct examples in a 2-dimensional space containing impassable regions for which the greedy approach can fail. Here are two examples, one trivial and one only slightly less trivial.

(1) Consider traveling by car or foot from Boulder, CO to London, England. Ignoring the possibility of taking a ferry, there is no land path that connects North America to England, and thus there is no path. (2) Now consider traveling by car or foot from Miami, FL to Merida, Mexico (on the tip of the Yucatan Peninsula, across the Gulf of Mexico from Florida). For this task, a path does exist: travel counter-clockwise around the Gulf of Mexico, following the coast, but a greedy approach will never find it. A greedy approach always seeks to minimize the remaining distance with every choice, while taking this path would at first necessarily increase the remaining distance. Instead, the greedy approach would take us straight to the southern tip of the Florida peninsula and then fail.

When a problem has these kinds of “local optima,” from which no greedy move can further improve the quality of our position, we must use other means to solve it because a greedy approach may get stuck. (And under the adversarial model of algorithm analysis, if there exists even a single input for which a greedy approach fails, then the greedy algorithm is not a correct algorithm.) That said, there are many tasks that admit a greedy solution because there exists a way to express the progress of the algorithm toward the goal so that repeatedly making the greedy choice will always yield the correct solution.

¹We use L_2 (pronounced “L2”) to denote the Euclidian distance because the general form of this distance measure is L_p , in which we take the p th power of each difference inside the sum and the p th root of the whole sum. In many problems, it is convenient to choose $p \neq 2$, which can produce a distance metric with slightly different mathematical properties.

1.1 Structure of a greedy algorithm

Greedy algorithms are perhaps the most common type of algorithm because they are so simple to formulate. In fact, we will see a number of greedy algorithms later in the course, when we cover graph algorithms. For a greedy algorithm to be correct, the problem definition itself must have certain mathematical properties. If we can prove that a problem has these properties, then there must exist a greedy algorithm for solving it.

1. Every solution to the problem can be assigned or translated in a numerical value or score. Let x be a candidate solution and let $\text{score}(x)$ be the value assigned to that solution.
2. There exists a “best” (optimal) solution, with the highest (or lowest) score among all solutions.
3. *Optimal substructure property*: the optimal solution contains optimal solutions to all of its subproblems.
4. *Greedy choice property*: a solution can be constructed incrementally (and a partial solution assigned a score) without reference to future decisions, past decisions, or the set of possible solutions to the problem.
5. At each intermediate step in constructing or finding a solution, there are a set of options for which piece to add next.

A greedy algorithm *always chooses the incremental option that yields the largest improvement in the intermediate solution's score.*

Proving that a greedy algorithm is correct can be done in the usual ways (e.g., proof by strong induction, proof by contradiction, proof by reduction), or by proving that the problem itself has the optimal substructure property and that the algorithm has the greedy choice property.

2 Huffman codes

A classic example of a greedy algorithm is Huffman encoding, a problem from compression.

In this problem, we aim to take an existing “message,” or rather a sequence of symbols, and translate it into a new message, composed with a different set of symbols, such that the new message is as small as possible while still containing all the information contained in the original one. You may be familiar with this idea from compressing files on your computer, e.g., making `.zip` or `.tgz` files. This form of compression is called *lossless* because the original message can be exactly reconstructed from the compressed one, with a little computation. If some information is lost, as in `.jpg` files, it is called *lossy compression*.

Let Σ denote the input alphabet, a set of $|\Sigma| = n$ distinct “symbols” that can be used to construct a string of symbols or message. The input is a message or file x that is an arbitrary sequence of symbols drawn from Σ . The output must be an encoding (mapping) g of words from Σ to codewords in a binary alphabet Γ , such that the encoding is *prefix-free* and the encoded message minimal in size. In transforming the input to output, we aim to be as efficient as possible both in time (computation) and bits (encoded message size).

2.1 Binary code trees and prefix-free codes

Before covering Huffman’s algorithm, we must first define terms like binary codes, prefix-free codes, and message size.

