Data scraping, computing IV, SVI model

Elezov Daniil

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Reference Forward

The forward of an equity at time T as seen from $t < T$ is given by:

$$F(t, T) = S(t)\frac{RF(t, T).DivF(t, T)}{DF(t, T)}$$

where:

DF(t,T) is the discount factor from t to T. If r_s is the instantaneous spot rate, **DF(t,T)** $= E_t^Q \left[\exp(-\int_t^T r_s ds)\right]$, Q being the risk neutral measure.

RF(t,T) is the repo factor from t to T. It correspond to the borrowing cost of the share. We can represent it as **RF(t,T)** $= \exp(-\int_t^T b_s ds)$

DivF(t,T) is expresssed as **DivF(t,T)** $= \exp(-\int_t^T q_s ds)$, with q the dividend yield

Dividend yield

$$Dividend\ yield = \frac{AverDiv \times FP}{SP}$$

where:
AverDiv is Average size of dividens for the last year,
FP is frequently of payments dividends,
SP is spot price.
Remark: Data scraped by Yahoo Finance

Discount Factor

Three-Month SOFR Futures with different expiration from yahoo finance.

$$r(t) = 100 - F(t)$$

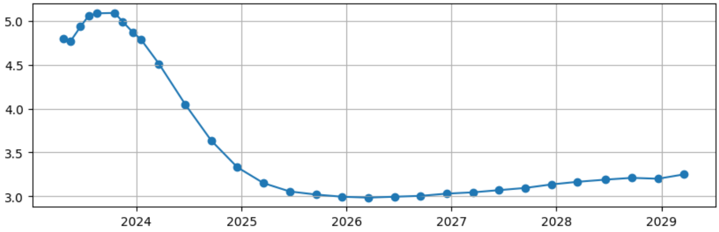
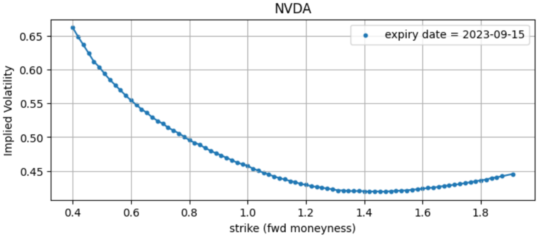


Figure: SOFR rate curve on 10:30 22, April 2023

For discount factor at time T used linear approximation based on SOFR curve

Some result



Implied Volatility for different listed companies on 22:06 21, April 2023

Some result

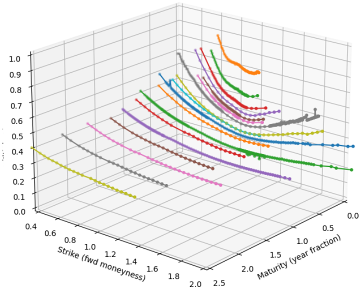


Figure: AMZN

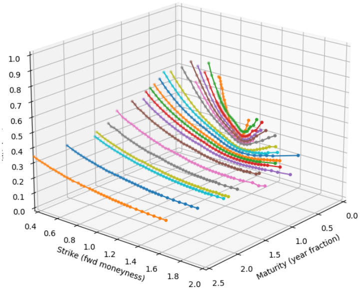


Figure: AAPL

Implied Volatility for different listed companies on 22:06 21, April 2023

SVI Model (Stochastic volatility inspired)

SVI model is some function that used to approximate the implied volatility curve. $x = ln(K/F_0)$ and $\omega = T\widetilde{\sigma}^2$ - full implied variance

Definition SVI model (raw parametrization)

$$\omega(x) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right)$$

a, b, ρ, m, σ - parameters, where$(a \in \mathbb{R}, b \geqslant 0, |\rho| < 1, m \in \mathbb{R}, \sigma > 0)$

Meaning of parameters:

