Computing volatility surface via SVI model (Stochastic Volatility Inspired)

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Abstract

The purpose of this work is to create an algorithm for constructing an implied volatility surface based on available market data using the SVI model. The paper considers the issues of effective scraping of market data on American options, determining corresponding discount rate and dividends. Paper considers the nuances of calculating the implied volatility using the Black-76 model for real market data, an algorithm for efficient calibration of the SVI model. Work presents an algorithm for checking the presence of static arbitrage (arbitrage on "butterflies", calendar arbitrage, the Lee formula). The paper also considers the possibility of constructing a regression based on the invariant behavior of Jump Wing parameters over time, to extrapolate the values of the volatility surface beyond the available expirations.

Keywords: Implied Volatility Surface, SVI, butterfly spread, calendar spread, Calibration, Scraping market data, Regressions, Extrapolation and interpolation over time.

JEL Classification: C61 · C5

Mathematical Subject Classification (2000): 91B70

Introduction

Banks and other financial institutions use a variety of models to price various types of derivative financial instruments, but most of them involve implied volatility. The concept of implied volatility first appeared in the work of F. Black and M. Sholes in 1973 as one of the parameters that determine the pricing of options and has become the cornerstone of financial mathematics. You can compute the value of the implied volatility by revers solving the Black-Sholes-73 equation [1] with respect to the implied volatility. In 1976 was introduced new model called Black-76 [2], the main difference of the model was that it used not the price of the asset, but the forward for this asset, which made it possible to take into account non-constant interest rates in pricing and extended this model to other derivatives.

In the real market and active exchanges, options are traded only with certain strikes and expirations, therefore we need to search for various models that would allow us to determine the value of the implied volatility for arbitrary strike and expiration values. There are a number of models for constructing the implied volatility surface, one of the simplest and at the same time quite well describing individual volatility smiles is the SVI (Stochastic Volatility Inspired) model. It was first presented by Jim Gatheral in 2004 at the Global Derivatives and Risk Management conference [8] and has been widely adopted.

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In 2013, Gatheral and Jacquier published a joint paper [14] in which they presented SVI as a parametric model describing the smiles of implied volatility, and also considered the presence of static arbitrage, including butterfly arbitrage, calendar arbitrage, asymptotic behavior for large and small strikes (the Lee formula), construction of an arbitrage-free surface, some issues of calibration of this model.

In 2020, Tahar Ferhati [12] in his work presented a robust calibration algorithm for the SVI model and proposed the most efficient initial parameters for calibration.

The main goal of the work is to apply the SVI model to real market data and implement a unified algorithm for constructing an implied volatility surface from collecting data on American options and converting them, to calibrating the model and building the corresponding surface.

The first part of the paper discusses the general theoretical issues of implied volatility calculation and describes the mechanism for scraping and converting the necessary market data.

The second part of the work considers application of the SVI model to the obtained data, some issues of model calibration and the presence of static arbitrage.

The third part of the work is devoted to finding the invariant behavior of Jump Wing parameters when moving from one expiration to another, building regressions and extrapolating values over time.

1 Options as derivatives and the Implied Volatility Surface

The **derivative financial instrument** in John C. Hull book [10] is understood as a financial document whose value depends on the value of other basic variables. As a rule, these variables are the prices of various market assets.

Option is a financial agreement between two parties, one of which is the buyer and the other the seller. There are two main types of options:

1) **Purchase option** (*Call option*) gives its owner the right to purchase the underlying asset at a fixed price at a certain point in time;

2) **Sell option** (*Put option*) gives its owner the right to sell the underlying asset at a fixed price at a certain point in time.

The value of an option depends on the price of the underlying asset, the risk-free interest rate, the expiration date, strike and the volatility of the asset. The price fixed in the contract is called the option exercise price or strike price. The date specified in the contract is called the option expiration date or maturity. The risk-free interest rate is the amount of guaranteed income of a hypothetical investment portfolio, usually the SOFR rate is used. Volatility is an indicator showing the level of variability in the price of an asset over a certain period of time[10].

There are several types of option on the market:

1) **American, European**. American options can be execute at any time before the specified expiration date (for example, options on the Nasdaq exchange), European options only on the date specified in the contract (for example, crypto options on Derebit Exchange).

2) **Delivery and Settlement**. With a delivery option at the time of expiration, the underlying asset is directly delivered. With settlement, the option owner receives only the monetary difference between the strike price and the market price of the underlying asset;

3) **Dividend and non-dividend** For example options on Apple and Amazon company respectively.

According to [1] **fair price** (*non arbitrage price*) of a payment obligation on complete market is the value of initial portfolio for a trading strategy that replicates a payment obligation. Complete market is a market with two conditions: negligible transaction costs and therefore also perfect information, every asset in every possible state of the world has a price. But in practice we haven't complete market and we need understand non arbitrage price as a value which don't create any opportunities for arbitrage.

Black – Sholes – Merton formula [1] for the fair price of an option on a risky asset, the price changing of which is define by geometric Brownian motion:

$$dS_t = S_t \mu dt + \sigma S_t dW_t \longleftrightarrow S_t = S_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W_t}.$$

Parameters: r – risk-free rate, μ – trend, $\sigma > 0$ – volatility, W_t – Brownian motion
Then the following formulas are right for option prices:

$$C(t, S_t) = e^{-rT} E^Q (S_T - K)^+,$$

$$P(t, S_t) = e^{-rT} E^Q (K - S_T)^+.$$

Where Q is a risk neutral measure in which process S_t have a drift equal risk free rate (μ = r).
Integrating them over the density of the normal distribution, we obtain:

$$C(t, S_t) = S_0 \phi(d_+) - K e^{-rT} \phi(d_-);$$

$$P(t, S_t) = K e^{-rT} \phi(-d_-) - S_0 \phi(-d_+);$$

$$d(\pm) = \frac{1}{\sigma \sqrt{T}} \left(ln \frac{s_0}{K} + \left(r \pm \frac{\sigma^2}{2} \right) T \right).$$

The Black – Scholes – Merton model doesn't take into account a number of aspects that are essential for option pricing [5]:

1) Presence of "jumps" in price processes;
2) Correlation between volatility and price.

The prices derived from the Black – Scholes – Merton model depend on strike, maturity, risk free rate, spot price and implied volatility. Since we know S_t, T, K, r and $V(t, S_t)$, then the inverse solving equation: $V(t, S_t) = V(S_t, K, T, r, \sigma)$ with respect to σ, we'll find the implied volatility.

Implied volatility is commonly derived from options pricing to indicate how much the market expects the price of the underlying asset to change over time. It is expressed as the percentage change in the underlying asset price over one year. Implied volatility is the solution of inverse problem respect to σ^* of the equation $V(t, S_t) = V(S_t, K, T, r, \sigma^*)$. In the Black – Scholes – Merton model, the value of σ^* would be constant for all traded options on the same stock (for all possible T, K). But in reality this does not happen.

Volatility surface is a graphical representation of the implied volatility for options with the same underlying asset and different strikes and expirations [9]. The implied volatility in options is not uniform distributed. Volatility is lower near the central strike and increases for smaller and higher strikes. When fixing the execution time, we get a slice called **volatility smile**. An example of a volatility surface and a volatility smile is shown in Fig. 1

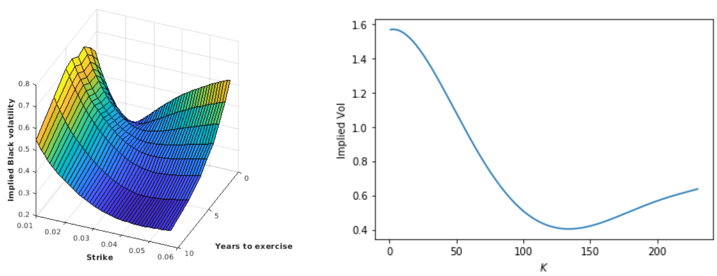

Figure 1: Volatility surface and smile example [9].

2 Black-76 model

In financial mathematics, the Black model (also known as the Black-76 model) is a variant of the Black–Scholes option pricing model. It has direct applications in the pricing of bond options, cap and flor agreements, and swaptions. The model was first presented in an article by Fisher Black in 1976 [2].

Black's model can be generalized to a class of models known as forward lognormal models, also known as LIBOR market models or the more modern SOFR market models.

Forward price per equity at expiration T where $t < T$ is equal to:

$$F(t,T) = S(t)\frac{RF(t,T)}{DF(t,T)} - \sum_{t<t_{ex_i}\leq T}^{T} d_i \frac{RF(t_{p_i},T)}{DF(t_{p_i},T)}.$$

Where:

- d_i is the dividend on the day t_{ex_i} (means that the owner of the equity receives dividend income if he owns it on the day $t_{ex_i} - 1$, "ex_i" is the exclusion date for the $dividend\ i$") the payment is made in $t_{p_i} \geq t_{ex_i}$. In this case, no assumptions are made about the size of the dividend, whether it is deterministic or proportional to the expected value of the equities at the time when dividend is announced. The determination of the dividend amount is described below.

- $DF(t,T)$ is the discount factor from t to T. If r_s isn't constant interest rate, then $DF(t,T) = E_t^Q\left[\exp(-\int_t^T r_s ds)\right]$, where Q will be a risk neutral measure.

- $RF(t,T)$ is the REPO factor from t to T, it corresponds to the cost of borrowing equities. We can express it as $RF(t,T) = \exp(-\int_t^T b_s ds)$.

The formula presented above works only for cases when the interest rate and dividends are known and non stochastic, which does not occur in practice. Therefore, the forward price in practice is a stochastic process. The randomness of this process is neutralized by choosing a new measure, in which forward price process is a martingale

From presented definition of the forward value, we can see that in practice the definition of the value of an underlying asset is much more complicated than in the formulation of Black-Sholes [1], where the trend of the underlying asset at a risk-neutral indicator is deterministic and equals r. The Black-76 model uses a different change of measure and introduces additional hypotheses allowing us to rewrite this model in a much more general but simple form, commonly known among practitioners as the Black model.

