Explanations

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1 Lab2

Explanations on Q1:

For this exercise we are going to study the QR factorization of a n*n (3*3) given Matrix and I decided to Perform QR decomposition of matrix A using Householder reflection. In this exercises we should make sure that the diagonal entries of triangular matrix R are non negative to have a nice matrix Q. and by this we would be able to make sure that our factorization unique

Explanations on Q2: For this exercise, we perform a function that creates random orthogonal matrices with $\sigma = 0.1$ and $\mu = 5$. Again for having a unique factorization and having Q, a nice matrix we should make sure that the diagonal entries of triangular Matrix R must be non-negative.

Explanations on Q3:

Next we used a function that performs QR factorization and we reduce it to Hessenberg form by Householder Reflection and also we should write a function that turn a real symmetric matrix into a tridiagonal Matrix.

Explanations on Q4: In this exercise we write a code that performs QR factorization on a real symmetric tridiagonal matrix and then we apply 3 shifts:
1. No shift or basic QR
2. Rayleigh Quotient shift
3. Wilkinson shift.

Explanations on Q5:

In this question we write a function that produces a symmetric random matrix with a given set of eigenvalues as $A = Q^T \Lambda Q$ where Q is a random orthogonal matrix. again we implement the QR algorithm on this matrix and we choose some special eigenvalues, the largest one, the middle-size and the smallest one in the sense of absolute values. then I plot the size of the smallest, middle-size and largest eigenvalue for some iteration and then the errors of these chosen (special) eigenvalues and the off-diagonal entries for some iterations .

1.1 Data analysis of magnitudes (QR basic):

1.1.1 The smallest eigenvalue:

as we iterate the QR algorithm at every iteration we compute the absolute value of the smallest one and we plot its size against the iteration , and we can see that as we go further, the size of the smallest eigenvalue starts from positive number very close to 10 to the power zero (not exact) and decreases dramatically until the iteration (not exact) 27 and then it becomes constant say,10^{-33}. I should add this point that 50 iteration is considered.

1.1.2 The Middle-sized eigenvalues:

for these middle-size eigenvalues at the beginning the size of this eigenvalue starts from a negative small number say $10^{-0.75}$ and then decreases until the very beginning iterations and then this size increases dramatically until it arrives the number $10^{0.50}$ and then it decreases slowly until it becomes 10^0 from the iteration 20 and then it stays constant at the amount 10^0 until the iteration 50.

1.1.3 The largest eigenvalue:

The more iteration goes further the size of the largest eigenvalues decreases starting from say, 10^1 at the iteration 0 the very first step and arriving to 10^{-11} and It illustrate a linear relationship between the logarithmic scale of the magnitude of the largest eigenvalues against the number of iterations.

1.2 Data analysis of errors (QR basic):

The next graphs shows the errors or the absolute value of difference of the smallest, middle and largest eigenvalues with the off-diagonal entries that are close in terms of position and are on the top of those special eigenvalues and in 3 versions (basic QR, Wilkinson shift, Rayleigh Quotient). The graph illustrates that for the Basic QR factorization with no shift, for the smallest eigenvalues it fluctuates with small ups and downs and then linearly decreases and then after the iteration say, 27 it stays constant.
On the other hand for the middle size eigenvalues it fluctuate very small and then it decreases linearly but with smaller slope comparing to the smallest eigenvalue. for the smallest eigenvalue it starts from 0 but for the middle-size eigenvalue starts from say, 1.5. Totally, for the smallest eigenvalue the error is much much smaller than the middle size eigenvalues.

1.3 Data analysis of magnitudes (Wilkinson shift):

1.3.1 The smallest eigenvalue:

The graph illustrates a fluctuating graph starting from $(1.8, 1.7)$ and at some iterations it goes ups and downs but the ups and downs don't have much difference except at iteration 20, and then after iteration say, 23 it starts to fluctuate a little until the size becomes constant.

1.3.2 The middle-size eigenvalue:

The graph again fluctuates but with more difference and sharp slopes, and it starts from $(1.50, 1.55)$ and it attains its maximum at iteration say, 6 and at iteration 17 and 18 it attains its minimum. after iteration 19 it stays constant say, 1.47 until the 50th iteration.

