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Crypto-assets portfolio optimization under the omega measure

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ABSTRACT

Crypto-currencies, or crypto-assets, represent a new class of investment assets. The traditional portfolio analysis approach of Markowitz is not appropriate for use with portfolios containing crypto-assets, as the model requires that the investor have a quadratic utility function or that the returns be normally distributed, which isn't the case for crypto-assets. We develop a portfolio optimization model based on the Omega measure which is more comprehensive than the Markowitz model, and apply this to four crypto-asset investment portfolios by means of a numerical application. The results indicate that these portfolios should favor traditional market assets over crypto-assets. In the case of portfolios formed only by crypto-assets, there is no clear preference in favor of any crypto-asset in particular.

Introduction

Nakamoto (2008) proposed the first electronic currency, denominated Bitcoin, as an electronic means of payment between individuals at low cost and without the need for the intermediation of financial institutions or governments. Unlike other electronic forms of payment such as debit and credit cards or internet banking, Bitcoin represents an electronic bearer asset, or equivalently, the digital equivalent of cash. Bitcoin transactions are written to a distributed decentralized database that relies on encryption algorithms, peer-to-peer communication and block chaining, which assures the immutability, transparency, and authenticity of any transaction in the system. Bitcoin was launched in 2009 and has a current market capitalization of over $110 billion (Coinmarketcap, 2018). After Bitcoin was established, other crypto-currencies known as altcoins were created, each with distinct characteristics from the original Bitcoin algorithm.

Although crypto-currencies were initially restricted to a group of technology enthusiasts, over the years their use has spread and expanded worldwide to include not only a means of payment but also a store of value. This is due to the fact that this class of assets has particular characteristics that make it uniquely suited for trading and investment purposes. While stock shares and other traditional securities are generally restricted to a particular country and operate only during working hours, crypto-

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currencies are continuously traded and easily accessible to anyone independent of their current location. Crypto-currencies can also be freely negotiated in very small values, which is not the case of many market investment funds which are not open to the public or have high minimum investment requirements. Therefore, given that crypto-currencies represent a new class of assets within the range of alternatives for the investor, it is relevant to assess the appropriate portfolio optimization methods that take into account their unique characteristics in order to determine the proportion that these assets should have in any investment portfolio.

The traditional portfolio analysis approach of Markowitz (1952) assumes that the investor has a quadratic utility function or that the returns of the assets in an investment portfolio are normally distributed, which is not the case for crypto-currencies. Thus, in this article, we propose a performance optimization model for portfolios of crypto-assets, which is more comprehensive than the Markowitz model. For this, we adopt the Omega (Ω) performance measure of Keating and Shadwick (2002) which considers all moments of the distribution, describing more appropriately the true distribution of asset returns.

This approach first determines the optimum composition of the asset portfolio by maximizing the Omega measure (Ω), which is defined as the ratio of weighted average gains to weighted average losses of portfolio returns distribution. This measure requires that a minimum return level (limit "L") desired by investors be predefined, which is the limit between the gains area and the losses area of the distribution. Finally, the optimal capital allocation of the portfolio, which also includes a risk-free asset, is done using the Sharpe-Omega ratio (Kazemi, Schneeweis, & Gupta, 2004), derived from the Omega measure, where risk and return are modeled through an investor's utility function.

To illustrate this approach, four investment portfolios were analyzed: Bitcoin × S&P500, Bitcoin × NASDAQ Composite, Bitcoin × Ethereum and Bitcoin × Ripple. The first two portfolios combine Bitcoin with two of the most traditional indexes in the US economy, while the latter two portfolios only combine crypto-assets with each other. The results indicate that one should hold a higher proportion of traditional economic assets than crypto-assets in the portfolio, which seems to be counterintuitive, considering the significant return that crypto-currencies have had in recent years. However, it can be verified that this result is coherent due to the greater volatility of crypto-assets. In the case of portfolios formed only by crypto-assets, the results suggest that there is a preference for the crypto-asset with better Sharpe-Omega ratio.

This paper is organized as follows. After this introduction, a review of the relevant crypto-currencies and portfolio optimization literature is presented. Next, the proposed portfolio optimization model is developed and in the following sections, a numerical application is provided to test the model and the results are presented and discussed. Finally, we conclude.

Literature review

Crypto-currencies

Crypto-currencies are virtual coinage systems that replicate the functioning of traditional currencies and allow users to make virtual payments for the purchase of goods and services without relying on third parties, regulation or control from a central monetary authority (Farell, 2015). The technology underlying this electronic payment system is known as blockchain, which records all the transactions carried out in the system through a cryptographic sequence of linked blocks stored in a public ledger that is visible to all users of the system (Antonopoulos, 2017). One of the main features of crypto-currencies is that this technology completely removes the need for a central authority or a trusted third party since the validation of the transactions is performed automatically and autonomously by a consensus algorithm (Bashir, 2018). In addition, the issuance of new stock of bitcoin is accomplished by a laborious cryptographic algorithm known as proof of work (Back, 2002), which contributes to guarantee the reliability and security of the system. These characteristics allowed Nakamoto (2008) to be the first to solve the problem of double spending that hitherto prevented the creation of a digital currency.

In January 2009, the first block of the system, known as the "genesis block", was officially created, and since then, the number of users of the system has been steadily increasing. Since the inception of Bitcoin, many other crypto-currencies known as altcoins loosely based on the Bitcoin concept have been created, each with their own distinct characteristics. More recently, the use of crypto-currencies has expanded to include its use as a store of value, in addition to a means of payment. Thus, crypto-currencies have also become a new class of investment asset, competing with any other traditional investment asset, such as stocks, fixed income securities and commodities (Baur, Hong, & Lee, 2017).

Portfolio assessment with crypto-assets

The literature on the optimization of investment portfolios that carry crypto-assets is scarce. Nonetheless, some works can be highlighted, such as that of Carpenter (2016), who analyzed a portfolio of Bitcoin as well as several indexes of the American market representing stocks, commodities, treasury bonds, real estate and foreign stocks. To optimize this portfolio, the Markowitz model was used, and the author showed that the inclusion of this crypto-asset improves portfolio performance. This occurs even when we penalize the expected average return of Bitcoin by a factor, considering the assumption that the high gains obtained in the past (January 2012 to May 2016) may not be repeated in the future.

Chan, Chu, Nadarajah, and Osterrieder (2017) analyzed the statistical properties of the prices of the main crypto-currencies between June 2014 and February 2017, and with this information, they modeled continuous probability distributions in an attempt to predict future price behavior. On the other hand, Trimborn, Li, and Härdle (2018) propose a methodology based on Markowitz (1952) to which they add a liquidity constraint. This is done by limiting the amount invested in crypto-currencies in order to be able to quickly adjust the position of the portfolio, given that the volume of trading in crypto-currencies for the American investor is much smaller compared to indexes such as S&P500. In addition, Klabbers (2017) assessed portfolios with US, European and Asian market indexes using historical data between 2010 and 2016, and through Monte Carlo simulation generated new distributions to apply Markowitz. They concluded that the inclusion of Bitcoin in portfolios has improved returns but does not necessarily reduce the risk.