Let's define the **Black - 76** model:

Suppose we are at time t_0. We define Q^T to time T as a measure, in which the forward price is a martingale:

$$\left.\frac{dQ^T}{dQ}\right|_t = \frac{B_{t_0}(t)DF(t,T)}{DF(t_0,T)},$$

where $B_{t_0}(t) = \exp(-\int_{t_0}^t r_s ds)$.

Then the price of the European option at the moment t_0 with strike K, with expiration T, on the specified asset $S(T)$ is equal to:

$$\mathrm{V}_{t_0}(K,T) = E^Q[B_{t_0}(T)\cdot \max(\epsilon\cdot(S(T)-K);0)] =$$

$$= DF(t_0,T)\cdot E^{Q^T}[\max(\epsilon\cdot(S(T)-K);0)],$$

where ϵ is 1 if it's a call option and -1 if it's a put.

If $F(t_0,T)$ is computed using the formula above, we define a forward measure Q^T as:

$$\frac{dF(t,T)}{F(t,T)} = \sigma_t^{F,T} dW_t^{F,T}.$$

Since $S_T = F(T, T)$, the undiscounted price at time t_0 of a European option with strike K, at expiration T in the Black-76 model, is expressed as:

$$\mathrm{V}(t_0, F(t_0,T), T, K, \Sigma^F(K,T), \epsilon) =$$
$$= \epsilon \cdot (F(t_0,T) \cdot \mathcal{N}(\epsilon \cdot d_1) - K \cdot \mathcal{N}(\epsilon \cdot d_2)),$$
$$d_1 = \frac{k}{\Sigma^F(K,T)\sqrt{T-t_0}} + \frac{1}{2}\Sigma^F(K,T)\sqrt{T-t_0},$$
$$d_2 = d_1 - \Sigma^F(K,T)\sqrt{T-t_0},$$
$$k = \ln\left(\frac{F(t_0,T)}{K}\right),$$

where:

- ϵ is 1 if it's a call option and -1 if it's a put;
- $\mathcal{N}()$ is the normal distribution function;
- $\Sigma^F(K,T)$ is the implied volatility for an option with strike K and expiration T. Then the market price of the European option at time t_0 is calculated as Black Formula $\cdot$ $DF(t_0,T)$.

Then the market price of the European option at time t_0:

$$\mathrm{PV}(t_0, F(t_0,T), T, K, \Sigma(K,T), \epsilon, DF(t_0,T_p))$$
$$= DF(t_0,T_p) \cdot \mathrm{V}(t_0, F(t_0,T), T, K, \Sigma^F(K,T), \epsilon).$$

This function is smooth, if you take market data about option prices, then you can solve this equation with respect to sigma using the Newton Rapson method and get the estimated volatility for various strikes and expiration.

It is important to note that the Black-76 model makes several assumptions, including that future prices are log-normally distributed and that the forward price process has constant volatility..

3 Determining forward value

Let's make a number of transformations and express the forward cost in a simpler form, given that there are no cash flows in the period from t to T, we get:

$$F(t,T) = S(t)\frac{RF(t,T)DivF(t,T)}{DF(t,T)},$$

where $DivF(t,T)$ is expressed as $\exp(-\int_t^T q_s ds)$, where q is the asset's proportional dividend yield.

To determine a more accurate forward value, we use call - put parity. We make an estimation according to the formula presented above, find option prices for closest strikes above and below given estimation and compute forward via call - put parity for them, after that we take average value between this two computed values.

To estimate the value of forward, we need to determine the amount of dividends and the discount factor.

3.1 Dividend size

Figure 2 shows the price chart of futures for the SP500 dividend index traded on the CME exchange. This graph clearly shows that in the coming years market expects either no increase in the size of dividends, or their decrease. So we can accept the following dividend hypothesis:

$$Dividend\ yield = \frac{AverDiv \times FP}{SP},$$

where: $AverDiv$ is average dividend for the last year, FP is dividend frequency of payments, SP is price of the underlying asset.

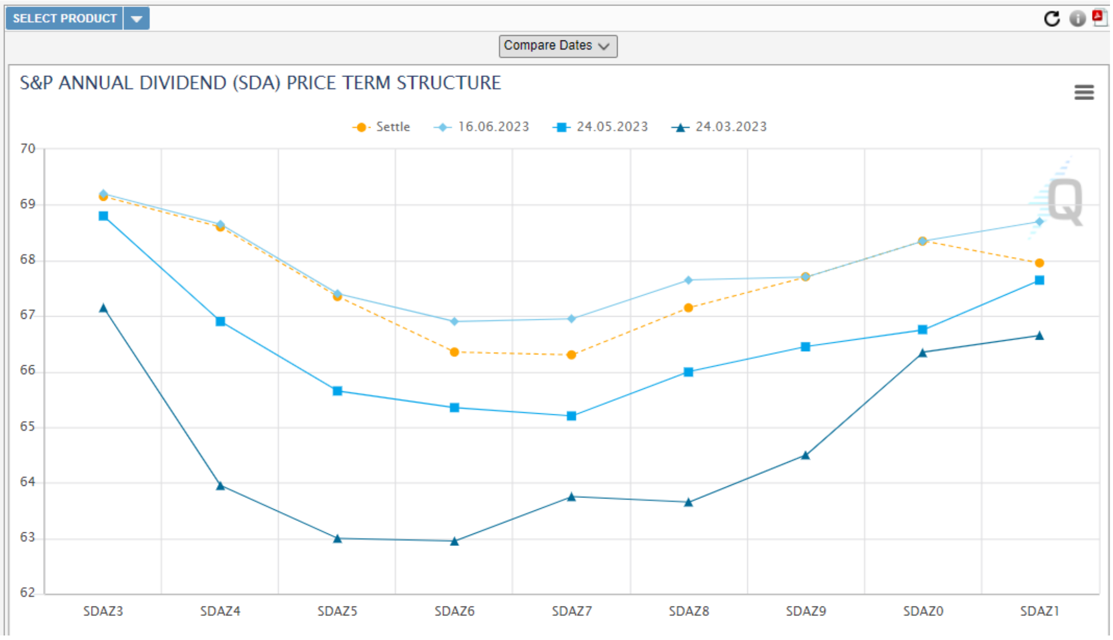

Figure 2: Futures quotes for the SP500 dividend index. Date: 24-06-2023.

3.2 Discount Factor

The paper uses 3-month SOFR forward contracts as a discount factor. They reflect the market's assumption of future overnight rates for various three-month periods. Only 3-month SOFR futures contracts are traded on the exchange markets, which forces us to calculate the forward rate based on the futures, taking into account the convexity adjustment.

The three-month SOFR futures contract is as close as possible to the Eurodollar futures contract. The settlement is made on the third Wednesday of the month according to the following principle: 100 minus the result of summing up the one-day overnignt SOFR rate for the previous three months. Trading in 3-month SOFR futures starts in one of 4 months: March, June, September and December. CME offers a total of 39 contracts for various months and years so that traders can hedge or speculate on SOFR rates in the future, up to 10 years. The contract is designed to hedge a 1 million dollar position (similar to a Eurodollar futures contract). When the quote changes by one basis point, the profit or loss per contract is 25 dollars. The 3-month SOFR future contracts only settles at the end of 3-month period (after the based day rates have been computed) [10] which results in a "convexity adjustment". The convexity adjustment is the difference between value of three-month SOFR futures and forwards and it can be explain by the fact that the futures are settled on a daily basis.

In Mark Henrard's article: Overnight futures convexity adjustment [16] presented an algorithm for calculating the amount of convexity adjustment. Define $T_s = t_0 < t_1 < \cdots t_i \cdots < t_n = T_e$, where T_s, T_e are the beginning and end of the contract. Accrual factor - δ_i for t_i, $\sum\limits_{i=0}^{n} \delta_i = \delta$ based on calculations based on 360 days.

Let's define r_i as the benchmark rate published in the period $[t_{i-1}, t_i]$, then the interest rate r for the futures will be calculated by the time T_e and will be equal to:

$$1 + \delta r = \prod_{i=0}^{n} (1 + \delta_i r_i).$$

The futures settlement price will be known by the time T_e and equal to $P(t_e) = 1 - r$. The convexity adjustment computing is based on the Jarrow Morton [17] Gaussian Heath model. The margin rate associated with the benchmark at time s is denoted as c_s. Then the discount rate:

$$N_t^c = \exp(\int\limits_0^t c_s ds).$$

The main object of modeling is the discount curve:

$$P_X^c(t,u) = N_t^c E^X[(N_u^c)^{-1} | \mathcal{F}_t].$$

Then the dynamics of the discount factor is determined as follows:

$$\frac{P^c(t,u)}{N_t^c} = \frac{P^c(s,u)}{N_s^c} \exp\left(-\int\limits_s^t v(\tau,u) dW_\tau^X - \frac{1}{2} \int\limits_s^t v^2(\tau,u) d\tau \right).$$