1. a and m changing the curve vertically and horizontally
2. b sets the corner between the left and right asymptotes
   $\omega(x) \sim a - b(1-\rho)(x-m)$ at $x \to -\infty$
   $\omega(x) \sim a + b(1+\rho)(x-m)$ at $x \to +\infty$
3. σ sets convexity the apex
4. ρ sets turn curve

SVI Model (Stochastic volatility inspired)

Definition SVI model (natural parametrization)

$$\omega(x) = \Delta + \frac{\omega}{2}\left(1 + \zeta\rho(x-\mu) + sqrt(\zeta(x-\mu)+\rho)^2 + 1 - \rho^2\right)$$

$\Delta, \mu, \rho, \omega, \zeta$ - parameters, where $(\omega \geqslant 0, \mu \in \mathbb{R}, \Delta \in \mathbb{R}|\rho| < 1, \zeta > 0)$

Parameters in raw and natural parametrization are equivalent and there is following ratio:

$$(a, b, \rho, m, \sigma) = \left(\Delta + \frac{\omega}{2}(1-\rho)^2, \frac{\omega\zeta}{2}, \rho, \mu - \frac{\rho}{\zeta}, \frac{\sqrt{1-\rho^2}}{\zeta}\right)$$

Remark: that if we make several changes of variables and consider $\lim_{T\to\infty}$ then the Heston model turn out the special case of the SVI model (natural parametrization)

SVI Model (Stochastic volatility inspired)

SVI-Jump-Wings parameterizati

For a given time to expiry $t > 0$ and a parameter set $[v_t, \phi_t, p_t, c_t, \widetilde{v}_t]$ the SVI-JW parameters are defined from the raw SVI parameters as follows:

$$v_t = \frac{a + b(-\rho m + \sqrt{m^2 + \sigma^2})}{t} \; - \; \textit{gives the ATM variance}$$

$$\phi_t = \frac{1}{\sqrt{\omega_t}} \frac{b}{2} \left(-\frac{m}{\sqrt{m^2 + \sigma^2}} + \rho \right) \; - \; \textit{gives the ATM skew}$$

$$p_t = \frac{1}{\sqrt{\omega_t}} b(1 - \rho) \; - \; \textit{gives the slope of the left (put) wing}$$

$$c_t = \frac{1}{\sqrt{\omega_t}} b(1 + \rho) : - \textit{gives the slope of the right (call) wing}$$

$$\widetilde{v}_t = \frac{1}{t} \left(a + b\sigma \sqrt{1 - \rho^2} \right) : - \textit{is the minimum implied varience}$$

The raw SVI parameters express from SVI-JW parameters as follows:

$$a = \widetilde{v}_t T - b\sigma\sqrt{1-\rho^2} \qquad \rho = 1 - \frac{2p_t}{c_t+p_t} \qquad b = \frac{1}{2}\sqrt{v_t T}(c_t + p_t)$$

$$\beta = \rho - \frac{2\phi\sqrt{v_t T}}{b}$$

$$if\ |\beta| \leqslant 1, \beta \neq 0 \begin{cases} \alpha = sgn(\beta)\sqrt{\beta^{-2} - 1} \\ m = \frac{(v_t - \widetilde{v}_t)T}{b\left(\rho + sgn(\alpha)\sqrt{1+\alpha^2} - \alpha\sqrt{1-\rho^2}\right)} \\ \sigma = \alpha m \end{cases}$$

$$if\ \beta = 0 \begin{cases} m = 0 \\ \sigma = \frac{(v_t - \widetilde{v}_t)T}{b\left(1 - \sqrt{1-\rho^2}\right)} \end{cases}$$