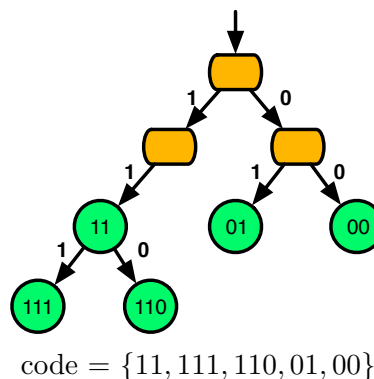
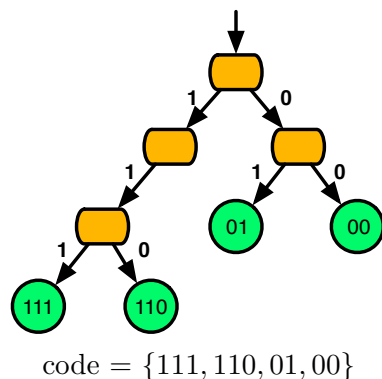
A binary code is simply a set of strings (set of codewords) all drawn from the binary alphabet $\Sigma_{01} = \{0, 1\}$. A prefix-free code is a set of codewords such that no codeword is a prefix of any other. That is, if x is in the code, then there can exist no codeword $y = x + z$, where $+$ denotes concatenation. Otherwise, x is a prefix of y . If and only if a (binary) code is prefix-free, can it be represented by a (binary) tree in which codewords are only located at the bottom of the tree, at the leaves.^{2,3}

To illustrate these ideas, below are two decoding trees, one that represents a prefix-free code, with codewords $\{111, 01, 110, 00\}$, and one that does not, which has codewords $\{11, 111, 01, 110, 00\}$. The second code is not prefix-free because 11 is a prefix of both 111 and 110, i.e., 11 is a codeword but is located at an internal node of the encoding tree.

In the coding trees, codewords are denoted by green circles, with the corresponding codeword contained within and non-codewords are denoted by orange oblongs. To decode a string, we treat the tree like a finite-state automaton: each character we read moves us from one state (node) to another; if we read a 1, we move to the left child; otherwise, we move to the right child. (This convention, of the left-daughter representing a 1 and the right-daughter representing a 0, is arbitrary, and an equivalent code can be obtained by reversing this convention.) When we reach a codeword node, we know what codeword we have just read, and we can return to the top of the tree to begin reading the next codeword. Clearly, any codeword that sits at an internal node in the tree is a prefix for all codewords in the subtree below it. We want to avoid prefix codes because they induce ambiguity in the decoding.

²Note that this is different from the *trie* data structure, which explicitly permits words to be prefixes of each other. Tries are very handy data structures for storing character strings in a space-efficient manner that allows fast lookups based on partial (prefix) matchings.

³Also note that this kind of tree is very different from a search tree. In fact, the ordering of the leaves in an encoding is very unlikely to follow any particular pattern.



2.2 Cost of coding

The length of a codeword is given by the depth of its associated node. A good compression scheme can be represented as a small tree, and the best compression is achieved by a minimal tree.⁴ However, the total *cost* of an encoding scheme is not just a function of the size of the tree, but also a function of the frequency of the words we encode. Let f_i be the frequency in the message x of the i th codeword in the original message. A minimal code minimizes the function

$$\text{cost}(x) = \sum_{i=1}^n f_i \cdot \text{depth}(i) .$$

In 1948, Claude Shannon proved that the theoretical lower bound on the cost per word (in bits) using a binary encoding is given by the entropy $H = -\sum_{i=1}^n p_i \log_2 p_i$, where $p_i = f_i/n$. Notice that the definition of entropy is similar the cost function we wrote down, when $\text{depth}(i) = \log_2 f_i$. This is not an accident!

2.3 Huffman's algorithm

In 1952 David Huffman developed a greedy algorithm that produces an encoding that minimizes this function and gets as close to Shannon's bound as possible for a finite-sized string. The key idea of Huffman's algorithm is remarkably compact: merge the two least frequent "symbols" and recurse.

Huffman encoding begins by first tabulating the frequencies f_i of each word in x ; this can be done quickly by constructing a histogram. We then create the n leaves of the coding tree. At the i th leaf, we store the i th word of the input alphabet Σ and its frequency f_i in the input message x .