The ratio between two discount factors, each of which is: $(s \le t \le u, v)$ is defined by the following equation:

$$\frac{P^c(t,u)}{P^c(t,v)} = \frac{P^c(s,u)}{P^c(s,v)} \exp\left(-\int\limits_s^t (v(\tau,u) - v(\tau,v)) dW_\tau^X - \frac{1}{2} \int\limits_s^t (v^2(\tau,u) - v^2(tau,v)) d\tau \right) =$$

$$= \frac{P^c(s,u)}{P^c(s,v)} \exp\left(-\alpha(s,t,u,v) X_{s,t,u,v} - \frac{1}{2} \alpha^2(s,t,u,v) \right) \gamma(t,s,u,v),$$

where $X_{s,t,u,v}$ is a random variable with normal distribution $\mathcal{F}_t$ - measurable and independent with respect to $\mathcal{F}_s$,

$$\alpha^2(s,t,u,v) = \int\limits_s^t (v(\tau,u) - v(\tau,v))^2 d\tau,$$

$$\gamma(s,t,u,v) = exp\left(\int\limits_s^t v(\tau,v)(v(\tau,v) - v(\tau,u)) d\tau \right).$$

Volatility is taken from the one-factor Hull-White model:

$$v(s,t) = \frac{\eta(s)}{a}(1 - \exp(-a(t-s))).$$

The general formula for estimating the value of V_u derivatives in period u with a discount rate of d.

$$V_u = N_t^d E^X[(N_v^d)^{-1} V_u | \mathcal{F}_t].$$

To calculate the forward value at time t = 0, we have to solve the following equation:

$$P(0) = E^X[P(Te)] = E^X\left[1 - \frac{1}{\delta}\left(\prod_{i=0}^{n}(1+\delta_i r_i) - 1\right)\right] = 1 - \frac{1}{\delta}\left(E^X\left[\prod_{i=0}^{n}(1+\delta_i r_i)\right] - 1\right).$$

After a series of transformations, we get:

$$E^X\left[\prod_{i=0}^{n}(1+\delta_i r_i)\right] = \frac{P^c(0,t_0)}{P^c(0,t_n)}\prod_{i=1}^{n}\gamma(t_{i-2},t_{i-1},t_{i-1},t_n).$$

taking into account that $t_{-1} = 0$

The forward value in this model is defined as:

$$F^c(t,u,v) = \frac{1}{\delta}\left(\frac{P^c(t,u)}{P^c(t,v)} - 1\right), \text{where } F^c \text{ – Forward rate.}$$

Substituting it into the formula for the moment of time when $t_0 < T_s$ we get the following formula:

$$F^c(0,t_s,t_e) = 1 - P(0) - \frac{1}{\delta}\frac{P^c(0,T_s)}{P^c(0,T_e)}\left(\prod_{i=1}^{n}\gamma(t_{i-2},t_{i-1},t_{i-1},t_n) - 1\right).$$

For a time greater than T_s and less than T_e, some part of the futures is already fixed, because some of the rates are already known. In this case, the correction for convexity is extremely small and, from a practical point of view, it can be neglected. Numerical results are described in more detail in the article by Mark Henrard [16]. Figure 3 shows a discount curve based on a three-month SOFR forward.

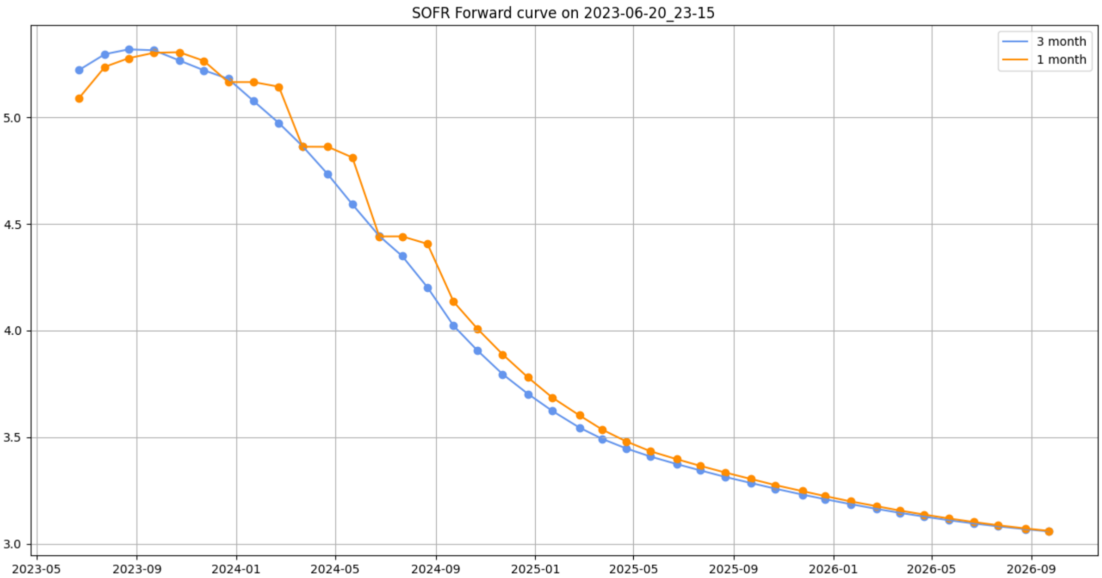

Figure 3: Discount curve for 1 and 3 month SOFR forwards. Date: 20-06-2023.

3.3 Call - Put parity with dividend

When the underlying asset pays a continuous dividend income of q. We, instead of deducting dividends from the price of the underlying asset, buy Se^{-qT} of the underlying asset today in order to have 1 unit of the asset by the time when the option expires.

Thus, the Call - Put formula for parity with continuous dividends is as follows:
$Call(K,T) - Put(K,T) = S_0 DivF(t_0,T) - KDF(t_0,T))$ and hence the forward is:

$$F(t_0,T) = \frac{Call(K,T) - Put(K,T)}{DF(t_0,T)},$$

where K - strike, T - expiration, S_0 is the value of the underlying asset

This model assumes that the received dividends are constantly reinvested. Thus, parity is maintained when for the underlying asset pays dividends.

3.4 Call - Put parity for American option

In the process of doing the work, I have encountered with the fact that the implied volatility of ITM options and OTM options don't match. The results of computing implied volatility for ITM and OTM for Apple are presented on fig. 4 .

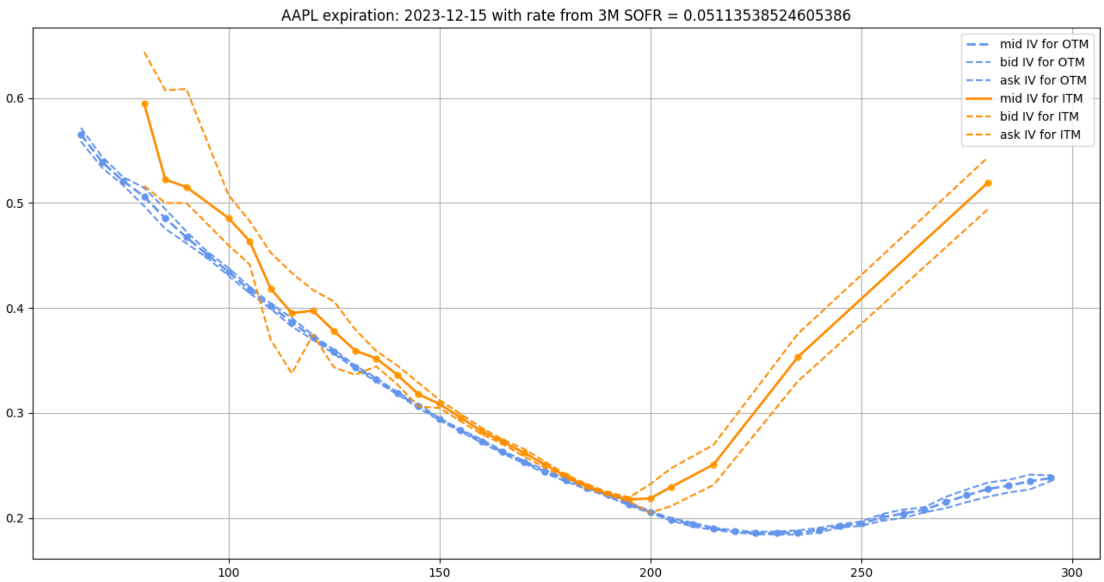

Figure 4: Implied volatility for Apple and expiration on 15.12.2023, with rate from 3M SOFR Forward. Date: 20-06-2023.

This chart clearly shows that on the left and right sides, implied volatility of ITM options differs from OTM options. The difference on the left wing for ITM call, is explained by using different interest rates as benchmarks, you can choose the value of discount factor and the interest rate directly from the option prices, based on the least square method, for example for expiration on 15.12.2023 for Apple company from this method we obtain the rate equal 0.0463, instead 0.0511 from 3 month SOFR Forward and we will get implied volatility shown on fig.5. But in practice this method is not used. The difference on the right wing is explained by the fact that call-put parity is generally strictly observed only for European options, and for American options there is only the following inequality:

$$S_0 - K \leq C - P \leq S_0 - Ke^{-rT}.$$

This inequality shows that the correctness of call put parity depends on the current interest rates in the market. This phenomenon was observed last time in 2008 - 2009 years. It depends on the high level of interest rates in dollars.