If $|\beta| > 1$ then the curve isn't convex, that doesn't happen in practice

Calibration of parameters in the SVI model

Consider a raw parameterization for a fixed expiration moment T with parameters a, b, ρ, m, σ. Let $\omega_m(x_i), i = 1, 2, .., n$ - denote the total market variances. Let's introduce a function:

$$y(x) = \frac{x - m}{\sigma}$$

Then the total variance in SVI model is equal:

$$\omega(x) = a + dy(x) + cz(x)$$

$$z(x) = \sqrt{y(x)^2 + 1}, \;\; d = \rho b \sigma, \;\; c = b\sigma$$

$$f(a, b, \rho, m, \sigma) = \sum_{i=1}^{n} (\omega_i - \omega(x_i))^2 \to min$$

Internal optimization issue

Internal optimization issue

$$\sum_{i=1}^{n}(a + dy(x_i) + cz(x_i) - \omega_m(x_i))^2 \to min,\ where\ (a, d, c) \in \mathcal{D}$$

where $\mathcal{D}$ given the following condition:

$$0 \leqslant c \leqslant 2\sigma,$$
$$|d| \leqslant c,$$
$$|d| \leqslant 2\sigma - c,$$
$$0 \leqslant a \leqslant max\, \omega(x_i)$$

These conditions follow from the correctness of the model and the asymptotic behavior of the volatility curve.
Remark: The field $\mathcal{D}$ is compact, the function is convex, hence the minimum is reached

External optimization issue, find m, σ

External optimization issue

$$\sum_{i=1}^{n} (\omega(x_i | m, \sigma, a^*(m, \sigma), b^*(m, \sigma), \rho^*(m, \sigma)) - \omega_m(x_i))^2 \to min,$$

where $(m, \sigma) \in \mathcal{E}$ and a^*, b^*, σ^* are computed by internal optimization.

where $\mathcal{E}$ is the region:

$$min(x_i) \leqslant m \leqslant max(x_i)$$

$$\sigma_{min} \leqslant \sigma \leqslant \sigma_{max}$$

Boundary conditions for $\sigma_{min}, \sigma_{max}$ set directly, for example: $\sigma_{min} = 10^{-4}, \sigma_{max} = 10$

Remark: The function isn't convex and it's necessary to use some global optimization methods.

Asymptotics for large and small strikes (Lee's formula)

Consider an arbitrary non arbitrage market model in which the initial measure P is a martingale and $r = 0$ Let $\widetilde{\sigma}(x)$ - variance of call option with In-moneyness $x = ln(\frac{K}{S_0})$ and $\omega(x = T\widetilde{\sigma}^2(x))$

Lee's formula for right wing

$$\widetilde{p} = sup(p > 0 : E(S_T^{1+p} < \infty), \quad \beta_R = \lim_{x \to +\infty} sup \frac{\omega(x)}{x} \text{ Then:}$$

$$\beta_R = 2 - 4(\sqrt{\widetilde{p}^2 + \widetilde{p}} - \widetilde{p}), \text{ where } \beta_R \in [0, 2]$$

Lee's formula for left wing

$$\widetilde{q} = sup(q > 0 : E(S_T^{-q} < \infty), \quad \beta_L = \lim_{x \to -\infty} sup \frac{\omega(x)}{x} \text{ Then:}$$

$$\beta_L = 2 - 4(\sqrt{\widetilde{q}^2 + \widetilde{q}} - \widetilde{q}), \text{ where } \beta_L \in [0, 2]$$

Corollary: The SVI model does not contradict the Lee formula for both the left and right wings if $(1 + |\rho|b \leqslant 2)$ in raw parametrization

Definition of static arbitration

Definition

Call option price surface $C(T, K) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ doesn't allows static arbitrage if there is a martingale Xt, given on some probability space such as

$$C(T, K) = E(X_T - K)^+, \; for \; each \; T, K$$

The absence of static arbitrage indicates that the prices we received are consistent with some model that does not have dynamic arbitrage.