Andrianto and Diputra (2017) assessed the effect of including crypto-currencies on foreign exchange (Forex) portfolios with commodities, stocks and Exchange Traded Funds (ETF). Using the Markowitz approach they found that in all three cases the inclusion of crypto-currencies improved the performance of the portfolio. Similarly, Brauneis and Mestel (2019) analyzed portfolios of crypto-currencies under the mean-variance theory, including also the CRIX index that measures various crypto-currencies, evidencing the risk reduction that occurs when several crypto-currencies are combined. Chuen, Guo, and Wang (2017) also explore the potential of cryptocurrencies as investment assets by analyzing the risk and return characteristic of a cryptocurrency portfolio. They equally conclude that although the correlation is low, adding cryptocurrencies to a traditional asset portfolio improves its risk-return characteristics.

On the other hand, Borri (2018) estimates the Conditional Value at Risk (CVaR) measure for several crypto-currencies, since this measure better captures the potential losses due to the high excess of kurtosis and skewness in the historical returns of its prices. He also notes that such returns are strongly correlated with each other but less with traditional economy asset returns. Thus, including crypto-assets in investor portfolios can provide more attractive returns and also act as a hedging mechanism. Stensås, Nygaard, Kyaw, and Treepongkaruna (2019) also discuss the role of Bitcoin in the financial markets and analyze whether this crypto-asset is held by investors for hedging, diversification purposes or as a safe haven tool. They conclude that the use depends on whether investors are from developing or developed countries. In a similar line Guesmi, Saadi, Abid, and Ftiti (2018) analyze the effectiveness of using Bitcoin as a portfolio diversification tool and find that it is an effective strategy for risk reduction for some type of portfolios. Both capitalization weighted and equally weighted market portfolios of altcoins are studied by Vidal-Tomás, Ibáñez, and Farinós (2019), who note that there has been an increasing interest of investors in cryptocurrencies.

None of these studies consider the fact that crypto-currency price returns have non-Gaussian distributions. In that sense, Chu, Nadarajah, and Chan (2015), using daily data from September 13, 2011, to May 8, 2014, analyzed Bitcoin prices using the 15 most common distributions in finance and concluded that the generalized hyperbolic distribution provides the best fit. Interestingly, the normal distribution provided the worst fit.

Thus, we propose the use of the Omega (Ω) performance measure (Keating & Shadwick, 2002) which considers all the high-order moments of the distribution for the optimization of crypto-assets portfolios. In addition, this work uses more recent price series that capture the effect of the increase in value that occurred in late 2017, and the significant price decline that followed in the first months of 2018. In addition, our portfolio model includes the capital allocation line based on the Sharpe-Omega ratio concept, which is novel. In this case, a risk-free asset and the investor's utility function are included to capture risk-return preferences of the investor in order to determine the optimal portfolio. Thus, this paper contributes to the literature by proposing an original model that better reflects the reality of investment decision makers by offering a more complete approach to optimal capital allocation theory in portfolios that include crypto-assets.

Model

Markowitz's traditional model for portfolio optimization assumes that the mean and variance are sufficient to define a distribution of returns. This is valid if the return are normally distributed or if the investor's utility function is quadratic. A normal distribution is characterized by having a skewness equal to zero and kurtosis equal to three. In the case of historical returns data, a normality test is recommended. In this article, we used the Jarque-Bera test (Jarque & Bera, 1980) where a result of zero indicates a standard normal distribution. In practice, however, it is common to define a maximum tolerance limit within a certain degree of confidence. For a confidence level of 95%, the maximum limit to accept normality would be 5.99, which is the 95th percentile of a chi-square distribution with two degrees of freedom.

Table 1 shows the statistics of the daily returns of the six assets that will be analyzed. The non-normality condition of each of them can be determined by observing their skewness, kurtosis and Jarque-Bera normality test values. We highlight the case of the Ripple crypto-asset, which presents a skewness of 3.83, kurtosis of 31.43 and Jarque-Bera test value of 22,504, which are significantly distant from the reference values of a standard normal distribution. Therefore, from the statistics shown, we verify that the traditional approach is not appropriate to assess portfolios with crypto-assets, including the traditional indexes S&P500 and NASDAQ. Thus, there is a need for a portfolio optimization method that overcomes the restriction of the normality of returns.

It is standard in the literature to define portfolio performance as the ratio of expected gain per unit of risk. Several metrics can be used to measure risk, such as the standard deviation of a series of returns, or the expected minimum, average or maximum loss for a certain level of confidence, the last being known as Value at Risk (JPMorgan, 1996).

The proper definition of a risk measure can be illustrated in Figure 1, which shows the importance of considering higher order moments of the distribution.

In this example, the two distributions (A and B) have the same mean ($E[R] = 10$) and variance ($Var[R] = 152$). However, it should be noted that they have very different shapes,

Table 1. Main statistics of daily returns (05/01/2016 to 10/19/2018).

	Bitcoin BTC	S&P500	NASDAQ	Ethereum ETH	Ripple XRP
Mean	0.55%	0.05%	0.07%	0.78%	1.16%
Standard deviation	4.91%	0.69%	0.85%	7.87%	10.76%
Minimum value	−21.24%	−4.1%	−4.12%	−25.3%	−42.16%
Maximum value	25.25%	2.72%	3.26%	47.9%	111.88%
Skewness	0.43	−1.13	−0.83	1.31	3.83
Kurtosis	7.42	9.52	6.86	9.51	31.43
Jarque-Bera Test Statistic	526	1,236	458	1,277	22,504

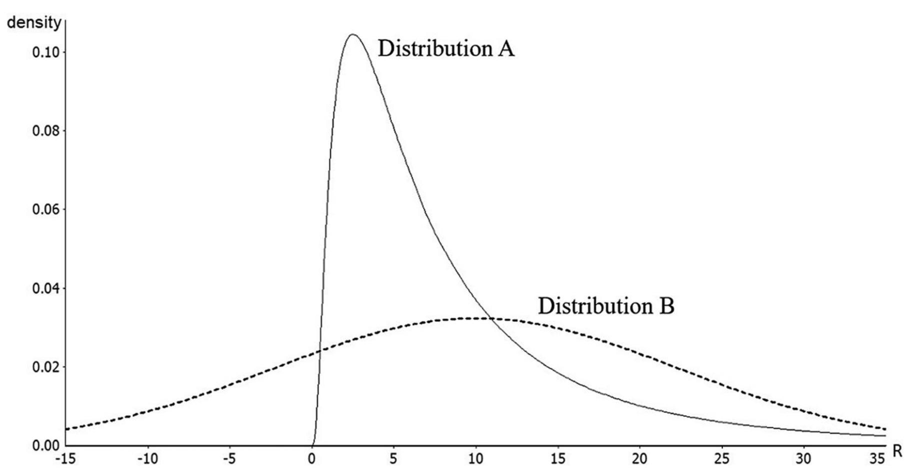

Figure 1. Distributions A and B with the same mean and variance ($E[R] = 10$; $Var[R] = 152$).

differing in skewness, kurtosis and all higher moments. However, some traditional performance indicators, such as the Sharpe ratio (Sharpe, 1966), which considers only the mean and the standard deviation of the returns, would indicate that performances of both distributions are equivalent. The Omega measure proposed by Keating and Shadwick (2002), on the other hand, is more comprehensive as it considers all moments of the distribution of return and thus is able to differentiate both these distributions.

The use of the Sharpe ratio (Sharpe, 1966) is standard in the literature and also widely used by practitioners in financial markets. It is based on the modern portfolio theory of Markowitz (1952) and indicates the points on the capital allocation line that correspond to optimal portfolios. The Sharpe ratio of a portfolio P (S_p) is defined by Equation (1).

$$S_p = \frac{E(R_p) - r_f}{\sigma_p} \tag{1}$$

where, $E(R_p)$ and σ_p represent respectively the expected return and standard deviation (volatility) of the portfolio P, and r_f is the risk-free interest rate.