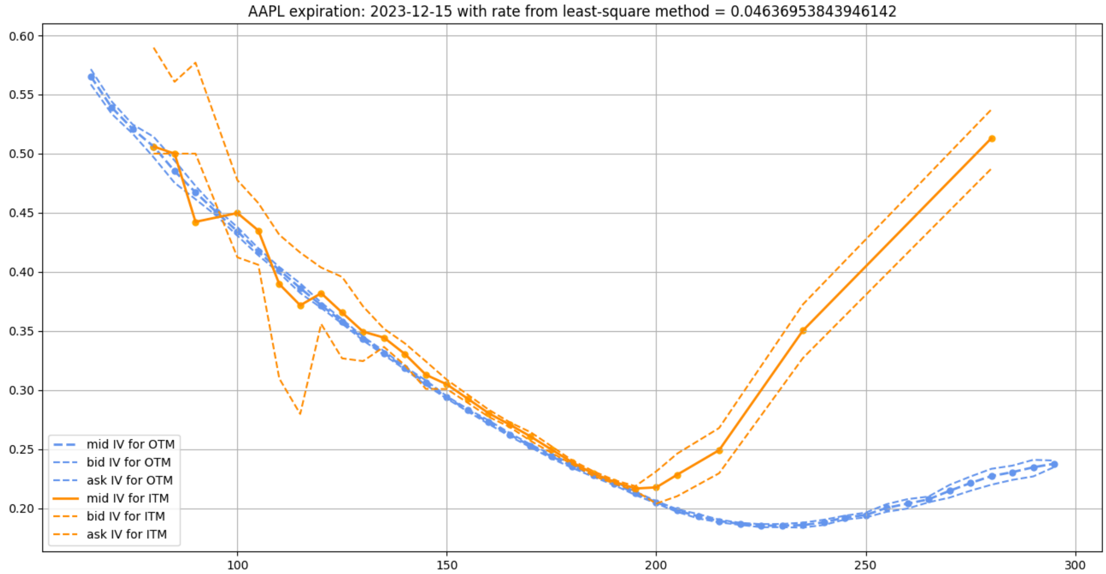

Figure 5: Implied volatility for Apple and expiration on 15.12.2023, with rate from least square method. Date: 20-06-2023

4 Implied Volatility from Black-76 model

The data for computing the implied volatility is taken from the Nasdaq exchange using the yfinance library of the Yahoo Finance portal. Options quotes are downloaded, including: last price, bid-ask spread, expiration dates, price of the underlying asset, date and time of the last transaction, option type, dividend amount. In work implement an algorithm that scraps all the necessary information on the specified ticker and converts the data for further calculation of implied volatility.

An algorithm for computing implied volatility has also been implemented, the average running time of the algorithm is 1.6 seconds. The following libraries are used in the algorithm: pandas, numpy, scipy, datetime, matplotlib, time, os and others.

Figure 6, 7, 8 shows the graphs of implied volatility for Apple and Amazon on 20.06.2023.

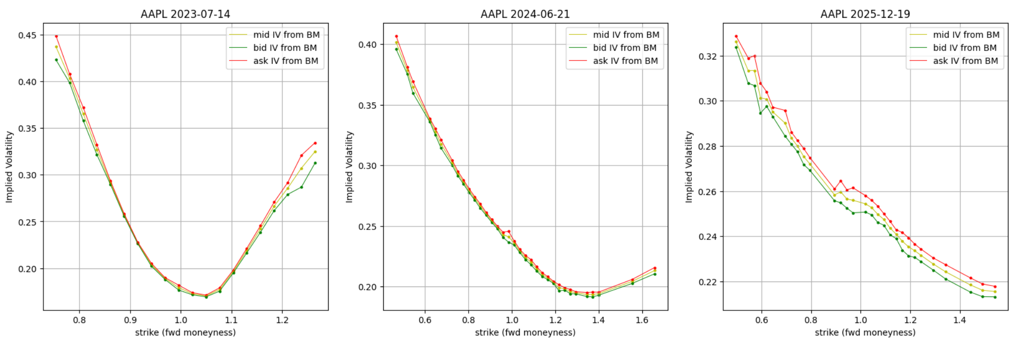

Figure 6: Volatility smiles for Apple on 20.06.2023 for different expirations.

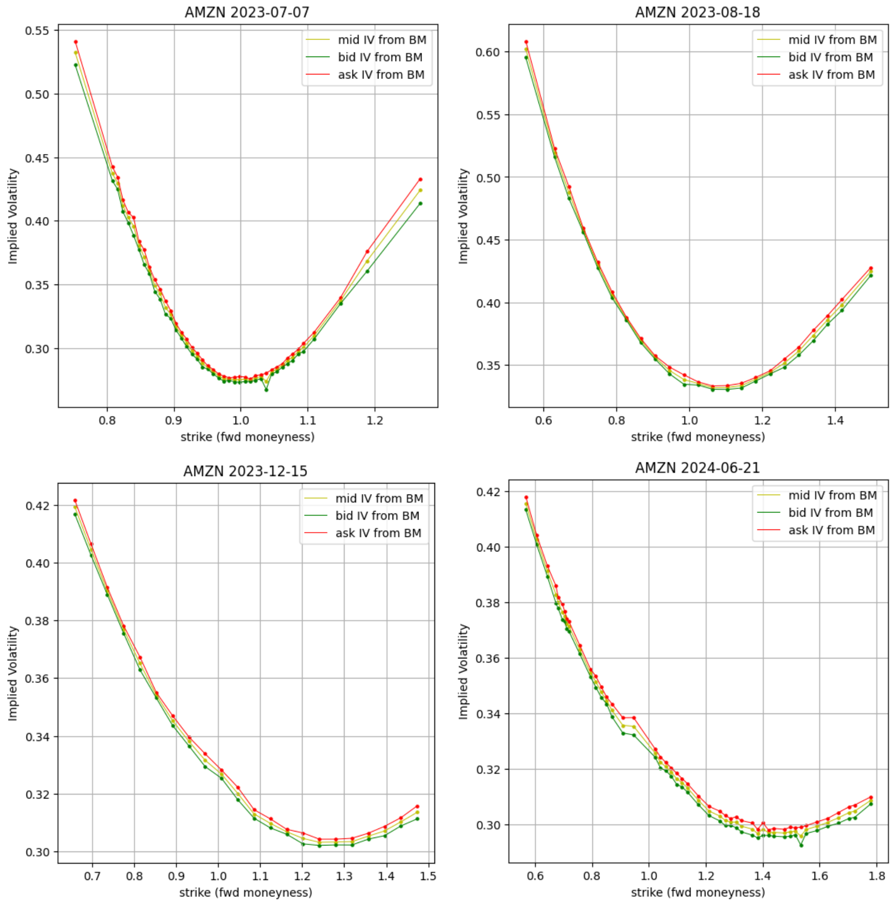

Figure 7: Volatility smiles for Amazon on 20.06.2023 for different expirations.

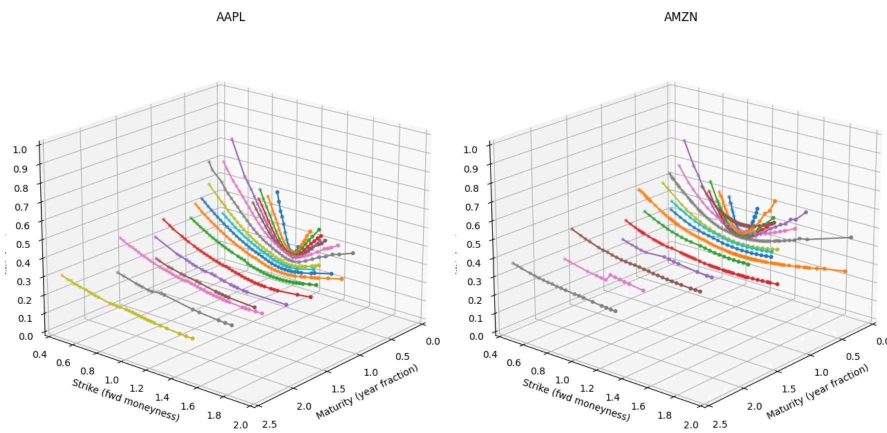

Figure 8: Volatility smiles for Apple and Amazon on 20.06.2023 for different expirations.

5 Stochastic volatility inspired model [14]

The SVI model is a non-symmetric parabolic function with a flexible set of different parameters. We calibrate these parameters,based on the least squares method, with purpose to get a function that describes our implied volatility curve in the most accurate way. There are 3 types of SVI model: raw, natural and Jump Wings parametrizations.

5.1 Raw parameterization

The raw parametrization is given by the following formula and parameters:

$$\omega(x) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right).$$

a, b, ρ, m, σ are parameters, where $(a \in R, b \geq 0, |\rho| < 1, m \in R, \sigma > 0)$ Parameter values:

1. a and m change the position of the curve vertically and horizontally;
2. b defines the angle between the left and right side:
   $\omega(x) \sim a - b(1-\rho)(x-m)$ at $x \to -\infty$,
   $\omega(x) \sim a + b(1+\rho)(x-m)$ at $x \to +\infty$.
3. σ defines the convexity at the top;
4. ρ defines the rotation of the curve.

It is important to note that we make a number of substitutions, because this simplifies the work with SVI model. First, let's go to the log moneyness: $x = \ln(K/F_0)$. If such a replacement is not made, then it will be necessary to somehow describe the behavior of the function when the asset value approaches to 0, but in this case, x can be from - infinity to + infinity. We also move on to the total implied variance - $\omega = T\tilde{\sigma}^2$.

This type of parameterization is very convenient for parameter calibration and well reflects the geometric meaning of this model.

5.2 Natural parametrization

The natural parametrization is given by the following formula and has the following parameters:

$$\omega(x) = \Delta + \frac{\omega}{2}\left(1 + \zeta\rho(x-\mu) + sqrt(\zeta(x-\mu)+\rho)^2 + 1 - \rho^2\right).$$

$\Delta, \mu, \rho, \omega, \zeta$ - parameters, where $(\omega \geq 0, \mu \in R, \Delta \in R|\rho| < 1, \zeta > 0)$.

The parameters in the raw and natural parameterizations are equivalent and have the following relationships:

$$(a, b, \rho, m, \sigma) = \left(\Delta + \frac{\omega}{2}(1-\rho)^2, \frac{\omega\zeta}{2}, \rho, \mu - \frac{\rho}{\zeta}, \frac{\sqrt{1-\rho^2}}{\zeta}\right).$$

The natural parametrization formula is obtained by transition to the limit in time from the Heston model by making a number of substitutions. This indicates that the SVI model corresponds at least to the Heston model. The natural parametrization is convenient from a theoretical point of view, but its parameters are extremely difficult to interpret. In practice, for trading, the third parameterization is used, the parameters of which give traders the greatest amount of information and are easily interpreted.