1.3.3 The largest eigenvalue:

for this case the graph decrease not too fast, not too slowly until it arrive at -17 and then it decreases too fast and it attains its minimum at -33

1.4 Data analysis of errors (Wilkinson shift):

The next graphs shows the errors or the absolute value of difference of the smallest, middle and largest eigenvalues with the off-diagonal entries that are close in terms of position and are on the top of those special eigenvalues and in 3 versions (basic QR, Wilkinson shift, Rayleigh Quotient). The graph illustrates that for the Basic QR factorization with Wilkinson shift, for this shift, both the smallest and the middle-size eigenvalues graphs, are fluctuating for the smallest eigenvalues, the peak is attained at iteration 7 , in the interval $(0, 1)$ and for the middle-size eigenvalue the peak is attained at iteration 12, equal to 1. both graphs stay constant after some iteration, for the smallest one at iteration 29 and after that it attains the error value of $(-6, -5)$ and for the middle size iteration, at iteration 19 it stays constant at $(-4, -3)$. both graphs are decreasing although they have lots of fluctuations.

1.5 Data analysis of magnitudes (Rayleigh shift):

1.5.1 The smallest eigenvalue:

As the iteration goes forward, the absolute value of this special eigenvalue becomes bigger, after iteration say 7, it starts to decrease, fluctuate, and then increases and after some fluctuation it stay constant.

1.5.2 The middle-size eigenvalue:

This graphs is also fluctuating between (1.39, 1.51) and after iteration 14 it stays constant attaining 1.41 approximately.

1.5.3 The largest eigenvalue:

The graph illustrates that the absolute value of largest eigenvalue starts too close to 0 , the positive side and then it decreases slowly, fluctuating between $(1, -9)$ and then at iteration 23 it decreases dramatically and it arrive to say, -33.

1.6 Data analysis of errors (Rayleigh shift):

The next graphs shows the errors or the absolute value of difference of the smallest, middle and largest eigenvalues with the off-diagonal entries that are close in terms of position and are on the top of those special eigenvalues and in 3 versions (basic QR, Wilkinson shift, Rayleigh Quotient). The graph illustrates that for the Basic QR factorization with Rayleigh shift, the error for these special cases are shown and as we look at the graph we can see that at first, at step (iteration 0), the errors for the largest one is less than the smallest one and the smallest's error is less than the middle size one. the smallest one fluctuate not too sharp, and at iteration 24 it stays constant and it shows that for this case the error attains its smallest amount. but for the middle size the error finally becomes constant but it is mire that the two other case. On the other hand for the largest eigenvalue, at first it has the smallest error, then decreases, and it has a peak at $(0, 2)$ and after the iteration 8 it stays constant but totally it behaves better.

Explanations on Q6:
In this exercise I did the same procedure as exercise 5 with some interesting eigenvalues, such as : negative eigenvalue, multiple eigenvalues, extremely small eigenvalues and Big eigenvalues. and again I did this for each special shift (No shift, Wilkinson shift, Rayleigh Quotient shift)

1.7 Data analysis of negative eigenvalues-QR basic (The smallest size):

This graph shows the size of the smallest eigenvalue decreases approximately linearly, but in some parts of that.

1.8 Data analysis of negative eigenvalues-QR basic (The middle-size):

Again, this graphs starts from $(-2, 0)$ decreases, and fluctuating between (-3,0) and then decreases dramatically until the 50th step.

1.9 Data analysis of negative eigenvalues-QR basic (The largest one):

As the graphs shows, it starts from (0,3) and then decreases slowly until the 3rd iteration and after that it decreases linearly until the 27th step and then it becomes constant.

1.10 Data analysis of negative eigenvalues-QR basic

: The graph shows us that the middle-size eigenvalue error is larger than the largest eigenvalue at all iterations. Both graphs decreases linearly and the slope of the middle-size is less than the slope of the largest one. for the largest one after iteration 28 it stays constant at (-17,-17.5).

1.11 Data analysis of negative eigenvalues-Wilkinson shift (The smallest size):

As it shows the graph has a increasing behaviour until the 8th iteration and then after that it stays constant but it has a small fluctuation and after that again it stays constant.

1.12 Data analysis of negative eigenvalues-Wilkinson shift (The middle-size):

This graph decreases linearly until the 2nd iteration then it increases and then after iteration 4 it stays constant at (1.38,1.40).

1.13 Data analysis of negative eigenvalues-Wilkinson shift (The largest-size):

This graph starts from positive numbers and then decreases slowly along with fluctuation. After a while or some iteration, it fluctuate with high differences and then after 35th iteration it decreases dramatically.

1.14 Data analysis of negative eigenvalues-Wilkinson shift:

The errors of the largest, the middle size eigenvalues are shown in the graph. Totally the middle-size eigenvalue has the smaller size that the largest one. the largest one is fluctuating between $(-4, 1)$ and then it goes to -7.5 approximately and then after a little increase it again decreases and goes to the minimum error and then stays constant.