Markowitz's Mean-Variance method determines the maximum expected return portfolios for a given level of risk, forming the efficient frontier. The portfolio with higher Sharpe ratio is at the efficient frontier, assuming normality in the distribution of returns. Keating and Shadwick (2002) present the universal performance measure Omega (Ω), which reflects all the statistical properties of the distribution of returns, incorporating all higher-order moments and not only the mean and the variance.

The omega (Ω) performance measure

The Omega (Ω) performance measure formulated by Keating and Shadwick (2002) defines exogenously a return level (L) that divides the probability distribution of returns into two areas: the gains area (to the right of L) , and the losses area (to the left of L). The level L is stipulated by the investor according to his preferences, being the minimum value that he considers as gain.

In its original form, the Omega measure (or Omega function) is defined as shown in Equation (2)

$$\Omega(L) = \frac{I_2(L)}{I_1(L)} = \frac{\int_L^b [1 - F(x)]dx}{\int_a^L F(x)dx} \tag{2}$$

where:

- $F(x)$ = Cumulative distribution function (CDF) of returns X with values x,
- (a,b) = The lower and upper limits, respectively, of the range of returns. In the more general case, a = -∞ and b=∞.
- L = Loss threshold or minimum return level of gains.
- $I_2(L)$ = Weighted average of gains above the return level L.
- $I_1(L)$ = Weighted average of losses below the return level L.

Areas $I_1(L)$ and $I_2(L)$ are illustrated in Figure 2.

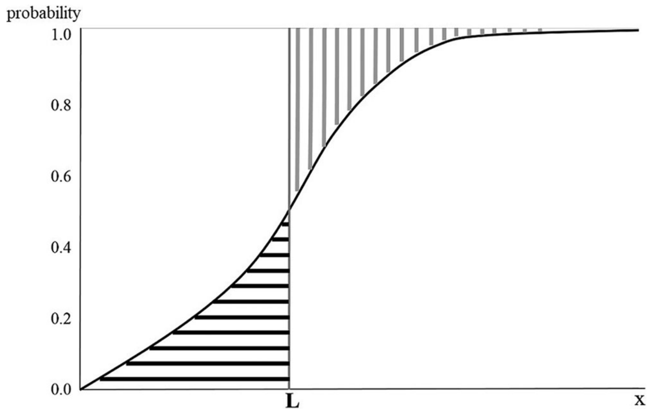

Figure 2. CDF of returns X and areas of gains $I_2(L)$ (upper) and losses $I_1(L)$ (lower).

Equation (2) can be represented in a more intuitive way. Kazemi et al. (2004) show that the Omega function $\Omega(L)$ can be written as the ratio of two expected values, or $\Omega(L) = \mathrm{EG}(L)/\mathrm{EL}(L)$, where EG($L$) (Expected Gain) is the expected value of excess gain (X–L) conditional to favorable results and EL(L) (Expected Loss) is the expected value of loss difference (L–X) conditional to unfavorable results. These definitions are illustrated in Figure 3, using the probability density function *f(x)* of returns *X*.

Figure 3 illustrates the basic idea of this measure, which can be stated as: "What is the Expected Gain compared to the Expected Loss?" The risk preferences are defined by the threshold *L*, so, a higher *L* denotes a higher appetite for risk. Regardless of the return level *L*, a higher Omega value is preferable.

Kazemi et al. (2004) detail how to get the alternative representation of Omega measure (Ω), as Equation (3):

$$\Omega(L) = \frac{\int_L^b [1 - F(x)]dx}{\int_a^L F(x)dx} = \frac{\int_L^b (X - L)f(x)dx}{\int_a^L (L - X)f(x)dx} = \frac{E[\max(X - L; 0)]}{E[\max(L - X; 0)]} = \frac{EG(L)}{EL(L)} \tag{3}$$

where X is the random variable that represents the asset return.

Equation (3) can also be expressed as Equation (4) or Equation (5):

$$\Omega(L) = \frac{E(X) - L}{EL(L)} + 1 \tag{4}$$

$$\Omega(L) = \left(1 - \frac{E(X) - L}{EG(L)}\right)^{-1} \tag{5}$$

where E(X) is the expected return of the return distribution of X. Thus, to compute the Omega measure, it is sufficient to compute only EG(L) or EL(L), since E (X) and L are constants. Equations (4) and (5) follow directly from

$$EG(L) - EL(L) = \int_L^{\infty} (x - L)f(x)dx - \int_{-\infty}^{L} (L - x)f(x)dx = \int_{-\infty}^{\infty} xf(x)dx - L\int_{-\infty}^{\infty} f(x)dx = E(X) - L$$

The risk measure used in Omega, the Expected Loss (EL(L)), indicates the average loss to the left of a limit, which in this case is the required minimum return level, L, of

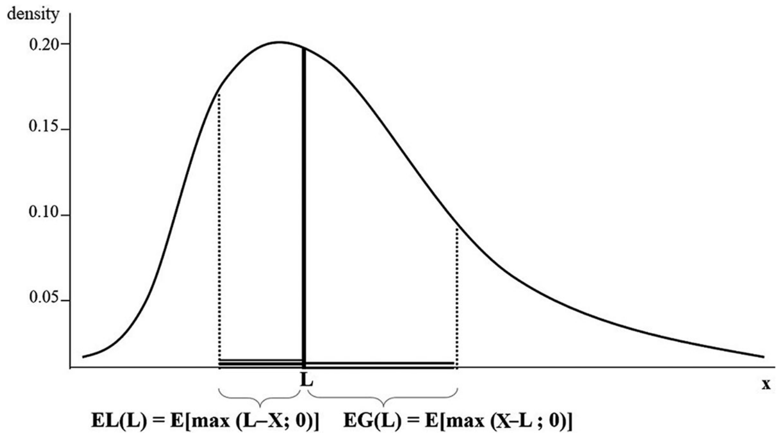

Figure 3. Numerator (EG(L)) and denominator (EL(L)) of Omega measure.

a return distribution, providing information about the tail of the distribution. Since the Omega measure relates the Expected Gain (EG) to the Expected Loss (EL), all the higher-order moments of the distribution are considered. This makes it more robust than conventional performance measures, since it considers the real shape of the return distribution, instead of simplifying it to its first two moments (mean and variance), as in the traditional method of Markowitz (1952).

Portfolio optimization with the omega measure

In order to determine the portfolio weights of n assets, we maximize the objective function shown in Equation (6) based on Omega performance measure (Ω).

$$max\Omega(L) = \frac{EG_p(L)}{EL_p(L)} = \frac{E[\max(R_p - L; 0)]}{E[\max(L - R_p; 0)]} \tag{6}$$

subject to $$\sum\nolimits_{j=1}^{n} w_j = 1; 0 \leq w_j \leq 1$$

where $EG_p(L)$ is the Expected Gain and $EL_p(L)$ is the Excess Loss of a portfolio P and R_p is the portfolio return random variable which is formed by the sum of the returns over time for each of the n assets considered in the portfolio P multiplied by their respective weights (w_j). Thus, a given portfolio return in time $t = i$ would be $R_{p_{t=i}} = \sum_{j=1}^{n} w_j R_{ij}$ where R_{ij} is the return of the asset j in time $t = i$. The sum of the weights of each asset in the portfolio w_j must be equal to 1. In this way, the optimization program finds the w_j weights of each asset that maximize the Omega (Ω) performance measure of the entire portfolio.