5.3 Jump Wing parametrization

For a certain expiration date $t > 0$ and a set of parameters$[v_t, \phi_t, p_t, c_t, \widetilde{v_t}]$, Jump Wings parameters are determined from the parameters of the raw parameterization using the following formulas:

$$v_t = \frac{a + b(-\rho m + \sqrt{m^2+\sigma^2})}{t} \ - \text{ ATM implied variance;}$$

$$\phi_t = \frac{1}{\sqrt{\omega_t}}\frac{b}{2}\left(-\frac{m}{\sqrt{m^2+\sigma^2}} + \rho\right) \ - \text{ ATM skew;}$$

$$p_t = \frac{1}{\sqrt{\omega_t}} b(1-\rho) \ - \text{ slope for put side;}$$

$$c_t = \frac{1}{\sqrt{\omega_t}} b(1+\rho) : - \text{ slope for call side;}$$

$$\widetilde{v_t} = \frac{1}{t}\left(a + b\sigma\sqrt{1-\rho^2}\right) : - \text{minimum implied variance.}$$

The main advantage of Jump Wings parameterization is that such parameters are more stable for different expiration dates and have a time dependence. These parameters are related to the raw parameterization as follows:

$$a = \widetilde{v_t}T - b\sigma\sqrt{1-\rho^2},\ \rho = 1 - \frac{2p_t}{c_t + p_t},\ b = \frac{1}{2}\sqrt{v_t T}(c_t + p_t),$$

$$\beta = \rho - \frac{2\phi\sqrt{v_t T}}{b},$$

$$|\beta| \le 1, \beta \ne 0 \begin{cases} \alpha = sgn(\beta)\sqrt{\beta^{-2} - 1}, \\ m = \frac{(v_t - \widetilde{v_t})T}{b\left(\rho + sgn(\alpha)\sqrt{1+\alpha^2} - \alpha\sqrt{1-\rho^2}\right)}, \\ \sigma = \alpha m. \end{cases}$$

$$\beta = 0 \begin{cases} m = 0, \\ \sigma = \frac{(v_t - \widetilde{v_t})T}{b\left(1 - \sqrt{1-\rho^2}\right)}. \end{cases}$$

If $|\beta| > 1$, then the curve will not be convex, which does not happen in practice.

6 SVI parameter calibration

Consider a raw parametrization for a fixed expiration T and parameters a, b, ρ, m, σ. Then $\omega_m(x_i), i = 1, 2, .., n$ is the total implied variance. Then we define the function $y(x)$:

$$y(x) = \frac{x - m}{\sigma}.$$

Then the implied total variance is:

$$\omega(x) = a + dy(x) + cz(x),$$

$$z(x) = \sqrt{y(x)^2 + 1},\ \ d = \rho b\sigma,\ \ c = b\sigma.$$

We define the calibration tusk as follows:

$$f(a, b, \rho, m, \sigma) = \sum_{i=1}^{n}(\omega_i - \omega(x_i))^2 \to min.$$

The main idea is that we are trying to choose the parameters in such a way that the values in the standard deviation are minimal. If we try to calibrate all 5 parameters at once, then the algorithm will converge poorly or show a bad result, so it is proposed to split the calibration into two steps: external and internal [12]:

At the **outer step** we choose a pair of values (m, σ) from the region $\mathcal{E}$

$$\sum_{i=1}^{n} (\omega(x_i|m, \sigma, a^*(m, \sigma), b^*(m, \sigma), \rho^*(m, \sigma)) - \omega_m(x_i))^2 \to min,$$

where $\mathcal{E}$ area is defined by the following conditions:

$$min(x_i) \leq m \leq max(x_i),$$

$$\sigma_{min} \leq \sigma \leq \sigma_{max}.$$

The boundary conditions for $\sigma_{min}, \sigma_{max}$ are as follows: $\sigma_{min} = 10^{-4}, \sigma_{max} = 10$

Parameters a^*, b^*, σ^* are calibrated at the **inner step**.

$$\sum_{i=1}^{n} (a + dy(x_i) + cz(x_i) - \omega_m(x_i))^2 \to min, \; where \; (a, d, c) \in \mathcal{D},$$

where the region $\mathcal{D}$ is bounded by the following conditions:

$$0 \leq c \leq 2\sigma,$$

$$|d| \leq c,$$

$$|d| \leq 2\sigma - c,$$

$$0 \leq a \leq max \, \omega(x_i).$$

These conditions follow from the correctness of the model and the asymptotic behavior of the volatility curve. The region $\mathcal{D}$ is compact, the function is smooth, convex, hence the minimum is easily attainable.

For parameters (m, σ) this function is not convex and it is necessary to use global optimization methods.

To implement the algorithm I use the optimaze library, for the parameters a, b, σ the use gradient descent method, for (m, σ) the SLSQP and Dual Annealing methods are used. The average running time of the calibration algorithm for one curve is 3.47 seconds.

It is appropriate to take the following [12] parameters as initial parameters for calibration: $a = \frac{1}{2}min(W_{SVI}), b = 0.1, \rho = -0.5, m = 0.1, \sigma = 0.1$. In practice, SVI parameters are calibrated not once, but at regular intervals, and a good approach would be to use the parameters calibrated the previous time as initial parameters. This algorithm is implemented in the work.

7 Implied volatility from SVI model

Figure 9 shows the plots of the implied volatility and error for Apple on 20.06.2023. The blue line on the left chartis the volatility from the calibrated SVI model. The right chart shows an error, which is considered if the volatility from the SVI model does not put within the bid ask spread.

Note that for small expiration periods, the bid ask spread is very narrow and it is quite difficult to put within it, but with an increase in the expiration date, the spread widens and the error decreases.

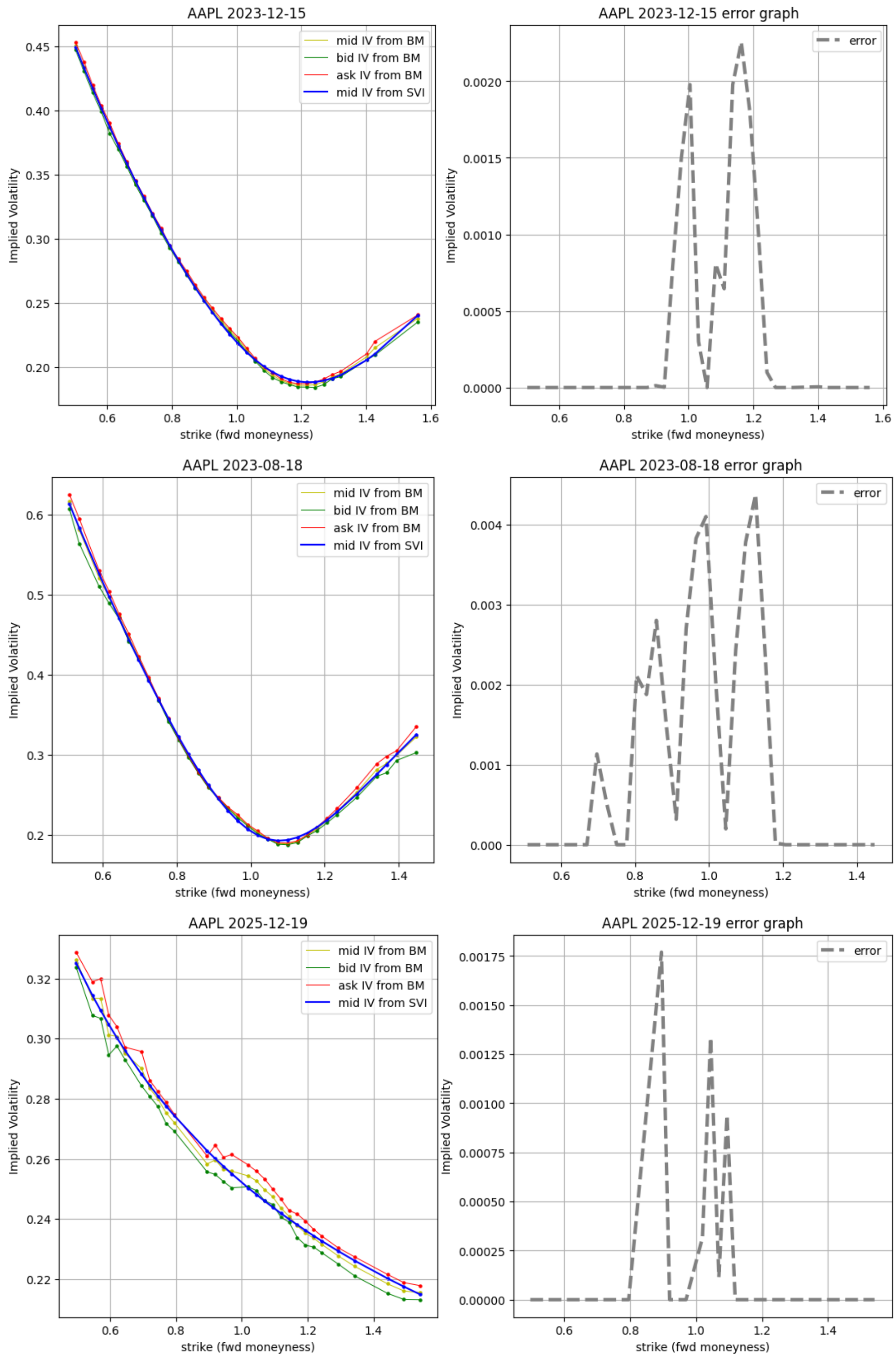

Figure 9: Volatility smiles and error graph for Apple on 20.06.2023 for different expirations from calibrated SVI.