⁴"A" minimal tree, rather than "the" minimal tree. There may be multiple distinct smallest trees for the same set of input symbols and frequencies.

Pseudocode for this procedure is straightforward, where we let f be an array of length n containing the given frequencies. The output will be a binary encoding tree with n leaves.

```
huffmanEncode(f) {  
    initialize H          // H, a priority queue of integers, ordered by f  
    for i = 1 to n {      insert(H,i)    }    // build queue  
    for k=n+1 to 2n-1 {    // merge two least-frequent symbols and recurse  
        i = deletemin(H), j = deletemin(H)  
        create a node numbered k with children i,j  
        f[k] = f[i] + f[j]  
        insert(H,k)  
    }  
}
```

Let Γ denote the set of symbols we are currently working with. Initially $\Gamma = \Sigma$. At the k th step of the algorithm, we select the two words in Γ with smallest frequencies, f_i and f_j . We create a new word $n+k$ with frequency $f_{n+k} = f_i + f_j$, and make it the parent of i and j . We then update Γ by removing words i and j and adding word $n+k$. The algorithm halts when $|\Gamma| = 1$, i.e., when no pair of symbols remains, and returns to us the constructed tree.

Because the size of our intermediate alphabet Γ decreases by one at each step, the algorithm must terminate after exactly $n - 1$ steps. Implementing the algorithm simply requires a data structure that allows us to efficiently find the two symbols in Γ with the smallest frequencies. This is typically done using a *priority queue* or *min heap* (Chapter 6). Each find, merge, and add operation in the priority queue takes $O(\log n)$ time. There are $\Theta(n)$ such operations in each pass of the **for** loop, and so the running time is $O(n \log n)$.⁵ (How much space does it take?)

2.4 A small example

To illustrate how Huffman encoding works, consider the following set of symbols and their frequencies in a message to be encoded; for the encoding, the message itself doesn't matter, only the symbols and their frequencies.

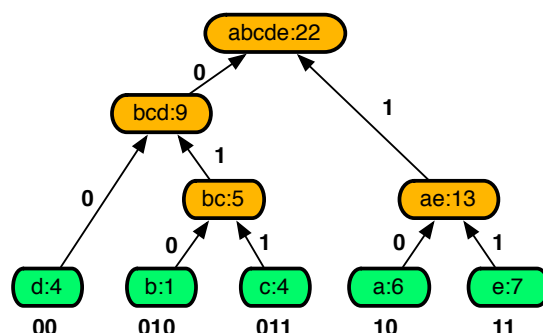
a:6 b:1 c:4 d:4 e:7

Huffman's algorithm proceeds by inserting each of these symbols into a priority queue data structure, in ascending order of their frequency. If two symbols have the same frequency, their order in the queue is arbitrary. Thus, the queue's initial contents are (b:1) (c:4) (d:4) (a:6) (e:7).

⁵It is possible to run this sequence in $O(n)$ time using a pair of cross-linked queues; however, this version is more complicated to implement and assumes the word frequencies f_i have already been computed and sorted, which takes $O(n \log n)$ time.

We then repeatedly dequeue the first two elements i and j , with frequencies f_i and f_j , merge them into a new symbol whose frequency is $f_i + f_j$, which we insert back into the queue. (Remember that in a priority queue, the inserted new symbol will move up from the back of the queue until it finds its correct location.)

We repeat this process until the queue is empty, in which case we have just dequeued the last two symbols and the newly merged symbol represents the entire set of symbols Σ and their number in the message. Applied to our example, we produce the following Huffman tree. Note that two symbols c and d have the same frequency; swapping their locations in the tree would produce a different optimal encoding. Similarly, we have arbitrarily labeled the left-child connection with 0 and the right-child connection with 1 to produce a set of codewords; making the symmetric choice would result in a different optimal encoding (in terms of the precise code words).