In addition to determining the w_j weights of the portfolio assets, in order to properly evaluate the risk, the optimization program calculates the Expected Loss that maximizes the Omega (Ω) performance measurement. This is important for capital allocation decision purposes.

Sharpe-omega capital allocation model

According to Sharpe (1966), based on the classical mean-variance theory of Markowitz (1952), both the risk-free asset and the risky portfolio can be represented by two points in the plane (μ, σ), where μ represents the mean and σ is the standard deviation. The line that connects the risk-free asset with the risky portfolio touches the point that makes it tangent to the efficient frontier. This is known as the Capital Allocation Line (CAL), and the angular coefficient of this line is known as the Sharpe ratio and is presented in Equation (1).

From a financial perspective, the Sharpe ratio measures the relationship between the excess return to the risk-free asset $(E(R_p) - r_f)$ and the volatility σ_p, that is, the Sharpe ratio measures the profitability per unit risk of the asset. Drawing a parallel with this ratio, Kazemi et al. (2004) propose an alternative performance measure, which they called Sharpe-Omega. First, from Equation (3), these authors arrive at the relation shown in Equation (7):

$$\Omega(L) = \frac{EG(L)}{EL(L)} = \frac{E[\max(X-L;0)]}{E[\max(L-X;0)]} = \frac{e^{-r_f}E[\max(X-L;0)]}{e^{-r_f}E[\max(L-X;0)]} = \frac{C(L)}{P(L)} \tag{7}$$

Through Equation (7) they show that $EG(L)$ and $EL(L)$ essentially represent a nominal value of a European *call* $(C(L))$ and *put* $(P(L))$ with a strike of L. When calculating the present values of these two values, multiplying both sides by e^{-r_f} where r_f is the risk-free rate, we get $C(L)$ and $P(L)$ which are the prices of European call and put options. Then:

$$P(L) = e^{-r_f}EL(L) \tag{8}$$

Thus, by relating Equations (4) and (8), the ratio $[E(X)\text{-}L]/P(L)$ found is what Kazemi et al. (2004) call Sharpe-Omega, which is shown in Equation (9):

$$\Omega(L){-}1 = \frac{E(X){-}L}{EL(L)} = \frac{e^{-r_f}\left(E(X)-L\right)}{e^{-r_f}E[\max(L-X;0)]} = \frac{e^{-r_f}\left(E(X)-L\right)}{P(L)}$$

$$\text{Sharpe}{-}\text{Omega} = \frac{\Omega(L){-}1}{e^{-r_f}} = \frac{E(X){-}L}{P(L)} \tag{9}$$

It can be seen that Sharpe-Omega is proportional to $\Omega(L)$-1 and, therefore, both measures evaluate the portfolios in the same way. But according to Kazemi et al. (2004), the Sharpe-Omega ratio may represent a more intuitive optimization measure than Omega. The denominator or put price can be seen as the cost of protecting the investment when returns are below the target (L). To ensure an excess return beyond L, one should pay a certain price (put) as if it were insurance. A higher put value would indicate that the portfolio is riskier. Thus, the Sharpe-Omega measures the relation of the excess return over the limit L and the risk in terms of a put option. Since P(L) is essentially EL(L) discounted, this risk measure considers the true shape of the distribution of returns below the limit L. Some interpretations of the Sharpe-Omega ratio are shown for the following cases:

- If E(X) is equal to the limit L, Sharpe-Omega is zero, that is, on average we expect to reach the minimum desired value.
- When $E(X) < L$, the Sharpe-Omega ratio will be negative. In this case of the higher put price, the investment improves. If the distribution of x has higher volatility, the value of the put becomes larger and the value of Sharpe-Omega

will also increase by becoming less negative. This is similar to the case where Omega is less than 1. In practice, this is an unrealistic scenario, since the limit (L) is usually defined below the mean $E(X)$.

- When $E(X)>L$, the Sharpe-Omega measure will be positive. Unlike the previous case, greater volatility will increase the selling price and, consequently, reduce the Sharpe-Omega ratio. This is similar to the case of Omega being greater than 1.

Similarly to the Sharpe ratio, the Sharpe-Omega ratio can also be used for capital allocation purposes. When considering $r_f = L$, that is, that the investor accepts having at least a gain equal to the risk-free asset, the Sharpe-Omega ratio presents a linear behavior by relating the surplus earnings ($E(X)$-L) with the risk measure ($P(L)$). The Sharpe ratio is a tangent between the surplus of gains (relative to the risk-free asset) and its risk measure (standard deviation). Thus, both indexes are linear and do not change along their capital allocation lines, which is relevant from the point of view of capital allocation. Since this ratio remains constant along the line, this allows investors to define their risk preferences as is done when using Sharpe's Capital Allocation Line. In the case of Sharpe-Omega, a different risk measure is used – a put, rather than the standard deviation.

Investor's optimal capital allocation model

Tobin's (1958) separation theorem states that optimal capital allocation occurs when investors allocate capital into a general portfolio (with investment return R_g consisting of a risk-free asset (with investment return r_f) and a portfolio composed of risky assets (with investment return R_p). The exact proportions of each depend on the risk preference of the investor. Thus, in order to determine the optimal investment decision, it is necessary to define a utility function that that reflects the investor's risk and return preferences.

A utility function that is widely used in the literature is the quadratic utility function (Bodie, Kane, & Marcus, 2013). In our model we consider this utility function simply to establish a criterion to define the proportion that the risk-free asset would have in the optimal general portfolio. The tangent point of the capital allocation line with the quadratic utility function will determine this proportion. It should be noted that it is possible to consider other utility functions that may be more appropriate to the profile of each investor, or simply to choose by some other criteria the proportion of risk-free assets in the portfolio - the Sharpe-Omega ratio would be the same in all cases.

The quadratic utility function is defined as a second-degree polynomial as shown in Equation (10):

$$U = E(R_g) - \frac{1}{2}\gamma\sigma_g^2 \tag{10}$$

where U is the utility value, γ is the investor's risk aversion coefficient, R_g and σ_g^2 are respectively the return and variance of the general investment portfolio. According to Equation (10), the coefficient γ penalizes the risk in such a way that higher values of γ

mean that the investor is risk averse, and vice versa. Consequently, the choice of larger values of γ results in the selection of conservative portfolios and smaller values of γ result in the selection of riskier portfolios.

The most common risk measure is the standard deviation, but in this work, we adopt the put value of the general portfolio $P_g(L)$ as the risk measure. Thus, in the utility function presented, one must change the standard deviation by $P_g(L)$, resulting in:

$$U = E(R_g) - \frac{1}{2}\gamma[P_g(L)]^2 \tag{11}$$

Now suppose that y is the ratio allocated to the risky portfolio which has a return R_p (for example, in a portfolio with two risky assets $R_p = w_1R_1 + w_2R_2$. Therefore, the remaining ratio, (1-y), should be allocated to the risk-free asset (with a fixed return r_f). The return of the general portfolio can be calculated by Equation (12):

$$R_g = yR_p + (1-y)r_f \tag{12}$$

Assuming that the risky portfolio follows a given distribution with mean μ_p, this is defined as shown in Equation (13):

$$\mu_p = E(R_p) = \sum_{j=1}^{n} E(R_j)w_j \tag{13}$$

where w_j are the asset weights determined using the optimization by Omega (Equation (6)). Then the return of the general portfolio μ_g will be: (Equation (14))

$$\mu_g = y\mu_p + (1-y)r_f \tag{14}$$

In relation to the $P(L)$ of the general portfolio $P_g(L)$, this is simply a linear proportion of the $P(L)$ of the risk portfolio $P_p(L)$. According to Equation (8):

$$P_p(L) = e^{-r_f}EL_p(L) = e^{-r_f}E[max(L - R_p; 0)] \tag{15}$$

and,

$$P_g(L) = yP_p(L) \tag{16}$$

Substituting Equations (14) and (16) into Equation (11) and maximizing the utility function as a function of y, one has:

$$\max_y\{U\} = \frac{\partial U}{\partial y} = \frac{\partial}{\partial y}\left(\mu_g - \frac{1}{2}\gamma[P_g(L)]^2\right) = \mu_p - r_f - \gamma y[P_p(L)]^2 = 0 \tag{17}$$