Definition

Volatility surface $\sigma(T, K) : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ doesn't allows static arbitrage if there is option surface $C(T, K)$, that don't allows static arbitrage such as:

$$C(T, K) = C_B(T, K, \sigma(T, K)), \; for \; each \; T, K$$

where $C_B(T, K, \sigma(T, K))$ are price of options in Black-Sholes or Black models

Conditions for no static arbitrage

Conditions for option price surface:

Let function $C(T, K)$ is define to R^2_+, such as $C(T, \cdot) \in C([0,\infty)) \cap C^1((0, \infty))$*for each* $T > 0$ and carry out following conditions:

1. $C(T, K)$ doesn't decrease by T
2. $C(T, K)$ is convex by K
3. $\lim \lim\limits_{K \to \infty} C(T, K) = 0$
4. $C(T, 0) = s$ where $s > 0$ - some constant
5. $C(0, K) = (s - K)^+$

Then there is a martingale $X_t \geqslant 0$ that $X_0 = s$ and $C(T, K) = E(X_T - K)^+$

Interpretation of conditions

Condition (1) is the absence of **calendar arbitrage**. If it is not done:

$$C(T_1, K) > C(T_2, K), \; \textit{for some } T_1 < T_2 \textit{ and } K > 0$$

Then we can get arbitrage by selling the option call $(T1; K)$ and buying the option call $(T2; K)$

Condition (2) is the absence of **butterflies arbitrage**. If it is not done:

$$C(T, K_1) + C(T, K_3 < 2C(T, K_2)), \; \textit{for some } K_1 < K_2 < K_3 \textit{ and } T > 0$$

We can buy one call option (T, K_1) and one (T, K_3) and sell two call option (T, K_2)

Condition (3) If lim $\lim_{K \to \infty} C(T, K) > 0$, we can sell option with strike$= K$, which will execute with probability tending to 0 (It's non arbitrage, but anyway is too good, that to be a truth)

Conditions (4, 5) come from the definition of the pricing options

Conditions for Volatility Surface

Let current futures price is f, then:

$x = ln(K/s) \qquad \theta = \sigma\sqrt{(}T) \qquad d_1 = -\frac{x}{\theta} + \frac{theta}{2} \qquad d_2 = d_1 - \theta$

Then the Black's formula is:

$$C_B(T, K, \sigma) = C_B(x, \theta) = f\Phi(d_1) - fe^x\Phi(d_2)$$

In what follow, $\sigma(T, K)$ we will define via functions $\theta(T, x)$

Let function $\theta(T,x) \in C^{1,2}((0,\infty)\times\mathbb{R})$ for each $T>0, x\in\mathbb{R}$ is satisfy following conditions:

1. $\theta > 0$
2. $\theta'_T \geqslant 0$
3. $g(T,x) := \left(1-\frac{x}{\theta}\theta'_x\right)^2 - \frac{\theta^2}{4}(\theta'_x)^2 + \theta\theta''_{xx} \geqslant 0$
4. $\lim\limits_{x\to\infty} d_1(x,\theta(T,x)) = -\infty$
5. $\lim\limits_{T\to 0} \theta(T,x) = 0$

Then follow price option surface isn't allow static arbitrage:

$$C(T,K) = \begin{cases} C_B(x,\theta(T,x))|_{x=ln\frac{K}{f}}, T>0, K>0 \\ (f-K)^+, T=0, K\geqslant 0 \\ f, T\geqslant 0, K=0 \end{cases}$$

Checking conditions in the SVI model

The SVI model in raw parameterization (one cross section of the volatility for fixed $T > 0$):

$$\theta^2(x) = a + b\left(\rho(x-m) + \sqrt{(x-m)^2 + \sigma^2}\right)$$

When calibration the parameters of the model SVI, conditions 1,4,5 don't cause problems, because these properties have observable option prices.

Condition 2 (lack of calendar arbitration) and **condition 3** (lack of Butterfly Arbitration) needs to be checked for the calibrated parametres for model, some of the combinations of parameters may not work.

Article: Gatheral, Jacquier, 'Arbitrage-free SVI volatility surfaces', 2013