1.15 Data analysis of negative eigenvalues-Rayleigh shift (The smallest size):

From this graph it can be seen that the graph is again fluctuating. the smallest or the minimum of the graph is attained when the iteration is started. Moreover the graph is totally decreasing as the iteration goes further. The Maximum of this graph is when the graph stays constant.

1.16 Data analysis of negative eigenvalues-Rayleigh shift (The middle-size):

The graph is decreasing and increasing linearly and in some iteration it stays constant. The peak occurs at iteration say,13 at the value of 1.49 and then dramatically decreases and after iteration 14 it stays constant until the end.

1.17 Data analysis of negative eigenvalues-Rayleigh shift (The largest size):

This graph illustrates that as the iterates goes further the size of off-diagonal close to the position of the largest eigenvalue decreases starting from very close to zero and finally it becomes -22 at the final steps. The graph is fluctuating but totally the graph is decreasing.

1.18 Data analysis of negative eigenvalues-Rayleigh shift:

In this graph the errors of off-diagonal entries and the 3 special eigenvalues are computed and shown. As the graph illustrates, at first step, the error for the largest eigenvalues and the off-diagonal entries are much bigger than the other two eigenvalues. for the smallest eigenvalue the error is always less than the two other cases until the 9th step. for the middle size eigenvalue, the error is constant until the 10th step or iteration and then increases and after the 11th step it again stays constant.The largest eigenvalue is the smallest value all over the iteration specialy after the 23th step.

1.19 Data analysis of multiple eigenvalues-Basic QR (The smallest size):

The graph which shows the size of the off-diagonals close to the smallest eigenvalues starts from 0 and then decreases and attains its minimum value which is -29 and then increases and fluctuating between zero and -6 and at the final steps it decreases linearly and it arrives at -11.

1.20 Data analysis of multiple eigenvalues-Basic QR (Middle-size):

This graph has an interesting shape, totally it decreases linearly with a gentle slope but it has two giant jumps at say, iteration 32 and 41. at the end steps it arrives at a very low value which is the minimum of the graph.

1.21 Data analysis of multiple eigenvalues-Basic QR (The largest eigenvalue):

This graph shows us a mix increasing decreasing behaviour. it starts from a very low value , fluctuate, and then after 25th steps it increases dramatically and it attains its maximum at > -17.5 and then it decreases linearly again with a sharp slope.

1.22 Data analysis of multiple eigenvalues-Basic QR:

In this case, both the smallest and middle-size cases have decreasing behaviour with sharp fluctuation for the middle-size case and for the smallest eigenvalues case it has some jumps and a little increases but finally it decreases totally and is larger that the middle-size case. For the largest eigenvalue case, it starts from a low value and then increases with some jumps and it arrives at -15.

1.23 Data analysis of multiple eigenvalues-Wilkinson (The smallest size):

This graph also shows a total increasing behaviour, it differs between (1.55,1.80) at the 13th step it stays constant.

1.24 Data analysis of multiple eigenvalues-Wilkinson (The Middle size):

This graph is increasing and decreasing or in other words it fluctuate linearly between 1 and 1.3. after the 6th step it decreases linearly and then increases and attains its maximum or peak. after that continues to decrease and after the 13th step it stays constant.

1.25 Data analysis of multiple eigenvalues-Wilkinson (The Largest-size):

This graph is a nice one. I doubt it is true. but if it is, then it's clear that this graph is always constant. which is interesting!

1.26 Data analysis of multiple eigenvalues-Wilkinson shift:

To compare the error for the middle-size and the smallest-size, we should note that the middle-size decreases until the 6th step and after that it again goes back and increase and arrives to its first value. The smallest eigenvalue , on the other hand, after some fluctuations, decreases fast and it attains its minimum value and stays constant.

1.27 Data analysis of extremely small eigenvalue eigenvalues-Basic QR (The smallest size):

for this graph, the size of the off-diagonal matrices starts at a very low value say, -4.5 and as the iteration goes further this value increases and at 6th step, it stays constant at the value -1.0.

1.28 Data analysis of extremely small eigenvalue eigenvalues-Basic QR (The middle-size):

This graph is totally decreasing although it has a giant jump. it has a minimum values occurs at 5th step. after this step it increases and then gradually it stays constant until the last step.