2.5 A less small but more cute example

Here is a cute “self-descriptive” Huffman example from Lee Sallows⁶

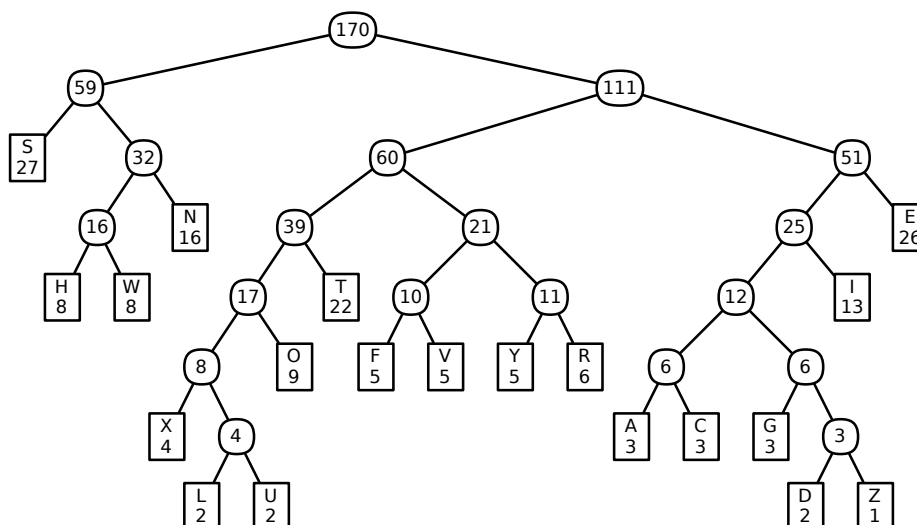
This sentence contains three a’s, three c’s, two d’s, twenty-six e’s, five f’s, three g’s, eight h’s, thirteen i’s, two l’s, sixteen n’s, nine o’s, six r’s, twenty-seven s’s, twenty-two t’s, two u’s, five v’s, eight w’s, four x’s, five y’s, and only one z.

To keep things simple, we will ignore capitalization, the spaces (44), apostrophes (19), commas (19), hyphens (3) and the one period in the sentence; instead, we will focus on encoding the letters alone. The frequencies of the 26 letters are the following.

⁶A. K. Dewdney. Computer recreations. *Scientific American*, October 1984. Other examples appeared a few years earlier in some of Douglas Hofstadter’s columns. My credit goes to Jeff Erickson, who also produced the figure on the next page (but, see the footnote on the next page, too).

A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1

Below is the encoding tree produced by applying Huffman's rule to this histogram: after 19 merges, all 20 characters have been merged and the history of merges gives the encoding tree. (Suppose there is a tie as to which pair of characters to merge; what does this tell us about the uniqueness of the Huffman code?)



To read the encoding of a character, place a 0 on each left-branch and a 1 on each right-branch, then write the sequence of 1s and 0s backwards as you read up the tree. To decode an encoded letter, do the reverse: start at the root, and take the left-right path given by the encoding down the tree to arrive at the decoded character. For example, the encoding of the letter “a” is 110000. In our example, the encoded message is 661 bits long. Below is a table showing how we arrive at that number, with each input symbol, its frequency f_i in the input, the length of its encoding d_i and the total cost for encoding those symbols $f_i d_i$. The total message length is just $\sum_i f_i d_i$.

At home exercise: verify that this tree is correct.⁷

	A	C	D	E	F	G	H	I	L	N	O	R	S	T	U	V	W	X	Y	Z
f_i	3	3	2	26	5	3	8	13	2	16	9	6	27	22	2	5	8	4	5	1
d_i	6	6	7	3	5	6	4	4	7	3	5	5	2	4	7	5	4	6	5	7
$f_i d_i$	18	18	14	78	25	18	32	54	14	48	45	30	54	88	14	25	32	24	25	7

⁷Hint: this tree is not the minimal tree, which represents an encoding that uses only 649 bits. Can you find the mistake?