For the calibrated SVI curves, the corresponding option prices are calculated and compared with the original bid-ask spread. For all traded strikes, the amount of points within the bid ask spread was: 68.06%, $7.66\% - higher$,24.27 $\%$ - lower. The error graph for all options is shown in Figure 10. Separately for in-the-money and out-of-the-money options, in Figure 12.

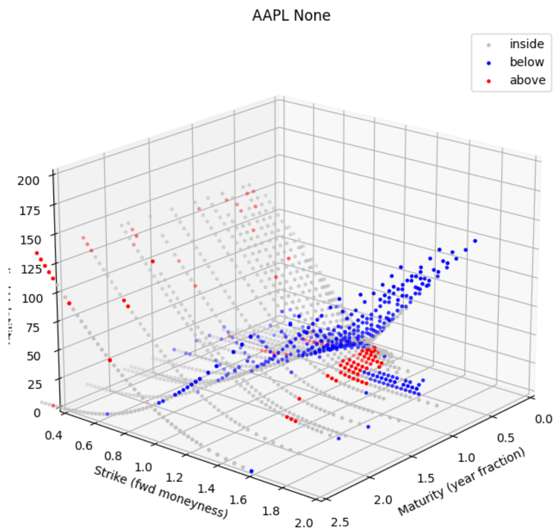

Figure 10: Amount of points within the initial bid ask spread with volatility from the SVI model for Apple on 20.06.2023 for different expirations.

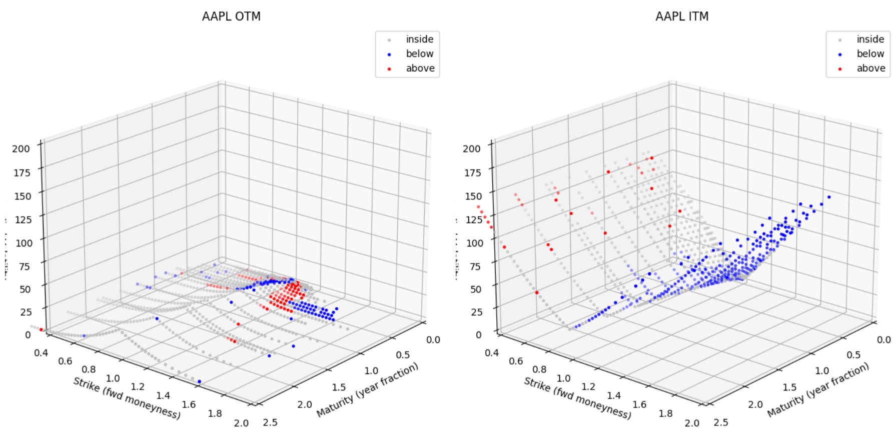

Figure 11: Amount of points within the initial bid ask spread with volatility from the SVI model for Apple on 20.06.2023 for different expirations fro ITM and OTM options.

Option prices for ITM puts on the right side below the original bid-ask spread is because the implied volatility of OTM calls is used in option pricing, while the implied volatility of ITM puts is higher than the implied volatility of OTM calls. This phenomenon has been described previously.

8 Static arbitrage [14]

In order to use the obtained curves in practice, we need to check the absence of static arbitrage in them.

Option price surface $C(T, K) : R_+ \times R_+ \to R_+$ does not contain static arbitrage if there exists a martingale X_t defined on a probability space such that:

$$C(T, K) = DF \cdot E(X_T - K)^+, \forall\, T, K,$$

where $DF(t, T) = E_t^Q \left[\exp(-\int_t^T r_s ds)\right]$.

The absence of static arbitrage indicates that the model is consistent with the underlying market laws. The volatility surface $\sigma(T, K) : R + \times R_+ \to R_+$ does not contain static arbitrage if the options surface $C(T, K)$ does not contain static arbitrage.

$$C(T, K) = DF(t, T) \cdot C_B(T, K, \sigma(T, K)), \forall\, T, K,$$

where $C_B(T, K, \sigma(T, K))$ is the option price according to the Black-76 model.

Let a function $C(T, K)$ be defined on R_+^2 such that $C(T, \cdot) \in C([0,\infty)) \cap C^1((0, \infty)) \forall\, T > 0$ and the following conditions are satisfied:

1. $C(T, K)$ does not decrease by T;
2. $C(T, K)$ is convex by K;
3. $\lim\limits_{K \to \infty} C(T, K) = 0$;
4. $C(T, 0) = s$ where $s > 0$ are some constants;
5. $C(0, K) = (s - K)^+$.

Then the martingale $X_t \geq 0$ exists and $X_0 = s$ and $C(T, K) = DF(t, T) \cdot E(X_T - K)^+$ Let's take a closer look at the above conditions:

Condition (1) absence of **calendar arbitration**. If this is not done then:

$$C(T_1, K) > C(T_2, K), T_1 < T_2, K > 0.$$

Then we can get arbitrage by selling the option $(T1; K)$ and buying the option $(T2; K)$

Condition (2) of the absence of **arbitrage on butterflies**. If this is not done then:

$$C(T, K_1) + C(T, K_3) < 2C(T, K_2)), K_1 < K_2 < K_3, T > 0.$$

Then we can get arbitrage by buying 1 option (T, K_1) and buying 1 option (T, K_3) and selling two options (T, K_2).

Condition (3) if $\lim\limits_{K \to \infty} C(T, K) > 0$, we can sell the option with strike $= K$, which will be exercised with probability tending to 0 (This is not arbitration, but too good to be true)

Conditions (4, 5) stem from the definition of options.

Based on the no-arbitrage conditions for the option price surface, we determine the conditions for the no-arbitrage volatility surface in term of total implied variance. $w(K, t)$ should satisfy the following conditions:

If the spot price is s and forward price is F then:

$$x = \ln(K/F), \theta = \sigma\sqrt{(T)}, d_1 = -\frac{x}{\theta} + \frac{\theta}{2}, d_2 = d_1 - \theta.$$

Then the Black-76 formula with no dividends is:

$$C(T, K) = DF(t, T) \cdot C_B(T, K, \sigma) = DF(t, T) \cdot C_B(x, \theta, T) = s\Phi(d_1) - se^x\Phi(d_2).$$

Further, $\sigma(T, K)$ will be defined as a function of $\theta(T, x)$.

The function $\theta(T, x) \in C^{1,2}((0, \infty) \times R)$ for all $T > 0, x \in R$ satisfies the following conditions:

1. $\theta > 0$;
2. $\theta'_T \geq 0$;
3. $g(T,x) := \left(1 - \frac{x}{\theta}\theta'_x\right)^2 - \frac{\theta^2}{4}(\theta'_x)^2 + \theta\theta''_{xx} \geq 0$;
4. $\lim_{x\to\infty} d_1(x, \theta(T,x)) = -\infty$;
5. $\lim_{T\to 0} \theta(T,x) = 0$.

Then the following option surface does not allow static arbitrage:

$$C(T,K) = \begin{cases} DF(t,T) \cdot C_B(x, \theta(T,x))|_{x=ln\frac{K}{s}}, T > 0, K > 0; \\ (s-K)^+, T = 0, K \geq 0; \\ DF(t,T) \cdot s, T \geq 0, K = 0. \end{cases}$$

For a calibrated SVI model, there are no difficulties with conditions 3,4,5, but conditions 1 and 2 should be checked in each individual case using the option price surface.

8.1 Checking Calendar Arbitrage

There is no calendar arbitrage only when $\partial_t W(x,t) \geq 0$, where $W(x,t)$ is the total implied variance, $x \in R$

This means that the total variance curves for different expiration dates should not intersect, as shown in Figure 12.

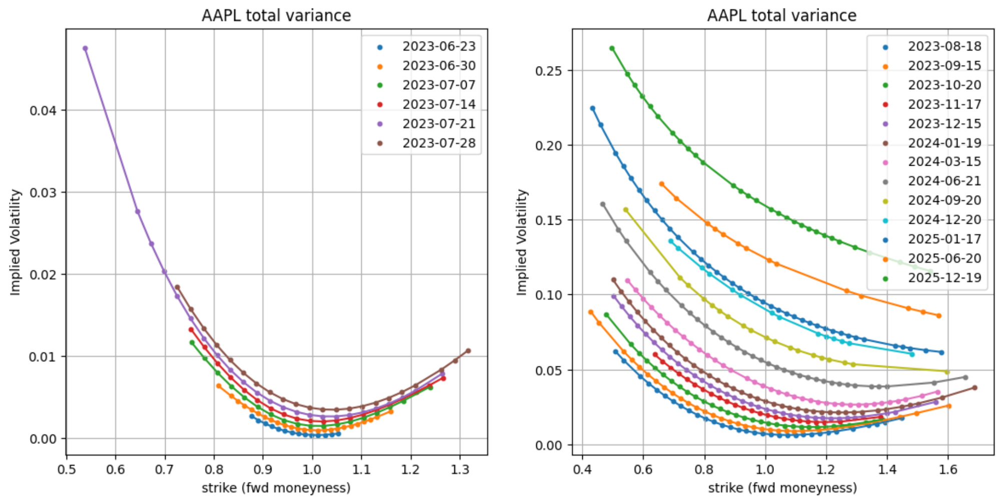

Figure 12: Total implied variance for Apple on 20.06.2023 for different expiration dates.

It is important to note that if these curves are extrapolated, they will intersect. We see this, because we don't have enough information for deep ITM and OTM options.

8.2 Checking Butterfly Arbitrage

Butterfly arbitrage is absent if and only if the function $g > 0$ for all x values and all expirations.

Function g: $g(x) := \left(1 - \frac{xw'(x)}{2w(x)}\right)^2 - \frac{w'(x)^2}{4}\left(\frac{1}{w(x)} + \frac{1}{4}\right) + \frac{w''(x)}{2}$.