1.29 Data analysis of extremely small eigenvalue eigenvalues-Basic QR (The Largest-size):

This graph is of the form $a^x anda < 1$. it is clear that it has concave up and so it is a decreasing graph. after the 25th step it starts to becomes constant until the end of the step.

1.30 Data analysis of extremely small eigenvalue eigenvalues-Basic QR

Both graphs are linearly decreasing. but as it is clear seen, the smallest eigenvalue case is decreasing with a sharper slope and totally its much less than the middle-size eigenvalue case. The smallest case after its giant decrease, fluctuate a little bit and then it stays constant after the 30th step until the last step.

1.31 Data analysis of extremely small eigenvalue eigenvalues-Wilkinson (The smallest size):

Again, it is a nice graph in my idea. it is a constant graph everywhere.

1.32 Data analysis of extremely small eigenvalue eigenvalues-Wilkinson (The Middle size):

This graph has 2 fluctuation. every decrease or increase is linearly, and after the second fluctuation it stays constant.

1.33 Data analysis of extremely small eigenvalue eigenvalues-Wilkinson (The Largest size):

This graph is again fluctuating, the increase and decrease is linearly. The total graph is increasing and it attains its maximum at the 8th step.

1.34 Data analysis of extremely small eigenvalue eigenvalues-Wilkinson:

This graph has 2 fluctuation. every decrease or increase is linearly, and after the second fluctuation it stays constant.at the 3rd iteration it attains its minimum value. And also at the 1st step it also attains its maximum.

1.35 Data analysis of Big eigenvalues-Basic QR (The smallest size):

This graph is decreasing dramatically and then it stays constant.

1.36 Data analysis of Big eigenvalue-Basic QR (The middle-size):

This graph is decreasing and it attains its minimum value and then it increases and attains its maximum value and then it rapidly decreases and after 21st step it stays constant.

1.37 Data analysis of Big eigenvalue-Basic QR (The Largest-size):

This graph shows us that the size of off-diagonal entries is increasing and after the 15th step it stays constant.

1.38 Data analysis of Big eigenvalue-Basic QR :

The two graphs are shown in the picture. the smallest one is totally decreasing with a sharper slope, and then it stays constant. The middle size graph is also decreasing but with a gentle slope.

1.39 Data analysis of Big eigenvalues-Wilkinson (The smallest size):

The graph is totally decreasing, it has many sharp fluctuation but at the end it stays constant.

1.40 Data analysis of Big eigenvalues-Wilkinson (The middle size):

This graph is also fluctuating it has a giant hole and after that it increases rapidly. after some fluctuations, it decreases dramatically and it stays constant.

1.41 Data analysis of Big eigenvalues-Wilkinson (The Largest size):

This graph is totally increasing, and then stays constant. after a small fluctuation it again stay constant.

1.42 Data analysis of Big eigenvalue-Wilkinson shift :

The graph of both cases (The smallest case and the middle-size case) are fluctuating rapidly. both graphs after a specific step stays constant. Totally, the middle-size case error is less than the other one.

Explanations on Q7 In this question we should remove the last row and column of matrix Am and then imply the QR factorization algorithm on the cropped matrix once the off-diagonal entry $\|A_{n,n-1}| < 10^{-10}$. I plotted the errors against the iteration for 3 versions : The smallest, middle-size, The largest eigenvalue in 2 cases: Basic QR, Wilkinson shift.

1.43 Data analysis-Basic QR (The smallest size):

In this case, The error is decreasing and then after some iteration it stays constant.

1.44 Data analysis -Basic QR (The Middle size):

For this case, The error graphs is totally increasing, but with some sharp fluctuation. after some steps, say, 22, it stays constant.

1.45 Data analysis-Basic QR (The Largest size):

In this case the error is increasing rapidly, after the 15th steps it stays constant.

1.46 Data analysis-Basic QR:

Both errors are decreasing, the smallest one and the middle one, they decreases linearly, totally the error of the smallest one is less than the middle-size one.

1.47 Data analysis-Wilkinson (The smallest size):

This error graph is totally decreasing, it has some sharp fluctuation, finally, it stays constant after the 22nd step.

1.48 Data analysis-Wilkinson (The middle-size):

This error graph is fluctuating sharply it attains its minimum when the iteration is 12. after 17th step it stays constant.

1.49 Data analysis-Wilkinson (The largest-size):

This graph is increasing until the 7th step and then the error stays constant.

1.50 Data analysis-Wilkinson QR:

Both cases (The smallest one and The middle-size cases) starts from positive side, fluctuating rapidly, and after some steps, 18th for the middle-size step and 22nd step for the smallest case stays constant.