2.6 Huffman codes are optimal

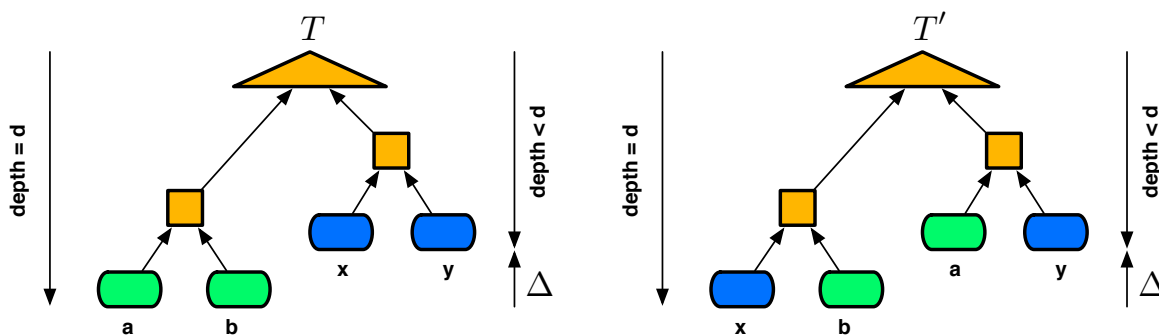
We can now prove that Huffman encoding is optimal, by proving that it has optimal substructure and that Huffman encoding makes the greedy choice.

Lemma 1: Let x and y be two least frequent symbols (with ties broken arbitrarily). There is an optimal code tree in which x and y are siblings and their parent is the largest depth of any leaf.

Proof: Let T be an optimal code tree with depth d . Because T is an optimal code tree, it must also be a “full” binary tree, in which every tree node has either 0 or 2 children. It must be a full binary tree because if there were a tree node with only 1 child, then we could delete that node, connecting its one child directly to its one parent, and reduce the cost of all codewords in its subtree by 1, implying that T was not optimal.

Thus, because T is a full binary tree, it must have at least two leaves at depth d that are siblings.

Assume that these leaves are not x and y , but rather some other pair a and b , with x and y being located at some other depth $d' = d - \Delta$, for $\Delta > 0$. The following figure (left panel) illustrates this situation.



Now let T' be the code tree if we swapped x and a (right panel in the figure). The depth of x increases by some amount at most Δ , and the depth of a decreases by the same amount. Thus,

$$\text{cost}(T') = \text{cost}(T) - (f_a - f_x)\Delta .$$

However, by assumption, x is one of two least frequent symbols while a is not. This implies $f_a \geq f_x$, and thus swapping x and a cannot increase the total cost of the code.

Moreover, since T was an optimal code tree in the first place, swapping x and a cannot decrease the total cost of the code.

Thus, T' must also be an optimal code tree, which implies $f_a = f_x$.

A similar argument applies for swapping y and b , and thus carrying out both swaps yields an optimal tree T'' in which x and y as siblings and at maximum depth. \square

We can now prove that the greedy encoding rule of Huffman is optimal.

Theorem 1: Huffman codes are optimal prefix-free binary codes.

Proof: If the message has only one or two different symbols, the theorem is trivially true.

Otherwise, let f_1, \dots, f_n be the frequencies in the original message. Without loss of generality, let f_1 and f_2 be the smallest frequencies.

When the Huffman algorithm makes a recursive call, it creates a new frequency at the end of the list, e.g., $f_{n+1} = f_1 + f_2$. By Lemma 1, we know that the optimal tree T has symbols 1 and 2 as siblings.

Let T' be the Huffman tree for f_3, \dots, f_{n+1} , which serves as our inductive hypothesis that T' is an optimal code tree for this smaller set of frequencies. We now prove that the tree T , on the larger set of frequencies, is also an optimal tree, in which we replace the tree node representing the frequency f_{n+1} with an internal node that has 1 and 2 as children.

To show that T is also optimal, we write down its cost and show that it is equal to the cost of T' , an optimal but smaller tree, plus the cost of replacing one leaf in T' with a subtree representing leaves for symbols 1 and 2.