Figure 13 shows a plot of total volatility and g-function for various expiration dates. Butterfly arbitrage is absent for listed market strikes. It exists for strikes higher than a given interval by more than 2 times, and this can be neglected.

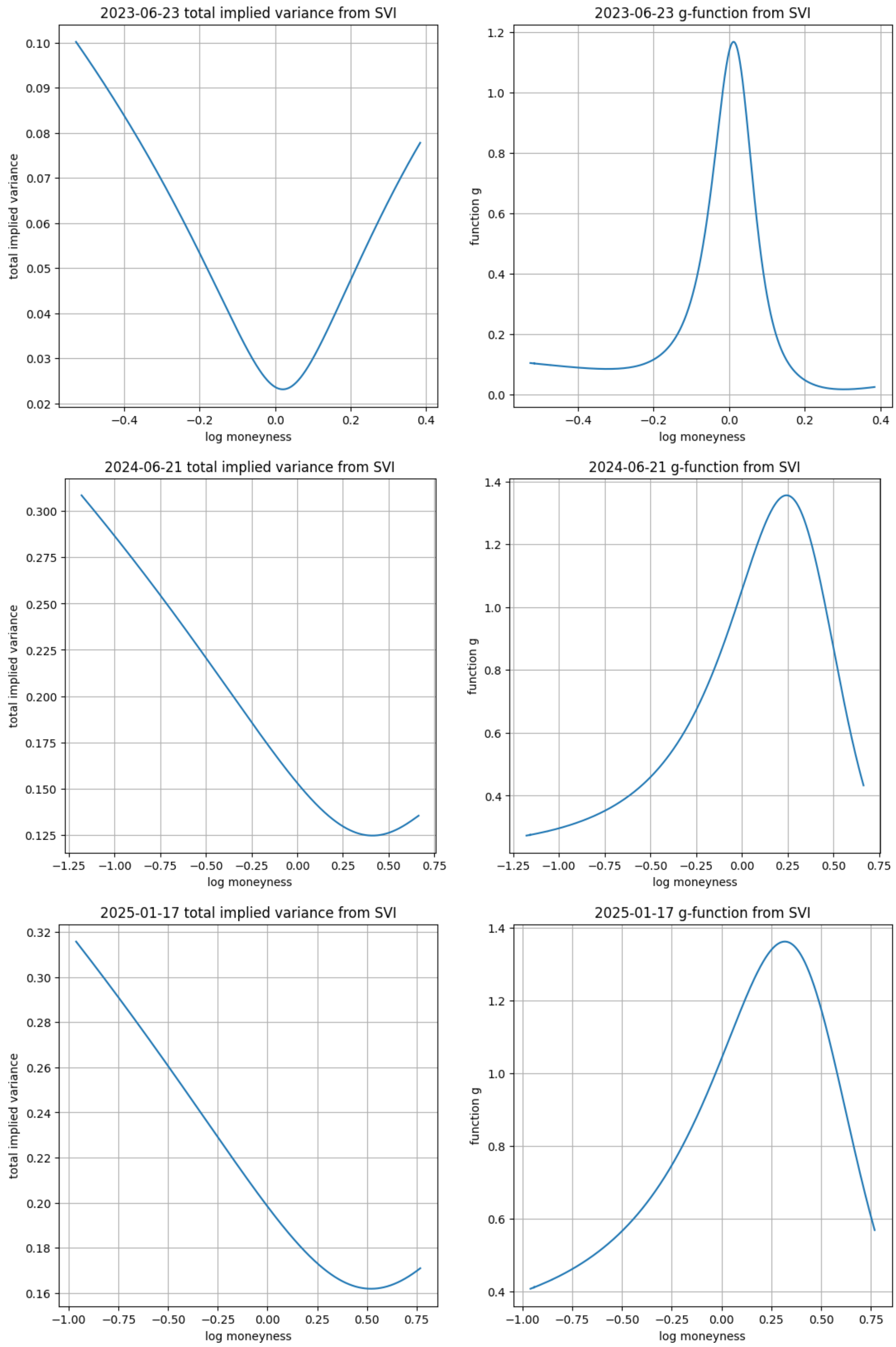

Figure 13: Total implied variance and g-function for Apple on 20.06.2023 for different expirations.

8.3 Asymptotic curve behavior, Lee formula [7]

Another important condition for the absence of arbitrage is the asymptotic behavior of the curve for large and small strikes.

Consider a possible arbitrage-free market model in which the initial measure P is martingale and $r = 0$. Let $\widetilde{\sigma}(x)$ be the variance of the call option in log-moneyness $x = ln(\frac{K}{S_0})$ and $\omega(x = T\widetilde{\sigma}^2(x))$ Then the Lee formula for the call side will be equal to:

$$\widetilde{p} = sup(p > 0 : E(S_T^{1+p} < \infty), \quad \beta_R = \lim_{x \to +\infty} sup \frac{\omega(x)}{x}.$$

Then: $\beta_R = 2 - 4(\sqrt{\widetilde{p}^2 + \widetilde{p}} - \widetilde{p})$, where $\beta_R \in [0, 2]$.

The Lee formula for the put side will be:

$$\widetilde{q} = sup(q > 0 : E(S_T^{-q} < \infty), \quad \beta_L = \lim_{x \to -\infty} sup \frac{\omega(x)}{x}.$$

Then: $\beta_L = 2 - 4(\sqrt{\widetilde{q}^2 + \widetilde{q}} - \widetilde{q})$, where $\beta_L \in [0, 2]$.

Therefore, SVI model does not contradict Lee's formula for call's and put's sides if the following inequality holds: $(1 + |\rho|b \leq 2)$ for a raw parameterization. During the work, this condition was always met, and almost never exceeded the value of 0.4. For example, you can see the results of Lee formula check for the data collected on 20.06.2023 in Table 1.

Expiration: Value:	2023.06.23 0.058	2023.06.30 0.051	2023.07.07 0.063	2023.07.14 0.075	2023.07.28 0.173	2023.09.15 0.142
Expiration: Value:	2023.12.15 0.123	2024.06.21 0.325	2024.09.20 0.324	2024.12.20 0.231	2025.01.17 0.089	2025.06.20 0.181

Table 1: Checking Lee formula for different expiration. Data on 20.06.2023.

Functions for checking the presence of static arbitrage conditions are implemented in the work.

9 Extrapolation implied volatility beyond known expirations

The main idea of extrapolating the values of the implied volatility beyond the available expiration is to find the invariant behavior of the parameters from one expiration to another. Parameters change in the raw parameterization and in the Jump Wing parameterization are shown on fig 14. Jump Wing parameters turned out to be much more stable when moving from one expiration to another, and we will use them to build regressions and extrapolate values for implied volatility.

Note that the presented parameters are also stable for data scraped on different days, it indicates that there is no necessity for often recalibration of the SVI model and we also can use previous calibrated parameters as initial parameters for new calibration. It is shown on fig. 15.

The following regression types were considered in work: linear, polynomial, exponential, logarithmic, reciprocal, the last two most accurately describe the dependence in the parameters changing. Note that the construction of regressions depends on the data sample, based on empirically observation on regressions for 57 companies, it was found that for ATM volatility it is worth throwing out the first 28% values, for a minimum of volatility 33%, for ATM skew 43 %, for right and left wing slopes 19%.

Figure 14: Changes in raw and Jump Wing parameters for Apple on 08.06.2023 for different expiration dates.

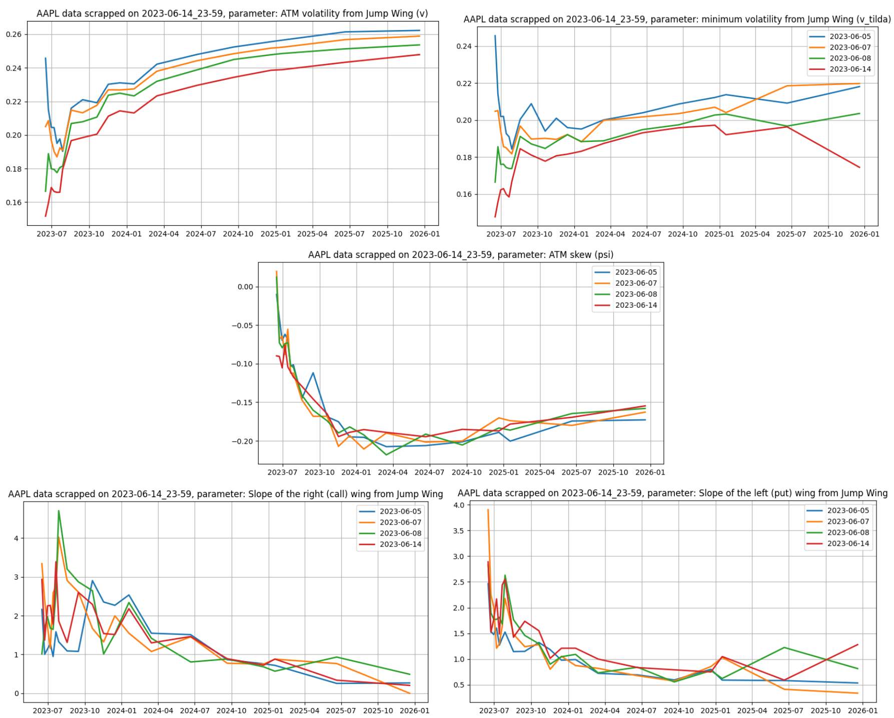

Figure 15: Changes in Jump Wing parameters for Apple for data scraped in different dates.