Therefore, the optimal proportion of investment in the risky portfolio is:

$$y^* = \frac{1}{\gamma}\frac{\mu_p - r_f}{[P_p(L)]^2} = \frac{\text{Sharpe}-\text{Omega}}{\gamma P_p(L)} \tag{18}$$

The remainder of the investment $(1-y^*)$ will be allocated to the risk-free asset r_f.

The weights w_j determined through optimization using Omega (Equation (6)) refer only to the weights of the risky assets. By including in the general portfolio the risk-free asset r_f with weight equal to (1-y*), then the weights w_j must be adjusted in proportion to the risky portfolio's weight y*. Therefore $w_j^* = y^*w_j$. At the end, $\sum_{j=1}^{n} w_j^* + (1 - y^*) = 1$, where n is the number of risky assets in the portfolio.

Numerical application

The following is a numerical application of the proposed portfolio optimization model. The three major crypto-currencies in market value were selected for this analysis. Considering a price history from May 1, 2016, to October 19, 2018, these were: Bitcoin (BTC), Ethereum (ETH) and Ripple (XRP). As of October 19, 2018, these crypto-assets together represented approximately 70% of the total value of the crypto-currencies market. Data were obtained from Coinmarketcap (2018), which considers closing price as the latest price in the day at UTC time. Since the negotiations are uninterrupted, UTC time was taken as reference.Also included are two major stock indexes in the US market: S&P500 and NASDAQ Composite. The historical series of these indexes was obtained from Yahoo Finance (2018) for the same time span as the crypto-coins. The main statistics of these assets are presented in Table 1. For the calculation of these statistics, the data were synchronized based on the trading days of the S&P500 index so that the number of days was the same for all assets. Thus, in the case of holidays or weekends, the crypto-assets prices for those days were excluded.

The daily arithmetic return considering 252 trading days in the year was calculated with the closing prices of trading days (close-to-close). In the case of crypto-assets, this same criterion was also used, although there were negotiations on weekends and holidays. In fact, calculating the return that way for a trading weekend, would be a 3 trading-days return. One could consider turning it into a daily return (equivalent daily return) or calculating the return with closing prices on Monday and Sunday. Our criterion was to preserve the same closing days of the S&P500 and not make adjustments.

Four portfolios are analyzed: (Bitcoin × S&P500), (Bitcoin × NASDAQ Composite), (Bitcoin × Ethereum) and (Bitcoin × Ripple). In all cases, a risk-free asset, which in this article was approximated by the risk-free rate of the US market, was also included.

In the cases presented, the risky optimal portfolio is first determined using the optimization program with the Omega measure (Equation (6)). We obtain the weights w_1 and w_2 for each asset, and the put value for the risky portfolio (Equation (15)). Then the Sharpe-Omega ratio is calculated using Equation (9). As verified in the Appendix, this ratio is linear.

Next, the investor's general portfolio (Equations (14) (16) and (18)) is determined considering the utility function (Equation (11)) and the Sharpe-Omega equation line (Equation (9)). For all the cases analyzed the investor's general portfolio was calculated using an aversion coefficient equal to $\gamma = 200$ (Equation (11)) only as an example, but larger values would result in a more conservative portfolio, and lower values would result in riskier portfolios. The γ coefficient represents the degree of risk aversion. In relation to the risk-free rate, the reference rate of the Federal Reserve was used, which at the date of the present analysis was 1.75% p.a. (0.007% per trading day). All analyses are performed considering daily returns (in trading days).

Bitcoin × S&P500

Figure 4 shows the portfolio formed between Bitcoin and the S&P500 index showing the risks (put) and expected returns (μ), Figure 5 shows the shape of returns distribution of the general portfolio where the non-normality of the returns can be seen clearly, and Figure 6 represents the capital allocation of the portfolios.

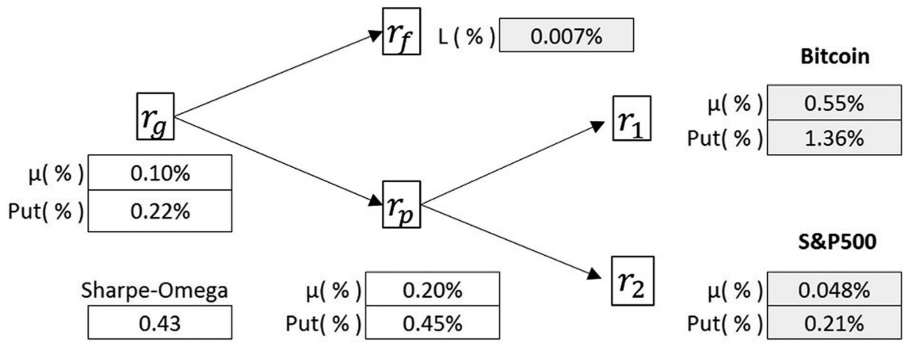

Figure 4. Portfolio BTC × S&P500.

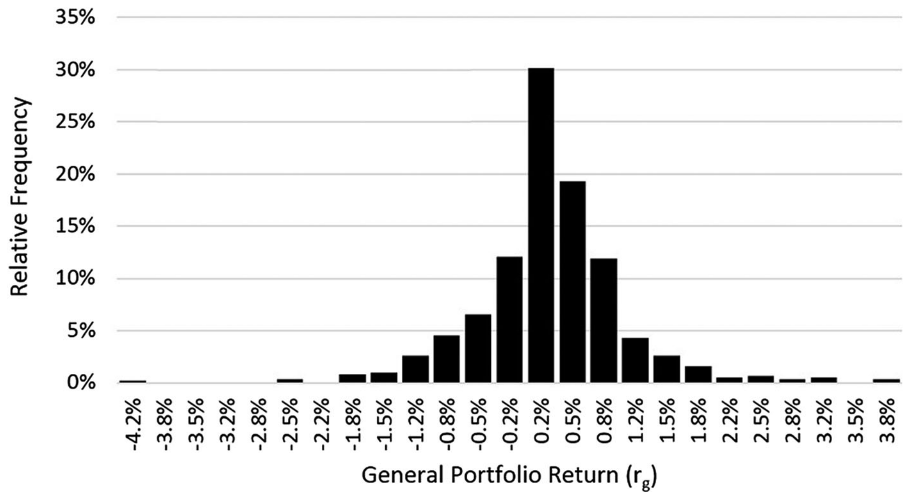

Figure 5. Distribution of the returns (mean = 0.10%) of the general portfolio (BTC, S&P500 and r_f).