Let d_i be the depth of node i :

$$\begin{aligned} \text{cost}(T) &= \sum_{i=1}^n f_i d_i \\ &= \left(\sum_{i=3}^{n+1} f_i d_i \right) + f_1 d_1 + f_2 d_2 - f_{n+1} d_{n+1} \\ &= \text{cost}(T') + f_1 d_1 + f_2 d_2 - f_{n+1} d_{n+1} \\ &= \text{cost}(T') + (f_1 + f_2) d_T - f_{n+1} (d_T - 1) \end{aligned}$$

$$\begin{aligned}\text{cost}(T) &= \text{cost}(T') + (f_1 + f_2)d_T - (f_1 + f_2)(d_T - 1) \\ &= \text{cost}(T') + (f_1 + f_2)(d_T - d_T + 1) \\ &= \text{cost}(T') + f_1 + f_2\end{aligned}$$

Thus, the cost of T is no more than the cost of T' itself, which is minimal by assumption, plus the cost of adding one binary symbol to each of the codewords represented by the $(n + 1)$ th node in T' . \square

3 On your own

1. Read Chapter 16

4 Two more greedy algorithms

4.1 Insertion sort is an optimal but inefficient greedy algorithm

A greedy algorithm can be optimal, but not efficient. To illustrate this, we will consider the behavior of *insertion sort*. Recall that insertion sort takes $\Theta(n^2)$ time to sort n numbers and that we know a number of sorting algorithms that are more efficient, taking only $O(n \log n)$ time.

Insertion sort is a simple loop. It starts with the first element of the input array A . For each subsequent element j , it then inserts $A[j]$ into the sorted list $A[1..j-1]$.

```
INSERTION-SORT(A)
  for j=2 to n
    // assert: A[1..j-1] is sorted
    insert A[j] into the sorted sequence A[1..j-1]
    // assert: A[1..j] is sorted
  }
```

To see that insertion sort is correct, we observe that the following loop invariant, i.e., a property of the algorithm (either its behavior or its internal state) that is true each time we begin or end the loop.

Note that at the start of the `for` loop, the subarray $A[1..j-1]$ contains the original elements of $A[1..j-1]$ but now in sorted order. When $j = 2$ (“initialization”) this fact is true because a list with one element is, by definition, sorted. When we insert the j th item into the subarray $A[1..j-1]$, we do so in a way that $A[1..j]$ is now sorted; thus, if the invariant is true at the beginning of the loop, it will also be true at the end of the loop (“maintenance”). Finally, when the loop terminates, $j = n$ and we have inserted the last element correctly into the sorted subarray (“termination”); thus, the entire array is sorted.

Now that we know insertion sort is correct, we can show that it is, in fact, an *optimal greedy algorithm*, meaning that (i) at any intermediate step, the algorithm always makes the choice that increases the quality of the intermediate state (greedy property) and (ii) it returns the correct answer upon termination (optimum behavior).

To begin, we first define a score function that allows us to build sorted sequences one element at a time. Clearly, if A is already sorted, $\text{score}(A)$ must yield a maximal value, but it must also give partial credit if A is partially sorted and that credit should be larger the more sorted A is.

A sufficient score function is to count the number of sequential comparisons that violate the sorting

requirement, i.e.,

$$\text{score}(A) = \sum_{i=1}^{n-1} (A[i] \leq A[i+1]) \quad ,$$

where we assume that the comparison operator is a binary function that returns 1 if the comparison yields true and 0 if it yields false. By definition, this property is true for all i in a fully sorted array, i.e., $A[1] \leq A[2] \leq \dots \leq A[n]$, and so a fully sorted sequence will receive the maximal score of $n - 1$. A sequence in reverse order will receive the lowest score of 0, and every other sequence can be assigned something between these two extremes.

With the score function selected, let's analyze the behavior of insertion sort under this function. An intermediate solution here is the partially sorted array. Given such an array, our “local” move is to take one element $A[j]$ and insert it into the subarray $A[1..j-1]$. Not all choices of where within $A[1..j-1]$ we put $A[j]$ will lead to a sorted list upon termination, and most of the possible choices of where to insert $A[j]$ are suboptimal.

Recall our loop invariant from above. For any input sequence A , when the loop initializes, it has $\text{score}(A) \geq 0$, where the lower bound is achieved by a strict reverse ordering. Because the sorted subarray's size grows by 1 each time we pass through the loop, so too does the number of correct sequential comparisons. That is, our loop invariant is equivalent to $\text{score}(A) \geq j - 1$ where j is the loop index. And, when the loop completes, $j = n$ and $\text{score}(A) \geq n - 1$. Thus, insertion sort is a kind of optimally greedy algorithm.