To check the quality of the regression and the obtained implied volatility values, the last expiration was used as a test. The sample values were taken without data about last expiration in order to build a regression on the data for each parameter and extrapolate it to the last expiration. Based on the obtained values of the extrapolated parameters, construct an SVI curve of implied volatility and compare it with the available bid-ask spread. The regression results for Apple with data on 23.06.2023 are shown on fig. 16, extrapolation results are presented on fig. 17.

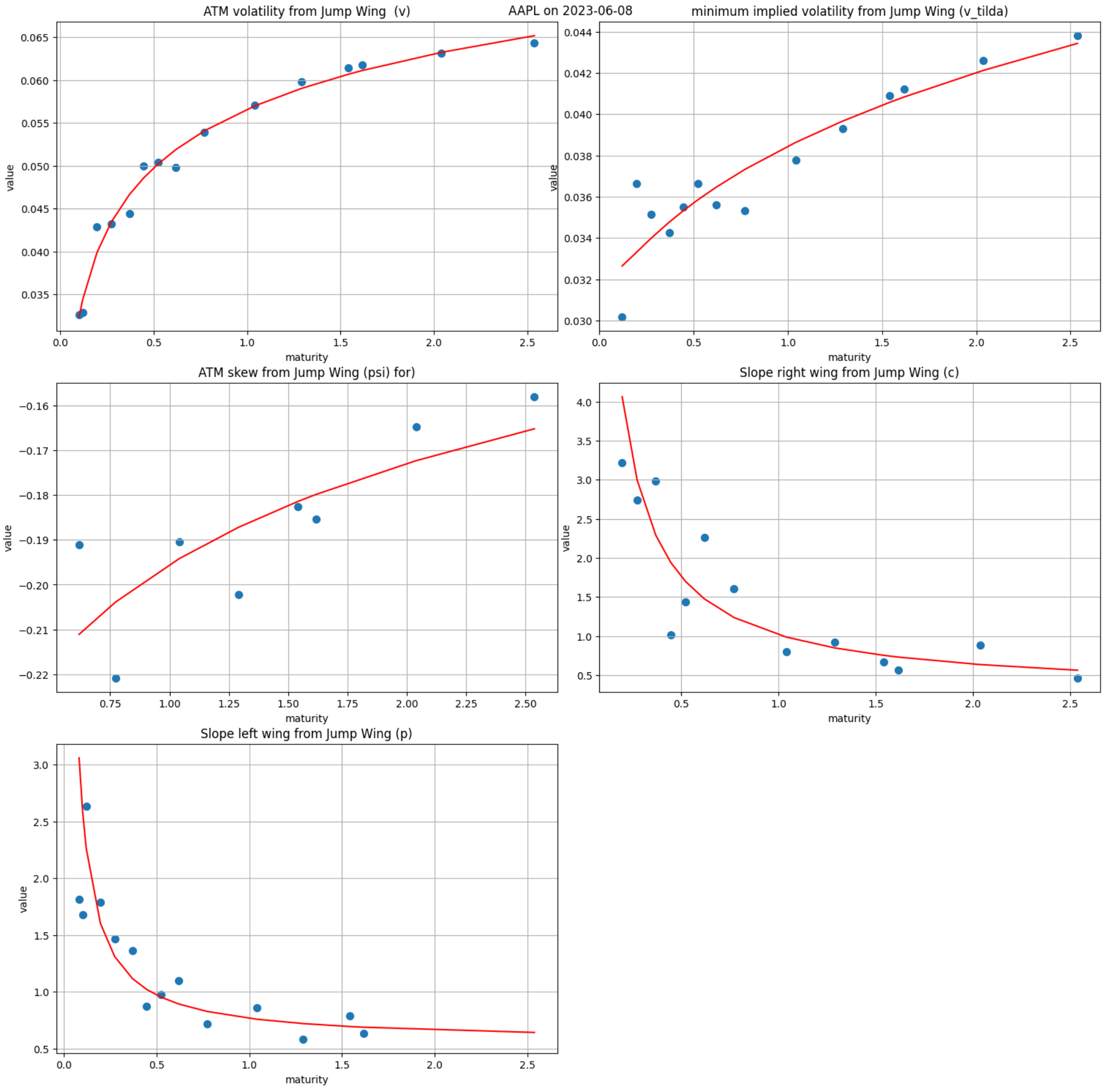

Figure 16: Regressions for Jump Wing parameters for Apple on 08.06.2023.

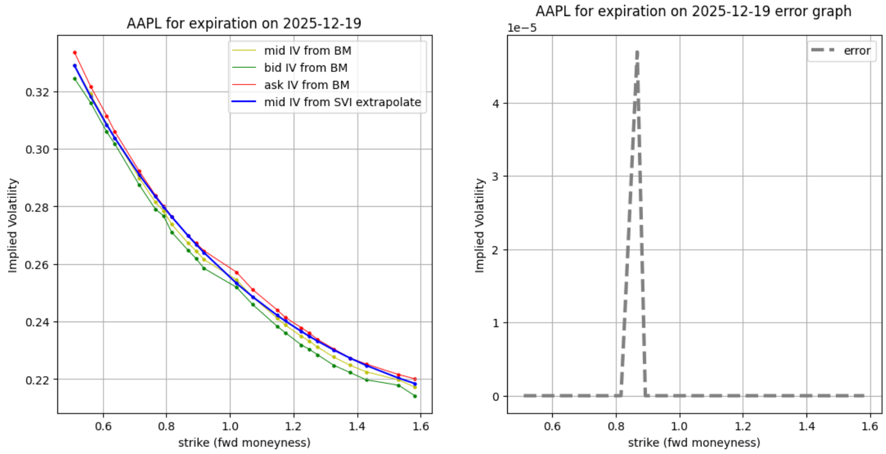

Figure 17: Extrapolation result for Jump Wing parameters for Apple with data on 08.06.2023. and expiration on 19.12.2025.

The result obtained from extrapolation the implied volatility smile for last expiration is quite good. And the obtained regressions can be used as a potential extrapolation and interpolation of the values for implied volatility surface, but such good estimates are not always observed.

The reason for this is that the first two parameters, namely ATM volatility and the minimum implied volatility, have good predictive power, unlike the remaining parameters. Regressions for these 3 parameters are built poorly, the resulting values are not stable. This can be solved by excluding a number of values from the sample. Changing the sample manually in some cases, you can achieve a good result and a high degree of prediction, but not always.

In general, it shows the instability of using this model for extrapolation of values and does not give any guarantees that this extrapolation will work correctly for the implied volatility surface for a given particular asset.

Figure 18 shows the static arbitrage free implied volatility surface. The black smiles on the graph are built using interpolation or extrapolation with the JUMP WING parameters. For example, the last expiration date for which data can be scraped on the market has a maturity equal to 2.5 years, using extrapolated parameters, it is possible to extend the implied volatility surface to any maturities, for example, to 5 years as on the figure.

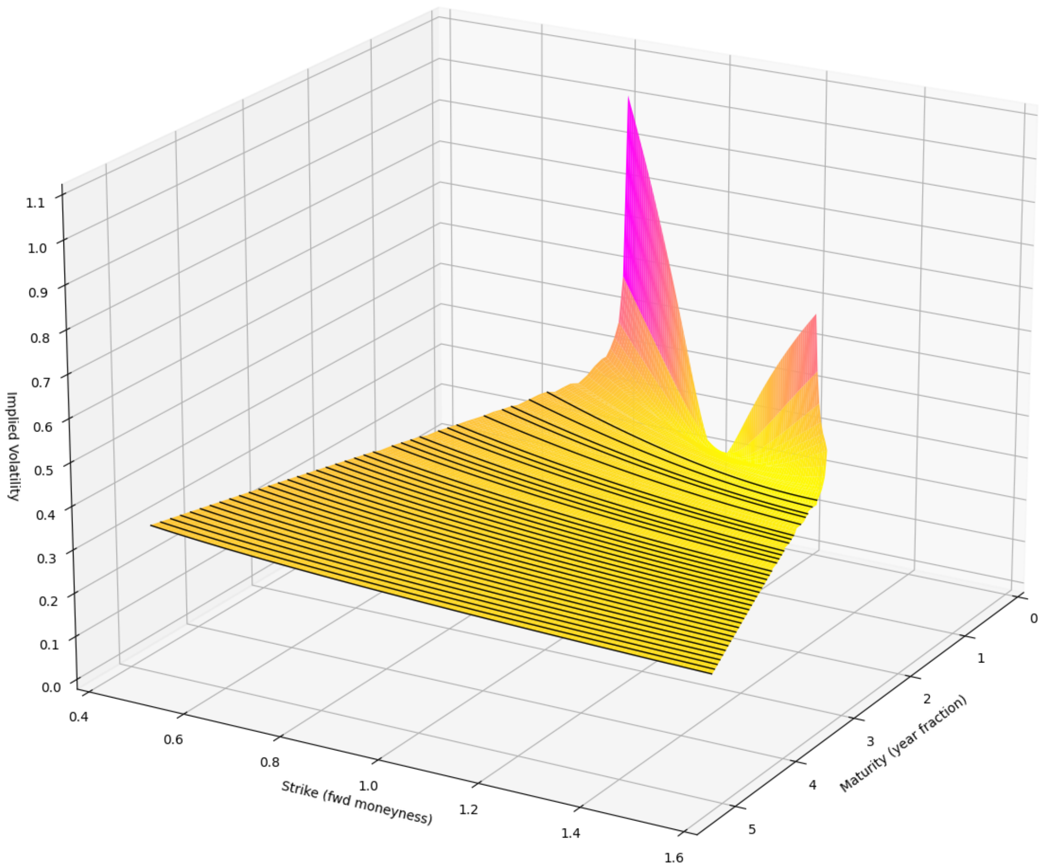

Figure 18: Implied volatility surface for Apple for Data on 08.06.2023 with extrapolation and interpolation.

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