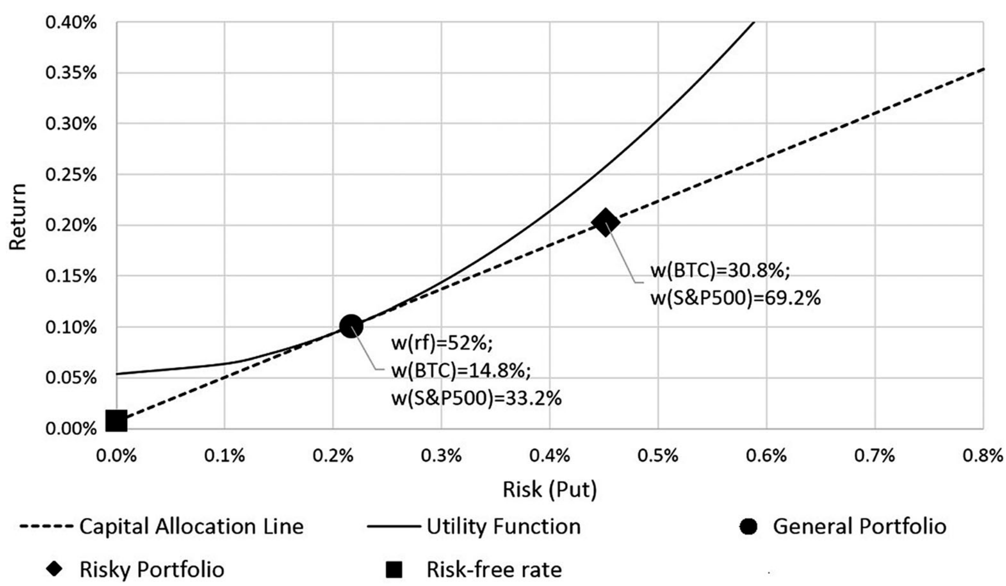

Figure 6. Portfolio Analysis BTC × S&P500.

As shown in Figure 6, the investor's general portfolio resulted in the following proportions: $w_{rf} = 52\%$, $w_{BTC} = 14.8\%$ and $w_{S\&P500} = 33.2\%$, with $\mu = 0.10\%$ and put $=$ 0.22%. In addition, as can be seen in Figure 5, the returns distribution of this portfolio in the period analyzed has long tails. This means there is the possibility of having very high and also very low returns in relation to the mean. This is a common feature that occurs in portfolios that have one or more crypto-assets, as we will see.

On the other hand, as explained in the previous section, the risk measure used is the put and not the standard deviation. The result of 0.22% can be interpreted as the cost (in percentage terms) of protecting the portfolio in one day when the expected return (μ) is below the strike price, or minimum required return (L). The higher that cost, the riskier the portfolio.

We note that the general portfolio was calculated using a utility function (Equation (11)) with a coefficient of aversion (γ) equal to 200, which indicates a certain degree of conservatism, since the general portfolio ($\mu = 0.10\%$, put $= 0.22\%$) lies to the left of the risky optimal portfolio ($\mu = 0.20\%$; put $= 0.45\%$), closer to the risk-free rate, with a Sharpe-Omega ratio of 0.43. The risky portfolio resulted in a weight of 30.8% for Bitcoin and 69.2% for the S&P500, a combination that provides the best Sharpe-Omega ratio. This result is apparently counter-intuitive from the point of view of the return but is consistent due to the high risk embedded in the crypto-currency.

Bitcoin × NASDAQ composite

Figure 7 shows the portfolio formed between Bitcoin and the NASDAQ Composite index showing the risks (put) and expected returns (μ), Figure 8 shows the shape of returns distribution of the general portfolio (showing non-normality), and Figure 9 represents the capital allocation of the portfolios.

In this case, the results obtained from the portfolio are similar to those found using the S&P500, mainly due to the fact that the NASDAQ Composite statistics (see Table 1) are similar to those of the S&P500. Figure 9 shows that the general portfolio ($\mu =$ 0.11%, put $= 0.23\%$) resulted in the following proportions: $w_{rf} = 50.4\%$, $w_{BTC} = 14.9\%$ and $w_{Nasdaq} = 34.7\%$, and the optimal portfolio ($\mu = 0.22\%$, put $= 0.46\%$) with a

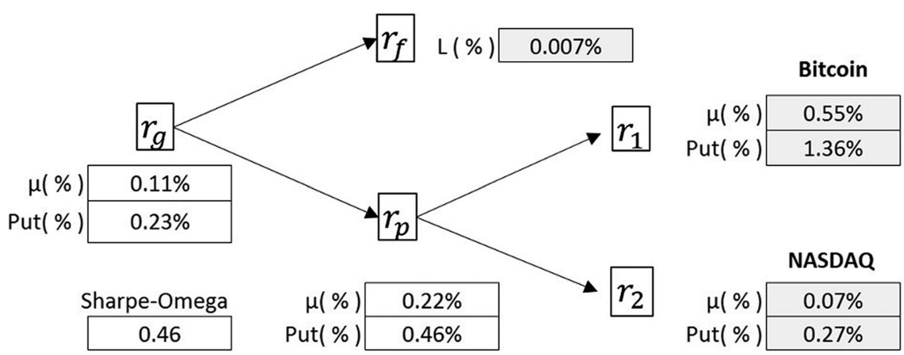

Figure 7. Portfolio BTC × NASDAQ Composite.

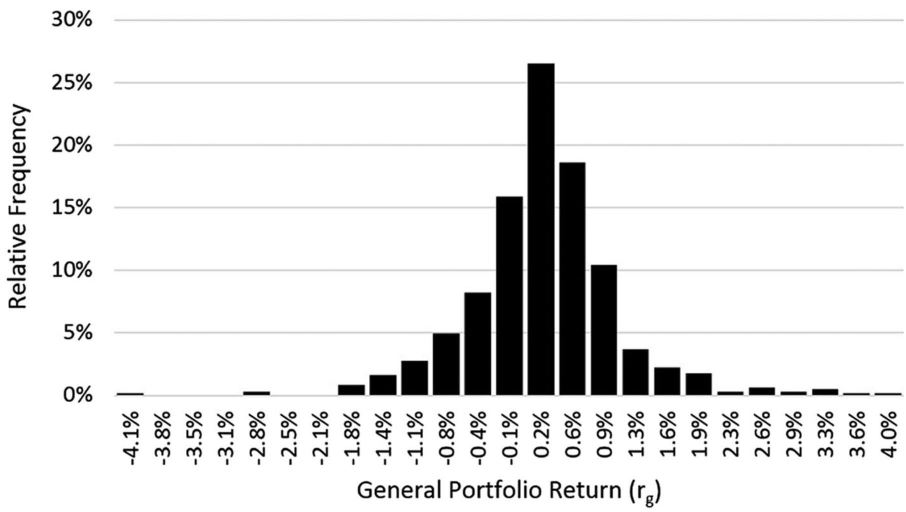

Figure 8. Distribution of the returns (mean = 0.11%) of the general portfolio (BTC, NASDAQ and r_f).