4.2 Linear storage media

Here's another good example of a simple greedy algorithm. Suppose we have a set of n files and that we want to store these files on a tape, or some other kind of linear storage media.⁸ Once we've written these files to the tape, the cost of accessing any particular file is proportional to the length of the files stored ahead of it on the tape. In this way, tape access is very slow and costly relative to either magnetic disks or RAM. Let $L[i]$ be the length of the i th file. If the files are stored on the tape in order of their indices, the cost of accessing the j th file is

$$\text{cost}(j) = \sum_{i=1}^j L[i] \quad .$$

⁸Although tape memory is not often used by individuals, it remains one of the most efficient storage media for both very large files and for archival purposes. A linked list can be used to simulate a linear storage medium if the amount of data that can be stored in each node is limited; in fact, this kind of abstraction is precisely how files are stored on magnetic media.

That is, we first have to scan past (which takes the same time as reading) the first $j - 1$ files, and then we read the j th file.

If files are requested uniformly at random, then the expected cost for reading one is

$$E[\text{cost}] = \sum_{j=1}^n \Pr(j) \cdot \text{cost}(j) = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^j L[i] .$$

What if we change the ordering of the files on the tape? If not all files are the same size, this will change the cost of accessing some files versus others. For instance, if the first file is very large, then the cost of accessing every other file will be larger by an amount equal to its length. We can formalize and analyze the impact of a given ordering by letting $\pi(i)$ give the index of the file stored at location i on the tape. The expected cost of accessing a file is now simply

$$E[\text{cost}(\pi)] = \frac{1}{n} \sum_{j=1}^n \sum_{i=1}^j L[\pi(i)] .$$

What ordering π should we choose to minimize the expected cost? Intuitively, we should order the files in order of their size, smallest to largest. Let's prove this.

Lemma 2: $E[\text{cost}(\pi)]$ is minimized when $L[\pi(i)] \leq L[\pi(i + 1)]$ for all i .

Proof: Suppose $L[\pi(i)] > L[\pi(i + 1)]$ for some i . If we swapped the files at these locations, then the cost of accessing the first file $\pi(i)$ increases by $L[\pi(i + 1)]$ and the cost of accessing $\pi(i + 1)$ decreases by $L[\pi(i)]$. Thus, the total change to the expected cost is $(L[\pi(i + 1)] - L[\pi(i)]) / n$, which is negative because, by assumption, $L[\pi(i)] \leq L[\pi(i + 1)]$. Thus, we can always improve the expected cost by swapping some out-of-order pair, and the globally minimum cost is achieved when the files are sorted. \square

Thus, any greedy algorithm that repeatedly swaps out-of-order pairs on the tape will lead us to the globally optimal ordering. (At home exercise: can you bound the expected cost in this case?)

Suppose now that files are not accessed with equal probability, but instead the i th file will be accessed $f(i)$ times over the lifetime of the tape. Now, the total cost of these accesses is

$$\text{total-cost}(\pi) = \sum_{j=1}^n \sum_{i=1}^j f(\pi(j)) \cdot L[\pi(i)] .$$

What ordering π should we choose now? Just as when the access frequencies were the same but the lengths were different we would sort the files by their lengths, if the lengths are all the same but the

access frequencies different, we should sort the files in decreasing order of their access frequencies. (Can you prove this by modifying Lemma 2?) But, what if the sizes and frequencies are both non-uniform? The answer is to sort by the length-frequency ratio L/f .

Lemma 3: total-cost(π) is minimized when $\frac{L[\pi(i)]}{f(\pi(i))} \leq \frac{L[\pi(i+1)]}{f(\pi(i+1))}$ for all i .

Proof: Suppose $\frac{L[\pi(i)]}{f(\pi(i))} > \frac{L[\pi(i+1)]}{f(\pi(i+1))}$. The proof follows the same structure as Lemma 2, but where we observe that the proposed swap changes the total cost by $L[\pi(i+1)] \cdot f(\pi(i)) - L[\pi(i)] \cdot f(\pi(i+1))$, which is negative. \square

Thus, the same class of greedy algorithms is optimal for non-uniform access frequencies.