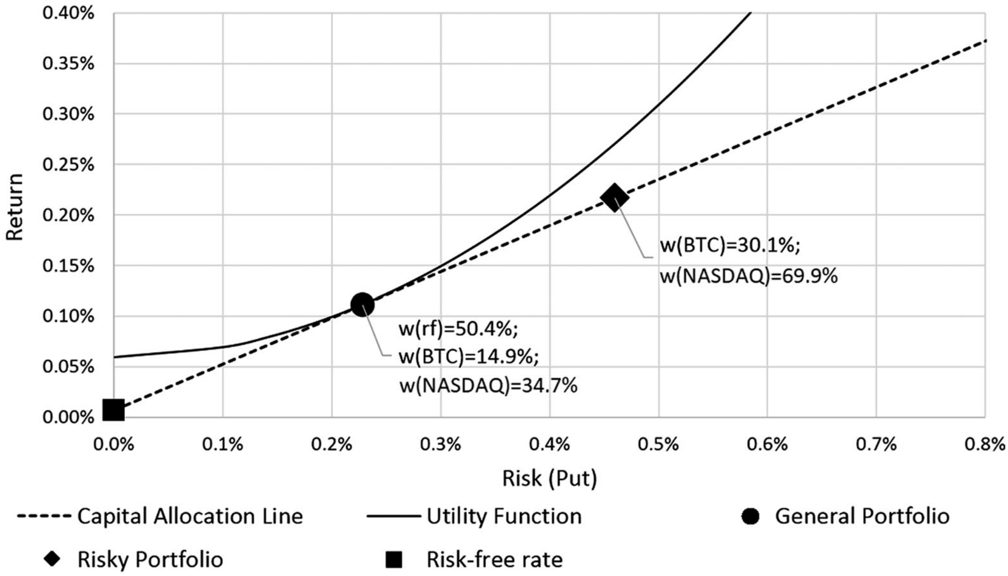

Figure 9. Portfolio Analysis BTC × NASDAQ Composite.

Sharpe-Omega ratio equal to 0.46, resulted in a weight of 30.1% for Bitcoin and 69.9% for NASDAQ.

Bitcoin × ethereum

Figure 10 shows the portfolio formed between Bitcoin and Ethereum showing the risks (put) and expected returns (μ), Figure 11 shows the shape of returns distribution of the general portfolio (showing non-normality) and Figure 12 represents the capital allocation of the portfolios.

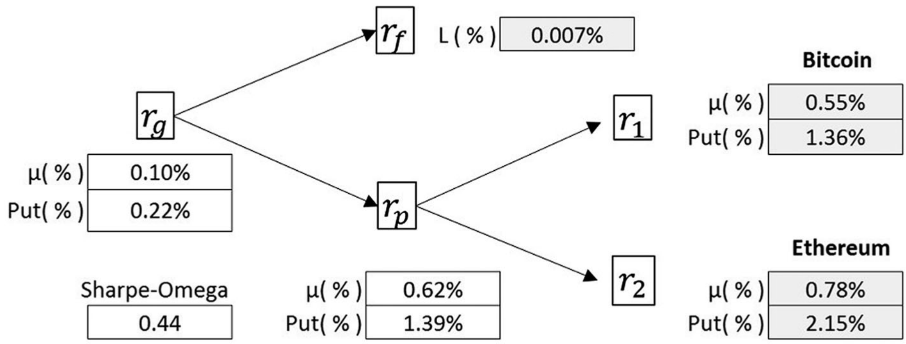

Figure 10. Portfolio BTC × ETH.

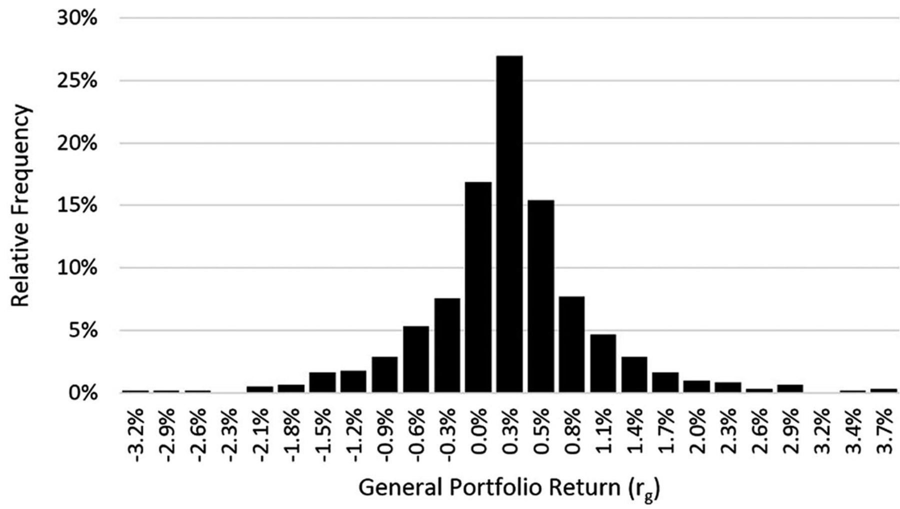

Figure 11. Distribution of the returns (mean = 0.10%) of the general portfolio (BTC, ETH and r_f).

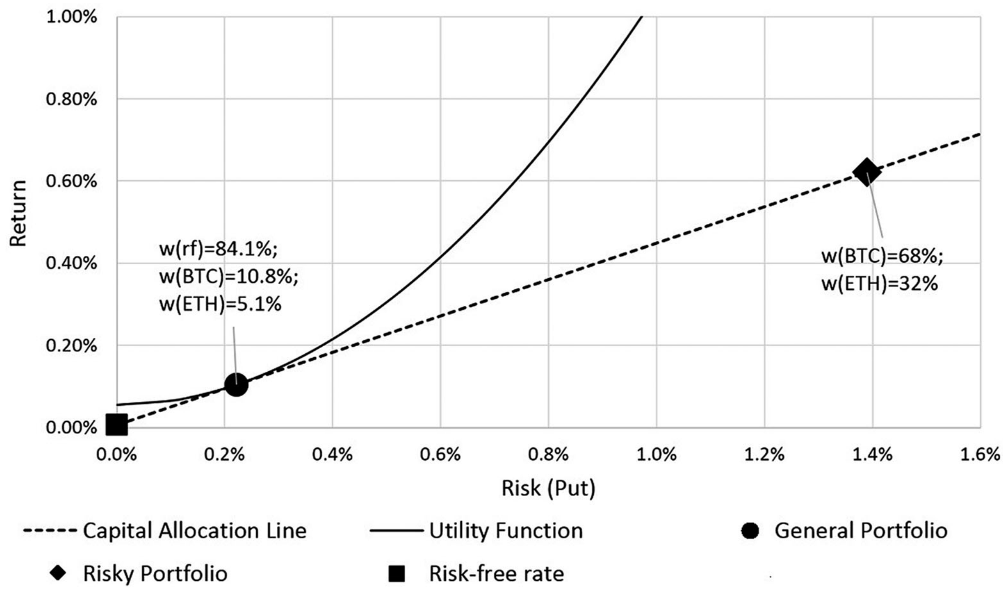

Figure 12. Portfolio Analysis BTC × ETH.

The general investor portfolio (μ = 0.10%, put = 0.22%) shown in Figure 12, resulted in the following ratios: w_{rf} = 84.1%, w_{BTC} = 10.8% and w_{ETH} = 5.1%. Also, the risky optimal portfolio (μ = 0.62%, put = 1.39%) resulted in a weight of 68% for Bitcoin and 32% for Ethereum, providing the best Sharpe-Omega ratio (0.44).

Bitcoin × ripple

Figure 13 shows the portfolio formed between Bitcoin and Ripple showing the risks (put) and expected returns (μ), Figure 14 shows the shape of returns distribution of the general portfolio (showing non-normality) and Figure 15 represents the capital allocation of the portfolios.

Figure 15 shows that the general portfolio (μ = 0.17%, put = 0.28%) has the following weights: w_{rf} = 82.6%, w_{BTC} = 6.8% and w_{XRP} = 10.6%. The optimal risk portfolio (μ = 0.92%, put = 1.62%), resulted in a weight of 39.2% for Bitcoin and 60.8% for Ripple, thus giving the best Sharpe-Omega ratio (0.56) for the portfolio. In this case,

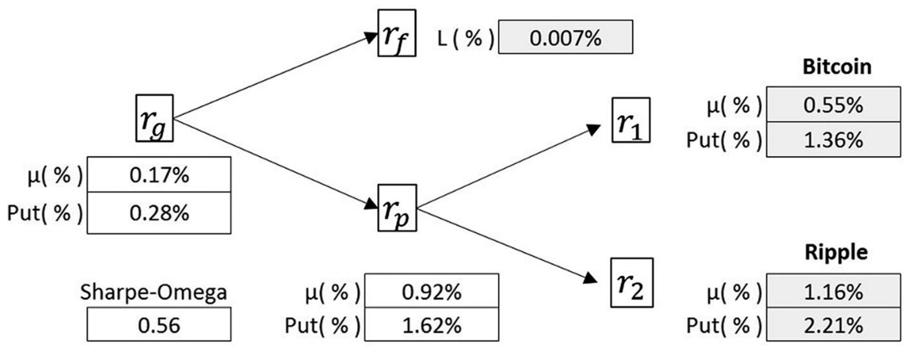

Figure 13. Portfolio BTC × XRP.

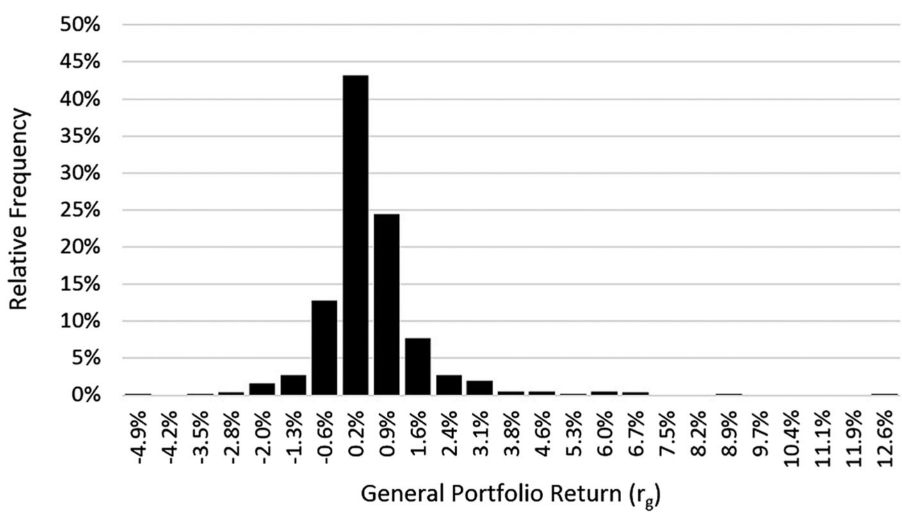

Figure 14. Distribution of the returns (mean = 0.17%) of the general portfolio (BTC, XRP and r_f).

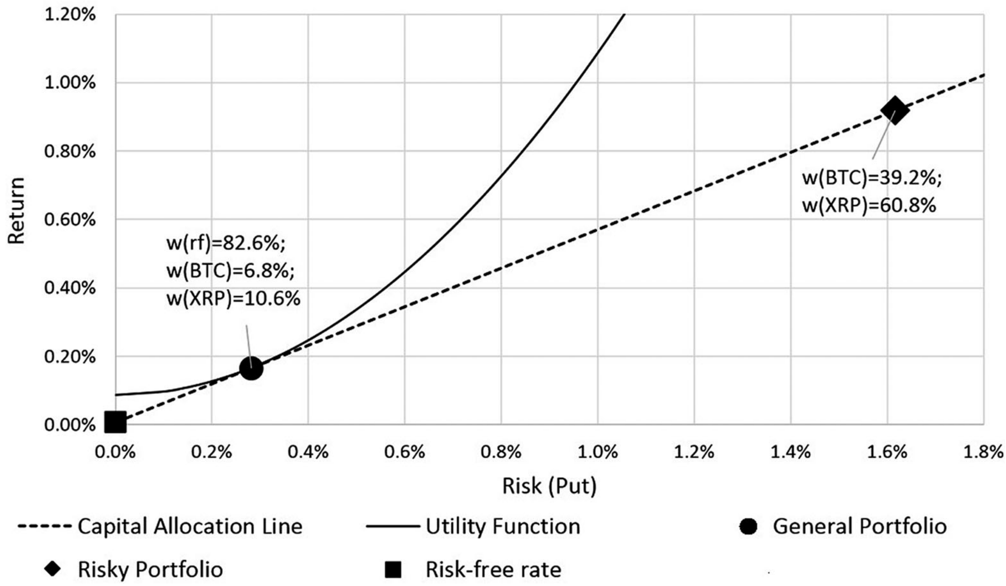

Figure 15. Portfolio Analysis BTC × XRP.

the optimization using the Omega measure gave greater weight to Ripple, different from the portfolio with Ethereum in which Bitcoin was privileged.

Conclusion

Crypto-currencies were initially developed to be used as a peer-to-peer payment method without the need for a central authority for control and issuing. However, crypto-currencies can also be used as a storage of value and therefore represent a new category of an investment asset. On the other hand, Markowitz's traditional portfolio analysis model may not be appropriate for the evaluation of portfolios containing crypto-currencies.

In this article, we propose a model for the evaluation of crypto-currencies portfolios, which is more comprehensive than the Markowitz model, as it incorporates higher-order moments of the distribution. For that, a method based on the Omega measure (Ω) with these characteristics was developed. Unlike the traditional model where risk is defined as the standard deviation of portfolio returns distribution, in the proposed model the risk is defined by a put option, which is the discounted Expected Loss ($EL(L)$).

In relation to the Sharpe-Omega capital allocation line used, we show that the proposed model has a linear behavior similar to that of the capital allocation line of the Shape ratio. In both cases, the investor may choose a point at which he or she combines risky assets and a proportion of his investment in fixed income. The difference between both ratios is the risk measure used.

For purposes of illustration and numerical application of the model, four investment portfolios were evaluated: Bitcoin × S&P500, Bitcoin × NASDAQ, Bitcoin × Ethereum and Bitcoin × Ripple. The results of the analysis indicate that the incorporation of crypto-assets improves the return of the portfolios, but on the other hand, also increase the risk exposure.

In the case of Bitcoin × S&P500 and Bitcoin × NASDAQ Composite portfolios, the results showed that a larger proportion should be invested in the respective indexes and a smaller proportion in Bitcoin, even though the latter had a much higher expected return than the traditional indexes. In the case of the Bitcoin × Ethereum and Bitcoin × Ripple portfolios, the results suggest that there is no clear-cut decision for or against Bitcoin and that the optimal weights of the crypto-assets are the ones that provide the best Sharpe-Omega ratio.

Due to the fact that the price history of crypto-currency is relatively short, this analysis was restricted to the last 18 months, where the transactions in this market were more relevant and representative in relation to the transacted volume when compared to the first years of the emergence of crypto-currencies. In the year 2017, in particular, prices showed significant growth, which is not guaranteed to be repeated in the coming years. Thus, a suggestion for future work is the replication of this study with a series of long-term data to verify if the results obtained here are confirmed.

Note

1. We thank the Editor in Chief for suggesting this proof

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We thank Felipe Van de Sande Araújo for helpful suggestions in the literature review on crypto-currencies.

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